Preconditioned implicit DG schemes for hyperbolic systems.
Application to linear and nonlinear wave problems.

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Summary

Multi-scale problems

Time Multi-scale problems:

- **Models**: hyperbolic systems able to model complex physics through nonlinear conservation laws
- **Properties propagation**: hyperbolic systems have finite propagation speed given by the wave velocities (eigenvalues of the Jacobian).
- **Stability**: the time step is constrained by the fastest waves.
- **Multi-scale problem**: $V_{max} << V_{min}$ and $T_f = O(V_{min})$.

- Morphodynamics flows: caused by the movement of a fluid in contact with the topography.
- Shallow Water + Exner equations:

$$\begin{cases} \partial_t h + \nabla \cdot (h\boldsymbol{u}) = 0 \\ \partial_t h\boldsymbol{u} + \nabla \cdot (h\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = -gh\nabla b \\ \partial_t b + \nabla \cdot \boldsymbol{Q} = 0 \end{cases}$$

where h is the height, $\boldsymbol{u}$ the velocity, b the topography, and $\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{u})$.

time scales:

- **time step** Δt: given by gravity waves' speeds $\lambda = \sqrt{hg}$.
- **simulation time** $T_f >> \Delta t$: given by the sedimentation behavior.

Project

Implicit scheme:

- To treat this problem, one good option: implicit scheme.
- In the case of implicit schemes we must invert a linear system. Two solutions:
  - **exact solvers**: too greedy for fine 2D or 3D problems.
  - **iterative solvers**: the stiff or multi-scale hyperbolic systems are ill-conditioned.
- For iterative solvers, we need to find a robust and efficient preconditioning.

Aim :

To design efficient and robust preconditioned implicit algorithm for hyperbolic systems with DG high-order method on complex geometries.

Objectives:

- Write implicit method (based on GMRES+ Free Jacobian method) for one macro-cell.
- Design and study the physics based preconditioning (based on physical or numerical approximations).
- Full model in **Discontinuous Galerkin** and preconditioning in **Continuous Galerkin**.
- Validate the methods on **Wave** and **Shallow Water** equations.

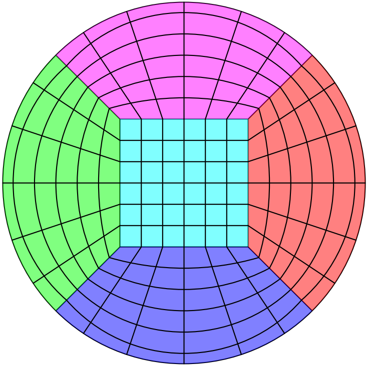
Schnaps code: DG method for hyperbolic scheme

DG schemes:

- High order method adapted to the discretization of hyperbolic systems.
- **Principle**: we discretize in each cell the weak form without enforcing continuity between the cells.
- **Reduction CPU**: quadrature using Gauss Lobatto points (diagonal mass matrix).
- **Conditioning**: High-order methods are ill-conditioned.

Complex geometries:

- **Idea**: we decompose the domain between curved macro-cells (GMSH).
- **Macro-cell**: Cartesian in the interior.



General discretization method

We consider systems of equation of the following shape:

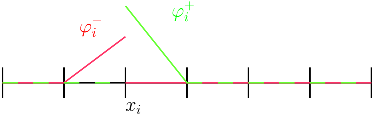
$$\partial_t \boldsymbol{U} + F(\boldsymbol{U}) = S,$$

with $\boldsymbol{U} \in R^N$ our unknowns, F a function acting on $\boldsymbol{U}$ and S source terms independent of $\boldsymbol{U}$.

- **Discretization in space :** Discontinuous Galerkin
  - Polynomial approximation of the solution $\boldsymbol{U}$ in each cell,
  - Weak formulation : $\partial_t(\boldsymbol{U}, \varphi)_{L^2} + (f(\boldsymbol{U}), \varphi)_{L^2} = (S, \varphi)_{L^2}$, $\forall \varphi$ basis function,
  - Discontinuous basis functions,

$$\varphi_i^+(x_j) = \begin{cases} 1, & \text{if } j = i^+, \\ 0, & \text{if } j \neq i^+, \end{cases}$$

where x_j is a Gauss-Lobatto point.



- **Discretization in time :** θ-scheme

$$\boldsymbol{U}^{n+1} + \Delta t \theta F(\boldsymbol{U}^{n+1}) = \boldsymbol{U}^n - \Delta t(1-\theta)F(\boldsymbol{U}^n) + \Delta t \theta S^{n+1} + \Delta t(1-\theta)S^n$$

Second-order and unconditionally-stable for $\theta = 0.5$ (Crank-Nicholson).

Wave equation

- Acoustic wave equations :

$$\begin{cases} \partial_t p + c\nabla\cdot \boldsymbol{u} = S_p, \\ \partial_t \boldsymbol{u} + c\nabla p = S_{\boldsymbol{u}}. \end{cases}$$

  with $\boldsymbol{u}$ the velocity, p the pressure and c the speed wave.

- We use the previously described discretization with

$$\boldsymbol{U} = \begin{pmatrix} p \\ \boldsymbol{u} \end{pmatrix}, \quad F(\boldsymbol{U}) = \begin{pmatrix} c\nabla\cdot\boldsymbol{u} \\ c\nabla p \end{pmatrix}, \quad S = \begin{pmatrix} S_p \\ S_{\boldsymbol{u}} \end{pmatrix}$$

- Linear hyperbolic system.

Wave propagation :

- Study of solutions given by an equilibrium state and an irrotational perturbation $p = p_0 + \delta p$ and $\boldsymbol{u} = \boldsymbol{u}_0 + \delta\boldsymbol{u}$.
- Propagation of the perturbation at the velocity $\pm c$.

Matrix equation

The weak formulation leads to the matrix equation

$$\begin{pmatrix} M & c\theta\Delta t U_1 & c\theta\Delta t U_2 \\ c\theta\Delta t L_1 & M & 0 \\ c\theta\Delta t L_2 & 0 & M \end{pmatrix} \begin{pmatrix} p^{n+1} \\ u^{n+1} \\ v^{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} M & -c(1-\theta)\Delta t U_1 & -c(1-\theta)\Delta t U_2 \\ -c(1-\theta)\Delta t L_1 & M & 0 \\ -c(1-\theta)\Delta t L_2 & 0 & M \end{pmatrix} \begin{pmatrix} p^{n} \\ u^{n} \\ v^{n} \end{pmatrix},$$

with

$$M = \left(\int_\Omega \varphi_i \varphi_j d\mathbf{x}\right)_{(i,j)\in[\![1,N]\!]^2}, \quad \text{the mass matrix}$$
$$U_1 = \left(\int_\Omega \partial_x \varphi_j \varphi_i\right)_{(i,j)\in[\![1,N]\!]^2}, \quad U_2 = \left(\int_\Omega \partial_y \varphi_j \varphi_i\right)_{(i,j)\in[\![1,N]\!]^2},$$
$$L_1 = \left(\int_\Omega \partial_x \varphi_j \varphi_i\right)_{(i,j)\in[\![1,N]\!]^2}, \quad L_2 = \left(\int_\Omega \partial_y \varphi_j \varphi_i\right)_{(i,j)\in[\![1,N]\!]^2}.$$

Aim

Preconditioning of the Jacobian matrix thanks to the Schur theory.

Schur complement

Property (Schur decomposition)

Let $\mathscr{A}$ be a matrix defined by blocks

$$\mathscr{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A invertible. Assume A, B, C, D are respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices, one has

$$\mathscr{A} = \begin{pmatrix} I_p & 0 \\ CA^{-1} & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I_p & A^{-1}B \\ 0 & I_q \end{pmatrix}$$

where I_p is the $p \times p$ identity matrix.

Definition (Schur complement)

If $\mathscr{A}$ is a matrix defined by blocks as in the previous property, the Schur complement of $\mathscr{A}$ is $D - CA^{-1}B$

Preconditioning for Wave equation

Applying this decomposition to our system yields

$$\begin{pmatrix} M & c\theta\Delta t U_1 & c\theta\Delta t U_2 \\ c\theta\Delta t L_1 & M & 0 \\ c\theta\Delta t L_2 & 0 & M \end{pmatrix} =$$

$$\begin{pmatrix} I_N & 0 \\ c\theta\Delta t L M^{-1} & I_{N^2} \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & \boldsymbol{M} - c^2\theta^2\Delta t^2 L M^{-1} U \end{pmatrix} \begin{pmatrix} I_N & c\theta\Delta t M^{-1} U \\ 0 & I_{N^2} \end{pmatrix}$$

with

$$\boldsymbol{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}, \quad L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 & U_2 \end{pmatrix},$$

and

$$P_{\text{schur}} = \boldsymbol{M} - c^2\theta^2\Delta t^2 L M^{-1} U.$$

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and

$$P_{\text{schur}} = \boldsymbol{M} - c^2\theta^2\Delta t^2 L M^{-1} U.$$

Hence, the preconditioning for the wave problem unfolds as the following splitting

$$\begin{cases} M p^* = M p^n - c(1-\theta)\Delta t M U \begin{pmatrix} u^n \\ v^n \end{pmatrix}, & \text{prediction step}, \\ P_{\text{schur}} \begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} u^n \\ v^n \end{pmatrix} - c(1-\theta)\Delta t \boldsymbol{M} L p^n - c\theta\Delta t L p^*, & \text{propagation step}, \\ M p^{n+1} = -c\theta\Delta t U \begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} + M p^*, & \text{correction step}. \end{cases}$$

Properties of P_{schur}

- To retrieve from those systems of equations the underlying physics, preconditioning has to follow some properties.
- The splitted system should keep as most physical properties from the original problem as possible.
- $P_{\text{schur}} \equiv I_2 - c^2\theta^2\nabla(\nabla \cdot I_2)$

Properties of P_{schur}

- P_{schur} should be easy to invert,
- P_{schur} is self adjoint,
- P_{schur} propagates an irrotational perturbation with the same speed $\pm c$ as the original problem.

Proof. P_{schur} is the discretization of the motion equation

$$\partial_{tt}\boldsymbol{\xi} - c^2\theta^2\nabla\left(\nabla \cdot \boldsymbol{\xi}\right) = 0,$$

where $\boldsymbol{u} = \partial_t\boldsymbol{\xi}$.

Remark : The preconditioning has the same propagation speed as the full model, which is equivalent at the spectral level.

Results

Here, we compile some results on different test-cases in Discontinuous Galerkin of fourth order for the acoustic wave equations.

Δt	Type of preconditioning	Mesh	Number of iteration / time-step
0.01	GMRES	20×20	103
	GMRES-PC	20×20	3
	GMRES	40×40	224
	GMRES-PC	40×40	3
0.05	GMRES	20×20	762
	GMRES-PC	20×20	14
	GMRES	40×40	1594
	GMRES-PC	40×40	20

Table: Results for a steady state.

Δt	Type of preconditioning	Mesh	Number of iteration / time-step
0.01	GMRES	30×30	150
	GMRES-PC	30×30	11
	GMRES	40×40	220
	GMRES-PC	40×40	12

Table: Results for a periodic wave problem.

Shallow water equation

- Shallow Water equation :

$$\begin{cases} \partial_t h + \nabla \cdot (h\boldsymbol{u}) = 0 \\ \partial_t h\boldsymbol{u} + \nabla \cdot (h\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = -gh\nabla b \end{cases}$$

with h the height, $\boldsymbol{u}$ the velocity, and the pressure $p = \frac{gh^2}{2}$.

- We can diagonalize the system to obtain the eigenvalues : $(\boldsymbol{u}, \boldsymbol{n}) \pm c$ and $(\boldsymbol{u}, \boldsymbol{n})$, with $c = \sqrt{hg}$ the sound speed
- **Linearized Shallow Water Homogeneous equation**: we consider that the solutions are given by an equilibrium and a perturbation $h = h_0 + \delta h$ and $\boldsymbol{u} = \boldsymbol{u}_0 + \delta \boldsymbol{u}$.
- The linearized system propagates these perturbations at the velocity $(\boldsymbol{u}_0, \boldsymbol{n}) \pm \sqrt{h_0 g}$ and $(\boldsymbol{u}_0, \boldsymbol{n})$.

Aim of the Physic-Based Preconditioner:

- To obtain a simpler operator (well-conditioned) which propagates the perturbations with velocities close to the original problem.

θ-scheme and free Jacobian method (I)

θ-scheme applied on Shallow Water equation yields

$$\begin{cases} h^{n+1} + \theta\Delta t \nabla\cdot\left(h^{n+1}\boldsymbol{u}^{n+1}\right) = h^n - \Delta t(1-\theta)\nabla\cdot\left(h^n\boldsymbol{u}^n\right), \\ h^{n+1}\boldsymbol{u}^{n+1} + \theta\Delta t h^{n+1}\left(\boldsymbol{u}^{n+1}\cdot\nabla\right)\boldsymbol{u}^{n+1} + \theta\Delta t h^{n+1} g\nabla h^{n+1} + \theta\Delta t g h^{n+1}\nabla b \\ \qquad = h^n\boldsymbol{u}^n - \Delta t(1-\theta)h^n\left(\boldsymbol{u}^n\cdot\nabla\right)\boldsymbol{u}^n - \Delta t(1-\theta) g h^n\nabla h^n - (1-\theta)\Delta t g h^n \nabla b. \end{cases}$$

This system can be rewritten in the form

$$G\begin{pmatrix} h^{n+1} \\ \boldsymbol{u}^{n+1} \end{pmatrix} = B\begin{pmatrix} h^n \\ \boldsymbol{u}^n \end{pmatrix},$$

with

$$G : \begin{pmatrix} h \\ \boldsymbol{u} \end{pmatrix} \mapsto \begin{pmatrix} h + \theta\Delta t\nabla\cdot(h\boldsymbol{u}) \\ h\boldsymbol{u} + \theta\Delta t h\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} + \theta\Delta t h g\nabla h + \theta\Delta t g h\nabla b \end{pmatrix},$$

and

$$B : \begin{pmatrix} h \\ \boldsymbol{u} \end{pmatrix} \mapsto \begin{pmatrix} h - \Delta t(1-\theta)\nabla\cdot(h\boldsymbol{u}) \\ h\boldsymbol{u} - \Delta t(1-\theta)h\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} - \Delta t(1-\theta)h g\nabla h - (1-\theta)\Delta t g h\nabla b \end{pmatrix}.$$

θ-scheme and free Jacobian method (II)

A linearization of G gives

$$Jac_G{}^n \begin{pmatrix} \delta h^n \\ \delta \boldsymbol{u}^n \end{pmatrix} = B \begin{pmatrix} h^n \\ \boldsymbol{u}^n \end{pmatrix} - G \begin{pmatrix} h^n \\ \boldsymbol{u}^n \end{pmatrix},$$

with $\delta h^n = h^{n+1} - h^n$, $\delta \boldsymbol{u}^n = \boldsymbol{u}^{n+1} - \boldsymbol{u}^n$ and Jac_G^n the Jacobian matrix of G at $\begin{pmatrix} h^n \\ \boldsymbol{u}^n \end{pmatrix}$,

$$Jac_G{}^n = \begin{pmatrix} D_1 & U \\ L & D_2 \end{pmatrix}$$

with

$$\begin{aligned} D_1 &= I_1 + \theta \Delta t \nabla \cdot (\boldsymbol{u}^n I_1), \quad U = \theta t \nabla \cdot (h^n I_2), \\ L &= \boldsymbol{u}^n I_1 + \theta \Delta t g \nabla (h^n I_1) + \theta \Delta t I_1 (\boldsymbol{u}^n \cdot \nabla) \boldsymbol{u}^n + \theta g I_1 \Delta t \nabla b, \\ D_2 &= h^n I_2 + \theta \Delta t h^n \left(\boldsymbol{u}^n \cdot \nabla\right) I_2 + \theta \Delta t h^n (I_2 \cdot \nabla) \boldsymbol{u}^n. \end{aligned}$$

Free Jacobian

The full jacobian matrix is not stored, on the contrary, the jacobian matrix is approximated by the relation

$$Jac_G^n X \approx \frac{G\left(\begin{pmatrix} h^n \\ \boldsymbol{u}^n \end{pmatrix} + \epsilon X\right) - G\begin{pmatrix} h^n \\ \boldsymbol{u}^n \end{pmatrix}}{\epsilon}$$

which requires the computation of G only.

Preconditioning Algorithm based on P_{schur}

The Schur decomposition gives the following algorithm

$$\begin{cases} D_1 \delta h^* = -\Delta t \nabla \cdot (h^n \boldsymbol{u}^n), \\ (D_2 - L D_1^{-1} U) \delta \boldsymbol{u}^{n+1} = -L \delta h^* - \Delta t h^n (\boldsymbol{u}^n \cdot \nabla) \boldsymbol{u}^n - \Delta t g h^n \nabla h^n - \Delta t \theta g h^n \nabla b, \\ D_1 \delta h^{n+1} = D_1 \delta h^* - U \delta \boldsymbol{u}^{n+1}, \end{cases}$$

while its complement for the Shallow Water system is

$$P_{\text{schur}} = D_2 - L D_1^{-1} U.$$

Different flows

We want to study different Schur approximations introduced by L. Chacón for MHD flows:

- slow flow,
- arbitrary flow,

in order to compute the linear wave propagation of P_{schur}.

L. Chacón: An optimal, parallel, fully implicit Newton-Krylov solver for three-dimensional viscoresistive magnetohydrodynamics, Physics of plasmas 2008.

Slow flow approximation of the Schur complement

Slow flow hypothesis

We assume that the flow is small, consequently $\Delta t|\boldsymbol{u}^n| << 1$.
Consequently we obtain that $D_1 \approx I_1$ in this regime.

For a constant velocity $\boldsymbol{u}^n$, P_{schur} becomes

$$P_{\text{schur}} = D_2 - L I_1 U = h^n I_2 + \theta t h^n \left(\boldsymbol{u}^n \cdot \nabla\right) I_2 + \theta \Delta t h^n \left(I_2 \cdot \nabla\right) \boldsymbol{u}^n - LU,$$

and

$$LU = \theta \Delta t \left(\boldsymbol{u}^n + \theta \Delta t \left(\boldsymbol{u}^n \cdot \nabla\right) \boldsymbol{u}^n\right) \nabla \cdot \left(h^n I_2\right) + \theta^2 \Delta t^2 \nabla \left[I_2 \cdot \nabla p^n + 2 p^n \nabla \cdot I_2\right].$$

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Hypothesis 1. We neglect the advection term in LU, to obtain the dispersion relation

$$\omega = \left(\theta \frac{\boldsymbol{u}^n \cdot \boldsymbol{n}}{2} \pm \theta \sqrt{h^n g - \frac{(\boldsymbol{u}^n \cdot \boldsymbol{n})^2}{4}}\right) ||\boldsymbol{k}||.$$

Hypthesis 2. We consider now the full LU operator, and we obtain

$$\omega = \pm \theta \sqrt{g h^n} ||\boldsymbol{k}||.$$

Proof. To prove those two results, we write the motion equation on ξ with $\partial_t \xi = \boldsymbol{u}$ and inject an irrotational linear plane wave.

Arbitrary flow approximation of the Schur

Arbitrary flow hypothesis

The approximation $D_1 \approx I_1$ is not valid anymore, we have to consider D_1^{-1} in P_{schur}.

We introduce the construction of an operator M such that $UM \approx D_1 U$ consequently we obtain that

$$P_{\text{schur}} = (D_2 M - LU) M^{-1}.$$

The solution of the equation $P_{schur} \delta \boldsymbol{u} = 0$ is given by

$$\begin{cases} (D_2 M - LU)\delta \boldsymbol{u}^* & = 0, \\ \delta \boldsymbol{u} = M \delta \boldsymbol{u}^*. \end{cases}$$

We choose

$$M = I_2 + \theta \Delta t \boldsymbol{u}^n \left(\nabla \cdot I_2 \right).$$

For a constant velocity $\boldsymbol{u}^n$,
Hypothesis 1. We neglect advection terms in LU to obtain the dispersion relation

$$\omega = \pm \theta \sqrt{g h^n} ||\boldsymbol{k}||.$$

Hypothesis 2. We consider each term of the LU operator to obtain the following dispersion relation

$$\omega = \left(-\theta \frac{\boldsymbol{u}^n \cdot \boldsymbol{n}}{2} \pm \theta \sqrt{h^n g - \frac{3}{4} (\boldsymbol{u}^n \cdot \boldsymbol{n})^2} \right) ||\boldsymbol{k}||.$$

Perspectives

Theoretical perspectives

- Propose two new approximations of the Schur that are
  - non-negative,
  - partly or fully symmetric,
  - spectrally close in the fast-flow regime:
    $\boldsymbol{u} \cdot \boldsymbol{n} \pm c$ for the first approximation, and $\pm(\boldsymbol{u} \cdot \boldsymbol{n} + c)$ for the second one.

Numerical perspectives

- Optimize preconditioning for the wave system,
- Validation of the Shallow Water preconditioning with Jacobian free method,
- Variation of the approximation degrees between the preconditioned and the full model.

We are grateful to our supervisors for their continuous support !

Thank you for your attention