Uniform asymptotic preserving and well-balanced schemes for hyperbolic systems with source terms

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Outline

Mathematic and physical context

Stiff hyperbolic systems

- **Stiff hyperbolic system with source terms**:

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon}\partial_x F(\mathbf{U}) + \frac{1}{\varepsilon}\partial_y G(\mathbf{U}) = \frac{1}{\varepsilon}S(\mathbf{U}) - \frac{\sigma}{\varepsilon^2}R(\mathbf{U}),\ \mathbf{U} \in \mathbb{R}^n$$

with $\varepsilon \in]0,1]$ et $\sigma > 0$.

- Subset of solutions given by the balance between the source terms and the convective part:
  - **Diffusion solutions** for $\varepsilon \to 0$ and $S(\mathbf{U}) = 0$:

$$\partial_t \mathbf{V} - \operatorname{div}\left(K(\nabla \mathbf{V}, \sigma)\right) = 0, \quad \mathbf{V} \in \operatorname{Ker} R.$$

  - **Steady-state** for $\sigma = 0$ et $\varepsilon \to 0$:

$$\partial_x F(\mathbf{U}) + \partial_y G(\mathbf{U}) = S(\mathbf{U}).$$

- Applications: biology, neutron transport, fluid mechanics, plasma physics, Radiative hydrodynamic (hydrodynamic + linear transport of photon).

Notion of WB and AP schemes

- Acoustic equation with damping and gravity:

$$\left\{\begin{array}{l} \partial_t p + \frac{1}{\varepsilon}\partial_x u = 0, \\ \partial_t u + \frac{1}{\varepsilon}\partial_x p = -\frac{1}{\varepsilon} g - \frac{\sigma}{\varepsilon^2} u, \end{array}\right. \qquad \longrightarrow \partial_t p - \partial_x\left(\frac{1}{\sigma}(\partial_x p + g)\right) = 0.$$

- Steady-state: $u = 0$, $\partial_x p = -g$.
- **Godunov-type** schemes give an error homogeneous to $O(\Delta x)$.
- For nearly uniform flows, spurious velocities larger that physical velocity.
- Important deviation of the steady-state.
- **WB scheme**: discretize the steady-state exactly of with high accuracy.
- Ref: S. Jin, A *steady-state capturing method for hyperbolic method with geometrical source terms*.

- Consistency of **Godunov-type** schemes: $O(\frac{\Delta x}{\varepsilon} + \Delta t)$.
- CFL condition: $\Delta t(\frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2}) \leq 1$.
- Consistency of AP schemes: $O\left(\Delta x + \Delta t\right)$.
- CFL condition: degenerate on parabolic CFL at the limit.
- Ref: S. Jin, D. Levermore *Numerical schemes for hyperbolic conservation laws with stiff relaxation*.

- To construct WB and AP schemes: incorporate the source in the fluxes to capture the balance between source and convective terms.

Reduced bibliography

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- Well balanced schemes for chemotaxis and Euler equations
  - R. Natalini and M. Ribot, *An asymptotic high order mass-preserving scheme for a hyperbolic model of chemotaxis*, (2012).
  - V. Desveaux, M. Zenk, C. Berthon, C. Klingenberg, *A well-balanced scheme to capture non-explicit steady states in the Euler equations with gravity*, (2015).
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- 2D asymptotic preserving schemes
  - A. Duran, F. Marche, R.Turpault, C. Berthon, *Asymptotic preserving scheme for the shallow water equations with source terms on unstructured meshes*, (2015).
  - C. Berthon, G. Moebs, C. Sarazin-Desbois and R. Turpault, *An AP scheme for systems of conservation laws with source terms on 2D unstructured meshes*, (2014).

Exemple of AP and WB Godunov schemes

- **Jin-Levermore (or Gosse-Toscani) scheme**. Plug the balance law $\partial_x Ep = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2)$ in the fluxes. We write

$$p(x_j) = p(x_{j+\frac{1}{2}}) + (x_j - x_{j+\frac{1}{2}})\partial_x p(x_{j+\frac{1}{2}})$$

$$p(x_j) = p(x_{j+\frac{1}{2}}) - (x_j - x_{j+\frac{1}{2}})\frac{\sigma}{\varepsilon} u(x_{j+\frac{1}{2}})$$

Coupling the previous relation (and the same for x_{j+1}) with the fluxes

$$\begin{cases} u_j + p_j = u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma \Delta x}{2\varepsilon} u_{j+\frac{1}{2}}, \\ u_{j+1} - p_{j+1} = u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma \Delta x}{2\varepsilon} u_{j+\frac{1}{2}}. \end{cases}$$

- To finish, we take the following source term $\frac{1}{2}(u_{j+\frac{1}{2}} + u_{j-\frac{1}{2}})$.

Gosse-Toscani scheme:

$$\begin{cases} \frac{p_j^{n+1} - p_j^n}{\Delta t} + M \frac{u_{j+1}^n - u_{j-1}^n}{2\varepsilon \Delta x} - M \frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{2\varepsilon \Delta x} = 0, \\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + M \frac{p_{j+1}^n - p_{j-1}^n}{2\varepsilon \Delta x} - M \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\varepsilon \Delta x} + M \frac{\sigma}{\varepsilon^2} u_j^n = 0, \end{cases}$$

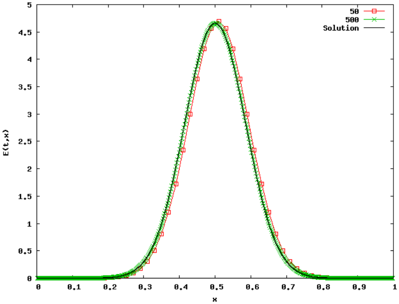
with $M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x}$.

- Consistency error of the **Gosse-Toscani** scheme: $O(\Delta x + \Delta t)$.
- Explicit CFL: $\Delta t \left(\frac{1}{\Delta x \varepsilon}\right) \leq 1$, Semi-implicit CFL : $\Delta t \left(\frac{1}{\Delta x \varepsilon + \Delta x^2}\right) \leq 1$.

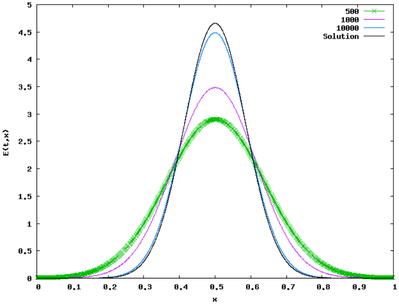
Numerical example

- **Validation test for the AP scheme**: the data are $p(0,x) = G(x)$ with $G(x)$ a Gaussian $u(0,x) = 0$ and $\sigma = 1$, $\varepsilon = 0.001$.

Ap scheme

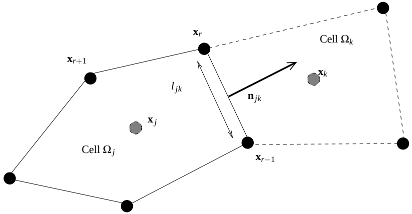


Godunov scheme



Scheme	L^1 error	CPU time
Godunov, 10000 cells	0.0366	1485m4.26s
Godunov, 500 cells	0.445	0m24.317s
AP, 500 cells	0.0001	0m15.22s
AP, 50 cells	0.0065	0m0.054s

- **Classical extension in 2D of the Jin-Levermore scheme** : modify the upwind fluxes (1D fluxes write in the normal direction) plugging the steady-state in the fluxes.



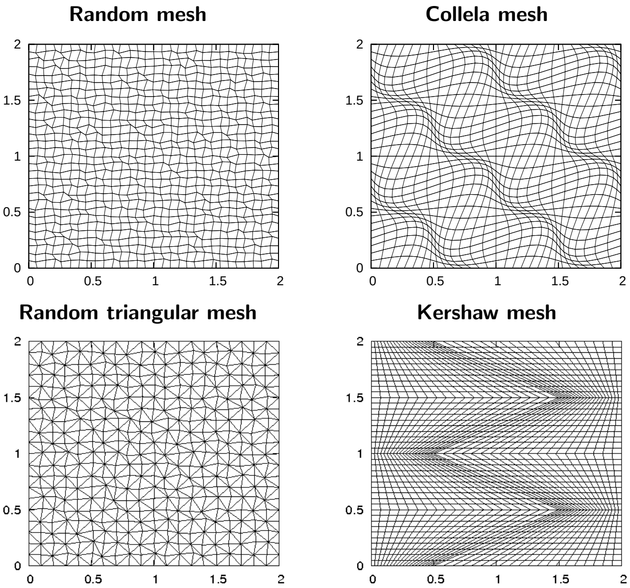
- l_{jk} and $\mathbf{n}_{jk}$ the normal and length associated with the edge $\partial\Omega_{jk}$.

Asymptotic limit of the hyperbolic scheme:

$$\mid \Omega_j \mid \partial_t p_j(t) - \frac{1}{\sigma}\sum_k l_{jk}\frac{p_k^n - p_j^n}{d(\mathbf{x}_j, \mathbf{x}_k)} = 0.$$

- $||P_h^0 - P_h|| \to 0$ only on strong geometrical conditions.
- **Additional difficulty in 2D**: The basic extension of AP schemes **do not converge** on 2D general meshes $\forall\varepsilon$.

Example of unstructured meshes



AP scheme for the P_1 model

Nodal scheme : linear case

- Linear case: P_1 model

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\,\text{div}(\mathbf{u}) = 0, \\ \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p = -\frac{\sigma}{\varepsilon^2}\mathbf{u}. \end{cases} \longrightarrow \partial_t p - \text{div}\left(\frac{1}{\sigma}\nabla p\right) = 0.$$

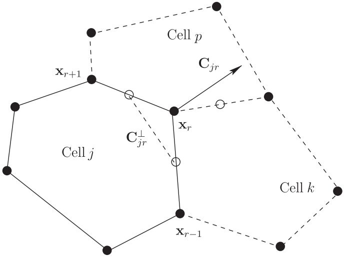
Idea:

Nodal finite volume methods for P_1 model + AP and WB method.

Nodal schemes:

The fluxes are localized at the nodes of the mesh (for the classical scheme this is at the edge).

Notations



- Nodal geometrical quantities $\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r}|\Omega_j|$.
- $\sum_j \mathbf{C}_{jr} = \sum_r \mathbf{C}_{jr} = \mathbf{0}$.

Nodal AP schemes

$$\begin{cases} \mid \Omega_j \mid \partial_t p_j(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\ \mid \Omega_j \mid \partial_t \mathbf{u}_j(t) + \dfrac{1}{\varepsilon} \displaystyle\sum_r \mathbf{pc}_{jr} = \mathbf{S}_j. \end{cases}$$

- Classical nodal fluxes:

$$\begin{cases} \mathbf{pc}_{jr} - p_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r), \\ \sum_j \mathbf{pc}_{jr} = \mathbf{0}, \end{cases}$$

with $\widehat{\alpha}_{jr} = \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{\|\mathbf{C}_{jr}\|}$.

- New fluxes obtained plugging steady-state $\nabla p = -\frac{\sigma}{\varepsilon}\mathbf{u}$ in the fluxes:

$$\begin{cases} \mathbf{pc}_{jr} - p_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \dfrac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\mathbf{u}_r, \\ \left(\displaystyle\sum_j \widehat{\alpha}_{jr} + \dfrac{\sigma}{\varepsilon}\sum_j \widehat{\beta}_{jr}\right)\mathbf{u}_r = \displaystyle\sum_j p_j \mathbf{C}_{jr} + \sum_j \widehat{\alpha}_{jr}\mathbf{u}_j. \end{cases}$$

with $\widehat{\beta}_{jr} = \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j)$.

- Source term: (1) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2} \mid \Omega_j \mid \mathbf{u}_j$ ou (2) $\mathbf{S}_j = -\frac{\sigma}{\varepsilon^2}\sum_r \widehat{\beta}_{jr}\mathbf{u}_r$, $\quad \sum_r \widehat{\beta}_{jr} = \hat{I}_d|\Omega_j|$.
- Using the second source term and rewriting the scheme we obtain an local semi implicit scheme with a CFL independent of ε.

Assumptions for the convergence proof

Geometrical assumptions

- $\left(\mathbf{u}, \left(\sum_r \frac{\mathbf{C}_{jr}\otimes\mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}\right)\mathbf{u}\right) \geq \beta h(\mathbf{u},\mathbf{u})$,
- $\left(\mathbf{u}, \left(\sum_j \frac{\mathbf{C}_{jr}\otimes\mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}\right)\mathbf{u}\right) \geq \gamma h(\mathbf{u},\mathbf{u})$,
- $\left(\mathbf{u}, \left(\sum_j \mathbf{C}_{jr}\otimes(\mathbf{x}_r-\mathbf{x}_j)\right)\mathbf{u}\right) \geq \alpha h^2(\mathbf{u},\mathbf{u})$.

- First and second assumptions: true on all non degenerated meshes.
- Last assumption: we have obtained sufficient but not necessary conditions on the meshes to satisfy this assumption.
- Example for triangles: all the angles must be larger that 12 degrees.

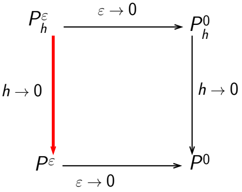
Assumption on regularity and initial data

- $\mathbf{u}(t=0,\mathbf{x}) = -\frac{\varepsilon}{\sigma}\nabla p(t=0,\mathbf{x})$
- Regularity for exact data: $\mathbf{V}(t,\mathbf{x}) \in H^4(\Omega)$
- Regularity for initial data of the scheme: $\mathbf{V}_h(t=0,\mathbf{x}) \in L^2(\Omega)$

Uniform convergence in space

- Naive convergence estimate : $||P_h^\varepsilon - P^\varepsilon||_{naive} \leq C\varepsilon^{-b}h^c$
- **Idea**: use triangular inequalities and AP diagram (Jin-Levermore-Golse).

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(||P_h^\varepsilon - P^\varepsilon||_{naive}, ||P_h^\varepsilon - P_h^0|| + ||P_h^0 - P^0|| + ||P^\varepsilon - P^0||)$$



- Intermediary estimations :
  - $||P^\varepsilon - P^0|| \leq C_a\varepsilon^a$,
  - $||P_h^0 - P^0|| \leq C_d h^d$,
  - $||P_h^\varepsilon - P_h^0|| \leq C_e\varepsilon^e$,
  - $d \leq c$, $e \geq a$.

- We obtain:

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq C\min(\varepsilon^{-b}h^c, \varepsilon^a + h^d + \varepsilon^e))$$

.

- Comparing ε and $\varepsilon_{threshold} = h^{\frac{ac}{a+b}}$ we obtain the final estimation:

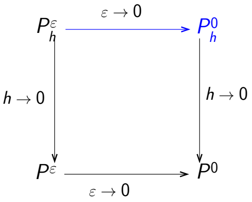
$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq h^{\frac{ac}{a+b}}$$

.

Diffusion scheme

Limit diffusion scheme (P_h^0)

$$\begin{cases} \mid \Omega_j \mid \partial_t p_j(t) - \sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\ \sum_r \hat{\alpha}_{jr} \mathbf{u}_j = \sum_r \hat{\alpha}_{jr} \mathbf{u}_r, \\ \sigma A_r \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr}, \quad A_r = -\sum_j \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \end{cases}$$



- **Problem**: estimate $||P_h^\varepsilon - P_h^0||$.
- In practice, we have obtained $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$.

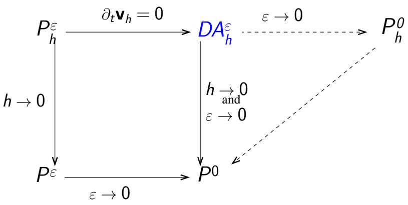
Condition H:

The discrete Hessian of P_h^0 can be bounded or the error estimate $\|P_h^\varepsilon - P_h^0\|$ can be obtained independently of the discrete Hessian.

Diffusion scheme

Limit diffusion scheme (P_h^0)

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t p_j(t) - \sum_r (\mathbf{u}_r, \mathbf{C}_{jr}) = 0, \\ \sum_r \hat{\alpha}_{jr} \mathbf{u}_j = \sum_r \hat{\alpha}_{jr} \mathbf{u}_r, \\ \sigma A_r \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr}, \quad A_r = -\sum_j \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \end{array}\right.$$



- **Problem**: estimate $||P_h^\varepsilon - P_h^0||$.
- In practice, we have obtained $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$.
- Introduction of an intermediary diffusion scheme DA_h^ε.
- DA_h^ε: P_h^ε scheme with $\partial_t \mathbf{F}_j = \mathbf{0}$.
- In the previous estimation we replace P_h^0 by DA_h^ε.

Condition H:

The discrete Hessian of P_h^0 can be bounded or the error estimate $||P_h^\varepsilon - P_h^0||$ can be obtained independently of the discrete Hessian.

Final results

Space result:

We assume that the assumptions are verified. There exist $C(T)>0$ such that:

$$\|\mathbf{V}^{\varepsilon}-\mathbf{V}_h^{\varepsilon}\|_{L^2([0,T]\times\Omega)}\leq Cf(h,\varepsilon)\parallel p_0\parallel_{H^4(\Omega)}\leq Ch^{\frac{1}{4}}\parallel p_0\parallel_{H^4(\Omega)}$$

with

$$f(h,\varepsilon)=\min\left(\sqrt{\frac{h}{\varepsilon}},\varepsilon\max\left(1,\sqrt{\frac{\varepsilon}{h}}\right)+h+(h+\varepsilon)+\varepsilon\right)$$

- Case $\varepsilon\leq h$: $\|\mathbf{V}^{\varepsilon}-\mathbf{V}_h^{\varepsilon}\|\leq C_1\min(\sqrt{\frac{\varepsilon}{h}},1)\leq C_1 h$
- Case $\varepsilon\geq h$: $\|\mathbf{V}^{\varepsilon}-\mathbf{V}_h^{\varepsilon}\|\leq C_1\min(\sqrt{\frac{h}{\varepsilon}},\sqrt{\frac{\varepsilon^3}{h}})$
- Introducing $\varepsilon_{thresh}=h^{\frac{1}{2}}$ we prove that the worst case is $\|\mathbf{V}^{\varepsilon}-\mathbf{V}_h^{\varepsilon}\|\leq C_2 h^{\frac{1}{4}}$.

Space-time result:

Wa assume that the assumptions are verified. There exist $C>0$ such that:

$$\|\mathbf{V}^{\varepsilon}(t_n)-\mathbf{V}_h^{\varepsilon}(t_n)\|_{L^2(\Omega)}\leq C\left(f(h,\varepsilon)+\Delta t^2\right)\parallel p_0\parallel_{H^4(\Omega)}$$

Remark: The condition H is not satisfied. The diffusion scheme used is DA_{ε}.

Intermediary results I

Estimation of $||\mathbf{V}^{\varepsilon} - \mathbf{V}_h^{\varepsilon}||$:

We assume that assumptions are verified. There exist $C > 0$ such that:

$$\|\mathbf{V}_h^{\varepsilon} - \mathbf{V}^{\varepsilon}\|_{L^{\infty}((0,T):L^2(\Omega))} \leq C\sqrt{\frac{h}{\varepsilon}}.$$

- Principle of proof:
  - Control the stability of the discrete quantities $\mathbf{u}_r$ and $\mathbf{u}_j$ by ε
  - We define the error $E(t) = ||\mathbf{V}^{\varepsilon} - \mathbf{V}_h^{\varepsilon}||_{L^2}$ and we estimate $E^{'}(t)$ using Young and Cauchy-Schwartz inequalities, stability estimates and integration in time.

Estimation of $||DA_h^{\varepsilon} - P^0||$:

Wa assume that the assumptions are verified. There exist $C_1 > 0$ such that:

$$||\mathbf{V}_h^0 - \mathbf{V}^0||_{L^2(\Omega)} \leq C_1(T)(h + \varepsilon), \qquad 0 < t \leq T.$$

- Principle of proof:
  - Control the stability of the discrete quantities $\nabla_r E$ and E_j.
  - Consistance study of Div and Grad discrete operators.
  - L^2 estimate using consistency error and Gronwall lemma.

Intermediary results II

Estimate $||P_h^\varepsilon - DA_h^\varepsilon||$:

We assume that the assumptions are verified. There exist $C_2(T) > 0$ such that:

$$||\mathbf{V}_h^\varepsilon - \mathbf{V}_h||_{L^2(\Omega)} \leq C_2(T)\varepsilon \max\left(1, \sqrt{\varepsilon h^{-1}}\right) + Ch, \qquad 0 < t \leq T.$$

Estimate $||P^\varepsilon - P^0||$:

We assume that the assumptions are verified. There exist $C_3(T) > 0$ such that:

$$||\mathbf{V}^\varepsilon - \mathbf{V}^0||_{L^2(\Omega)} \leq C_3(T)\varepsilon, \qquad 0 < t \leq T.$$

- Principe of proof:
  - Write $P^0 = P^\varepsilon + R$ (resp $DA_h^\varepsilon = P_h^\varepsilon + R$) with R a residue.
  - Find a bound with ε of the residue.
  - L^2 estimate of the difference between the two models and between the two schemes.

Analysis of AP schemes: modified equations

- To understand the behavior of the scheme, we use the modified equations method.

- The modified equation associated with the Upwind scheme is

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\partial_x u - \frac{\Delta x}{2\varepsilon}\partial_{xx} p = 0, \\ \partial_t u + \frac{1}{\varepsilon}\partial_x p - \frac{\Delta x}{2\varepsilon}\partial_{xx} u = -\frac{\sigma}{\varepsilon^2} u. \end{cases}$$

- Plugging $\varepsilon\partial_x p + O(\varepsilon^2) = -\sigma u$ in the first equation, we obtain the diffusion limit

$$\partial_t p - \frac{1}{\sigma}\partial_{xx} p - \frac{\Delta x}{2\varepsilon}\partial_{xx} p = 0.$$

- **Conclusion**: the regime is captured only on fine grids.

- The modified equation associated to the Gosse-Toscani scheme is

$$\begin{cases} \partial_t p + M\frac{1}{\varepsilon}\partial_x u - M\frac{\Delta x}{2\varepsilon}\partial_{xx} p = 0, \\ \partial_t u + M\frac{1}{\varepsilon}\partial_x p - M\frac{\Delta x}{2\varepsilon}\partial_{xx} u = -M\frac{\sigma}{\varepsilon^2} u. \end{cases}$$

- Plugging $M\varepsilon\partial_x p + O(\varepsilon^2) = -M\sigma u$ in the first equation, we obtain the diffusion limit

$$\partial_t p - \frac{M}{\sigma}\partial_{xx} p - \frac{1-M}{\sigma}\partial_{xx} p = 0$$

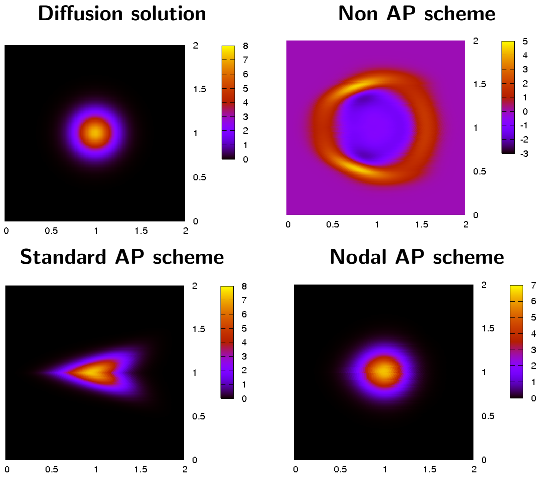
- **Conclusion**: the regime is capture only on all grids.

Construction of the AP scheme in 2D

- We must modify the viscosity to a consistent diffusion scheme with the good coefficient on coarse grids.
- We must also discretize correctly the source term and the gradient of pressure to obtain a consistent diffusion scheme on fine grids (WB schemes).

AP scheme vs classical scheme

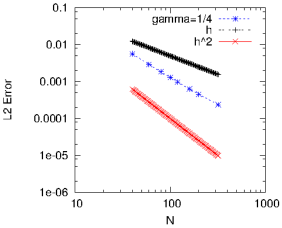
- Test case: heat fundamental solution. Results for different hyperbolic scheme with $\varepsilon = 0.001$ on Kershaw mesh.



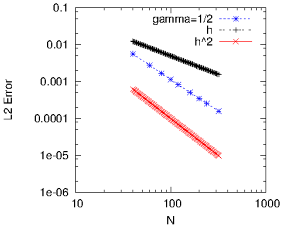
Uniform convergence

- ε dependent periodic solution for the P_1 model.
- $p(t, \mathbf{x}) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x)\cos(\pi y)$
- $\mathbf{u}(t, \mathbf{x}) = (-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\cos(\pi y), \quad -\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi y)\cos(\pi x))$
- Convergence study for $\varepsilon = h^\gamma$ on random mesh.

$\gamma = \frac{1}{4}$



$\gamma = \frac{1}{2}$

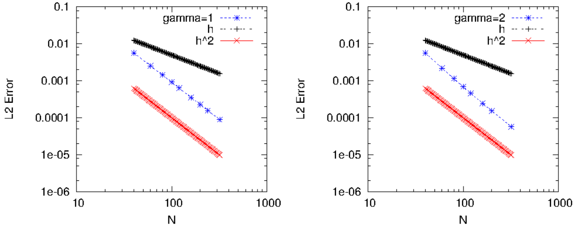


- Numerical results show that the error is homogenous to $O(h\varepsilon + h^2)$.
- Theoretical estimate that we can hope: $O((h\varepsilon)^{\frac{1}{2}} + h)$.
- Non optimal estimation in the intermediary regime.

Uniform convergence

- ε dependent periodic solution for the P_1 model.
- $p(t, \mathbf{x}) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha^{'}(t))\cos(\pi x)\cos(\pi y)$
- $\mathbf{u}(t, \mathbf{x}) = (-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\cos(\pi y), \quad -\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi y)\cos(\pi x))$
- Convergence study for $\varepsilon = h^\gamma$ on random mesh.

$\gamma = \frac{1}{\ }$ $\qquad$ $\gamma = 2$



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Extension to the Euler model

Euler equation with external forces

- Euler equation with gravity and friction:

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon^\alpha}\,\mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho \mathbf{u} + \dfrac{1}{\varepsilon^\alpha}\,\mathrm{div}(\rho \mathbf{u}\otimes \mathbf{u}) + \dfrac{1}{\varepsilon^\alpha}\nabla p = -\dfrac{1}{\varepsilon^\alpha}\left(\rho\nabla\phi + \dfrac{\sigma}{\varepsilon^\beta}\rho\mathbf{u}\right), \\ \partial_t \rho e + \dfrac{1}{\varepsilon^\alpha}\,\mathrm{div}(\rho \mathbf{u} e) + \mathrm{div}(p\mathbf{u}) = -\dfrac{1}{\varepsilon^\alpha}\left(\rho(\nabla\phi, \mathbf{u}) + \dfrac{\sigma}{\varepsilon^\beta}\rho(\mathbf{u},\mathbf{u})\right). \end{cases}$$

- with ϕ the gravity potential, σ the friction coefficient.

Subset of solutions :

- Hydrostatic Steady-state ($\alpha = 1,\ \beta = 0$):

$$\begin{cases} \mathbf{u} = \mathbf{0}, \\ \nabla p = -\rho\nabla\phi. \end{cases}$$

- High friction limit ($\alpha = 0,\ \beta = 1$), no gravity: $\mathbf{u} = \mathbf{0}$
- Diffusion limit ($\alpha = 1,\ \beta = 1$):

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho\mathbf{u}) = 0, \\ \partial_t \rho e + \mathrm{div}(\rho\mathbf{u}e) + p\,\mathrm{div}\,\mathbf{u} = 0, \\ \mathbf{u} = -\dfrac{1}{\sigma}\left(\nabla\phi + \dfrac{1}{\rho}\nabla p\right). \end{cases}$$

Design of AP nodal scheme I

Idea :

Modify the Lagrange+remap classical scheme with the Jin-Levermore method

- Classical Lagrange+remap scheme (LP scheme):

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon^\alpha}\left(\sum_{R_+} \mathbf{u}_{jr}\rho_j + \sum_{R_-} \mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j \mathbf{u}_j + \frac{1}{\varepsilon^\alpha}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho \mathbf{U})_j + \sum_{R_-} \mathbf{u}_{jr}(\rho \mathbf{U})_{k(r)} + \sum_r \mathbf{p}\mathbf{C}_{jr}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j e_j + \frac{1}{\varepsilon^\alpha}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho e)_j + \sum_{R_-} \mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r (\mathbf{p}\mathbf{C}_{jr}, \mathbf{u}_r)\right) = 0 \end{array}\right.$$

with Lagrangian fluxes

$$\left\{\begin{array}{l} \mathbf{G}_{jr} = p_j \mathbf{C}_{jr} + \rho_j c_j \hat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) \\ \sum_j \rho_j c_j \hat{\alpha}_{jr} \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \sum_j \rho_j c_j \hat{\alpha}_{jr} \mathbf{u}_j \end{array}\right.$$

- Advection fluxes: $\mathbf{u}_{jr} = (\mathbf{C}_{jr}, \mathbf{u}_r)$, $R_+ = (r/\mathbf{u}_{jr} > 0)$, $R_- = (r/\mathbf{u}_{jr} < 0)$ et $\rho_{k(r)} = \frac{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}\rho_j}{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}}$.

Jin Levermore method:

Plug the relation $\nabla p + O(\varepsilon^2) = -\rho\nabla\phi - \frac{\sigma}{\varepsilon}\rho\mathbf{u}$ in the Lagrangian fluxes

- The modified scheme is given by

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon^\alpha}\left(\sum_{R_+} \mathbf{u}_{jr}\rho_j + \sum_{R_-} \mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j \mathbf{u}_j + \frac{1}{\varepsilon^\alpha}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho\mathbf{u})_j + \sum_{R_-} \mathbf{u}_{jr}(\rho\mathbf{u})_{k(r)} + \sum_r \mathbf{pC}_{jr}\right) \\ = -\frac{1}{\varepsilon^\alpha}\left(\sum_r \hat{\beta}_{jr}(\rho\nabla\phi)_r + \frac{\sigma}{\varepsilon^\beta}\sum_r \rho_r\hat{\beta}_{jr}\mathbf{u}_r\right) \\ \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon^\alpha}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho e)_j + \sum_{R_-} \mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r (\mathbf{pC}_{jr}, \mathbf{u}_r)\right) \\ = -\frac{1}{\varepsilon^\alpha}\left(\sum_r (\hat{\beta}_{jr}(\rho\nabla\phi)_r, \mathbf{u}_r) + \frac{\sigma}{\varepsilon^\beta}\sum_r \rho_r(\mathbf{u}_r, \hat{\beta}_{jr}\mathbf{u}_r)\right) \end{array}\right.$$

with the new Lagrangian fluxes

$$\left\{\begin{array}{l} \mathbf{pC}_{jr} = p_j\mathbf{C}_{jr} + \rho_j c_j \hat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \hat{\beta}_{jr}(\rho\nabla\phi)_r - \frac{\sigma}{\varepsilon^\beta}\rho_r\hat{\beta}_{jr}\mathbf{u}_r \\ \left(\sum_j \rho_j c_j \hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon^\beta}\rho_r \sum_j \hat{\beta}_{jr}\right)\mathbf{u}_r = \sum_j p_j\mathbf{C}_{jr} + \sum_j \rho_j c_j \hat{\alpha}_{jr}\mathbf{u}_j - \left(\sum_j \hat{\beta}_{jr}\right)(\rho\nabla\phi)_r \end{array}\right.$$

- and $(\rho\nabla\phi)_r$ a discretization of $\rho\nabla\phi$ at the interface.

Properties

Limit diffusion scheme:

If the local matrices are invertible then the LR-AP scheme tends to the following scheme

$$\begin{cases} \mid \Omega_j \mid \partial_t \rho_j + \left(\sum_{R_+} (\mathbf{C}_{jr}, \mathbf{u}_r)\rho_j + \sum_{R_-} (\mathbf{C}_{jr}, \mathbf{u}_r)\rho_{k(r)} \right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j + \left(\sum_{R_+} (\mathbf{C}_{jr}, \mathbf{u}_r)(\rho e)_j + \sum_{R_-} (\mathbf{C}_{jr}, \mathbf{u}_r)(\rho e)_{k(r)} + p_j \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r) \right) = 0 \\ \sigma \rho_r \left(\sum_j \hat{\beta}_{jr} \right) \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} - \left(\sum_j \hat{\beta}_{jr} \right) (\rho \nabla \phi)_r \end{cases}$$

- The nodal gradient formula $\nabla_r p = \left(\sum_j \hat{\beta}_{jr} \right)^{-1} (\sum_j p_j \mathbf{C}_{jr})$ is a consistent and convergent approximation of the gradient on unstructured meshes (Consistency study+Gronwall's lemma).
- For $p = K\rho$, numerically the schemes converge at the first scheme.
- If we use a second order advection scheme for the remap part. The full scheme converges with the second order.
- **Open question**: Verify this for a non isothermal pressure law as perfect gas law.

Well balanced property

Well balanced property

- We define the discrete gradient $\nabla_r p = -(\sum_j \hat{\beta}_{jr})^{-1} \sum_j p_j \mathbf{C}_{jr}$ and ρ_r an average of ρ_j around $\mathbf{x}_r$.
- If the initial data are given by the discrete steady-state $\nabla_r p = -(\rho \nabla \phi)_r$, $\rho_j^{n+1} = \rho_j^n$, $\mathbf{u}_j^{n+1} = \mathbf{u}_j^n$ and $e_j^{n+1} = e_j^n$,
- **Remark**: The spatial error for a steady-state is only governed by the error between discrete steady-state and the continuous steady-state

High order reconstruction of steady-state

- **Aim**: Conserve the stability property of the first order scheme, but discretize the steady-state with a high order accuracy or exactly.
- **Method**: design high order discrete steady-state

- The discrete steady-state is given $(\sum_j \hat{\beta}_{jr})^{-1} \sum_j p_j \mathbf{C}_{jr} = -\rho_r (\sum_j \hat{\beta}_{jr})^{-1} \sum_j \phi_j \mathbf{C}_{jr}$.
- If ρ_r is an arithmetic average around a node r, this discrete steady-state is a second order approximation of the continuous one.

High order discretization of the steady-state

- To begin we consider the steady-state $\nabla p = -\rho \nabla \phi$
- we integrate on the dual cell Ω_r^* (volume V_r) to obtain

$$V_r \left(\frac{1}{V_r} \int_{\Omega_r^*} \nabla p(\mathbf{x}) \right) = -V_r \left(\frac{1}{V_r} \int_{\Omega_r^*} \rho(\mathbf{x}) \nabla \phi(\mathbf{x}) \right).$$

- We introduce 3 polynomials $\overline{\rho}_r(\mathbf{x})$ (order q), $\overline{p}_r(\mathbf{x})$ and $\overline{\phi}_r(\mathbf{x})$ (q+1 order) with

$$\int_{\Omega_r^*} \overline{\rho}_r(\mathbf{x}) = \mid \Omega_l \mid \rho_l, \quad \int_{\Omega_r^*} \overline{p}_r(\mathbf{x}) = \mid \Omega_l \mid p_l, \quad \int_{\Omega_r^*} \overline{\phi}_r(\mathbf{x}) = \mid \Omega_l \mid \phi_l$$

  and $l \in S(r)$ ($S(r)$ a subset of cell around the node r).
- Now we incorporate this high-order reconstruction in the scheme. For this we need to have a pressure gradient which corresponds to the viscosity of the scheme.
- We obtain a *q*-order steady-state:

$$\underbrace{-\left(\sum_j \hat{\beta}_{jr} \right)^{-1} \sum_j p_j \mathbf{C}_{jr}}_{\nabla p_r} = -(\rho \nabla \phi)_r^{HO}$$

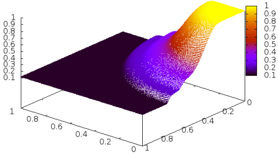
  with

$$(\rho \nabla \phi)_r^{HO} = \frac{1}{V_r} \left(\left(\int_{\Omega_r^*} \nabla p(\mathbf{x}) \right) + \left(\int_{\Omega_r^*} \rho(\mathbf{x}) \nabla \phi(\mathbf{x}) \right) \right) + \left(\sum_j \hat{\beta}_{jr} \right)^{-1} \sum_j p_j \mathbf{C}_{jr}$$

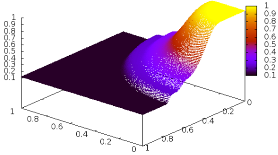
Numerical result : large opacity

- Test case: sod problem with $\sigma > 0$, $\varepsilon = 1$ and $\nabla \phi = \mathbf{0}$.
- $\sigma = 1$

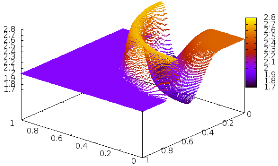
AP scheme, ρ



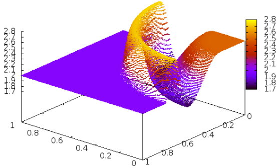
non-AP scheme, ρ



AP scheme, ϵ



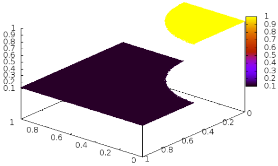
non-AP scheme, ϵ



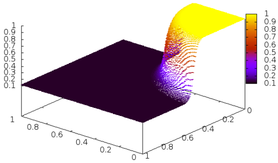
Numerical result : large opacity

- Test case: sod problem with $\sigma > 0$, $\varepsilon = 1$ and $\nabla \phi = \mathbf{0}$.
- $\sigma = 10^6$

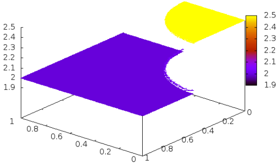
AP scheme, ρ



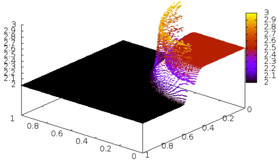
non-AP scheme, ρ



AP scheme, ϵ



non-AP scheme, ϵ



Result for steady-state

- **1D Steady-state**: $\rho(t,x) = 3 + 2\sin(2\pi x)$, $u(t,x) = 0$
- $p(t,x) = 3 + 3\sin(2\pi x) - \frac{1}{2}\cos(4\pi x)$ and $\phi(x) = -\sin(2\pi x)$. **Random 1D Grid**.

Cells	LR		LR-AP(2)		LR-AP O(3)		LR-AP O(4)	
	Error	q	Error	q	Error	q	Error	q
20	0.8335	-	0.0102	-	0.0079	-	0.0067	-
40	0.4010	1.05	0.0027	1.91	8.4E-4	3.23	1.5E-4	5.48
80	0.2065	0.96	7.0E-4	1.95	7.7E-5	3.45	4.1E-6	5.19
160	0.1014	1.02	1.7E-4	2.04	7.0E-6	3.46	1.0E-7	5.36

- **2D Steady-state**: $\rho(t,\mathbf{x}) = e^{-\mathbf{x},\mathbf{g}}$, $u(t,\mathbf{x}) = 0$, $p(t,\mathbf{x}) = e^{-\mathbf{x},\mathbf{g}}$ ans $\phi = (\mathbf{x},\mathbf{g})$.

	Cells	LR		LR-AP O(2)		LR-AP O(3)	
		Error	q	Error	q	Error	q
Cartesian Mesh	16×16	0.04132	1.07	0.00147	2.34	5.47E-6	3.8
	32×32	0.02013	1.04	3.28E-4	2.16	3.67E-7	3.9
	64×64	0.00993	1.02	7.65E-5	2.1	2.38E-8	3.95
	128×128	0.00493	1.01	1.90E-5	2.1	1.52E-9	3.96
Random Cartesian Mesh	16×16	0.05465	0.86	0.00155	2.7	8.25E-6	3.47
	32×32	0.02940	0.89	3.4E-4	2.18	7.55E-7	3.45
	64×64	0.01488	0.98	7.98E-5	2.09	8.5E-8	3.15
	128×128	0.00742	1.00	2.06E-5	1.95	2.37E-8	1.84

Result for steady-state

- **1D Steady-state**: $\rho(t,x)=3+2\sin(2\pi x)$, $u(t,x)=0$
- $p(t,x)=3+3\sin(2\pi x)-\frac{1}{2}\cos(4\pi x)$ and $\phi(x)=-\sin(2\pi x)$. **Random 1D Grid**.

Cells	LR		LR-AP(2)		LR-AP O(3)		LR-AP O(4)	
	Error	q	Error	q	Error	q	Error	q
20	0.8335	-	0.0102	-	0.0079	-	0.0067	-
40	0.4010	1.05	0.0027	1.91	8.4E-4	3.23	1.5E-4	5.48
80	0.2065	0.96	7.0E-4	1.95	7.7E-5	3.45	4.1E-6	5.19
160	0.1014	1.02	1.7E-4	2.04	7.0E-6	3.46	1.0E-7	5.36

- **2D Steady-state**: $\rho(t,\mathbf{x})=e^{-\mathbf{x},\mathbf{g}}$, $u(t,\mathbf{x})=0$, $p(t,\mathbf{x})=e^{-\mathbf{x},\mathbf{g}}$ ans $\phi=(\mathbf{x},\mathbf{g})$.

	Cells	LR		LR-AP O(2)		LR-AP O(3)	
		Error	q	Error	q	Error	q
Collela Mesh	16×16	0.08902	0.45	0.00197	2.44	2.97E-5	1.9
	32×32	0.05725	0.63	5.9E-4	1.74	5.43E-6	2.45
	64×64	0.03232	0.82	1.6E-4	1.88	5.93E-7	3.19
	128×128	0.01711	0.92	4.5E-5	1.86	4.68E-8	3.66
Kershaw Mesh	16×16	0.08376	0.83	3.38E-4	2.36	6.13E-6	3.84
	32×32	0.04253	0.98	7.29E-5	2.24	3.97E-7	3.95
	64×64	0.02060	1.05	7.87E-5	2.13	2.03E-8	4.3
	128×128	0.00988	1.06	4.34E-6	1.9	1.77E-9	3.52

Conclusion and perspectives

- Conclusion
  - P_1 **model**: First AP scheme on unstructured meshes (now other schemes have been developed).
  - P_1 **model**: Uniform proof of convergence on unstructured meshes in 1D and 2D for the implicit scheme.
  - An extension for general Friedrich's systems have been also studied (algebraic micro-macro decomposition)
  - **Euler model with external force**: AP schemes for the high friction regime.
  - **Euler model with external force**: new high-order reconstruction of the hydrostatic steady-state.
  - **Problem for all the schemes** : spurious mods in few cases (example: Cartesian mesh + Dirac Initial data).
- Possible perspectives
  - P_1 **model**: Theoretical study of the explicit and semi-implicit scheme (CFL independent of ε).
  - **Euler model**: Entropy study for the AP-WB scheme.
  - **Euler model**: Validate on analytic case the convergence of the diffusion scheme for nonlinear pressure law.
  - Find a generic procedure to stabilize the nodal schemes (B. Després and E. Labourasse for the Lagrangian Euler equations).

Stage CEA DAM

- **Project**: "implicit scheme and preconditioning for radiative transfer" models" with Xavier Blanc, Emmanuel Labourasse + Master student ?

Transport equation (photonics neutronic):

- The distribution function $f(t, \mathbf{x}, \mathbf{\Omega})$ with $\mathbf{\Omega}$ the direction, c the light speed satisfy

$$\partial_t f + c\mathbf{\Omega} \cdot \nabla f = c\sigma \left(\int_{S^2} f d\mathbf{\Omega} - f \right)$$

- The kinetic equations are approximated by linear hyperbolic P_n systems:

$$\partial_t \mathbf{U} + cA_x \partial_x \mathbf{U} + cA_y \partial_y \mathbf{U} + cA_z \partial_Z \mathbf{U} = -c\sigma R\mathbf{U}$$

- Important regimes: free transport regime ($\sigma \rightarrow 0$) : exact transport of the solution and diffusion regime ($\sigma \rightarrow \infty$).
- **Problems for explicit scheme**: Very large and stiff hyperbolic systems. Stiff hyperbolic CFL for explicit schemes, Stiff parabolic CFL condition for the AP schemes.
- **Problems for implicit scheme**: the large hyperbolic system (bad structure) and the large ratio between wave velocities ($\{\lambda_{min} c, ..., \lambda_{max} c\}$ with $\lambda_{min} \approx -1$, $\lambda_{max} \approx 1$).
- **Aim**: Test a physic-based preconditioning + GMRES for the P_1 model. Extend this preconditioning to the P_n models and the transport regime.

Thanks

Thank you