Modified finite volume nodal for hyperbolic equations with external forces on unstructured meshes

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Outline

Mathematical context

Euler equations with friction and gravity

- Euler equations with gravity and friction:

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho \mathbf{u} + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}\nabla p = \frac{1}{\varepsilon}(\rho \mathbf{g} - \frac{\sigma}{\varepsilon}\rho \mathbf{u}), \\ \partial_t \rho e + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} e) + \operatorname{div}(p\mathbf{u}) = \frac{1}{\varepsilon}(\rho(\mathbf{g}, \mathbf{u}) - \frac{\sigma}{\varepsilon}\rho(\mathbf{u}, \mathbf{u})). \end{cases}$$

Properties :

- Entropy inequality: $\partial_t \rho S + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} S) \geq 0$.
- Steady states :

$$\begin{cases} \mathbf{u} = 0, \\ \nabla p = \rho \mathbf{g}. \end{cases}$$

- Diffusion limit:

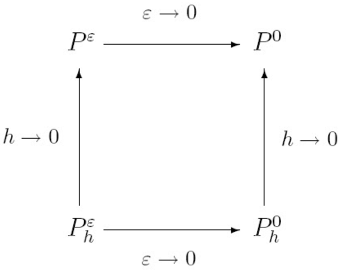
$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho e + \operatorname{div}(\rho \mathbf{u} e) + p \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u} = \frac{1}{\sigma}\left(\mathbf{g} - \frac{1}{\rho}\nabla p\right). \end{cases}$$

Ap scheme

- P_1 model:

$$\left\{\begin{array}{l}\partial_t p + \frac{1}{\varepsilon}\partial_x u = 0,\\ \partial_t u + \frac{1}{\varepsilon}\partial_x p = -\frac{\sigma}{\varepsilon^2} u,\end{array}\right. \qquad \longrightarrow \partial_t p - \partial_x\left(\frac{1}{\sigma}\partial_x p\right) = 0.$$

Ap scheme



- Consistency **Godunov-type** schemes: $O(\frac{\Delta x}{\varepsilon} + \Delta t)$.
- CFL condition: $\Delta t(\frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2}) \leq 1$.
- Consistency AP schemes: $O\left(\Delta x + \Delta t\right)$.
- CFL condition: $\Delta t(\frac{1}{\Delta x \varepsilon + \frac{\Delta x^2}{\sigma}}) \leq 1$.
- AP vs non AP schemes: Important reduction of CPU cost.

- Classical extension (1D fluxes in the normal direction) of AP schemes in 2D are not convergent on general meshes $\forall \varepsilon$ (limit diffusion scheme non convergent).

Well Balanced schemes

- Discretization of physical steady states is important (Lack at rest for Shallow water equations, hydrostatic equilibrium for astrophysical flows ..)
- **Classical scheme**: the physical steady states or a good discretization of the steady states are not the equilibrium of the schemes.
- **Consequence**: Spurious numerical velocities larger than physical velocities for nearly or exact uniform flows.

WB scheme: definitions

- **Exact Well-Balanced scheme**: scheme exact for continuous steady states.
- **Well-Balanced scheme**: scheme exact for discrete steady states at the interfaces.

- **For shallow water model**: in general the schemes are exact WB schemes.
- **For Euler model**: in general the schemes are WB schemes.

Linear case

Nodal scheme : principle for linear case

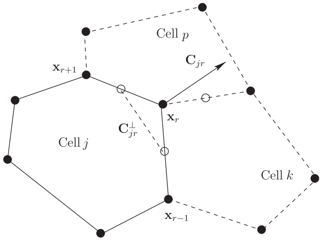
- Linear case : P_1:

$$\left\{\begin{array}{l}\partial_t p + \frac{1}{\varepsilon}\operatorname{div}(\mathbf{u}) = 0,\\[2ex] \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p = -\frac{\sigma}{\varepsilon^2}\mathbf{u}.\end{array}\right. \longrightarrow \partial_t p - \operatorname{div}\left(\frac{1}{\sigma}\nabla p\right) = 0.$$

Idea: nodal Finite Volume method for the P_1 model + AP method.

Nodal scheme: fluxes at the node and not at the middle of the edge (Bruno talk). Introduced for Lagrangian scheme.

Notations



- Geometrical quantities defined by $\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r}|\Omega_j|$.
- $\sum_j \mathbf{C}_{jr} = \sum_r \mathbf{C}_{jr} = \mathbf{0}$.

2D AP schemes

Nodal AP schemes:

$$\begin{cases} \mid \Omega_j \mid \partial_t p_j(t) + \frac{1}{\varepsilon} \sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\ \mid \Omega_j \mid \partial_t \mathbf{u}_j(t) + \frac{1}{\varepsilon} \sum_r \mathbf{p}\mathbf{C}_{jr} = \mathbf{S}_j. \end{cases}$$

- Classical nodal fluxes:

$$\begin{cases} \mathbf{p}\mathbf{C}_{jr} - p_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r), \\ \sum_j \mathbf{p}\mathbf{C}_{jr} = \mathbf{0}, \end{cases}$$

  with $\widehat{\alpha}_{jr} = \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}$.

- Modified fluxes obtained plugging the balance equation $\nabla p = -\frac{\sigma}{\varepsilon}\mathbf{u}$:

$$\begin{cases} \mathbf{p}\mathbf{C}_{jr} - p_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \frac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\mathbf{u}_r, \\ \left(\sum_j \widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_j \widehat{\beta}_{jr}\right) \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \sum_j \widehat{\alpha}_{jr} \mathbf{p}_j. \end{cases}$$

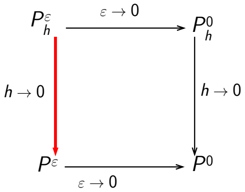
  with $\widehat{\beta}_{jr} = \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$.

- Source term: $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2} \sum_r \widehat{\beta}_{jr}\mathbf{u}_r, \quad \sum_r \widehat{\beta}_{jr} = \hat{I}_d |\Omega_j|$.

Uniform convergence in space: idea of proof

- Naive convergence estimate : $||P_h^\varepsilon - P^\varepsilon||_{naive} \leq C\varepsilon^{-b}h^c$.
- **Idea**: intermediary estimates and triangle inequalities (Jin-Levermore-Golse).

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(||P_h^\varepsilon - P^\varepsilon||_{naive}, ||P_h^\varepsilon - P_h^0|| + ||P_h^0 - P^0|| + ||P^\varepsilon - P^0||)$$



- Intermediary estimates :
  - $||P^\varepsilon - P^0|| \leq C_a\varepsilon^a$,
  - $||P_h^0 - P^0|| \leq C_d h^d$,
  - $||P_h^\varepsilon - P_h^0|| \leq C_e\varepsilon^e$,
  - $d > c$, $e = a$.

Final result: We assume that some assumptions about regularity and meshes are satisfied. There exist a constant $C(T) > 0$ such that:

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}, \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + h + (h+\varepsilon) + \varepsilon\right) \leq Ch^{\frac{1}{4}}.$$

Euler equations with friction and gravity

Design of new finite volume nodal scheme I

Idea: Modify the classic one step Lagrangian+remap scheme with the Jin-Levermore AP method

- The classic Lagrange+remap scheme (LR scheme) is

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}\rho_j + \sum_{R_-} \mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j \mathbf{u}_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho \mathbf{u})_j + \sum_{R_-} \mathbf{u}_{jr}(\rho \mathbf{u})_{k(r)} + \sum_r \mathbf{p}\mathbf{C}_{jr}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho e)_j + \sum_{R_-} \mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r (\mathbf{p}\mathbf{C}_{jr}, \mathbf{u}_r)\right) = 0 \end{array}\right.$$

with the Lagrangian fluxes

$$\left\{\begin{array}{l} \mathbf{G}_{jr} = p_j \mathbf{C}_{jr} + \rho_j c_j \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) \\ \sum_j \rho_j c_j \widehat{\alpha}_{jr} \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \sum_j \rho_j c_j \widehat{\alpha}_{jr} \mathbf{u}_j \end{array}\right.$$

- Advection fluxes: $\mathbf{u}_{jr} = (\mathbf{C}_{jr}, \mathbf{u}_r)$, $R_+ = (r/\mathbf{u}_{jr} > 0)$, $R_- = (r/\mathbf{u}_{jr} < 0)$ and $\rho_{k(r)} = \dfrac{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}\rho_j}{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}}$.

Design of new finite volume nodal scheme II

Jin Levermore method: plug the balance equation $\nabla p + O(\varepsilon^2) = \rho \mathbf{g} - \frac{\sigma}{\varepsilon}\rho \mathbf{u}$ in the Lagrangian fluxes

- The modified scheme is

$$\begin{cases} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}\rho_j + \sum_{R_-} \mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j \mathbf{u}_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho \mathbf{u})_j + \sum_{R_-} \mathbf{u}_{jr}(\rho \mathbf{u})_{k(r)} + \sum_r \mathbf{pC}_{jr}\right) \\ = \frac{1}{\varepsilon}\left(\sum_r \rho_r \widehat{\beta}_{jr}\mathbf{g} - \sum_r \rho_r \widehat{\beta}_{jr}\frac{\sigma}{\varepsilon}\mathbf{u}_r\right) \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho e)_j + \sum_{R_-} \mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r (\mathbf{pC}_{jr}, \mathbf{u}_r)\right) \\ = \frac{1}{\varepsilon}\left(\sum_r \rho_r(\widehat{\beta}_{jr}\mathbf{g}, \mathbf{u}_r) - \frac{\sigma}{\varepsilon}\sum_r \rho_r(\mathbf{u}_r, \widehat{\beta}_{jr}\mathbf{u}_r)\right) \end{cases}$$

with the new Lagrangian fluxes

$$\begin{cases} \mathbf{pC}_{jr} = p_j \mathbf{C}_{jr} + \rho_j c_j \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) + \rho_r \widehat{\beta}_{jr}\mathbf{g} - \rho_r \widehat{\beta}_{jr}\frac{\sigma}{\varepsilon}\mathbf{u}_r \\ \left(\sum_j \rho_j c_j \widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\rho_r \sum_j \widehat{\beta}_{jr}\right)\mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \sum_j \rho_j c_j \widehat{\alpha}_{jr}\mathbf{u}_j + \rho_r\left(\sum_j \widehat{\beta}_{jr}\right)\mathbf{g} \end{cases}$$

AP properties

Limit diffusion scheme: If the local matrices are invertibles then the scheme LR-AP tends formally to the following diffusion scheme

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t \rho_j + \left(\sum_{R_+} \mathbf{u}_{jr} \rho_j + \sum_{R_-} \mathbf{u}_{jr} \rho_{k(r)} \right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \left(\sum_{R_+} \mathbf{u}_{jr} (\rho e)_j + \sum_{R_-} \mathbf{u}_{jr} (\rho e)_{k(r)} + p_j \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r) \right) = 0 \\ \sigma \rho_r \left(\sum_j \widehat{\beta}_{jr} \right) \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \rho_r \left(\sum_j \widehat{\beta}_{jr} \right) \mathbf{g} \end{array}\right.$$

- Remarks about limit diffusion scheme.
  - We obtain a nonlinear positive diffusion scheme.
  - For $p = K\rho$, we observe that the scheme converge with the first order.
  - **Open question**: Verify these properties for the full Euler scheme.
- Remarks about time scheme.
  - Another formulation gives a local source term for the momentum equation.
  - Using an implicit discretization of the local term source we verify numerically that the CFL is independent of ε.

WB properties

Result:

- We define $\nabla_r p = -(\sum_j \widehat{\beta}_{jr})^{-1} \sum_j p_j$ and ρ_r a mean of ρ_j around the node $\mathbf{x}_r$.
- If the initial data are given by the discrete steady state $\nabla_r p = \rho_r \mathbf{g}$ there are preserved exactly by the time scheme.

Conclusion:

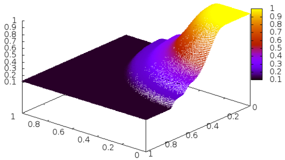
- The numerical error is governed only by the error between discrete and continuous steady states.

- Question: what is the error between the discrete steady states and the real steady states ?
  - for ρ constant: the discrete steady state is exact.
  - for ρ variable: the discrete steady state is not exact, but the error is homogeneous to $O(h^2)$.

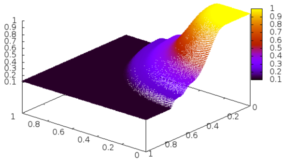
Numerical results : short time limit

- Test case: Sod problem with $\sigma > 0$, $\varepsilon = 1$ and $g = 0$ (short time limit).
- $\sigma = 1$

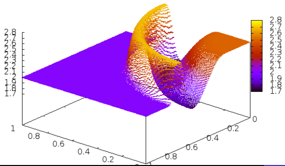
AP scheme, ρ



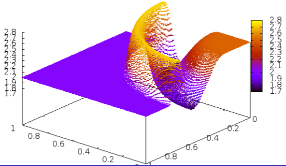
non-AP scheme, ρ



AP scheme, ϵ



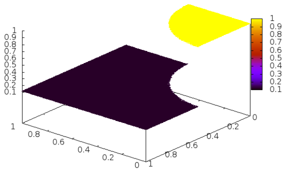
non-AP scheme, ϵ



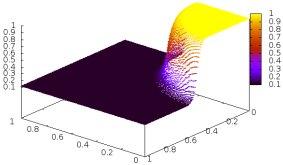
Numerical results : short time limit

- Test case: Sod problem with $\sigma > 0$, $\varepsilon = 1$ and $g = 0$ (short time limit).
- $\sigma = 10^6$

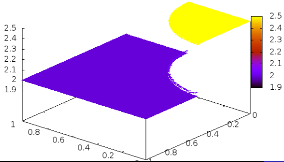
AP scheme, ρ



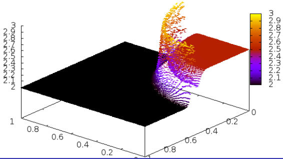
non-AP scheme, ρ



AP scheme, ϵ



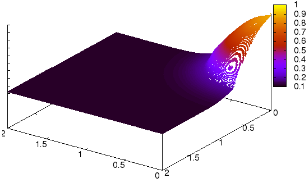
non-AP scheme, ϵ



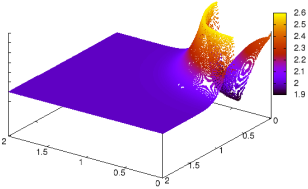
Numerical results : long time limit

- Test case: Sod problem with $\sigma > 0$, and $g = 0$ (non longer time limit).
- Non AP scheme, $\varepsilon = 0.005$, mesh 480×480

ρ

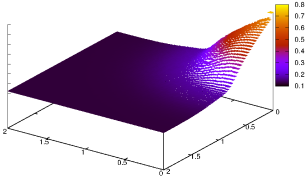


Energy

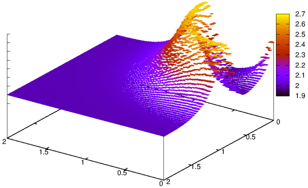


- Non AP scheme, $\varepsilon = 0.005$, mesh 60×60

ρ



Energy



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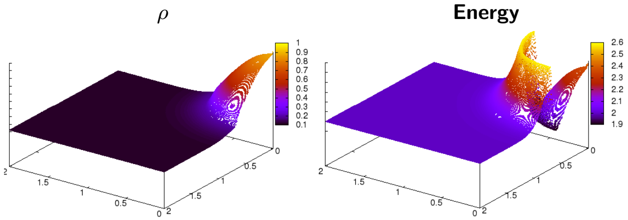
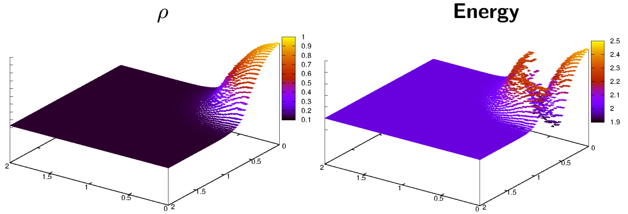


- AP scheme, $\varepsilon = 0.005$, mesh 60×60



Numerical results: WB properties

- Validation of the Well-Balanced properties.
- The gravity vector is $\mathbf{g} = (0, -1)$.
- First test case is defined by $\rho_j = 1$, $\mathbf{u}_j = \mathbf{0}$ and $e_j = \frac{1}{\gamma-1}(\mathbf{x}_j, \mathbf{g}) + C$ with C a constant.

Schemes	LP-AP			LP		
Meshes/cells	40	80	160	40	80	160
Cartesian	5.9×10^{-17}	1×10^{-16}	7.1×10^{-17}	0.00470	0.00239	0.00121
Random	1.1×10^{-16}	1.5×10^{-16}	3×10^{-16}	0.01519	0.00947	0.00526
Kershaw	1.4×10^{-16}	2.2×10^{-16}	3.2×10^{-16}	0.08503	0.050	0.02908

- **Classical scheme**: convergence with $O(h)$.
- **AP scheme**: preserve exactly the steady states.

Numerical results: WB properties

- Validation of the Well-Balanced properties.
- The gravity vector is $\mathbf{g} = (0, -1)$.
- The initial data for the second test case are defined by $\rho_j(t, \mathbf{x}) = y + b$, $\mathbf{u}_j = \mathbf{0}$ and $p_j(t, \mathbf{x}) = -(\frac{y^2}{2} + by)g$.

Schemes	LP-AP			LP		
Meshes/cells	80	160	320	80	160	320
Cartesian	2.3×10^{-15}	9.4×10^{-15}	3.4×10^{-14}	0.003407	0.00167	0.00008
Random	3.4×10^{-5}	1×10^{-5}	2.8×10^{-6}	0.00967	0.00529	0.00282
Kershaw	1.1×10^{-6}	1.8×10^{-7}	2.6×10^{-8}	0.03687	0.008363	0.00215

- **Classical scheme**: convergence with $O(h)$.
- **AP scheme**: convergence with $O(h^2)$.

Ongoing works and conclusion

Local Very high order scheme around equilibrium

- **Aim**: converse the classical properties of stability associated with the first order scheme and obtain a very high order discretization of the equilibrium.
- **Method** : construct a very high order discrete steady state.

- 1D Discrete steady state: $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho g)_{j+\frac{1}{2}}$ with $(\rho g)_{j+\frac{1}{2}} = \frac{1}{2}(\rho_{j+1} + \rho_j) g$.
- To begin we consider the following simple steady state

$$\partial_x p = -\rho g$$

- Integrating on the diamond cell $[x_j, x_{j+1}]$ we obtain

$$\Delta x_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \partial_x p(x) \right) = -g \Delta x_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \rho(x) \right)$$

Local Very high order scheme around equilibrium

- **Aim**: converse the classical properties of stability associated with the first order scheme and obtain a very high order discretization of the equilibrium.
- **Method** : construct a very high order discrete steady state.

- 1D Discrete steady state: $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho g)_{j+\frac{1}{2}}$ with $(\rho g)_{j+\frac{1}{2}} = \frac{1}{2}(\rho_{j+1} + \rho_j) g$.
- We introduce two polynomials $\overline{\rho}_{j+\frac{1}{2}}(x) = \sum_{k=1}^{q} r_k x^k$ and $\overline{p}_{j+\frac{1}{2}}(x) = \sum_{k=1}^{q+1} p_k x^k$ with

$$\int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \overline{\rho}_{j+\frac{1}{2}}(x) = \Delta x_l \rho_l, \quad \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \overline{p}_{j+\frac{1}{2}}(x) = \Delta x_l p_l$$

and $l \in S(j)$ ($S(j)$ is a subset of cell around j). Using these polynomials we obtain the new discrete steady states

$$\Delta x_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \partial_x \overline{p}_{j+\frac{1}{2}}(x) \right) = -g \Delta x_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \overline{\rho}_{j+\frac{1}{2}}(x) \right)$$

Local Very high order scheme around equilibrium

- **Aim**: converse the classical properties of stability associated with the first order scheme and obtain a very high order discretization of the equilibrium.
- **Method** : construct a very high order discrete steady state.

- 1D Discrete steady state: $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho g)_{j+\frac{1}{2}}$ with $(\rho g)_{j+\frac{1}{2}} = \frac{1}{2}(\rho_{j+1} + \rho_j) g$.
- To obtain a scheme which preserves the discrete steady state, it is necessary to have the numerical pressure viscosity is the discrete steady state.
- We obtain following the *q*-order steady state:

$$p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho g)^{HO}_{j+\frac{1}{2}}$$

with

$$(\rho g)^{HO}_{j+\frac{1}{2}} = \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \left(\int_{x_j}^{x_{j+1}} \partial_x \overline{p}_{j+\frac{1}{2}}(x) \right) + g \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \overline{\rho}_{j+\frac{1}{2}}(x) \right) - \frac{p_{j+1} - p_j}{\Delta x_{j+\frac{1}{2}}} \right)$$

Results for local Very high order WB scheme

- **Test case**: $\rho(x) = p(x) = e^{-gx}$, $u(x) = 0$.
- AP scheme with three order equilibrium

Meshes	Cartesian		Random	
cells	error	order	error	order
40	3×10^{-6}		4.1×10^{-6}	
80	5×10^{-7}	2.6	5×10^{-7}	3
160	6.3×10^{-8}	3	6×10^{-8}	3.1

- AP scheme with fourth order equilibrium

Meshes	Cartesian		Random	
cells	error	order	error	order
40	1×10^{-7}		8.74×10^{-8}	
80	5.5×10^{-9}	4.17	4.6×10^{-9}	4.25
160	2.85×10^{-10}	4.25	2.6×10^{-10}	4.15

Conclusion and future works

Conclusion:

- P_1 **model**: AP nodal scheme on distorted meshes with CFL independent of ε.
- P_1 **model**: Uniform convergence for the semi discrete scheme on unstructured meshes.
- **Euler equations with friction** : AP scheme with a CFL independent to ε.
- **Euler equations with friction** : Well-Balanced scheme which converges with the second order.
- **All models** : Spurious mods in few cases (Cartesian mesh + initial Dirac data).

Future works:

- Validation of the LR-AP scheme with analytical test cases.
- Analysis of the Euler AP discretization: **entropy stability**.
- Local high order Well-Balanced scheme for hydrostatic equilibrium in 2D
- Generic stabilization procedure for the nodal schemes.

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