Hierarchy of fluids model for plasma and Adaptive Physic-Based Preconditioning

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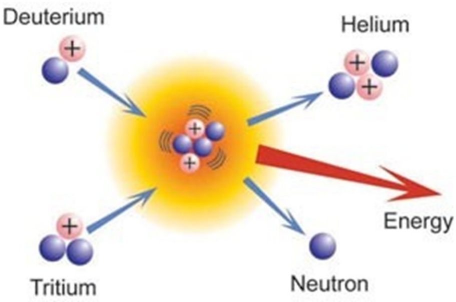
Outline

Mathematical context and JOREK code

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. A neutron and 17.6 MeV of free energy are released. At those energies, the atoms are ionized forming a plasma.
- **Plasma**: For very high temperature, the gas are ionized and gives a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroidal room where the plasma is confined using powerful magnetic fields.
- **ITER**: International project of fusion nuclear plant to validate the nuclear fusion as a power source.



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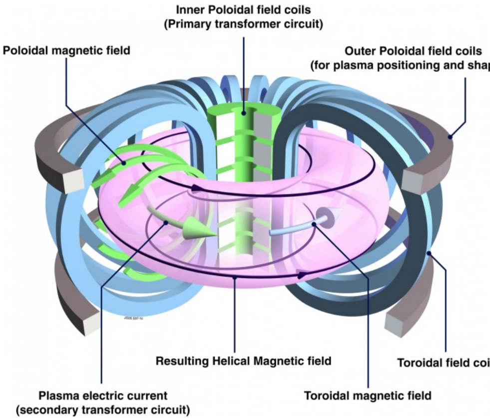
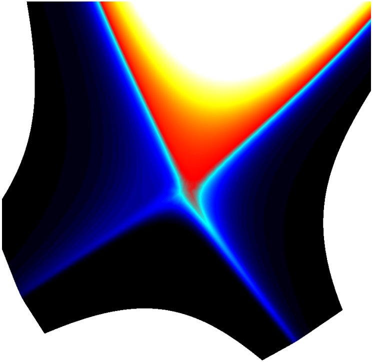


Figure: Tokamak

Physical context : MHD and ELM's

- In the tokamak some instabilities can appear at the edge of the plasmas.
- The simulation to these instabilities is an important subject for ITER.
- Exemple of Edge Instabilities in the tokamak :
  - **Disruptions**: Violent edge instabilities which can damage critically the tokamak.
  - **Edge Localized Modes (ELMs')**: Periodic edge instabilities which can damage the Tokamak.
- These instabilities are linked to the very large gradient of pressure and very large current at the edge.
- These instabilities are described by fluid models (MHD resistive and diamagnetic or extended).

- ELM's simulation



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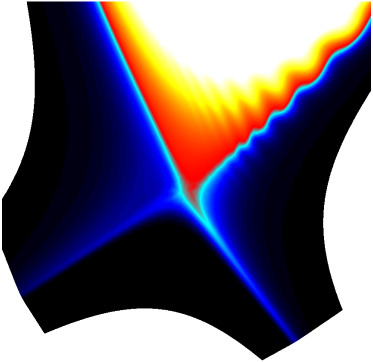

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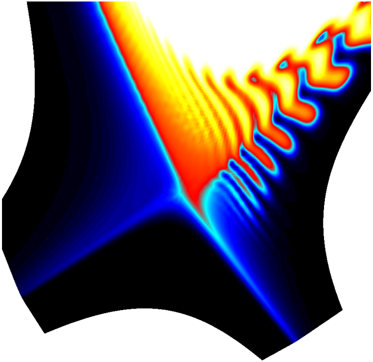

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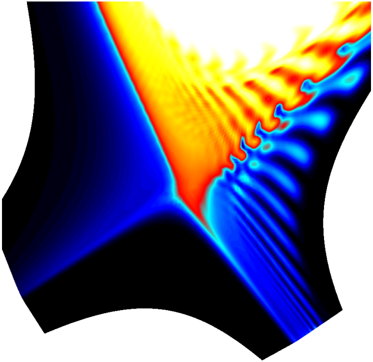

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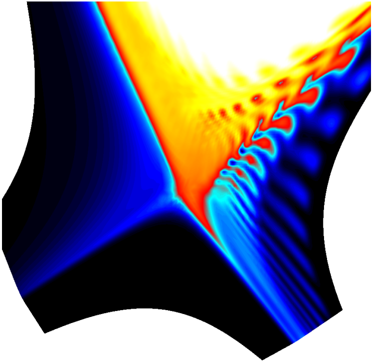

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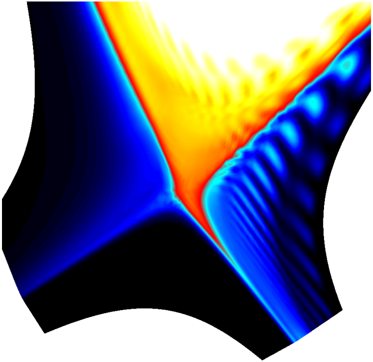

Vlasov equation

- First model to describe a plasma : **Two species Vlasov-Maxwell** kinetic equation.
- We define $f_s(t,\mathbf{x},\mathbf{v})$ the distribution function associated with the species s. $\mathbf{x}\in D_{\mathbf{x}}$ and $\mathbf{v}\in R^3$.

Two fluids Vlasov equation

$$\begin{cases} \partial_t f_s + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \dfrac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f_s = C_s = \sum_t C_{st}, \\ \frac{1}{c^2}\partial_t \mathbf{E} - \nabla\times\mathbf{B} = -\mu_0 \mathbf{J}, \\ \partial_t \mathbf{B} = -\nabla\times\mathbf{E}, \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\cdot\mathbf{E} = \frac{\sigma}{\varepsilon_0}. \end{cases}$$

- Derivation of two fluid model :
  - We apply this operator $\int_{R^3} g(\mathbf{v})(\cdot)$ on the equation.
  - $g(\mathbf{v})_s = 1, m_s\mathbf{v}, m_s|\mathbf{v}|^2$.
- Using
  - $\int_{D_{\mathbf{v}}} m_s\mathbf{v}C_{ss}d\mathbf{v} = 0, \quad \int_{D_{\mathbf{v}}} m_s|\mathbf{v}|^2 C_{ss}d\mathbf{v} = 0,$
  - $\int_{D_{\mathbf{v}}} g(\mathbf{v})_s C_{st}d\mathbf{v} + \int_{D_{\mathbf{v}}} g(\mathbf{v})_t C_{ts}d\mathbf{v} = 0.$

Two fluid model

- Computing the moment of the Vlasov equations we obtain the following two fluid model

Two fluid moments

$$
\left\{
\begin{array}{l}
\partial_t n_s + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s) = 0, \\
\partial_t (m_s n_s \mathbf{u}_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s \otimes \mathbf{u}_s) + \nabla_{\mathbf{x}} p_s + \nabla_{\mathbf{x}} \cdot \overline{\overline{\mathbf{\Pi}}}_s = \sigma_s \mathbf{E} + \mathbf{J}_s \times \mathbf{B} + \mathbf{R}_s, \\
\partial_t (m_s n_s \epsilon_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \mathbf{u}_s \epsilon_s + p_s \mathbf{u}_s) + \nabla_{\mathbf{x}} \cdot \left(\overline{\overline{\mathbf{\Pi}}}_s \cdot \mathbf{u}_s + \mathbf{q}_s \right) \\
= \sigma_s \mathbf{E} \cdot \mathbf{u}_s + Q_s + \mathbf{R}_s \cdot \mathbf{u}_s, \\
\\
\frac{1}{c^2} \partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}, \\
\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \\
\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \frac{\sigma}{\varepsilon_0}.
\end{array}
\right.
$$

- $n_s = \int_{D_{\mathbf{v}}} f_s d\mathbf{v}$ the particle number , $m_s n_s \mathbf{u}_s = \int_{D_{\mathbf{v}}} m_s \mathbf{v} f_s d\mathbf{v}$ the momentum, ϵ_s the energy.
- The isotropic pressure are p_s, $\overline{\overline{\mathbf{\Pi}}}_s$ the stress tensors and $\mathbf{q}_s$ the heat fluxes.
- $\mathbf{R}_s$ and Q_s associated with the interspecies collision (force and energy transfer).
- The current is given by $\mathbf{J} = \sum_s \mathbf{J}_s = \sum_s \sigma_s \mathbf{u}_s$ with $\sigma_s = q_s n_s$.

Extended MHD: assumptions and generalized Ohm law

Extended MHD: assumptions

- **quasi neutrality assumption**: $n_i = n_e$
  - Since $m_e << m_i$ therefore $\rho = m_i n_i + m_e n_e \approx m_i n_i$
  - Since $m_e << m_i$ therefore $\mathbf{u} = \frac{m_i n_i \mathbf{u}_i + m_e n_e \mathbf{u}_e}{\rho} \approx \mathbf{u}_i$
- **Magnetostatic assumption** : $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (characteristic velocity $<< c$)

- Taking the electronic density and momentum equations we obtain

$$m_e\left(\partial_t(n_e\mathbf{u}_e) + \nabla\cdot(n_e\mathbf{u}_e\otimes\mathbf{u}_e)\right) + \nabla p_e = -en_e\mathbf{E} + \mathbf{J}_e\times\mathbf{B} - \nabla\cdot\overline{\overline{\Pi}}_e + \mathbf{R}_e,$$

- We multiply the previous equation by $-e$ and we define $\mathbf{J}_e = -en_e\mathbf{u}_e$, we obtain

$$\frac{m_e}{e^2 n_e}\left(\partial_t\mathbf{J}_e + \nabla\cdot(\mathbf{J}_e\otimes\mathbf{u}_e)\right) = \mathbf{E} + \mathbf{u}_e\times\mathbf{B} + \frac{1}{en_e}\nabla p_e + \frac{1}{en_e}\nabla\cdot\overline{\overline{\Pi}}_e - \frac{1}{en_e}\mathbf{R}_e,$$

- Using the quasi neutrality, $m_e << m_i$ and $\mathbf{R} = -\mathbf{R}_e = -\eta\frac{e}{m_i}\rho\mathbf{J}$, we obtain

Generalized Ohm law

$$\mathbf{E} + \mathbf{u}\times\mathbf{B} = \eta\mathbf{J} - \frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\Pi}}_e + \frac{m_i}{\rho e}\mathbf{J}\times\mathbf{B} - \frac{m_i}{\rho e}\nabla p_e.$$

Extended MHD

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B} - \nabla \cdot \overline{\overline{\mathbf{\Pi}}}, \\ \frac{1}{\gamma - 1} \partial_t p + \frac{1}{\gamma - 1} \mathbf{u} \cdot \nabla p + \frac{\gamma}{\gamma - 1} p \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{q} = \frac{1}{\gamma - 1} \frac{m_i}{e\rho} \mathbf{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) \\ - \overline{\overline{\mathbf{\Pi}}} : \nabla \mathbf{u} + \overline{\overline{\mathbf{\Pi}}}_e : \nabla \left(\frac{m_i}{e\rho} \mathbf{J} \right) + \eta |\mathbf{J}|^2, \\ \partial_t \mathbf{B} = -\nabla \times \left(-\mathbf{u} \times \mathbf{B} + \eta \mathbf{J} - \frac{m_i}{\rho e} \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_e - \frac{m_i}{\rho e} \nabla p_e + \frac{m_i}{\rho e} (\mathbf{J} \times \mathbf{B}) \right), \\ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mathbf{J}. \end{cases}$$

- The total energy for the MHD is given by $E = \rho \frac{|\mathbf{u}|^2}{2} + \frac{|\mathbf{B}|^2}{2} + \frac{1}{\gamma - 1} p$ with $p = \rho T$ and $\gamma = \frac{5}{3}$. The conservation law for the total energy is given by

$$\partial_t E + \nabla \cdot \left[\mathbf{u} \left(\rho \frac{|\mathbf{u}|^2}{2} + \frac{\gamma}{\gamma - 1} p \right) - (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \right] + + \nabla \cdot \mathbf{q} + \nabla \cdot (\overline{\overline{\mathbf{\Pi}}} \cdot \mathbf{u}) + \eta \nabla \cdot (\mathbf{J} \times \mathbf{B})$$

$$+ \nabla \cdot \left[\frac{m_i}{\rho e} \left((\mathbf{J} \times \mathbf{B}) \times \mathbf{B} - \nabla p_e \times \mathbf{B} - \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_e \times \mathbf{B} - \frac{\gamma}{\gamma - 1} p_e \mathbf{J} - \mathbf{J} \cdot \overline{\overline{\mathbf{\Pi}}}_e \right) \right] = 0$$

Reduced MHD: assumptions and principle of derivation

- **Aim**: Reduce the number of variables and eliminate the fast waves in the reduced MHD model.
- We consider the cylindrical coordinate $(R, Z, \phi) \in \Omega \times [0, 2\pi]$.

Reduced MHD: Assumption

$$\mathbf{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi \quad \mathbf{u} = -R\nabla u \times \mathbf{e}_\phi + v_{||}\mathbf{B} + \tau_{IC}\frac{R}{\rho}\left(\mathbf{e}_\phi \times \nabla p\right)$$

with u the electrical potential, ψ the magnetic poloidal flux, $v_{||}$ the parallel velocity.

- To avoid high order operators, we introduce the vorticity $w = \Delta_{pol} u$ and the toroidal current $j = \triangle^* \psi = R^2 \nabla \cdot (\frac{1}{R^2}\nabla_{pol}\psi)$.
- Derivation: we plug $\mathbf{B}$ and $\mathbf{u}$ in the equations + some computations. For the equations on u and $v_{||}$ we use the following projections

$$\mathbf{e}_\phi \cdot \nabla \times R^2 \left(\rho\partial_t \mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{J}\times\mathbf{B} + \nu\Delta\mathbf{u}\right)$$

and

$$\mathbf{B}\cdot \left(\rho\partial_t \mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{J}\times\mathbf{B} + \nu\Delta\mathbf{u}\right).$$

Description of the JOREK code

- JOREK: Fortran 90 code, parallel (MPI+OpenMP)
- Determine the equilibrium
  - Define the boundary of the computational domain
  - Compute $\psi(R,Z)$ on a first poloidal grid.
- Compute equilibrium solving Grad-Shafranov equation

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}=-R^2\frac{\partial p}{\partial\psi}-F\frac{\partial F}{\partial\psi}$$

- Computation of aligned grid
  - Identification of the magnetic flux surfaces
  - Create the aligned grid (with X-point)
  - Interpolate $\psi(R,Z)$ in the new grid and recompute the equilibrium
- **Perturbation of the equilibrium** (small perturbations of non principal harmonics).
- Time-stepping (full implicit)
  - **Poloidal discretization**: 2D Cubic Bezier finite elements.
  - **Toroidal discretization**: Fourier expansion.

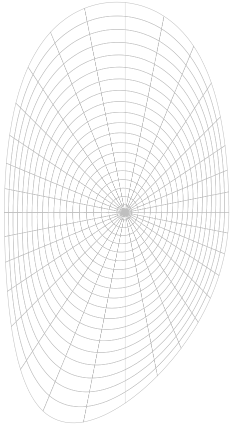

Figure: unaligned grid

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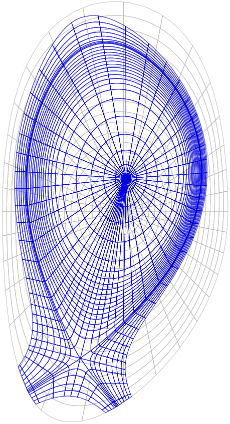

Figure: Aligned grid

Linear Solvers

- We solve a nonlinear problem $G(\mathbf{U}^{n+1}) = b(\mathbf{U}^n, \mathbf{U}^{n-1})$.
- First order linearization

$$\left(\frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}\right)\delta\mathbf{U}^n = -G(\mathbf{U}^n) + b(\mathbf{U}^n, \mathbf{U}^{n-1}) = R(\mathbf{U}^n),$$

  with $\delta\mathbf{U}^n = \mathbf{U}^{n+1} - \mathbf{U}^n$, and $J_n = \frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}^n}$ the Jacobian matrix of $G(\mathbf{U}^n)$.
- Linear solver in JOREK: Left Preconditioning + GMRES iterative solver.
- Principle of the preconditioning step:
  - Replace the problem $J_k\delta\mathbf{U}_k = R(\mathbf{U}^n)$ by $P_k(P_k^{-1}J_k)\delta\mathbf{U}_k = R(\mathbf{U}^n)$.
  - Solve the new system with two steps $P_k\delta\mathbf{U}_k^* = R(\mathbf{U}^n)$ and $(P_k^{-1}J_k)\delta\mathbf{U}_k = \delta\mathbf{U}_k^*$
- If P_k is easier to invert than J_k and $P_k \approx J_k$ the linear solving step is more robust and efficient.

Physic-based Preconditioning of JOREK

- Extraction of the blocks which are associated with each toroidal harmonic.
- Solve exactly with LU decomposition each subsystem associated with a block
- Reconstruction of the solution of $P_k\mathbf{x} = \mathbf{b}$

- **Principle of Physic-based preconditioning**: We neglect in the Jacobian the physical effect associated to the coupling between the Fourier mods (non diagonal block).

Physic based preconditioning for Waves equations

Implicit scheme for Damped waves equations

- Damping wave equation (baby problem used for Inertial fusion confinement)

$$\left\{\begin{array}{l}\partial_t p + c\nabla\cdot\mathbf{u} = 0\\ \\ \partial_t \mathbf{u} + c\nabla p = -c\sigma\mathbf{u}\end{array}\right. \iff \left\{\begin{array}{l}\partial_t p + \dfrac{1}{\varepsilon}\nabla\cdot\mathbf{u} = 0\\ \\ \partial_t \mathbf{u} + \dfrac{1}{\varepsilon}\nabla p = -\dfrac{\sigma}{\varepsilon^2}\mathbf{u}\end{array}\right.$$

- with σ opacity, c light speed and $\varepsilon \approx \frac{1}{c} \approx \frac{1}{\sigma}$
- When $\varepsilon \longrightarrow 0$ the model can be approximated by $\partial_t p - \nabla\cdot(\frac{1}{\sigma}\nabla p) = 0$.
- This problem is stiff in time. CFL condition is $\Delta t \leq C_1\varepsilon h + C_2\varepsilon^2$.
- Simple way to solve this: implicit scheme but the model is ill-conditioned.
- Two sources of ill-conditioning: the stiff terms (which depend of ε) and the hyperbolic structure.

We propose a preconditioning (work of L. Chacon) which

- allows to treat the stiffness using a reformulation,
- rewrites the hyperbolic system as a second order equation (well-conditioned) which can be solved easily,
- can be extend to the nonlinear hyperbolic system as MHD (and resistive MHD with additional splitting steps).

Construction of the preconditioning I

- First we implicit the equation

$$\begin{cases} p^{n+1} + \theta\dfrac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^{n+1} = p^n - (1-\theta)\dfrac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^n \\ \mathbf{u}^{n+1} + \theta\dfrac{\Delta t}{\varepsilon}\nabla p^{n+1} + \theta\dfrac{\Delta t\sigma}{\varepsilon^2}\mathbf{u}^{n+1} = \mathbf{u}^n - (1-\theta)\dfrac{\Delta t}{\varepsilon}\nabla p^n - (1-\theta)\dfrac{\Delta t\sigma}{\varepsilon^2}\mathbf{u}^n \end{cases}$$

- The implicit system is given by

$$\begin{pmatrix} M & U \\ L & D \end{pmatrix}\begin{pmatrix} p^{n+1} \\ \mathbf{u^{n+1}} \end{pmatrix} = \begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

with $M = I_d$, $D = \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix}$, $U = \begin{pmatrix} \theta\dfrac{\Delta t}{\varepsilon}\partial_x & \dfrac{\Delta t}{\varepsilon}\partial_y \end{pmatrix}$ and $L = \begin{pmatrix} \alpha\theta\dfrac{\Delta t}{\varepsilon}\partial_x \\ \alpha\theta\dfrac{\Delta t}{\varepsilon}\partial_y \end{pmatrix}$

- The solution of the system is given

$$\begin{pmatrix} p^{n+1} \\ \mathbf{u^{n+1}} \end{pmatrix} = \begin{pmatrix} M & U \\ L & D \end{pmatrix}^{-1}\begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

$$= \begin{pmatrix} I & M^{-1}U \\ 0 & I \end{pmatrix}\begin{pmatrix} M^{-1} & 0 \\ 0 & P_{schur}^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -LM^{-1} & I \end{pmatrix}\begin{pmatrix} R_p \\ R_{\mathbf{u}} \end{pmatrix}$$

with $P_{schur} = D - LM^{-1}U$.

Construction of the preconditioning I

- Secondly we rewrite the equation

$$\begin{cases} p^{n+1}+\theta\dfrac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^{n+1}=p^n-(1-\theta)\dfrac{\Delta t}{\varepsilon}\nabla\cdot\mathbf{u}^n \\ \mathbf{u}^{n+1}+\theta\dfrac{\alpha\Delta t}{\varepsilon}\nabla p^{n+1}=\alpha\mathbf{u}^n-(1-\theta)\dfrac{\alpha\Delta t}{\varepsilon}\nabla p^n-\alpha(1-\theta)\dfrac{\alpha\Delta t\sigma}{\varepsilon^2}\mathbf{u}^n \end{cases}$$

- with $\alpha=\dfrac{\varepsilon^2}{\varepsilon^2+\theta\sigma\Delta t}$
- The implicit system is given by

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Principle of the preconditioning II

- Using the previous Schur decomposition we can solve the implicit wave equation with the algorithm.

$$\begin{cases} \text{Predictor}: \quad M_h \mathbf{p}^* = R_p \\ \text{Velocity evolution}: \quad P_h \mathbf{u}^{n+1} = (-L_h \mathbf{p}^* + R_{\mathbf{u}}) \\ \text{Corrector}: \quad M \mathbf{p}^{n+1} = M_h \mathbf{p}^* - U_h \mathbf{u}_{n+1} \end{cases}$$

- with the matrices:
  - M_h the mass matrix which discretize the Identity operator
  - U_h discretize the operator U and L_h the discretization of the L operator.
  - P_h discretize the positive and symmetric operator :

$$P_{Schur} = I_d - \theta^2 \frac{\alpha \Delta t^2}{\varepsilon^2} \begin{pmatrix} \partial_{xx} & \partial_{xy} \\ \partial_{yx} & \partial_{yy} \end{pmatrix}$$

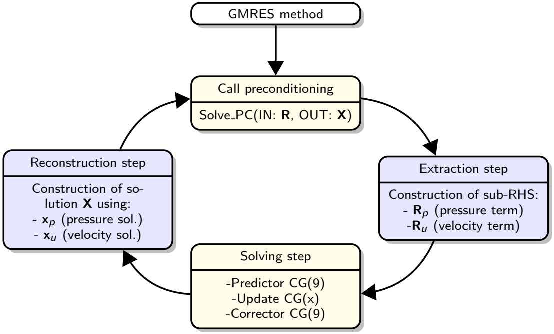
- The physic based preconditioning $PB(x)$ solves the previous algorithm with Conjugate-Gradient with $\varepsilon = 10^{-x}$ and Jacobi PC.

Future study

- The weak form of the Schur operator is not coercive. Study Mix methods.
- The Mass matrix are not not easy to invert for B-Splines. Specific PC based on $M \approx A \otimes B$ with A and B one 1D matrices

Algorithm of the PhyBas Preconditioning step

- Algorithm and implementation of the $PB(x)$ preconditioning:



- In this case we solve the sub-steps with a GC solver
- We can use also Multi-grid (MG) methods or other methods efficient for symmetric and diagonal dominant matrix.

Results for Waves equation

- Comparison between iterative solver for test case in the diffusion limit $\sigma = 1$.

Mesh / solvers		GC	GC-PC	Gmres	Gmres-PC-Jacobi
Mesh 4*4, ε_1	cv iter	✗ -	✗ -	✗ -	✓ 27
Mesh 16*16, ε_1	cv iter	✗ -	✗ -	✗ -	✓ 1.5E+4
Mesh 4*4, ε_2	cv iter	✗ -	✗ -	✗ -	✓ 21000
Mesh 16*16, ε_2	cv iter	✗ -	✗ -	✗ -	✗ -

- $\varepsilon_1 = 10^{-5}$ and $\varepsilon_2 = 10^{-10}$.
- The solver tolerance is 10^{-10} for convergence and iter_max=100000. We compute the average on ten time iterations.
- The GC solver is iunstable and cannot solve this type of problem.

A conclusion

- The results show that it is necessary to use a good preconditioning + robust solver (for general matrix).

Results for Waves equation

- Comparison between GMRES method with different preconditioning

Mesh / solvers		Jac	ILU(0)	ILU(4)	MG(2)	SOR	PB
Mesh4*4, ε_1	cv iter time	✓ 27 7.2 E-4	✓ 11 1.3E-3	✓ 7.7E-3	✓ 38 1.5E-2	✓ 8 1.4E-3	✓ 1 2.1E-3
4*4, ε_2	cv iter time	✓ 2.1E+4 3.6E-1	✓ 11 1.3E-3	✓ 1 7.7E-3	✗ - -	✓ 8 1.5E-3	✓ 1 2.1E-3
16*16, ε_1	cv iter time	✓ 1.5E+4 5.0E-0	✓ 18 2.3E-2	✓ 9 4.0E-1	✗ 140 5.0E-1	✓ 20 5.0E-2	✓ 1 2.1E-2
16*16, ε_2	cv iter time	✗ - -	✓ 18 2.3E-2	✓ 9 4.0E-1	✗ - -	✓ 20 5.0E-2	✓ 1 2.1E-2
64*64, ε_2	cv iter time	✗ - -	✗ - -	✓ 632 2.0E+1	✗ - -	✗ - -	x 1 4.2E-1

- ILU (Incomplete LU), MG (Multi-grids), SOR, PB (our physic based PC).

A conclusion

- On fine grid our method is the fastest (and the current implementation is not optimal).

Physic based preconditioning for MHD equations

Current Hole and preconditioning associated

- Current Hole : reduced problem in cartesian coordinates.
- The model

$$\begin{cases} \partial_t \psi = [\psi, u] + \eta \Delta \psi \\ \partial_t \Delta u = [\Delta u, u] + [\psi, \Delta \psi] + \nu \Delta^2 u \end{cases}$$

  with $w = \Delta u$ and $j = \Delta \psi$.
- In this formulation we split evolution and elliptic equations.
- For the time discretization we use a Cranck-Nicholson scheme and linearized the nonlinear system to obtain

$$\begin{pmatrix} M & U \\ L & D \end{pmatrix} \begin{pmatrix} \Delta \psi^n \\ \Delta u^n \end{pmatrix} = \begin{pmatrix} R_\psi \\ R_u \end{pmatrix}$$

  or

$$\begin{pmatrix} I_d - \Delta t \theta [\cdot, u^n] - \Delta t \theta \Delta & -\Delta \theta [\psi^n, \cdot] \\ -\Delta t \theta [\psi^n, \Delta \cdot] - \Delta t \theta [\cdot, \Delta \psi^n] & \Delta - \Delta t \theta ([\Delta \cdot, u^n] + [\cdot, \Delta u^n] + \Delta^2) \end{pmatrix} \begin{pmatrix} \delta \psi^n \\ \delta u^n \end{pmatrix} = \begin{pmatrix} R_\psi \\ R_u \end{pmatrix}$$

Design of the preconditioning for reduced MHD

PB-PC for Current Hole

$$\begin{cases} \text{Predictor :} \quad M\delta\psi_p^n = R_\psi \\ \text{potential update :} \quad P_{schur}\delta u^n = \left(-L\delta\psi_p^n + R_u\right) \\ \text{Corrector :} \quad M\delta\psi^n = M\delta\psi_p^n - U\delta u^n \\ \text{Current update :} \quad \delta z_j^n = \Delta\delta\psi^n \\ \text{Vorticity update :} \quad \delta w^n = \Delta\delta u^n \end{cases}$$

- The schur complement is given by $P_{schur} = D - LM^{-1}U$
- Two approximations for M^{-1}:
  - **Slow flow**: $M^{-1} = \Delta t$
  - **Arbitrary flow**: find M^* such that $UM^* \approx MU$. Consequently

$$P^{-1} = (D - LM^{-1}U)^{-1} \approx M^*(DM^* - LU)^{-1},$$

  we obtain

$$\begin{cases} \text{potential update I :} \quad (DM^* - LU)\delta u^{**} = \left(-L\delta\psi_p^n + R_u\right) \\ \text{potential update II :} \quad \delta u^n = M^*\delta u^{**} \end{cases}$$

- **Last question** : Computation of the operator LU (second order form of the coupling hyperbolic operators).

Approximation of the Schur complement I

- Computation of Schur complement for (slow flow approximation $M^{-1} \approx \Delta t$)

$$P_{schur} = \frac{\Delta \delta u}{\Delta t} + \mathbf{u}^n \cdot \nabla(\Delta \delta u) + \delta \mathbf{u} \cdot \nabla(\Delta u^n) - \theta \nu \Delta^2 \delta u - \theta^2 \Delta t L U$$

- Operator $LU = \mathbf{B}^n \cdot \nabla(\Delta(\mathbf{B}^n \cdot \nabla \delta u)) + \frac{\partial j^n}{\partial \psi^n} \mathbf{B}^n_{pol} \cdot \nabla(\mathbf{B}^n \cdot \nabla \delta u)$.
- $\mathbf{B}^n \cdot \nabla \delta u = -[\psi^n, \delta u]$ and $\mathbf{u}^n \cdot \nabla \delta u = -[\delta u, u^n]$ et $\delta \mathbf{u} \cdot \nabla u^n = -[u^n, \delta u]$.

- **Remark**: the LU operator is the parabolization of coupling hyperbolic terms which contains only the Alfvén waves (rigorous proof missing).

Properties of LU operator

- We consider the L^2 space. The operator LU is not self adjoint and not positive for all δu

$$< LU\delta u, \delta u >_{L^2} = \int |\nabla(\mathbf{B}^n \cdot \nabla \delta u)|^2 - \int \frac{\partial j^n}{\partial \psi^n} (\mathbf{B}^n_{pol} \cdot \nabla \delta u)(\mathbf{B}^n \cdot \nabla \delta u)$$

- We propose the following approximation $LU^{approx} = \mathbf{B}^n \cdot \nabla(\Delta(\mathbf{B}^n \cdot \nabla \delta u))$.
- The operator LU^{approx} is positive and self-adjoint.

- There are different methods to solve the Schur complement using splitting to solve smaller and more simple operators.

Results for Current Hole Model

- Comparison between GMRES method with different preconditioning
- 50 time step in the linear phase (kink instability ?). $tol = 10^{-8}$, $iter\_max = 10000$.

Mesh / solvers		Jac	ILU(0)	ILU(4)	MG	SOR	PB(6)	PB(4)
16*16 dt=0.5	cv	✗	✓	✓	✗	✓	✓	✓
	iter	-	14	6	-	12	1	1
	time	-	1.2E-1	1.4E+0	-	1.8E-1	2.6E+0	2.3E+0
32*32 dt=1	cv	✗	✓	✓	✗	✗	✓	✓
	iter	-	26	9	-	-	1	1
	time	-	6.8E-1	7.2E+0	-	-	9.8E+0	8.9E+0
64*64 dt=4	cv	✗	✓	✓	✗	✗	✓	✓
	iter	-	404	84	-	-	1	1
	time	-	2.4E+1	3.9E+1	-	-	3.9E+1	3.8E+1

- On fine grid our method is the more robust and competitive
- This is not optimal because :
  - The matrices (7 in this case) are assembled one by one and not at the same time.
  - The extraction and reconstruction are made one by one.
  - The assembly of the matrices in Django are not optimal (PETSC configuration).
  - We solve each sub-system with a GMRES-MG(2) and not just a MG solver.
- 75% of the solving time comes from to the construction of the sub-matrices. In the future we will assume that it is possible to decrease this part by 5-6.

Adaptive PhyBas preconditioning

Idea

- The PhyBas PC is based on physical approximations of the equations. We can also add approximations of the discretization in space.
- Indeed, we can use a less order approximation in the PC to reduce the size of the matrices and the storage and keep a good efficiency.

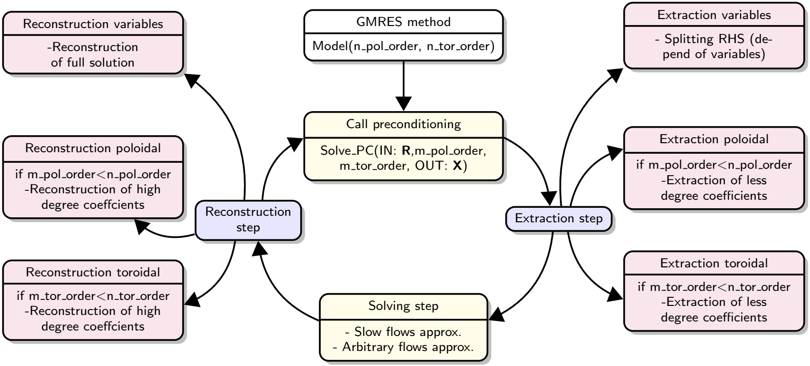
Applications to MHD PC

- We can call the preconditioning with
  - poloidal and toroidal orders of the B-Splines smaller than the orders used for the full model.
  - poloidal and toroidal regularity of the B-Splines different than the regularity used for the full model.
  - less Fourier harmonics than for the full model (we keep the coupling terms but neglect harmonics).
- Some restriction and interpolation steps must be added in the "extraction" and "reconstruction" steps.

- **Remark**: At the end, the user could choose the order and number of Harmonics for the PC (different that for the model) and adapt these parameters during the simulation.

Algorithm of the adaptive PhyBas Preconditioning step

- Algorithm and implementation of the $APB(x)$ preconditioning:



- In the future it is important to perform the extraction and reconstruction parts.

Conclusion

Conclusion:

- The idea to design a PC is to write the solving step as a suitability of simple operators (easy to invert) using splitting and reformulation (second order formulation) methods.
- The possible approximations gives the PC algorithm.
- **Problem**: the proposed method is dependent of the problem and use a lot of methods (CG, MG, GMRES etc) $\Longrightarrow$ lot of work to treat all the models.

Possible approximations:

- **Solving approximation**: each sub step can be solved with a small accuracy.
- **Physical approximation**: each subsystem can be simplified to obtain well-conditioned operators (necessary in the MHD case).
- **Discretization approximation**: the systems associated with the PC can be solved with less order numerical methods or coarser grids.
- **Multi-discretization approximation**: the PC models and the model can be discretized with different methods (finite element for PC and DG for the full system).

Others applications:

- **Shallow water equations and ocean flows**: Cemracs 2015 Project.
- **Radiative transfer**: project with CEA (DAM).