Nodal finite volumes for hyperbolic systems with source terms on unstructured meshes

E. Franck[1], C. Buet[2], B. Després [3]
T.Leroy[2,3], L. Mendoza [4]

August 17, 2015

[1]INRIA Nancy Grand-Est and IRMA Strasbourg, TONUS team, France
[2]CEA DAM, Arpajon, France
[3]LJLL, UPMC, France
[4]IPP, Garching bei München, Germany

Outline

Mathematic and physical context

Stiff hyperbolic systems

- **Stiff hyperbolic system with source terms**:

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon}\partial_x F(\mathbf{U}) + \frac{1}{\varepsilon}\partial_y G(\mathbf{U}) = \frac{1}{\varepsilon} S(\mathbf{U}) - \frac{\sigma}{\varepsilon^2} R(\mathbf{U}),\ \mathbf{U} \in R^n$$

with $\varepsilon \in]0,1]$ et $\sigma > 0$.

- Subset of solutions given by the balance between the source terms and the convective part:
  - **Diffusion solutions** for $\varepsilon \to 0$ and $S(\mathbf{U}) = 0$:

$$\partial_t \mathbf{V} - \text{div}\left(K(\nabla \mathbf{V}, \sigma)\right) = 0, \quad \mathbf{V} \in \text{Ker}\, R.$$

  - **Steady states** for $\sigma = 0$ et $\varepsilon \to 0$:

$$\partial_x F(\mathbf{U}) + \partial_y G(\mathbf{U}) = S(\mathbf{U}).$$

- Applications: biology, neutron transport, fluid mechanics, plasma physics, Radiative hydrodynamic for inertial fusion (hydrodynamic + linear transport of photon).

Well-Balanced schemes

- Discretization of physical steady states is important (Lack at rest for Shallow water equations, hydrostatic equilibrium for astrophysical flows ..)
- **Classical scheme**: the physical steady states or a good discretization of the steady states are not the equilibriums of the scheme.
- **Consequence**: Spurious numerical velocities larger than physical velocities for nearly or exact uniform flows.

WB scheme: definitions

- **Exact Well-Balanced scheme**: is a scheme exact for continuous steady-states.
- **Well-Balanced scheme**: is a scheme exact for discrete steady-states at the interfaces.

- **For shallow water model**: in general the schemes are exact WB schemes.
- **For Euler model**: in general the schemes are WB schemes.

Schémas "Asymptotic preserving"

- P_1 model:

$$\begin{cases} \partial_t E + \dfrac{1}{\varepsilon}\partial_x F = 0, \\ \partial_t F + \dfrac{1}{\varepsilon}\partial_x E = -\dfrac{\sigma}{\varepsilon^2} F, \end{cases} \longrightarrow \partial_t E - \partial_x \left(\frac{1}{\sigma} \partial_x E \right) = 0.$$

$$\begin{tikzcd} P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\ P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0 \end{tikzcd}$$

Figure: AP diagram

- Consistency of **Godunov-type** schemes: $O(\dfrac{\Delta x}{\varepsilon} + \Delta t)$.
- CFL condition: $\Delta t(\dfrac{1}{\Delta x \varepsilon} + \dfrac{\sigma}{\varepsilon^2}) \leq 1$.
- Consistency of AP schemes: $O\left(\Delta x + \Delta t\right)$.
- CFL condition: $$\Delta t \left(\frac{1}{\Delta x \varepsilon + \frac{\Delta x^2}{\sigma}} \right) \leq 1.$$
- AP vs non AP schemes: Important reduction of CPU cost.

- AP schemes are obtained plugging the source term into the fluxes (WB technic).

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon} F + O(\varepsilon^2)$ in the fluxes.

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon} F + O(\varepsilon^2)$ in the fluxes.

we write the relations

$$\begin{cases} E(x_j) = E(x_{j+\frac{1}{2}}) + (x_j - x_{j+\frac{1}{2}})\partial_x E(x_{j+\frac{1}{2}}), \\ E(x_{j+1}) = E(x_{j+\frac{1}{2}}) + (x_{j+1} - x_{j+\frac{1}{2}})\partial_x E(x_{j+\frac{1}{2}}). \end{cases}$$

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon} F + O(\varepsilon^2)$ in the fluxes.

we write the relations

$$\left\{ \begin{array}{l} E(x_j) = E(x_{j+\frac{1}{2}}) - (x_j - x_{j+\frac{1}{2}}) \frac{\sigma}{\varepsilon} F(x_{j+\frac{1}{2}}), \\ E(x_{j+1}) = E(x_{j+\frac{1}{2}}) - (x_{j+1} - x_{j+\frac{1}{2}}) \frac{\sigma}{\varepsilon} F(x_{j+\frac{1}{2}}). \end{array} \right.$$

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon}F + O(\varepsilon^2)$ in the fluxes.

we write the relations

$$\left\{\begin{array}{l} E(x_j) = E(x_{j+\frac{1}{2}}) - (x_j - x_{j+\frac{1}{2}})\frac{\sigma}{\varepsilon}F(x_{j+\frac{1}{2}}), \\ E(x_{j+1}) = E(x_{j+\frac{1}{2}}) - (x_{j+1} - x_{j+\frac{1}{2}})\frac{\sigma}{\varepsilon}F(x_{j+\frac{1}{2}}). \end{array}\right.$$

We couple these relations with the fluxes

$$\left\{\begin{array}{l} F_j + E_j = F_{j+\frac{1}{2}} + E_{j+\frac{1}{2}}, \\ F_{j+1} - E_{j+1} = F_{j+\frac{1}{2}} - E_{j+\frac{1}{2}}. \end{array}\right.$$

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon} F + O(\varepsilon^2)$ in the fluxes.

we write the relations

$$\left\{\begin{array}{l} E(x_j) = E(x_{j+\frac{1}{2}}) - (x_j - x_{j+\frac{1}{2}})\frac{\sigma}{\varepsilon} F(x_{j+\frac{1}{2}}), \\ E(x_{j+1}) = E(x_{j+\frac{1}{2}}) - (x_{j+1} - x_{j+\frac{1}{2}})\frac{\sigma}{\varepsilon} F(x_{j+\frac{1}{2}}). \end{array}\right.$$

$$\left\{\begin{array}{l} F_j + E_j = F_{j+\frac{1}{2}} + E_{j+\frac{1}{2}} + \frac{\sigma \Delta x}{2\varepsilon} F_{j+\frac{1}{2}}, \\ F_{j+1} - E_{j+1} = F_{j+\frac{1}{2}} - E_{j+\frac{1}{2}} + \frac{\sigma \Delta x}{2\varepsilon} F_{j+\frac{1}{2}}. \end{array}\right.$$

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon} F + O(\varepsilon^2)$ in the fluxes.

Jin Levermore scheme:

$$\left\{ \begin{array}{l} \frac{E_j^{n+1}-E_j^n}{\Delta t} + M \frac{F_{j+1}^n - F_{j-1}^n}{2\varepsilon \Delta x} - M \frac{E_{j+1}^n - 2E_j^n + E_{j-1}^n}{2\varepsilon \Delta x} = 0, \\ \frac{F_j^{n+1}-F_j^n}{\Delta t} + \frac{E_{j+1}^n - E_{j-1}^n}{2\varepsilon \Delta x} - \frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{2\varepsilon \Delta x} + \frac{\sigma}{\varepsilon^2} F_j^n = 0, \end{array} \right.$$

with $M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x}$.

AP Godunov schemes

- **Jin-Levermore scheme**
- **Principle**: plug the balance law $\partial_x E = -\frac{\sigma}{\varepsilon} F + O(\varepsilon^2)$ in the fluxes.

Gosse-Toscani scheme:

$$\begin{cases} \frac{E_j^{n+1}-E_j^n}{\Delta t} + M \frac{F_{j+1}^n - F_{j-1}^n}{2\varepsilon \Delta x} - M \frac{E_{j+1}^n - 2E_j^n + E_{j-1}^n}{2\varepsilon \Delta x} = 0, \\ \frac{F_j^{n+1}-F_j^n}{\Delta t} + M \frac{E_{j+1}^n - E_{j-1}^n}{2\varepsilon \Delta x} - M \frac{F_{j+1}^n - 2F_j^n + F_{j-1}^n}{2\varepsilon \Delta x} + M \frac{\sigma}{\varepsilon^2} F_j^n = 0, \end{cases}$$

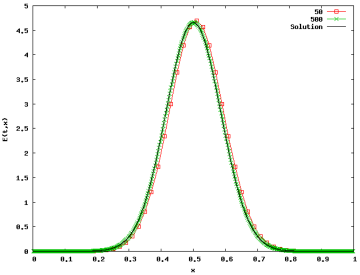
avec $M = \frac{2\varepsilon}{2\varepsilon + \sigma \Delta x}$.

- consistency error for the **Jin-Levermore** scheme:
  - first equation: $O\left(\Delta x^2 + \varepsilon \Delta x + \Delta t\right)$,
  - second equation: $O\left(\frac{\Delta x^2}{\varepsilon} + \Delta x + \Delta t\right)$.
- Explicit CFL: $\Delta t \left(\frac{1}{\Delta x \varepsilon} + \frac{\sigma}{\varepsilon^2}\right) \leq 1$.
- Semi-implicit CFL: $\Delta t \left(\frac{1}{\Delta x \varepsilon}\right) \leq 1$.

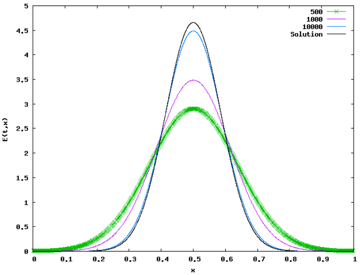
- **Principle of GT scheme**: JL-scheme with the source term $\frac{1}{2}(F_{j+\frac{1}{2}} + F_{j-\frac{1}{2}})$ gives the Gosse-Toscani scheme.
- Consistency error of the **Gosse-Toscani** scheme: $O\left(\Delta x + \Delta t\right)$.
- Explicit CFL: $\Delta t \left(\frac{1}{\Delta x \varepsilon}\right) \leq 1$.
- Semi-implicit CFL : $\Delta t \left(\frac{1}{\Delta x \varepsilon + \Delta x^2}\right) \leq 1$.

- **Validation test for AP scheme**: the data are $E(0,x)=G(x)$ with $G(x)$ a Gaussian $F(0,x)=0$ and $\sigma=1$, $\varepsilon=0.001$.

Ap scheme



Godunov scheme



Scheme	L^1 error	CPU time
Godunov, 10000 cells	0.0366	1485m4.26s
Godunov, 500 cells	0.445	0m24.317s
AP, 500 cells	0.0001	0m15.22s
AP, 50 cells	0.0065	0m0.054s

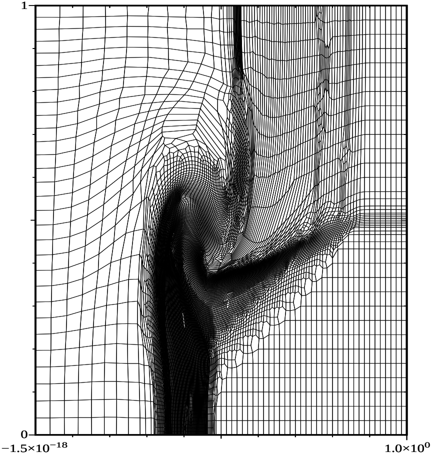
Non complete state of art

- S. Jin, D. Levermore, *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms*, (1996).
- C. Berthon, R. Turpault, *Asymptotic preserving HLL schemes*, (2012).
- L. Gosse, G. Toscani, *An asymptotic-preserving well-balanced scheme for the hyperbolic heat equations*, (2002).
- C. Berthon, P. Charrier and B. Dubroca, *An HLLC scheme to solve the M_1 model of radiative transfer in two space dimensions*, (2007).
- C. Chalons, M. Girardin, S. Kokh, *Large time step asymptotic preserving numerical schemes for the gas dynamics equations with source terms*, (2013).
- C. Chalons, F. Coquel, E. Godlewski, P-A. Raviart, N. Seguin, *Godunov-type schemes for hyperbolic systems with parameter dependent source*, (2010).

- R. Natalini and M. Ribot, *An asymptotic high order mass-preserving scheme for a hyperbolic model of chemotaxis*, (2012).
- M. Zenk, C. Berthon et C. Klingenberg, *A well-balanced scheme for the Euler equations with a gravitational potential*, (2014).
- J. Greenberg, A. Y. Leroux, *A well balanced scheme for the numerical processing of source terms in hyperbolic equations*, (1996).
- R. Kappeli, S. Mishra, *Well-balanced schemes for the Euler equations with gravitation*, (2013).

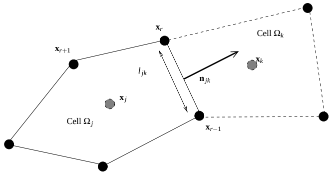
Why unstructured meshes ?

- **Applications** : coupling between radiation and hydrodynamic
- **In some hydrodynamic codes**: Lagrangian or ALE scheme cell-centered for multi-material problems.
- Example of meshes obtained using a ALE code.
- **Aim**: Design and analyze AP cell-centered for linear transport on general meshes.



Schémas "Asymptotic preserving" 2D

- **Classical extension in 2D of the Jin-Levermore scheme** : modify the upwind fluxes (1D fluxes write in the normal direction) plugging the steady states in the fluxes.



- l_{jk} and $\mathbf{n}_{jk}$ the normal and length associated with the edge $\partial\Omega_{jk}$.

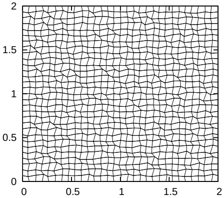
Asymptotic limit of the scheme:

$$\mid \Omega_j \mid \partial_t E_j(t) - \frac{1}{\sigma}\sum_k l_{jk}\frac{E_k^n - E_j^n}{d(\mathbf{x}_j, \mathbf{x}_k)} = 0.$$

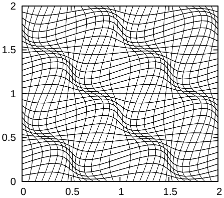
- $||P_h^0 - P_h|| \to 0$ only on strong geometrical conditions.
- These AP schemes **do not converge** on 2D general meshes $\forall \varepsilon$.

Example of unstructured meshes

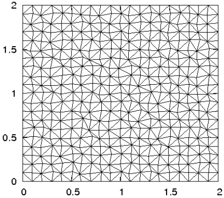
Random mesh



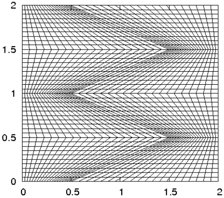
Collela mesh



Random triangular mesh



Kershaw mesh



AP scheme for the P_1 model

Nodal scheme : linear case

- Linear case: P_1 model

$$\begin{cases} \partial_t E + \frac{1}{\varepsilon}\,\mathrm{div}(\mathbf{F}) = 0, \\ \partial_t \mathbf{F} + \frac{1}{\varepsilon}\nabla E = -\frac{\sigma}{\varepsilon^2}\mathbf{F}. \end{cases} \longrightarrow \partial_t E - \mathrm{div}\left(\frac{1}{\sigma}\nabla E\right) = 0.$$

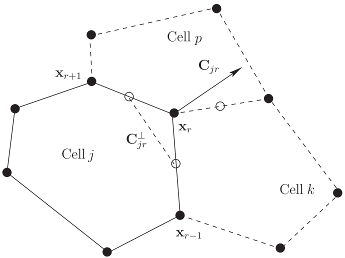
Idea:

Nodal finit evolume methods for P_1 model + AP and WB method.

Nodal schemes:

The fluxes are localized at the nodes of the mesh (for the classical scheme this is at the edge).

Notations



- Nodal geometrical quantities $\mathbf{C}_{jr} = \nabla_{\mathbf{x}_r}|\Omega_j|$.
- $\sum_j \mathbf{C}_{jr} = \sum_r \mathbf{C}_{jr} = \mathbf{0}$.

Nodal AP scheme

$$\left\{\begin{array}{l} \mid\Omega_j\mid\partial_t E_j(t)+\dfrac{1}{\varepsilon}\displaystyle\sum_r(\mathbf{F}_r,\mathbf{C}_{jr})=0,\\ \mid\Omega_j\mid\partial_t \mathbf{F}_j(t)+\dfrac{1}{\varepsilon}\displaystyle\sum_r\mathbf{Ec}_{jr}=\mathbf{S}_j. \end{array}\right.$$

- Classical nodal fluxes:

$$\left\{\begin{array}{l} \mathbf{Ec}_{jr}-E_j\mathbf{C}_{jr}=\widehat{\alpha}_{jr}(\mathbf{F}_j-\mathbf{F}_r),\\ \sum_j\mathbf{Ec}_{jr}=\mathbf{0}, \end{array}\right.$$

with $\widehat{\alpha}_{jr}=\frac{\mathbf{C}_{jr}\otimes\mathbf{C}_{jr}}{\|\mathbf{C}_{jr}\|}$.

- New fluxes obtained plugging steady-state $\nabla E=-\frac{\sigma}{\varepsilon}\mathbf{F}$ in the fluxes:

$$\left\{\begin{array}{l} \mathbf{Ec}_{jr}-E_j\mathbf{C}_{jr}=\widehat{\alpha}_{jr}(\mathbf{F}_j-\mathbf{F}_r)-\dfrac{\sigma}{\varepsilon}\widehat{\beta}_{jr}\mathbf{F}_r,\\ \left(\displaystyle\sum_j\widehat{\alpha}_{jr}+\dfrac{\sigma}{\varepsilon}\sum_j\widehat{\beta}_{jr}\right)\mathbf{F}_r=\displaystyle\sum_j E_j\mathbf{C}_{jr}+\sum_j\widehat{\alpha}_{jr}\mathbf{F}_j. \end{array}\right.$$

with $\widehat{\beta}_{jr}=\mathbf{C}_{jr}\otimes(\mathbf{x}_r-\mathbf{x}_j)$.

- Source term: (1) $\mathbf{S}_j=-\frac{\sigma}{\varepsilon^2}\mid\Omega_j\mid\mathbf{F}_j$ ou (2) $\mathbf{S}_j=-\frac{\sigma}{\varepsilon^2}\sum_r\widehat{\beta}_{jr}\mathbf{F}_r$, $\quad\sum_r\widehat{\beta}_{jr}=\hat{I}_d|\Omega_j|$.

Time AP scheme

- New formulation of the scheme + semi discrete scheme.

Local semi-implicit scheme

$$\begin{cases} \mid \Omega_j \mid \dfrac{E_j^{n+1} - E_j^n}{\triangle t} + \dfrac{1}{\varepsilon} \sum_r (M_r \mathbf{F}_r, \mathbf{C}_{jr}) = 0, \\ \mid \Omega_j \mid \dfrac{\mathbf{F}_j^{n+1} - \mathbf{F}_j^n}{\triangle t} + \dfrac{1}{\varepsilon} \sum_r \mathbf{Ec}_{jr} = -\dfrac{1}{\varepsilon} \left(\sum_r \widehat{\alpha}_{jr} (\widehat{I}_d - M_r) \right) \mathbf{F}_j^{n+1}. \end{cases}$$

with

$$\begin{cases} \mathbf{Ec}_{jr} - E_j \mathbf{C}_{jr} = \widehat{\alpha}_{jr} M_r (\mathbf{F}_j - \mathbf{F}_r), \\ \left(\sum_j \widehat{\alpha}_{jr} \right) \mathbf{F}_r = \sum_j E_j \mathbf{C}_{jr} + \sum_j \widehat{\alpha}_{jr} \mathbf{F}_j. \end{cases}$$

$$M_r = \left(\sum_j \widehat{\alpha}_{jr} + \frac{\sigma}{\varepsilon} \sum_j \widehat{\beta}_{jr} \right)^{-1} \left(\sum_j \widehat{\alpha}_{jr} \right)$$

- The scheme is stable under a CFL condition which is the sum to the parabolic and hyperbolic CFL conditions (verified numerically).
- The full implicit version is unconditionally stable.

Assumptions for the convergence proof

Geometrical assumptions

- $\left(\mathbf{u}, \left(\sum_r \frac{\mathbf{C}_{jr}\otimes\mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}\right)\mathbf{u}\right) \geq \beta h(\mathbf{u},\mathbf{u})$,
- $\left(\mathbf{u}, \left(\sum_j \frac{\mathbf{C}_{jr}\otimes\mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}\right)\mathbf{u}\right) \geq \gamma h(\mathbf{u},\mathbf{u})$,
- $\left(\mathbf{u}, \left(\sum_j \mathbf{C}_{jr}\otimes(\mathbf{x}_r-\mathbf{x}_j)\right)\mathbf{u}\right) \geq \alpha h^2(\mathbf{u},\mathbf{u})$.

- First and second assumptions: true on all non degenerated meshes.
- Last assumption: sufficient (not necessary) conditions on the meshes obtained.
- Example for triangles: all the angles must be larger that 12 degrees.

Assumption on regularity and initial data

- $\mathbf{F}(t=0,\mathbf{x}) = -\frac{\varepsilon}{\sigma}\nabla E(t=0,\mathbf{x})$
- Regularity for exact data: $\mathbf{V}(t,\mathbf{x}) \in H^4(\Omega)$
- Regularity for initial data of the scheme: $\mathbf{V}_h(t=0,\mathbf{x}) \in L^2(\Omega)$

Uniform convergence in space

- Naive convergence estimate : $||P_h^\varepsilon - P^\varepsilon||_{naive} \leq C\varepsilon^{-b}h^c$
- **Idea**: use triangular inequalities and AP diagram (Jin-Levermore-Golse).

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(||P_h^\varepsilon - P^\varepsilon||_{naive}, ||P_h^\varepsilon - P_h^0|| + ||P_h^0 - P^0|| + ||P^\varepsilon - P^0||)$$

$$\begin{tikzcd} P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\ P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0 \end{tikzcd}$$

- Intermediary estimations :
  - $||P^\varepsilon - P^0|| \leq C_a\varepsilon^a$,
  - $||P_h^0 - P^0|| \leq C_d h^d$,
  - $||P_h^\varepsilon - P_h^0|| \leq C_e\varepsilon^e$,
  - $d \leq c$, $e \geq a$.

- We obtain:

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq C\min(\varepsilon^{-b}h^c, \varepsilon^a + h^d + \varepsilon^e))$$

.

- Comparing ε and $\varepsilon_{threshold} = h^{\frac{ac}{a+b}}$ we obtain the final estimation:

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq h^{\frac{ac}{a+b}}$$

.

Limit diffusion scheme

Limit diffusion scheme (P_h^0):

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t E_j(t) - \sum_r (\mathbf{F}_r, \mathbf{C}_{jr}) = 0, \\ \sum_r \hat{\alpha}_{jr} \mathbf{F}_j = \sum_r \hat{\alpha}_{jr} \mathbf{F}_r, \\ \sigma A_r \mathbf{F}_r = \sum_j E_j \mathbf{C}_{jr}, \quad A_r = -\sum_j \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j). \end{array}\right.$$

$$\begin{tikzcd} P_h^\varepsilon \arrow[r, "\varepsilon \to 0"] \arrow[d, "h \to 0"'] & P_h^0 \arrow[d, "h \to 0"] \\ P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0 \end{tikzcd}$$

- **Problem**: estimation on $||P_h^\varepsilon - P_h^0||$.
- In practice we obtain $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$ (not sufficient for the proof).

H Condition:

The Hessian matrix of the scheme P_h^0 can be upper-bounded or the error estimate $\|P_h^\varepsilon - P_h^0\|$ can be obtained independently of the discrete Hessian matrix.

Limit diffusion scheme

Limit diffusion scheme (P_h^0):

$$
\begin{cases}
\mid \Omega_j \mid \partial_t E_j(t) - \sum_r (\mathbf{F}_r, \mathbf{C}_{jr}) = 0, \\
\sum_r \hat{\alpha}_{jr} \mathbf{F}_j = \sum_r \hat{\alpha}_{jr} \mathbf{F}_r, \\
\sigma A_r \mathbf{F}_r = \sum_j E_j \mathbf{C}_{jr}, \quad A_r = -\sum_j \mathbf{C}_{jr} \otimes (\mathbf{x}_r - \mathbf{x}_j).
\end{cases}
$$

$$
\begin{tikzcd}
P_h^\varepsilon \arrow[r, "\partial_t \mathbf{v}_h = 0"] \arrow[d, "h \to 0"'] & {\color{blue} DA_h^\varepsilon} \arrow[r, dashed, "\varepsilon \to 0"] \arrow[d, "h \to 0 \text{ and } \varepsilon \to 0"] & P_h^0 \arrow[dl, dashed] \\
P^\varepsilon \arrow[r, "\varepsilon \to 0"'] & P^0 &
\end{tikzcd}
$$

- **Problem**: estimation on $||P_h^\varepsilon - P_h^0||$.
- In practice we obtain $||P_h^\varepsilon - P_h^0|| \leq C\frac{\varepsilon}{h}$ (not sufficient for the proof).
- Introduction of a intermediary diffusion scheme DA_h^ε.
- DA_h^ε: P_h^ε scheme with $\partial_t \mathbf{F}_j = \mathbf{0}$.
- In the previous estimate we replace P_h^0 by DA_h^ε.

H Condition:

The Hessian matrix of the scheme P_h^0 can be upper-bounded or the error estimate $\|P_h^\varepsilon - P_h^0\|$ can be obtained independently of the discrete Hessian matrix.

Final result in space

- H condition obtained : we use P_h^0 in the estimates.
- H condition not obtained : we use DA_h^ε in the estimates.
- The H condition is obtained in 1D (grid uniform or not) and in 2D Cartesian grids.

Final result:

We assume that the assumptions are verified. There are some constant $C > 0$ such that

- $||P^\varepsilon - P_h^\varepsilon||_{naive} \leq C_0\sqrt{\frac{h}{\varepsilon}} \parallel p_0 \parallel_{H^4(\Omega)}$,
- $||DA_h^\varepsilon - P^0|| \leq C_1(h+\varepsilon) \parallel p_0 \parallel_{H^4(\Omega)}$,
- $||P_h^\varepsilon - DA_h^\varepsilon|| \leq C_2 \left(h^2 + \varepsilon \max\left(1, \sqrt{\varepsilon h^{-1}}\right)\right) \parallel p_0 \parallel_{H^4(\Omega)}$,
- $||P^\varepsilon - P^0|| \leq C_3\varepsilon, \qquad 0 < t \leq T$.

and

$$\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq C \min\left(\sqrt{\frac{h}{\varepsilon}}, h^2 + \varepsilon \max\left(1, \sqrt{\frac{\varepsilon}{h}}\right) + (h+\varepsilon) + \varepsilon\right) \parallel p_0 \parallel_{H^4} \leq C h^{\frac{1}{4}}.$$

- Using $\varepsilon_{thresh} = h^{\frac{1}{2}}$ we prove that the worst case is $\|\mathbf{V}^\varepsilon - \mathbf{V}_h^\varepsilon\| \leq C_2 h^{\frac{1}{4}}$.

Time estimation

- **Time scheme**: implicit scheme (the estimate for explicit scheme is an open question). We obtain

$$\frac{\mathbf{U}_h^{n+1}-\mathbf{U}_h^n}{\Delta t}=A_h\mathbf{U}_h^{n+1}$$

with A_h the matrix which discretized the space scheme.

- **Discrete stability**: We have $(\mathbf{U}_h, A_h\mathbf{U}_h)\leq 0$. Consequently $\parallel \mathbf{U}_h^{n+1} \parallel \leq \parallel \mathbf{U}_h^n \parallel$

Final result for the full discrete scheme

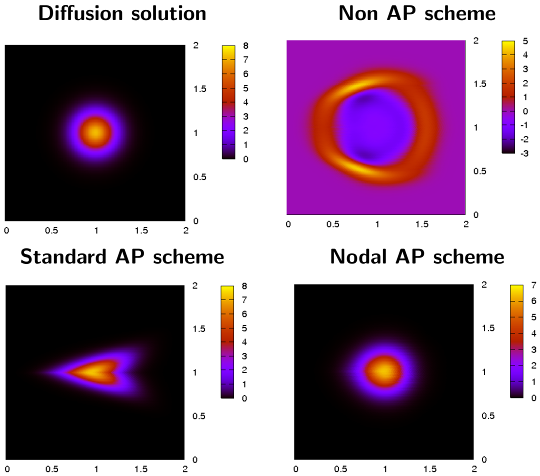
We assume that the regularity and geometrical assumptions are verified. There is a constant $C(T)>0$ such that:

$$\|\mathbf{V}^{\varepsilon}(t_n)-\mathbf{V}_h^{\varepsilon}(t_n)\|_{L^2(\Omega)}\leq C\left(f(h,\varepsilon)+\Delta t^{\frac{1}{2}}\right)\parallel p_0 \parallel_{H^4(\Omega)} .$$

- Idea of proof: Stability result + Duhamel formula (B. Després).

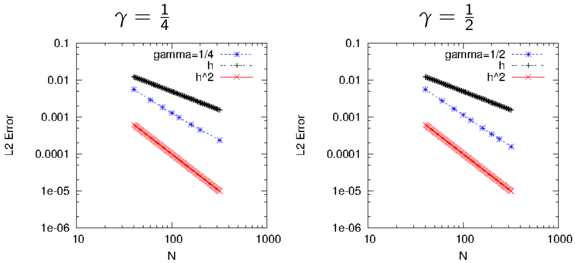
AP scheme vs classical scheme

- Test case: heat fundamental solution. Results for different P_1 scheme with $\varepsilon = 0.001$ on Kershaw mesh.



Uniform convergence for the P_1 model

- Periodic solution for the P_1which depend of ε.
- $E(t,\mathbf{x}) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x)\cos(\pi y)$
- $\mathbf{F}(t,\mathbf{x}) = \left(-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\cos(\pi y), \quad -\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi y)\cos(\pi x)\right)$
- Convergence study for $\varepsilon = h^\gamma$ on random mesh.

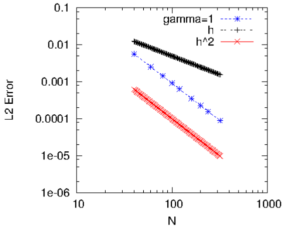


- Numerical results show that the error is homogenous to $O(h\varepsilon + h^2)$.
- Theoretical estimate that we can hope: $O((h\varepsilon)^{\frac{1}{2}} + h)$.
- Non optimal estimation in the intermediary regime.

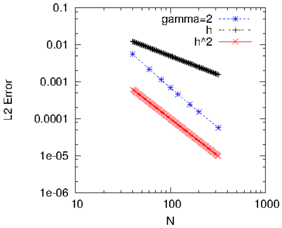
Uniform convergence for the P_1 model

- Periodic solution for the P_1which depend of ε.
- $E(t,\mathbf{x}) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x)\cos(\pi y)$
- $\mathbf{F}(t,\mathbf{x}) = \left(-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\cos(\pi y), \quad -\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi y)\cos(\pi x)\right)$
- Convergence study for $\varepsilon = h^\gamma$ on random mesh.

$\gamma = 1$



$\gamma = 2$



- Numerical results show that the error is homogenous to $O(h\varepsilon + h^2)$.
- Theoretical estimate that we can hope: $O((h\varepsilon)^{\frac{1}{2}} + h)$.
- Non optimal estimation in the intermediary regime.

Extension to the Euler model

Euler equation with external forces

- Euler equation with gravity and friction:

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho \mathbf{u} + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}\nabla p = -\frac{1}{\varepsilon}\left(\rho \nabla \phi + \frac{\sigma}{\varepsilon}\rho \mathbf{u}\right), \\ \partial_t \rho e + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} e) + \operatorname{div}(p\mathbf{u}) = -\frac{1}{\varepsilon}\left(\rho(\nabla \phi, \mathbf{u}) + \frac{\sigma}{\varepsilon}\rho(\mathbf{u}, \mathbf{u})\right). \end{cases}$$

- with ϕ the gravity potential, σ the friction coefficient.

Properties :

- Entropy inequality $\partial_t \rho S + \frac{1}{\varepsilon}\operatorname{div}(\rho \mathbf{u} S) \geq 0$.
- Steady-state :

$$\begin{cases} \mathbf{u} = \mathbf{0}, \\ \nabla p = -\rho \nabla \phi. \end{cases}$$

- Diffusion limit:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t \rho e + \operatorname{div}(\rho \mathbf{u} e) + p \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u} = -\frac{1}{\sigma}\left(\nabla \phi + \frac{1}{\rho}\nabla p\right). \end{cases}$$

Design of AP nodal scheme I

Idea :

Modify the Lagrange+remap classical scheme with the Jin-Levermore method

- Classical Lagrange+remap scheme (LP scheme):

$$\begin{cases} \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}\rho_j + \sum_{R_-} \mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j \mathbf{u}_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho \mathbf{U})_j + \sum_{R_-} \mathbf{u}_{jr}(\rho \mathbf{U})_{k(r)} + \sum_r \mathbf{pC}_{jr}\right) = 0 \\ \mid \Omega_j \mid \partial_t \rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+} \mathbf{u}_{jr}(\rho e)_j + \sum_{R_-} \mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r (\mathbf{pC}_{jr}, \mathbf{u}_r)\right) = 0 \end{cases}$$

with Lagrangian fluxes

$$\begin{cases} \mathbf{G}_{jr} = p_j \mathbf{C}_{jr} + \rho_j c_j \hat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) \\ \sum_j \rho_j c_j \hat{\alpha}_{jr} \mathbf{u}_r = \sum_j p_j \mathbf{C}_{jr} + \sum_j \rho_j c_j \hat{\alpha}_{jr} \mathbf{u}_j \end{cases}$$

- Advection fluxes: $\mathbf{u}_{jr} = (\mathbf{C}_{jr}, \mathbf{u}_r)$, $R_+ = (r / \mathbf{u}_{jr} > 0)$, $R_- = (r / \mathbf{u}_{jr} < 0)$ et $\rho_{k(r)} = \frac{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}\rho_j}{\sum_{j/\mathbf{u}_{jr}>0} \mathbf{u}_{jr}}$.

Design of AP nodal scheme II

Jin Levermore method:

Plug the relation $\nabla p + O(\varepsilon^2) = -\rho\nabla\phi - \frac{\sigma}{\varepsilon}\rho\mathbf{U}$ in the Lagrangian fluxes

- The modified scheme is given by

$$\left\{\begin{array}{l} \mid \Omega_j \mid \partial_t\rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+}\mathbf{u}_{jr}\rho_j + \sum_{R_-}\mathbf{u}_{jr}\rho_{k(r)}\right) = 0 \\ \mid \Omega_j \mid \partial_t\rho_j\mathbf{u}_j + \frac{1}{\varepsilon}\left(\sum_{R_+}\mathbf{u}_{jr}(\rho\mathbf{U})_j + \sum_{R_-}\mathbf{u}_{jr}(\rho\mathbf{U})_{k(r)} + \sum_r \mathbf{pC}_{jr}\right) \\ = -\frac{1}{\varepsilon}\left(\sum_r \hat{\beta}_{jr}(\rho\nabla\phi)_r + \frac{\sigma}{\varepsilon}\sum_r \rho_r\hat{\beta}_{jr}\mathbf{u}_r\right) \\ \mid \Omega_j \mid \partial_t\rho_j + \frac{1}{\varepsilon}\left(\sum_{R_+}\mathbf{u}_{jr}(\rho e)_j + \sum_{R_-}\mathbf{u}_{jr}(\rho e)_{k(r)} + \sum_r(\mathbf{pC}_{jr},\mathbf{u}_r)\right) \\ = -\frac{1}{\varepsilon}\left(\sum_r(\hat{\beta}_{jr}(\rho\nabla\phi)_r,\mathbf{u}_r) + \frac{\sigma}{\varepsilon}\sum_r \rho_r(\mathbf{u}_r,\hat{\beta}_{jr}\mathbf{u}_r)\right) \end{array}\right.$$

with the new Lagrangian fluxes

$$\left\{\begin{array}{l} \mathbf{pC}_{jr} = p_j\mathbf{C}_{jr} + \rho_j c_j\hat{\alpha}_{jr}(\mathbf{u}_j - \mathbf{u}_r) - \hat{\beta}_{jr}(\rho\nabla\phi)_r - \dfrac{\sigma}{\varepsilon}\rho_r\hat{\beta}_{jr}\mathbf{u}_r \\ \left(\displaystyle\sum_j \rho_j c_j\hat{\alpha}_{jr} + \frac{\sigma}{\varepsilon}\rho_r\sum_j\hat{\beta}_{jr}\right)\mathbf{u}_r = \displaystyle\sum_j p_j\mathbf{C}_{jr} + \sum_j \rho_j c_j\hat{\alpha}_{jr}\mathbf{u}_j - \left(\sum_j\hat{\beta}_{jr}\right)(\rho\nabla\phi)_r \end{array}\right.$$

- and $(\rho\nabla\phi)_r$ a discretization of $\rho\nabla\phi$ at the interface .

Properties

Limit diffusion scheme:

If the local matrices are invertible then the LR-AP scheme tends to the following scheme

$$\left\{\begin{array}{l} \mid\Omega_j\mid\partial_t\rho_j+\left(\sum_{R_+}\mathbf{u}_{jr}\rho_j+\sum_{R_-}\mathbf{u}_{jr}\rho_{k(r)}\right)=0\\ \mid\Omega_j\mid\partial_t\rho_j+\left(\sum_{R_+}\mathbf{u}_{jr}(\rho e)_j+\sum_{R_-}\mathbf{u}_{jr}(\rho e)_{k(r)}+p_j\sum_r(\mathbf{C}_{jr},\mathbf{u}_r)\right)=0\\ \sigma\rho_r\left(\sum_j\hat{\beta}_{jr}\right)\mathbf{u}_r=\sum_j p_j\mathbf{C}_{jr}-\left(\sum_j\hat{\beta}_{jr}\right)(\rho\nabla\phi)_r \end{array}\right.$$

- For $p=K\rho$, numerically the scheme converge at the order of the advection scheme.
- **Open question**: Verify this for a non isothermal pressure law as perfect gas law.

Well balanced property

- We define the discrete gradient $\nabla_r p=-(\sum_j\hat{\beta}_{jr})^{-1}\sum_j p_j\mathbf{C}_{jr}$ and ρ_r an average of ρ_j around $\mathbf{x}_r$.
- If the initial data are given by the discrete steady-state $\nabla_r p=-(\rho\nabla\phi)_r$, $\rho_j^{n+1}=\rho_j^n$, $\mathbf{u}_j^{n+1}=\mathbf{u}_j^n$ and $e_j^{n+1}=e_j^n$,
- **Remark**: if you initialize your scheme with a continuous steady-state the final space error is given by the consistency error between the continuous and discrete steady-state.

High order reconstruction of steady-state

- **Aim**: Conserve the stability property of the first order scheme but discretize the steady-state with a high order accuracy or exactly.
- **Method** : construct high order discrete steady-state

- 1D discrete steady state: $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}}$ with $(\rho \partial_x \phi)_{j+\frac{1}{2}} = \frac{1}{2}(\rho_{j+1} + \rho_j)(\phi_{j+1} - \phi_j)$.
- To begin we consider the steady state

$$\partial_x p = -\rho \partial_x \phi$$

- we integrate on the dual cell $[x_j, x_{j+1}]$ to obtain

$$\Delta x_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \partial_x p(x) \right) = -\Delta x_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \rho(x) \partial_x \phi(x) \right).$$

High order reconstruction of steady-state

- **Aim**: Conserve the stability property of the first order scheme but discretize the steady-state with a high order accuracy or exactly.
- **Method** : construct high order discrete steady-state

- We introduce 3 polynomials $\overline{\rho}_{j+\frac{1}{2}}(x)=\sum_{k=1}^{q} r_k x^k$ et $\overline{p}_{j+\frac{1}{2}}(x)=\sum_{k=1}^{q+1} p_k x^k, \overline{\phi}_{j+\frac{1}{2}}(x)=\sum_{k=1}^{q+1} \phi_k x^k$ with

$$\int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \overline{\rho}_{j+\frac{1}{2}}(x)=\Delta x_l \rho_l, \quad \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \overline{p}_{j+\frac{1}{2}}(x)=\Delta x_l p_l, \quad \int_{x_{l-\frac{1}{2}}}^{x_{l+\frac{1}{2}}} \overline{\phi}_{j+\frac{1}{2}}(x)=\Delta x_l \phi_l$$

and $l\in S(j)$ ($S(j)$ a subset of cell around j). Using these polynomials we obtain the new discrete steady-state

$$\Delta x_{j+\frac{1}{2}}\left(\frac{1}{\Delta x_{j+\frac{1}{2}}}\int_{x_j}^{x_{j+1}} \partial_x \overline{p}_{j+\frac{1}{2}}(x)\right)=-\Delta x_{j+\frac{1}{2}}\left(\frac{1}{\Delta x_{j+\frac{1}{2}}}\int_{x_j}^{x_{j+1}} \overline{\rho}_{j+\frac{1}{2}}(x)\partial_x\overline{\phi}_{j+\frac{1}{2}}(x)\right)$$

High order discretization of the steady-state

High order reconstruction of steady-state

- **Aim**: Conserve the stability property of the first order scheme but discretize the steady-state with a high order accuracy or exactly.
- **Method** : construct high order discrete steady-state

- To incorporate the discrete steady state in the scheme we need to have a pressure gradient which correspond to the viscosity of the scheme.
- We obtain a *q*-order steady-state:

$$p_{j+1}-p_j=-\Delta x_{j+\frac{1}{2}}(\rho\partial_x\phi)^{HO}_{j+\frac{1}{2}}$$

with

$$(\rho g)^{HO}_{j+\frac{1}{2}}=\frac{1}{\Delta x_{j+\frac{1}{2}}}\left(\left(\int_{x_j}^{x_{j+1}}\partial_x\overline{p}_{j+\frac{1}{2}}(x)\right)+\left(\int_{x_j}^{x_{j+1}}\overline{\rho}_{j+\frac{1}{2}}(x)\partial_x\overline{\phi}_{j+\frac{1}{2}}(x)\right)-(p_{j+1}-p_j)\right)$$

High order discretization of the steady-state

High order reconstruction of steady-state

- **Aim**: Conserve the stability property of the first order scheme but discretize the steady-state with a high order accuracy or exactly.
- **Method** : construct high order discrete steady-state

- To incorporate the discrete steady state in the scheme we need to have a pressure gradient which correspond to the viscosity of the scheme.
- We obtain a *q-order steady-state*:

$$p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)^{HO}_{j+\frac{1}{2}}$$

with

$$(\rho g)^{HO}_{j+\frac{1}{2}} = \frac{1}{\Delta x_{j+\frac{1}{2}}} \left(\left(\int_{x_j}^{x_{j+1}} \partial_x \overline{p}_{j+\frac{1}{2}}(x) \right) + \left(\int_{x_j}^{x_{j+1}} \overline{\rho}_{j+\frac{1}{2}}(x) \partial_x \overline{\phi}_{j+\frac{1}{2}}(x) \right) - (p_{j+1} - p_j) \right)$$

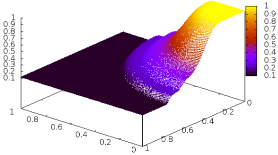
2D extension

- The method is the same. Just we use a constant stencil and a least square method to determinate the coefficient of the polynomials

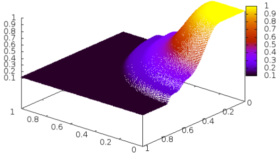
Numerical result : large opacity

- Test case: sod problem with $\sigma > 0$, $\varepsilon = 1$ and $\nabla\phi = \mathbf{0}$.
- $\sigma = 1$

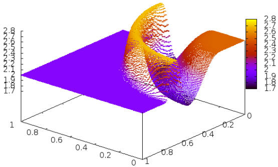
AP scheme, ρ



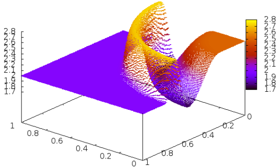
non-AP scheme, ρ



AP scheme, ϵ

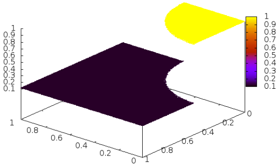


non-AP scheme, ϵ

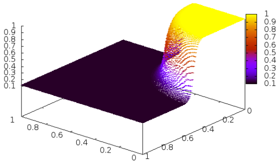


- Test case: sod problem with $\sigma > 0$, $\varepsilon = 1$ and $\nabla \phi = \mathbf{0}$.
- $\sigma = 10^6$

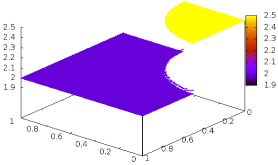
AP scheme, ρ



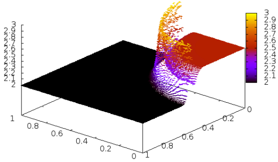
non-AP scheme, ρ



AP scheme, ϵ



non-AP scheme, ϵ



Result for steady-state

- **Steady-state**: $\rho(t,x) = 3 + 2\sin(2\pi x)$, $u(t,x) = 0$
- $p(t,x) = 3 + 3\sin(2\pi x) - \frac{1}{2}\cos(4\pi x)$ and $\phi(x) = -\sin(2\pi x)$. Random mesh.

Schemes	LR		LR-AP (2)		LR-AP (3)		LR-AP (4)	
cells	Err	q	Err	q	Err	q	Err	q
20	0.8335	-	0.0102	-	0.0079	-	0.0067	-
40	0.4010	1.05	0.0027	1.91	8.4E-4	3.23	1.5E-4	5.48
80	0.2065	0.96	7.0E-4	1.95	7.7E-5	3.45	4.1E-6	5.19
160	0.1014	1.02	1.7E-4	2.04	7.0E-6	3.46	1.0E-7	5.36

- **Steady-state**: $\rho(t,x) = e^{-gx}$, $u(t,x) = 0$, $p(t,x) = e^{-gx}$ et $\phi = gx$. Random mesh

Schemes	LR		LR-AP (2)		LR-AP (3)		LR-AP (4)	
cells	Err	q	Err	q	Err	q	Err	q
20	0.0280	-	6.5E-4	-	1.8E-5	-	8.0E-7	-
40	0.0152	0.88	1.4E-4	2.21	2.0E-6	3.17	3.8E-8	4.4
80	0.0072	1.08	3.3E-5	2.08	2.0E-7	3.32	2.0E-9	4.25
160	0.0038	0.92	8.8E-6	1.90	2.8E-8	2.84	1.1E-10	4.18

Conclusion and perspectives

- Conclusion
  - P_1 **model**: First AP scheme (time and space) on unstructured meshes (now other schemes have been developed).
  - P_1 **model**: Uniform proof of convergence on unstructured meshes in 1D and 2D.
  - AP schemes for general linear systems with source terms using previous schemes and "micro-macro" method.
  - **Euler model with external force** AP schemes with a new high order reconstruction of the steady states
  - **Problem for all the schemes** : spurious mods in few cases (example: Cartesian mesh + Dirac Initial data).

- Possible perspectives
  - P_1 model: Theoretical study of the explicit and semi implicit scheme.
  - Euler model: Entropy study for scheme.
  - Find a generic procedure to stabilize the nodal scheme (exist for the Lagrangian nodal scheme for the Euler equations).

Thanks

Thank you