Physic-based Preconditioning and B-Splines finite elements method for Tokamak MHD

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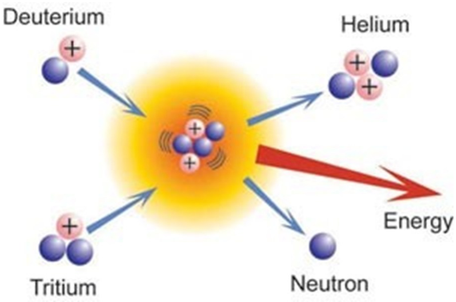
Outline

Mathematical and physical context

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. A neutron and 17.6 MeV of free energy are released. At those energies, the atoms are ionized forming a plasma.
- **Plasma**: For very high temperature, the gas are ionized and give a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroidal room where the plasma is confined using powerful magnetic fields.
- **ITER**: International project of fusion nuclear plant to validate the nuclear fusion as a power source.



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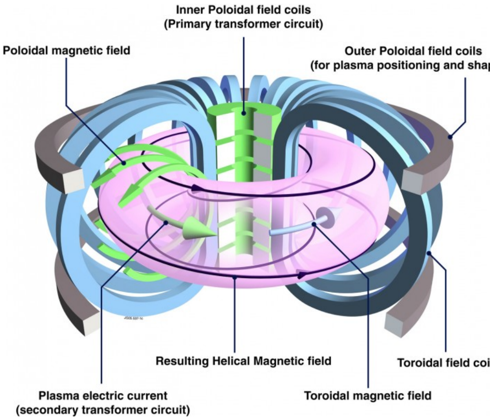
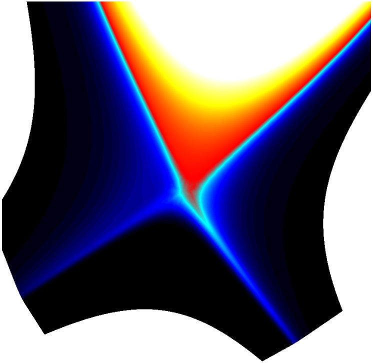


Figure: Tokamak

Physical context : MHD and ELM

- In the tokamak some instabilities can appear in the plasma.
- The simulation of these instabilities is an important subject for ITER.
- Exemple of Instabilities in the tokamak :
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- These instabilities are linked to the very large gradient of pressure and very large current at the edge.
- These instabilities are described by fluid models (MHD resistive and diamagnetic or extended).

- ELM's simulation



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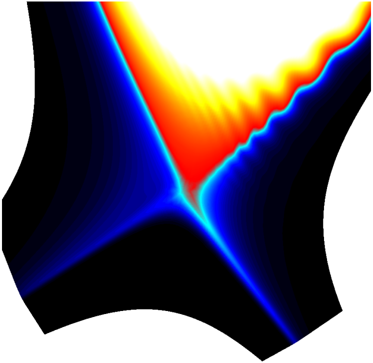

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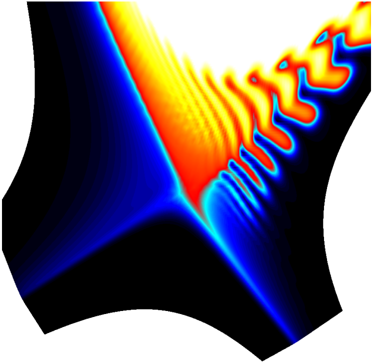

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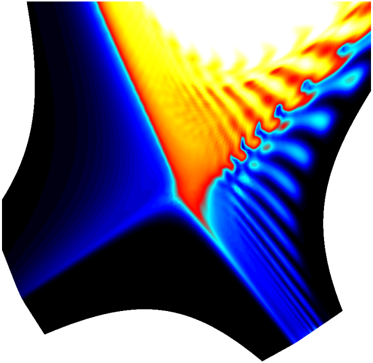

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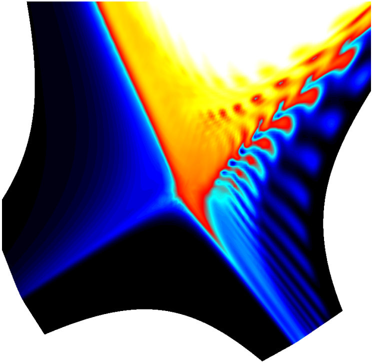

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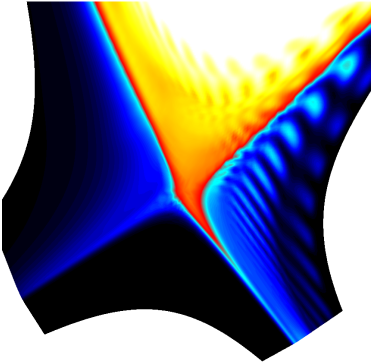


Extended MHD: model

- To simulate instabilities we solve the Extended MHD model.

Simplify Extended MHD

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} = \boldsymbol{J} \times \boldsymbol{B}, \\ \partial_t p + \boldsymbol{u} \cdot \nabla p + p \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q} = \frac{m_i}{\rho e} \nabla \left(\frac{p}{\rho} \right) + \eta \mid \boldsymbol{J} \mid^2 \\ \partial_t \boldsymbol{B} = -\nabla \times \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla p + \frac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right), \\ \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, p the pressure, $\mathbf{u}$ the velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the current, $\overline{\overline{\boldsymbol{\Pi}}}$ stress tensor and $\mathbf{q}$ the heat flux.
- In black: ideal MHD. In black and blue: Viscous-resistive MHD. All the terms: Hall or extended MHD.

JOREK code and spatial discretization

Spatial method

- **Mixed Parabolic-Hyperbolic problem** : Finite element method + Stabilization.
- **Strong anisotropic problem**: Aligned grids + High-order method ===> IsoGeometric / IsoParametric analysis.

JOREK

- **Jorek code** : (physical code for MHD simulations).
- IsoParametric approach for Flux Surface Aligned mesh (Hermite-Bézier element) + Fourier.

DJANGO

- **Django** : (New code for MHD simulations).
- IsoGeometric approach for Flux Surface/Field Aligned meshes (Arbitrary order B-Splines).

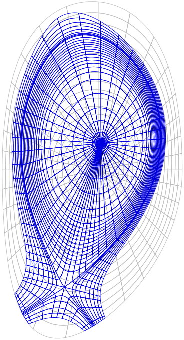


Figure: Flux-Surface Aligned grid

Time Numerical methods for MHD

Wave Structure of the MHD

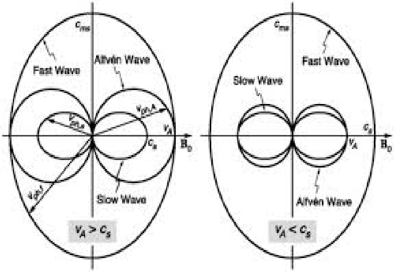
- We linearized the MHD around $\mathbf{B}_0 = \mathbf{B}\mathbf{e}_z$, ρ_0, p_0 and $\mathbf{u}_0 = 0$.
- **Alfvén velocity** and **Sound velocity** :

$$V_a = \sqrt{\frac{\mathbf{B}_0^2}{\mu_0\rho_0}} \text{ and } c = \sqrt{\frac{\gamma p_0}{\rho_0}}$$

- Waves in plasma (toroidal $\boldsymbol{B}$): V_a and

$$V_{\pm} = \left(\frac{1}{2}\left(V^2 \pm \sqrt{(V^4 - 4V_a^2 c^2 \cos^2\theta)}\right)\right)^{\frac{1}{2}}$$

  with $V^2 = V_a^2 + c^2$,θ the angle between $\mathbf{B}_0$ and the direction of the wave.
- **Tokamak regime**: $V_a >> c >> \| \mathbf{u} \|$.



Numerical context for time discretization

- Stiff fast wave + diffusion (resistivity and viscosity) ====> Implicit or semi-implicit methods.
- Nonlinear 3D problem ====> Iterative nonlinear implicit methods.
- $\lambda_{max} >> \lambda_{min}$ ====> Preconditioning.

Linear Solvers and preconditioning

- We solve a nonlinear problem $G(\boldsymbol{U}^{n+1}) = b(\boldsymbol{U}^n, \boldsymbol{U}^{n-1})$. First order linearization

$$\left(\frac{\partial G(\boldsymbol{U}^n)}{\partial \boldsymbol{U}^n}\right)\delta \boldsymbol{U}^n = -G(\boldsymbol{U}^n) + b(\boldsymbol{U}^n, \boldsymbol{U}^{n-1}) = R(\boldsymbol{U}^n),$$

  with $\delta \boldsymbol{U}^n = \boldsymbol{U}^{n+1} - \boldsymbol{U}^n$, and $J_n = \frac{\partial G(\boldsymbol{U}^n)}{\partial \boldsymbol{U}^n}$ the Jacobian matrix of $G(\boldsymbol{U}^n)$.

- Principle of the preconditioning step:
  - Replace the problem $J_k \delta \boldsymbol{U}_k = R(\boldsymbol{U}^n)$ by $P_k(P_k^{-1} J_k)\delta \boldsymbol{U}_k = R(\boldsymbol{U}^n)$.
  - Solve the new system with two steps $P_k \delta \boldsymbol{U}_k^* = R(\boldsymbol{U}^n)$ and $(P_k^{-1} J_k)\delta \boldsymbol{U}_k = \delta \boldsymbol{U}_k^*$
- If P_k is easier to invert than J_k and $P_k \approx J_k$ the solving step is more robust and efficient.

Physic-based Preconditioning

- In the GMRES context if we have a algorithm to solve $P_k \mathbf{U} = \mathbf{b}$, we have a preconditioning.
- **Principle**: construct an algorithm to solve $P_k \mathbf{U} = \mathbf{b}$ (not necessary to construct the matrix)

Preconditioning for Linearized Euler and MHD models

Physic-based: operator splitting

Idea:

- Coupled hyperbolic problems are ill-conditioned contrary to simple diffusion and advection operators.
- **Idea**: Use operator splitting and a reformulation to approximate the Jacobian by a series of suitable simple problems (advection or diffusion).
- For each subproblem we use an adapted solver as Multigrid solver.

- Implicit scheme for wave equation: we solve

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x u \end{cases} \longrightarrow \begin{cases} u^{n+1} = u^n + \Delta t \partial_x v^{n+1} \\ v^{n+1} = v^n + \Delta t \partial_x u^{n+1} \end{cases}$$

- which is strictly equivalent to solving one parabolic problem

$$\begin{cases} (1 - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n \\ v^{n+1} = v^n - \Delta t \partial_x u^{n+1} \end{cases}$$

Conclusion

- This algorithm gives a very good preconditioning, which is easy to invert (just one elliptic operator to invert).

Linearized Euler equation

- We consider the $2D$ Euler equation in the conservative form,

$$\left\{\begin{array}{ll} \partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) & = 0 \\ \partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p & = 0 \\ \partial_t p + \nabla\cdot(p\boldsymbol{u}) & = 0 \end{array}\right.$$

- Due to the isothermal assumption, we have $p = c^2\rho$ with $c = \sqrt{T_0}$.
- **Linearization**: $\boldsymbol{u} = \boldsymbol{u}_0 + \delta\boldsymbol{u}$, $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$ with $p_0 = c^2\rho_0$.
- Using the linear relation between p_0 and ρ_0 we obtain

$$\left\{\begin{array}{ll} \partial_t\delta\boldsymbol{u} + \boldsymbol{u}_0\cdot\nabla\delta\boldsymbol{u} + \frac{1}{\rho_0}\nabla\delta p & = 0 \\ \partial_t\delta p + \boldsymbol{u}_0\cdot\nabla\delta p + c^2\rho_0\nabla\cdot\delta\boldsymbol{u} & = 0 \end{array}\right.$$

- To simplify, we assume that $\rho_0 = \frac{1}{c}$. Defining a normalized velocity $\boldsymbol{a}$ and Mach number $M = \frac{|\boldsymbol{u}_0|}{c}$ we obtain the final model

Final model

$$\left\{\begin{array}{ll} \partial_t\boldsymbol{u} + cM\boldsymbol{a}\cdot\nabla\boldsymbol{u} + c\nabla p & = 0 \\ \partial_t p + cM\boldsymbol{a}\cdot\nabla p + c\nabla\cdot\boldsymbol{u} & = 0 \end{array}\right.$$

with $M \in]0,1]$, and $\mid \mathbf{a} \mid = 1$.

First preconditioning

Implicit scheme:

$$\begin{pmatrix} I_d + M\lambda \mathbf{a}\cdot\nabla & \lambda\nabla\cdot \\ \lambda\nabla & I_d + M\lambda\mathbf{a}\cdot\nabla \end{pmatrix}\begin{pmatrix} p^{n+1} \\ \mathbf{u}^{n+1}\end{pmatrix} = \begin{pmatrix} I_d - M\lambda_e\mathbf{a}\cdot\nabla & \lambda_e\nabla\cdot \\ \lambda_e\nabla & I_d - M\lambda_e\mathbf{a}\cdot\nabla \end{pmatrix}\begin{pmatrix} p^{n} \\ \mathbf{u}^{n}\end{pmatrix}$$

- with $\lambda = \theta c\Delta t$ and λ_e the numerical acoustic length.

- **Idea for preconditioning**: split the systems between some triangular problems to decouple the variables

$$A = \begin{pmatrix} I_d + \lambda AD_p & \lambda Div \\ \lambda Grad & I_d + \lambda AD_u \end{pmatrix} \approx (I_d + \lambda L_1)(I_d + \lambda L_2)(I_d + \lambda L_3)$$

- First choice SPC(1): $L_1 = L_1^0$, $L_1 = L_2^0$ and $L_1 = L_3^0$ with

$$L_1^0 = \begin{pmatrix} MAD_p & 0 \\ 0 & 0 \end{pmatrix}, \quad L_2^0 = \begin{pmatrix} 0 & 0 \\ Grad & MAD_u \end{pmatrix}, \quad L_3^0 = \begin{pmatrix} 0 & Div \\ 0 & 0 \end{pmatrix}$$

- Using the previous decomposition, we can approximate the wave solution solving the following algorithm:

$$\begin{cases} \text{Predictor}: & (I_d + M\lambda AD_p)p^* = R_p \\ \text{Velocity evolution}: & (I_d + M\lambda AD_u)\mathbf{u}^{n+1} = (-\lambda Grad\, p^* + R_{\mathbf{u}}) \\ \text{Corrector}: & p^{n+1} = p^* - \lambda Div\,\mathbf{u}_{n+1} \end{cases}$$

Others preconditoning

- Formal analysis of SPC(1) approximation:

$$E = A - (I_d + \lambda L_1)(I_d + \lambda L_2)(I_d + \lambda L_3) = O\left(\lambda^2\left(1 + M\right)\right)$$

- In the explicit splitting theory we kill the second order terms in λ (2 order differential operators) in the error adding step.
- However the 2nd order operators are easy to invert consequently we propose .

- Second choice SPC(1): $L_1 = L_1^0$, $L_1 = L_2^0 - \lambda L_2^0 L_3^0$ and $L_1 = L_3^0$ with

$$L_1^0 = \begin{pmatrix} MAD_p & 0 \\ 0 & 0 \end{pmatrix}, \quad L_2^0 = \begin{pmatrix} 0 & 0 \\ Grad & MAD_u - \lambda GradDiv \end{pmatrix}, \quad L_3^0 = \begin{pmatrix} 0 & Div \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \text{Predictor :} \quad (I_d + M\lambda AD_p)p^* = R_p \\ \text{Velocity evolution :} \quad (I_d + M\lambda AD_u - \lambda^2 GradDiv)\mathbf{u}^{n+1} = (-\lambda Grad\, p^* + R_{\mathbf{u}}) \\ \text{Corrector :} \quad p^{n+1} = p^* - \lambda Div\mathbf{u}_{n+1} \end{cases}$$

- Formal analysis of SPC(2) approximation:

$$E = A - (I_d + \lambda L_1)(I_d + \lambda L_2)(I_d + \lambda L_3) = O\left(\lambda^2 M\right)$$

Remarks:

- The SPC(2): the method corresponds to the physic-based PC of L. Chacon
- Spatial discretization gives additional error between the PC and A depending of h.
- We can construct SPC(3) with $E = O(\lambda^3(M + M^2))$

Results

- **Test case**: propagation of pressure perturbation (order 10^{-3}).
- The explicit time step is approximatively between 10^{-3} and 10^{-4}.
- We fixe the Mach number $M = 10^{-3}$ and we compare different PC for GMRES

	Δt PC	Jacobi	ILU(4)	MG(2)	SP(1)	SP(2)
	32*32 P3	1.1E+2	1	20	2	1
$\Delta t = 0.01$	$32 * 32P5$	1.3E+2	1	60	2	1
	32*32 P3	5.0E+2	3	2.0E+3	9	6
$\Delta t = 0.1$	$32 * 32P5$	1.4E+3	3	nc	9	6
	32*32 P3	4.0E+3	nc	nc	85	42
$\Delta t = 1$	$32 * 32P5$	3.5E+4	nc	nc	86	43

- Secondly we compare the effect on the mesh on the SPC methods.

	Δt mesh	16*16	32*32	64*64
SPC(1)	$\Delta t = 0.1$	5	8	14
	$\Delta t = 1$	40	90	>100
SPC(2)	$\Delta t = 0.1$	4	5	2
	$\Delta t = 1$	30	42	27

- Same effect with Hermite-Bézier scheme. The SP(2) method is better on fine grids.

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- To finish we consider the dependency of the Mach Number.

	PC mesh	$M = 0$	$M = 10^{-4}$	$M = 10^{-2}$	$M = 10^{-1}$	$M = 1$
SPC(1)	$\Delta t = 0.1$	15	15	15	22	80
SPC(2)	$\Delta t = 0.1$	2	2	2	4	10
	$\Delta t = 0.5$	15	15	17	40	>200
SPC(3)	$\Delta t = 0.1$	2	2	2	4	11
	$\Delta t = 0.5$	15	15	17	42	> 200

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Elliptic operators

Operators of the PB-PC

$$I_d + (M\lambda)\boldsymbol{a}\cdot\nabla I_d, \quad I_d + M\lambda\boldsymbol{a}\cdot\nabla I_d - \beta\lambda^2\nabla(\nabla\cdot I_d)$$

with $\mid \boldsymbol{a} \mid = 1,\ M << 1.$

- When $M\lambda = O(1)$ the transport operator is ill-conditioned. To invert this operator we can add stabilization terms, or design a specific preconditioning.
- We will focus on the low-mach regime and the elliptic operator.

Acoustic elliptic operator

- Here we consider the elliptic operator

$$\left\{\begin{array}{l}\boldsymbol{u} - \lambda^2\nabla(\nabla\cdot\boldsymbol{u}) = \mathbf{f} \\ M(\mathbf{n})\boldsymbol{u} = 0, \quad \partial\Omega\end{array}\right. \xrightarrow[\lambda\rightarrow\infty]{} \left\{\begin{array}{l}-\nabla(\nabla\cdot\boldsymbol{u}) = 0 \\ M(\mathbf{n})\boldsymbol{u} = 0, \quad \partial\Omega\end{array}\right.$$

Problem

- The limit operator is non-coercive. Indeed we can find $\parallel \boldsymbol{u} \parallel \neq 0$ (with the good BC) such that

$$\int_\Omega \mid \nabla\cdot\boldsymbol{u} \mid^2 = 0$$

- **Numerical problem**: conditioning number in $O(\lambda)$ (and also of h and the order).

Results

- **Test case**: Solution for the following operator with homogeneous Dirichlet on mesh 32*32

$$\boldsymbol{u} - \lambda^2 \Delta \boldsymbol{u} = \mathbf{f}$$

	Cells	*HB* 32	*HB* 64	Splines O3 32	Splines O3 64	Splines O5 32	Splines O5 64
$\lambda = 0.1$	Jacobi	3	4	29	55	110	100
	ILU(8)	5	7	2	3	1	2
	MG(2)	8	9	8	7	20	19
$\lambda = 1$	Jacobi	3	4	30	35	120	110
	ILU(8)	7	11	2	5	1	4
	MG(2)	10	11	8	9	20	21
$\lambda = 10$	Jacobi	3	4	30	34	120	110
	ILU(8)	7	12	3	5	1	4
	MG(2)	10	12	8	9	20	21

Strategy to solve acoustic operator

- **Step 1**: R. Hiptmair, J. Xu Using discrete B-Splines H(Div) space + Auxiliary space pc, split the kernel to the rest
- **Step 2:** We treat the orthogonal of the kernel with multi-grids+GLT method
- **GLT:** Generalized locally Toeplitz (S. Serra Capizzano) method which allows by a generalized Fourier analysis to modify the multi-grid method in the high-frequency.

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$$\boldsymbol{u} - \lambda^2 \nabla(\nabla \cdot \boldsymbol{u}) = \mathbf{f}$$

	Cells	*HB*		Splines O3		Splines O5	
		32	64	32	64	32	64
$\lambda = 0.1$	Jacobi	300	750	110	230	290	520
	ILU(8)	10	nc	3	6	1	nc
	MG(2)	45	80	15	25	45	55
$\lambda = 1$	Jacobi	nc	nc	6.3E+2	1.2E+3	1.7E+3	3.6E+3
	ILU(8)	nc	nc	nc	nc	nc	nc
	MG(2)	300	600	1.0E+2	2.0E+2	1.8E+2	3.5E+2
$\lambda = 10$	Jacobi	nc	nc	1.2E+5	5.0E+5	1.7E+5	6.6E+5
	ILU(8)	nc	nc	nc	nc	nc	nc
	MG(2)	3.0E+3	1.5E+4	6.8E+2	1.8E+3	2.2E+3	3.8E+3

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Linearized 3D MHD

- We consider the $3D$ Isothermal MHD equation in the non-conservative form,

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) & = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \otimes \boldsymbol{u} + \nabla p & = \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \partial_t p + \boldsymbol{u} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{u} & = 0 \\ \partial_t \boldsymbol{B} + \nabla \times (\boldsymbol{B} \times \boldsymbol{u}) & = \frac{\eta}{\mu_0} \nabla \times \boldsymbol{B} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- **Linearization**: $\boldsymbol{u} = \boldsymbol{u}_0 + \delta \boldsymbol{u}$, $\rho = \rho_0 + \delta \rho$, $p = p_0 + \delta p$ with $p_0 = c^2 \rho_0$, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta \boldsymbol{B}$.
- We define three important parameters: the Mach number M, the pressure ratio of the plasma $\beta = \frac{c^2}{V_a^2}$, the Alfvén speed $V_a^2 = \frac{|\boldsymbol{B}_0|^2}{\rho \mu_0}$ and the magnetic Reynolds $R_m = \frac{\mu_0 L |U_0|}{\eta}$.

Final model

$$\begin{cases} \partial_t \boldsymbol{u} + (M\sqrt{\beta} V_a) \boldsymbol{a} \cdot \nabla \boldsymbol{u} + \nabla p & = \frac{V_a^2}{|\boldsymbol{B}_0|} ((\nabla \times \boldsymbol{B}) \times \boldsymbol{b}_0) \\ \partial_t p + (M\sqrt{\beta} V_a) \boldsymbol{a} \cdot \nabla p + \gamma \beta V_a^2 \nabla \cdot \boldsymbol{u} & = 0 \\ \partial_t \boldsymbol{B} + (M\sqrt{\beta} V_a) \boldsymbol{a} \cdot \nabla \boldsymbol{B} + \mid \boldsymbol{B}_0 \mid \nabla \times (\boldsymbol{b}_0 \times \boldsymbol{u}) & = \frac{M\sqrt{\beta} V_a}{R_m} \nabla \times (\nabla \times \boldsymbol{B}) \end{cases}$$

with $M \in]0, 1]$, $\beta \in]10^{-6}, 10^{-1}]$, $\mid \mathbf{a} \mid = \mid \mathbf{b_0} \mid = 1$.

Implicit scheme for linear MHD equation

Implicit scheme:

$$\begin{cases} \delta \boldsymbol{u}^n + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla \boldsymbol{u}^n + \nabla p & = \frac{\lambda^2}{|\boldsymbol{B}_0|}\left((\nabla\times \boldsymbol{B}^n)\times \boldsymbol{b}_0\right) \\ \delta p^n + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla p^n + \beta\lambda^2\nabla\cdot \boldsymbol{u}^n & = 0 \\ \delta \boldsymbol{B}^n + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla \boldsymbol{B}^n + \mid \boldsymbol{B}_0 \mid \nabla\times(\boldsymbol{b}_0\times \boldsymbol{u}^n) & = \frac{M\sqrt{\beta}\lambda}{R_m}\nabla\times(\nabla\times \boldsymbol{B}^n) \end{cases}$$

- with $\lambda = V_a\Delta t$ the numerical Alfvén length, and $\delta\rho^n = \rho^{n+1}-\rho^n$.

- We propose to apply the SPC(2) method splitting the velocity to the other variables.
- In the end of the preconditioning we must invert three operators

Operators of the PB-PC

$$I_d + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla I_d - \frac{M\sqrt{\beta}\lambda}{R_m}\Delta I_d, \quad I_d + (M\sqrt{\beta}\lambda)\boldsymbol{a}\cdot\nabla I_d$$

$$P = \left(I_d + M\sqrt{\beta}\lambda\boldsymbol{a}\cdot\nabla I_d - \beta\lambda^2\nabla(\nabla\cdot I_d) - \lambda^2\left(\boldsymbol{b}_0\times(\nabla\times\nabla\times(\boldsymbol{b}_0\times I_d))\right)\right)$$

with $\mid \boldsymbol{a} \mid = 1$, $M << 1$, $\beta \in \left]10^{-4}, 10^{-1}\right]$

Remarks

- **First case**: We consider the regime $M << 1$ and $\beta << 1$.

Dominant Schur operator

- The Schur operator in this regime is mainly

$$P = \left(I_d - \lambda^2 \left(\boldsymbol{b}_0 \times \left(\nabla \times \nabla \times \left(\boldsymbol{b}_0 \times I_d\right)\right)\right)\right)$$

- The limit operator is non-coercive ($\lambda >> 1$). Indeed we can find $\parallel \boldsymbol{u} \parallel \neq 0$ such that

$$\int_\Omega \mid \nabla \times (\boldsymbol{b}_0 \times \boldsymbol{u}) \mid^2 = 0$$

- **Example**: all the velocity proportional to the magnetic field.

- **Second case**: We consider the regime $M << 1$ and $\beta < 1$.

Multis-cale operator

- Using a Fourier analysis and Diagonalizing the operator in the Fourier space we denote that the eigenvalues are the MHD speed waves
- When M and β is not so small, the different velocities (Alfvén, magneto-sonic slow and fast) have very different scales.

Alfven elliptic operator

Magnetic field

- $\boldsymbol{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi$
- Poloidal flux ψ satisfy equilibrium equation

$$\Delta^*\psi = -\mu R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

  with F_0 an approximation of F.
- Test case: $\boldsymbol{b}$ given by equilibrium for $\beta \approx 10^{-4}$

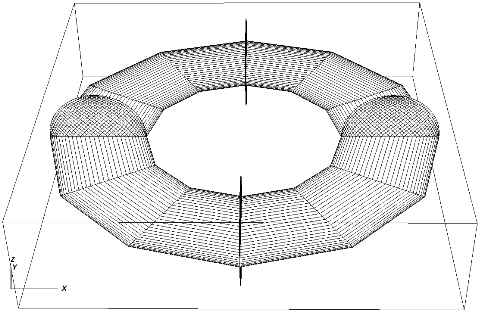

Figure: Mesh

- Example of convergence problem (Hermite-Bezier finite elements):

	Jacobi PC		MG(2)	
	32*32	64*64	32*32	64*64
$\lambda = 0.5$	60	55	12	11
$\lambda = 2$	nc	nc	nc	nc

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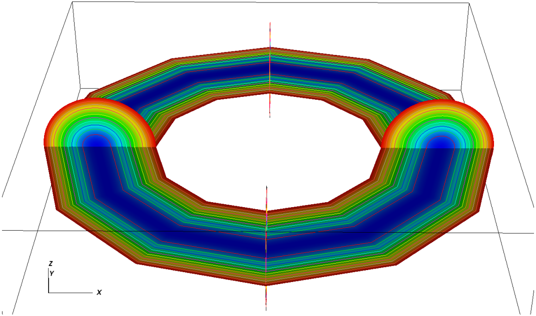

Figure: ψ equilibrum

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Other example Reduced Low beta MHD

- Current Hole : 2D reduced MHD in cartesian geometry in low β limit.
- We define ψ the poloidal magnetic flux and u the electrical potential. The model is given by

$$\begin{cases} \partial_t \psi = [\psi, u] + \eta(\Delta\psi - j_e) \\ \partial_t \Delta u = [\Delta u, u] + [\psi, \Delta\psi] + \nu\Delta^2 u \end{cases}$$

with the vorticity $w = \Delta u$ and the current $j = \Delta\psi$.

- After linearization we can use SPC method to design a preconditioning for the Jacobian.
- **Test case Kink Instability**: growth of a linear instability and non linear saturation phase.

Δt and mesh	iteration
$\Delta t = 1$ Mesh=32*32	1-3
$\Delta t = 10$ Mesh=32*32	4-25
$\Delta t = 10$ Mesh=64*64	1-20

- For this test case the GMRES tolerance is $\varepsilon = 10^{-9}$. **Remark**: The ILU(k), MG(2) and Jacobi PC tested are not able to treat this problem.

Conclusion

Physic-based pc

- If we are able to invert the sub-systems, then the physic-based pc is
  - very efficient in the Low-Mach regime for large time step.
  - less efficient in the sonic-regime, however we can treat large time step than the explicit one.
  - The efficiency does not decrease when the h decreases.

Euler equation

- **Euler equation**: at the end the main difficulty is to invert quickly the div-div operator.
- **Ongoing work:** Construct and validate preconditioning for div-div using $H(div)$ discrete space (R. Hiptmair, J. Xu) + GLT
- **Future work:** find a better version of the method for Mach close to one.

MHD equation

- In the low-Beta regime, the main difficulty is to invert quickly the curl-curl operator.
- **Ongoing work:** find a good preconditioning for curl-curl using compatible-space (difficulty: the dependance in the magnetic field).
- For β not so small, we have a multi-scale operator. **Future work**: find a strategy to separate the scales.