Physic-based Preconditioning and "multiscale" elliptic operators for fluid models

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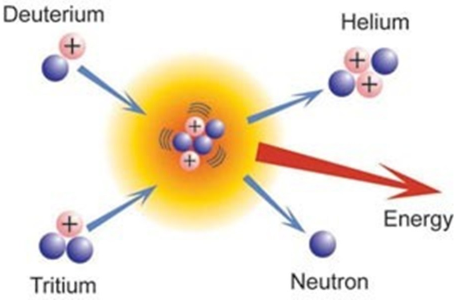
Outline

Model and physical context

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. A neutron and 17.6 MeV of free energy are released. At those energies, the atoms are ionized forming a plasma.
- **Plasma**: For very high temperature, the gas are ionized and give a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroidal room where the plasma is confined using powerful magnetic fields.
- **ITER**: International project of fusion nuclear plant to validate the nuclear fusion as a power source.



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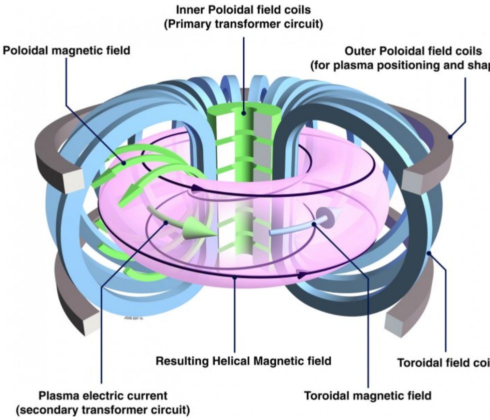


Figure: Tokamak

Physical context : MHD and ELM

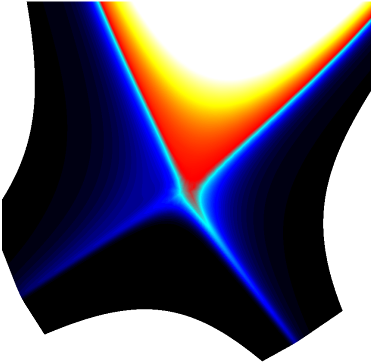
- In the tokamak some instabilities can appear in the plasma.
- The simulation of these instabilities is an important subject for ITER.

- Exemple of Instabilities in the tokamak :
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.

- These instabilities are linked to the very large gradient of pressure and very large current at the edge.

- These instabilities are described by fluid models (MHD resistive and diamagnetic or extended).

- ELM's simulation



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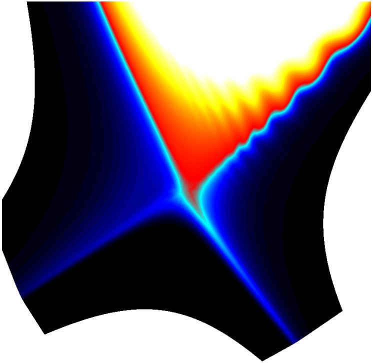

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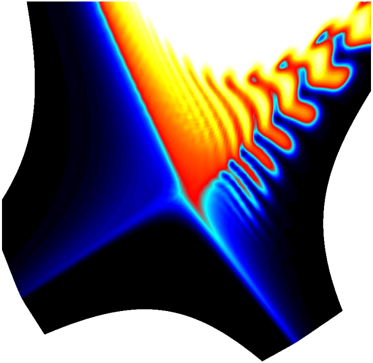

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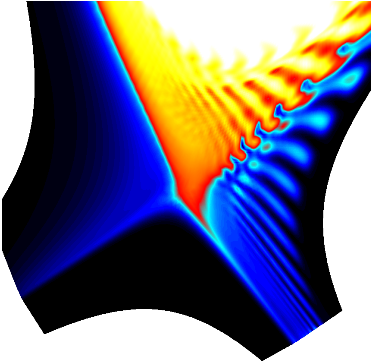

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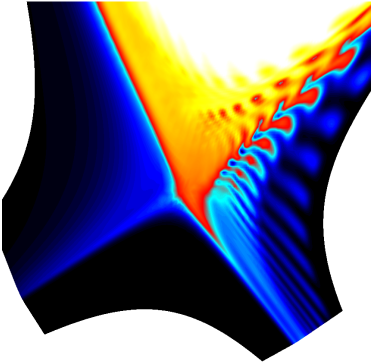

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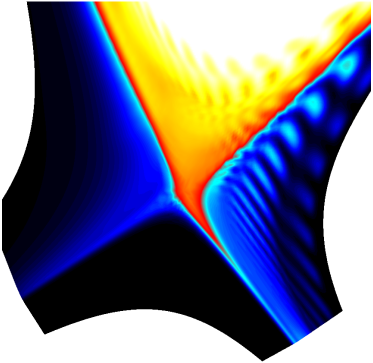


Extended MHD: model

- To simulate instabilities we solve the Extended MHD model (collisional and quasi-neutral limit of two species Vlasov-Maxwell equation.

Simplify Extended MHD

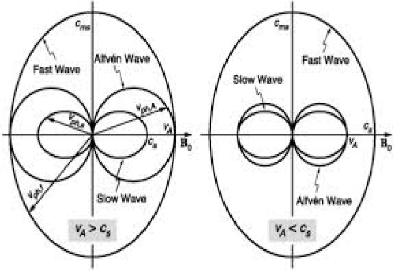
$$
\left\{
\begin{array}{l}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\
\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} = \boldsymbol{J} \times \boldsymbol{B}, \\
\partial_t p + \boldsymbol{u} \cdot \nabla p + p \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q} = \frac{m_i}{\rho e} \nabla \left(\frac{p}{\rho} \right) + \eta \mid \boldsymbol{J} \mid^2 \\
\partial_t \boldsymbol{B} = -\nabla \times \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla p + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right), \\
\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \\
\nabla \cdot \boldsymbol{B} = 0
\end{array}
\right.
$$

- with ρ the density, p the pressure, $\mathbf{u}$ the velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the current, $\overline{\overline{\boldsymbol{\Pi}}}$ stress tensor and $\mathbf{q}$ the heat flux. m_i the ion mass, e the charge, η the resistivity and μ_0 the permeability.
- In Black: ideal MHD. In Black and blue: Viscous-resistive MHD. All the term: Hall or extended MHD.

Wave structure of the MHD and time method

Wave Structure of the MHD

- We linearized the MHD around $\mathbf{B}_0 = \mathbf{B}\mathbf{e}_z$, ρ_0, p_0 and $\mathbf{u}_0 = 0$.
- **Alfvén velocity** and **Sound velocity** :
  $$V_a = \sqrt{\frac{\mathbf{B}_0^2}{\mu_0\rho_0}} \text{ and } c = \sqrt{\frac{\gamma p_0}{\rho_0}}$$
- Waves in plasma (toroidal $\boldsymbol{B}$): V_a and
  $$V_{\pm} = \left(\frac{1}{2}\left(V^2 \pm \sqrt{(V^4 - 4V_a^2 c^2\cos^2\theta})\right)\right)^{\frac{1}{2}}$$
  with $V^2 = V_a^2 + c^2$,θ the angle between $\mathbf{B}_0$ and the direction of the wave.
- **Tokamak regime**: $V_a >> c >> \parallel \mathbf{u} \parallel$.



Numerical context for time discretization

- Stiff fast wave + diffusion (resistive and viscous) ====> Implicit or semi-implicit methods.
- Nonlinear 3D problem ====> Iterative nonlinear implicit methods.
- $\lambda_{max} >> \lambda_{min}$ ====> Preconditioning.

JOREK code and typical test case

JOREK

- A reduced MHD (full MHD in the future) code which simulate instabilities with 2 numerical blocks:
  - Computation of the equilibrium and the aligned grid
  - Computation of the MHD instabilities perturbing equilibrium.
- **Spatial discretization**: 2D Cubic Bezier finite elements + Fourier expansion.
- **Time discretization**: implicit + Gmres with Fourier Block Jacobi.
- Problems with the JOREK code:
  - We need new numerical methods to solve huge cases.

grids2.pdf

Figure: Aligned grid

New code : DJANGO

- Modular code based of general finite elements (B-Splines, Lagrange, Powel-Sabin) and Physic-Based preconditioning

Preconditioning and Physic-Based PC

Linear Solvers and preconditioning

- We solve a nonlinear problem $G(\boldsymbol{U}^{n+1}) = b(\boldsymbol{U}^n, \boldsymbol{U}^{n-1})$. First order linearization

$$\left(\frac{\partial G(\boldsymbol{U}^n)}{\partial \boldsymbol{U}^n}\right)\delta \boldsymbol{U}^n = -G(\boldsymbol{U}^n) + b(\boldsymbol{U}^n, \boldsymbol{U}^{n-1}) = R(\boldsymbol{U}^n),$$

  with $\delta \boldsymbol{U}^n = \boldsymbol{U}^{n+1} - \boldsymbol{U}^n$, and $J_n = \frac{\partial G(\boldsymbol{U}^n)}{\partial \boldsymbol{U}^n}$ the Jacobian matrix of $G(\boldsymbol{U}^n)$.

- Principle of the preconditioning step:
  - Replace the problem $J_k \delta \boldsymbol{U}_k = R(\boldsymbol{U}^n)$ by $P_k(P_k^{-1}J_k)\delta \boldsymbol{U}_k = R(\boldsymbol{U}^n)$.
  - Solve the new system with two steps $P_k \delta \boldsymbol{U}_k^* = R(\boldsymbol{U}^n)$ and $(P_k^{-1}J_k)\delta \boldsymbol{U}_k = \delta \boldsymbol{U}_k^*$
- If P_k is easier to invert than J_k and $P_k \approx J_k$ the solving step is more robust and efficient.

Physic-based Preconditioning

- In the GMRES context if we have a algorithm to solve $P_k \mathbf{U} = \mathbf{b}$, we have a Preconditioning.
- **Principle**: construct an algorithm to solve $P_k \mathbf{U} = \mathbf{b}$ approximating and splitting the equations and approximating the discretizations.

Physic-based: operator splitting

Idea:

- Coupled hyperbolic problems are ill-conditioned contrary to simple diffusion and advection operators.
- **Idea**: Use operator splitting and a reformulation to approximate the Jacobian by a suitability of simple problems (advection or diffusion).
- For each subproblem we use an adapted solver as multi-grid solver.

- Implicit scheme for wave : we solve

$$\left\{ \begin{array}{l} \partial_t u = \partial_x v \\ \partial_t v = \partial_x u \end{array} \right. \longrightarrow \left\{ \begin{array}{l} u^{n+1} = u^n + \Delta t \partial_x v^{n+1} \\ v^{n+1} = v^n + \Delta t \partial_x u^{n+1} \end{array} \right.$$

- which is strictly equivalent to solve one parabolic problem

$$\left\{ \begin{array}{l} (1 - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n \\ v^{n+1} = v^n - \Delta t \partial_x u^{n+1} \end{array} \right.$$

Conclusion

- This algorithm gives a very good preconditioning, that is easy to invert (just one elliptic operator to invert).

Application: Linearized Euler equation

Linearized Euler equation

- We consider the 3D MHD equation in the conservative form,

$$\left\{\begin{array}{ll} \partial_t\rho+\nabla\cdot(\rho\boldsymbol{u}) & =0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\nabla p & =0\\ \partial_t p+\nabla\cdot(p\boldsymbol{u}) & =0 \end{array}\right.$$

- Due to the isothermal assumption, we have $p=c^2\rho$ with $c=\sqrt{T_0}$.
- **Linearization**: $\boldsymbol{u}=\boldsymbol{u}_0+\delta\boldsymbol{u}$, $\rho=\rho_0+\delta\rho$, $p=p_0+\delta p$ with $p_0=c^2\rho_0$.
- Using the linear relation between p_0 and ρ_0 we obtain

$$\left\{\begin{array}{ll} \partial_t\delta\boldsymbol{u}+\boldsymbol{u}_0\cdot\nabla\delta\boldsymbol{u}+\frac{1}{\rho_0}\nabla\delta p & =0\\ \partial_t\delta p+\boldsymbol{u}_0\cdot\nabla\delta p+c^2\rho_0\nabla\cdot\delta\boldsymbol{u} & =0 \end{array}\right.$$

- To simplify, we assume that $\rho_0=\frac{1}{c}$. Defining a normalized velocity $\boldsymbol{a}$ and Mach number $M=\frac{|\boldsymbol{u}_0|}{c}$ we obtain the final model

Final model

$$\left\{\begin{array}{ll} \partial_t\boldsymbol{u}+cM\boldsymbol{a}\cdot\nabla\boldsymbol{u}+c\nabla p & =0\\ \partial_t p+cM\boldsymbol{a}\cdot\nabla p+c\nabla\cdot\boldsymbol{u} & =0 \end{array}\right.$$

with $M\in]0,1]$, and $\mid\mathbf{a}\mid=1$.

Implicit scheme for wave equation

Implicit scheme:

$$\begin{pmatrix} I_d + M\lambda \mathbf{a}\cdot\nabla & \lambda^2\nabla\cdot \\ \lambda^2\nabla & I_d + M\lambda\mathbf{a}\cdot\nabla \end{pmatrix}\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{n+1}\end{pmatrix} = \begin{pmatrix} I_d - M\lambda_e \mathbf{a}\cdot\nabla & \lambda_e^2\nabla\cdot \\ \lambda_e^2\nabla & I_d - M\lambda_e\mathbf{a}\cdot\nabla \end{pmatrix}\begin{pmatrix} p^{n} \\ \mathbf{u}^{n}\end{pmatrix}$$

- with $\lambda = \theta c \Delta t$ and λ_e the numerical acoustic length.

- The implicit system is given by

$$\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{\mathbf{n}+1}\end{pmatrix} = \begin{pmatrix} AD_p & Div \\ Grad & AD_u \end{pmatrix}^{-1}\begin{pmatrix} R_p \\ R_{\mathbf{u}}\end{pmatrix}$$

- The solution of the system is given by

$$\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{\mathbf{n}+1}\end{pmatrix} = \begin{pmatrix} I & AD_p^{-1}Div \\ 0 & I \end{pmatrix}\begin{pmatrix} AD_p^{-1} & 0 \\ 0 & P_{schur}^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ -GradAD_p^{-1} & I \end{pmatrix}\begin{pmatrix} R_p \\ R_{\mathbf{u}}\end{pmatrix}$$

  with $P_{schur} = AD_u - Grad(AD_p^{-1})Div$.

- Using the previous Schur decomposition, we can solve the implicit wave equation with the following algorithm:

$$\begin{cases} \text{Predictor}: & AD_p p^* = R_p \\ \text{Velocity evolution}: & P\mathbf{u}^{n+1} = (-Grad\, p^* + R_{\mathbf{u}}) \\ \text{Corrector}: & AD_p p^{n+1} = AD_p p^* - Div\,\mathbf{u}_{n+1} \end{cases}$$

PC for linearized Euler equations

- The preconditioning is given by the algorithm of L. Chacon (2007-2008)

Low Mach approximation:

- We assume that $M << 1$, therefore we use the approximation

$$AD_p^{-1} = (I_d + M\lambda \mathbf{a} \cdot \nabla)^{-1} \approx I_d$$

in the second and third step.

- We obtain

$$\begin{cases} \text{Predictor :} \quad AD_p p^* = R_p \\ \text{Velocity evolution :} \quad P\mathbf{u}^{n+1} = (-Grad\, p^* + R_{\mathbf{u}}) \\ \text{Corrector :} \quad p^{n+1} = p^* - Div\mathbf{u}_{n+1} \end{cases}$$

with two small operators

PC-operators :

- Advection

$$AD_p = I_d + M\lambda \mathbf{a} \cdot \nabla$$

- Advection-Diffusion

$$P_{schur} = I_d + M\lambda \mathbf{a} \cdot \nabla - \lambda^2 \nabla(\nabla \cdot)$$

Results

- **Test case**: propagation of pressure perturbation (not an easy test).
- **Capture of acoustic phenomena**. We consider $\Delta t_{max} = 0.5\frac{L}{c} = 0.5$.

	Δt /Mach	10^{-3}	10^{-2}	0.1	1
16*16	0.005	1	1	1	2
	0.05	2	2	3	6
	0.5	10	11	24	$O(10^2)$
32*32	0.005	1	1	1	2
	0.05	2	2	3	5
	0.5	7	9	23	$O(10^2)$
64*64	0.005	1	1	1	1
	0.05	1	2	2	4
	0.5	2	3	15	$O(10^2)$

- Number of iterations for different PC with Mesh 32×32.

$\Delta t/PC$	Jacobi	ILU(0)	ILU(4)	Pb-PC
$\Delta t = 0.1$	x	70	20	1
$\Delta t = 1$	x	x	x	1

Results

- **Test case**: propagation of pressure perturbation (not an easy test).
- **Capture of material wave**. We consider $\Delta t_{max} = O(\frac{L}{|\mathbf{u}_0|})$

	Δt /Mach	10^{-4}	10^{-3}
16*16	$\Delta t = 2$	15-25	20-30
	$\Delta t = 10$	60-70	90-110
32*32	$\Delta t = 2$	10-15	10-15
	$\Delta t = 10$	15-25	15-25
64*64	$\Delta t = 2$	2	3
	$\Delta t = 10$	8	11

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Elliptic operators

- When $M\lambda = O(1)$ the transport operator is ill-conditioned. To invert this operator we can
  - add stabilization terms,
  - design a specific preconditioning.
- We will focus on low-mach regime and the elliptic operator.

Acoustic elliptic operator

- Here we consider the elliptic operator

$$\left\{\begin{array}{l} \boldsymbol{u} - \lambda^2 \nabla(\nabla\cdot\boldsymbol{u}) = \mathbf{f} \\ M(\mathbf{n})\boldsymbol{u} = 0, \quad \partial\Omega \end{array}\right. \xrightarrow[\lambda\rightarrow\infty]{} \left\{\begin{array}{l} -\nabla(\nabla\cdot\boldsymbol{u}) = 0 \\ M(\mathbf{n})\boldsymbol{u} = 0, \quad \partial\Omega \end{array}\right.$$

Problem

- The limit operator is non-coercive. Indeed we can find $\parallel \boldsymbol{u} \parallel \neq 0$ (with the good BC) such that

$$\int_\Omega \mid \nabla\cdot\boldsymbol{u} \mid^2 = 0$$

- For exemple: $\boldsymbol{u} = \nabla \times \boldsymbol{\psi}$.
- **Numerical problem**: conditioning number in $O(\lambda)$ (which depend also of h and the order).

Results

- **Test case**: Solution for the
$$\boldsymbol{u} - \lambda^2 \Delta \boldsymbol{u} = \mathbf{f}$$
- operator with homogeneous Dirichlet on mesh 32*32

$\Delta t / PC$	Jacobi	ILU(4)	ILU(8)	MG(2)
$\lambda = 0.05$	3 ma	5	3	8
$\lambda = 0.1$	3	7	5	8
$\lambda = 0.5$	3	11	7	10
$\lambda = 1$	3	11	7	10
$\lambda = 2$	3	11	7	10
$\lambda = 5$	3	11	7	10

Strategy to solve acoustic operator

- **Step 1**: Hiptmair, Xu Using discrete B-Splines H(Div) space + Auxiliary space pc, split the kernel to the rest
- **Step 2:** We treat the orthogonal of the kernel with multi-grids+GLT method
- **GLT:** Generalized locally Toeplitz method which allows by a generalized Fourier analysis to correct the multi grid method in the high-frequency (problem for high order discretization).

Results

- **Test case**: Solution for the
$$\boldsymbol{u}-\lambda^2\nabla(\nabla\cdot\boldsymbol{u})=\mathbf{f}$$
- operator with homogeneous Dirichlet on mesh 32*32

$\Delta t/PC$	Jacobi	ILU(4)	ILU(8)	MG(2)
$\lambda=0.05$	135	8	5	22
$\lambda=0.1$	310	20	10	44
$\lambda=0.5$	1800	nc	nc	135
$\lambda=1$	nc	nc	nc	300
$\lambda=2$	nc	nc	nc	500
$\lambda=5$	nc	nc	nc	2100

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Application: Linearized 3D MHD

Linearized 3D MHD

- We consider the 3D Isothermal MHD equation in the non-conservative form,

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) & = 0 \\ \rho\partial_t \boldsymbol{u} + \rho \boldsymbol{u}\cdot\nabla \boldsymbol{u} + \nabla p & = \frac{1}{\mu_0}(\nabla\times \boldsymbol{B})\times \boldsymbol{B} \\ \partial_t p + \boldsymbol{u}\cdot\nabla p + \gamma p \nabla\cdot \boldsymbol{u} & = 0 \\ \partial_t \boldsymbol{B} + \nabla\times(\boldsymbol{B}\times \boldsymbol{u}) & = \frac{\eta}{\mu_0}\nabla\times \boldsymbol{B} \\ \nabla\cdot \boldsymbol{B} = 0 \end{cases}$$

- **Linearization**: $\boldsymbol{u} = \boldsymbol{u}_0 + \delta\boldsymbol{u}$, $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$ with $p_0 = c^2\rho_0$, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$.
- We define three important parameters: the Mach number M, the pressure ratio of the plasma $\beta = \frac{c^2}{V_a^2}$, the Alfvén speed $V_a^2 = \frac{|\boldsymbol{B}_0|^2}{\rho_0\mu_0}$ and the magnetic Reynolds $R_m = \frac{\mu_0 L |\mathbf{u}_0|}{\eta}$.

Final model

$$\begin{cases} \partial_t \boldsymbol{u} + (M\sqrt{\beta}V_a)\boldsymbol{a}\cdot\nabla \boldsymbol{u} + \nabla p & = \frac{V_a^2}{|\boldsymbol{B}_0|}\left((\nabla\times \boldsymbol{B})\times \boldsymbol{b}_0\right) \\ \partial_t p + (M\sqrt{\beta}V_a)\boldsymbol{a}\cdot\nabla p + \gamma\beta V_a^2 \nabla\cdot \boldsymbol{u} & = 0 \\ \partial_t \boldsymbol{B} + (M\sqrt{\beta}V_a)\boldsymbol{a}\cdot\nabla \boldsymbol{B} + \mid \boldsymbol{B}_0 \mid \nabla\times(\boldsymbol{b}_0\times \boldsymbol{u}) & = \frac{M\sqrt{\beta}V_a}{R_m}\nabla\times(\nabla\times \boldsymbol{B}) \end{cases}$$

with $M \in]0,1]$, $\beta \in \left]10^{-6}, 10^{-1}\right]$, $\mid \mathbf{a} \mid = \mid \mathbf{b_0} \mid = 1$.

Implicit scheme for linear MHD equation

Implicit scheme:

$$\begin{cases} \delta \boldsymbol{u}^n + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla \boldsymbol{u}^n + \nabla p & = \frac{\lambda^2}{|\boldsymbol{B}_0|}\left((\nabla\times\boldsymbol{B}^n)\times\boldsymbol{b}_0\right) \\ \delta p^n + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla p^n + \beta\lambda^2\nabla\cdot\boldsymbol{u}^n & = 0 \\ \delta \boldsymbol{B}^n + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla\boldsymbol{B}^n + \mid \boldsymbol{B}_0 \mid \nabla\times(\boldsymbol{b}_0\times\boldsymbol{u}^n) & = \frac{M\sqrt{\beta}\lambda}{R_m}\nabla\times(\nabla\times\boldsymbol{B}^n) \end{cases}$$

- with $\lambda = V_a\Delta t$ the numerical Alfvén length, and $\delta\rho^n = \rho^{n+1} - \rho^n$.

- As before we apply the preconditioning splitting between the velocity and the other variables with the low Mach approximation.
- In the end of the preconditioning we must invert three operators

Operators of the PB-PC

$$I_d + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla I_d - \frac{M\sqrt{\beta}\lambda}{R}\Delta I_d, \quad I_d + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla I_d$$

$$P = \left(I_d + M\sqrt{\beta}\lambda\boldsymbol{a}\cdot\nabla I_d - \beta\lambda^2\nabla(\nabla\cdot I_d) - \lambda^2\left(\boldsymbol{b}_0\times(\nabla\times\nabla\times(\boldsymbol{b}_0\times I_d))\right)\right)$$

with $\mid \boldsymbol{a} \mid = 1$, $M \in]0,1]$, $\beta \in \left]10^{-6}, 10^{-1}\right]$

Remarks

- As for the Euler equation we can solve the advection equation adding stabilization or using specific preconditioning.
- **First case**: We consider the regime $M << 1$ and $\beta << 1$.

Dominant Schur operator

- The Schur operator in this regime is mainly

$$P = \left(I_d - \lambda^2 \left(\boldsymbol{b}_0 \times \left(\nabla \times \nabla \times \left(\boldsymbol{b}_0 \times I_d\right)\right)\right)\right)$$

- The limit operator is non-coercive ($\lambda >> 1$). Indeed we can find $\parallel \boldsymbol{u} \parallel \neq 0$ such that

$$\int_\Omega \mid \nabla \times (\boldsymbol{b}_0 \times \boldsymbol{u}) \mid^2 = 0$$

- **Second case**: We consider the regime $M < 1$ and $\beta < 1$.

Multis-cale operator

- Using a Fourier analysis and Diagonalizing the operator in the Fourier space we denote that the eigenvalues are the MHD velocities
- When M and β is not so small, the different velocities (Alfvén, magneto-sonic slow and fast) have very different scales.

Conclusion

Physic-based pc

- If we are able to invert the sub-systems, then the physic-based pc is
  - very efficient in the Low-Mach regime for large time step.
  - less efficient in the sonic-regime, however we can treat large time step than the explicit one.
  - The efficiency does not decrease when the h decreases.

Euler equation

- For the Euler equation, in the end the main difficulty is to invert quickly the div-div operator.
- **Ongoing work:** find a good preconditioning for div-div using $H(div)$ discrete space (Hiptmair, Xu +GLT)

MHD equation

- In the low-Beta regime, the main difficulty is to invert quickly the curl-curl operator.
- **Ongoing work:** find a good preconditioning for curl-curl using compatible-space (difficulty: the dependance in the magnetic field).
- When β is not so small, we have a multi-scale operator.
- **Future work**: find a strategy to separate the scales.