Fluid models for Tokamak Plasmas : Magneto-Hydrodynamic

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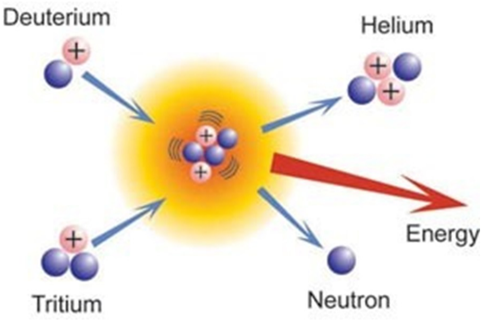
Outline

Introduction

Plasma Physics

- **Plasma**: For very high temperatures, the gas are ionized and gives a plasma which can be controlled by magnetic and electric fields.
- **Thermonuclear fusion**: The MHD allows to describe some configuration where the collision are not so small or for long time behavior.
- **Astrophysics**: The MHD describe a lot of astrophysics configuration: supernovae explosion, solar wind and instabilities etc.
- **Context**: in the case we consider the application of the MHD to the simulation of Tokamak instabilities.



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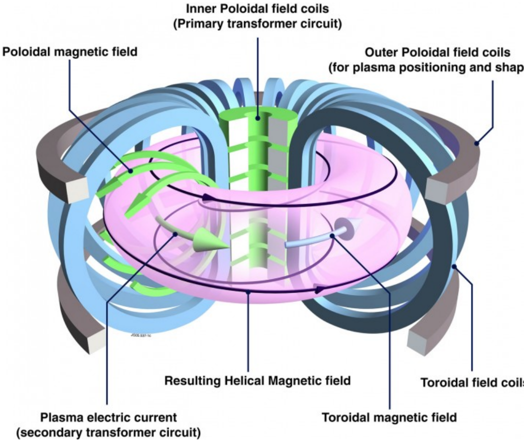


Figure: Tokamak

Hierarchy of Models

- **Microscopic model**: N-Body model. We write the dynamical Newton equation for each particle:

$$\left\{ \begin{array}{l} \frac{d\gamma m \boldsymbol{v}_i}{dt} = \sum_j q(\boldsymbol{E}_j + \boldsymbol{v}_i \times \boldsymbol{B}_j) \\ \frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i \end{array} \right.$$

- Unrealistic approach: we must solve N coupled equations with $N \approx 10^{16} - 10^{20}$.

- **Mesoscopic model**: Kinetic Vlasov model. Taking the limit of the N-Body model we obtain an equation on the distribution of the particles:

$$\partial_t f(t, \boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla f + \boldsymbol{F}_{ext} \cdot \nabla_{\boldsymbol{v}} f = Q(f, f)$$

  with $\boldsymbol{F}_{ext}$ the external force (gravity, Lorentz force, etc).

- **Macroscopic model**: Fluid models (moment models). If we are close to the equilibrium, taking the three first moments of the distribution function we obtain:

$$\partial_t \mathbf{U} + \nabla \cdot F(\mathbf{U}) + \epsilon \nabla \cdot (D(\nabla \mathbf{U})) = 0$$

- Examples : hyperbolic models (Euler, Euler-Lorentz, ideal MHD), parabolic models (Navier-Stokes, Resistive MHD).

Example of Application : MHD and ELM

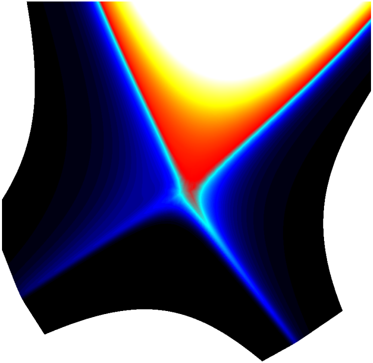
- In the tokamak some instabilities can appear in the plasma.
- The simulation of these instabilities is an important subject for ITER.

- Example of Instabilities in the tokamak :
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.

- These instabilities are linked to the very large gradient of pressure and very large current at the edge.

- Many aspects of these instabilities are described by fluid models (MHD resistive and diamagnetic or extended)

- ELM simulation



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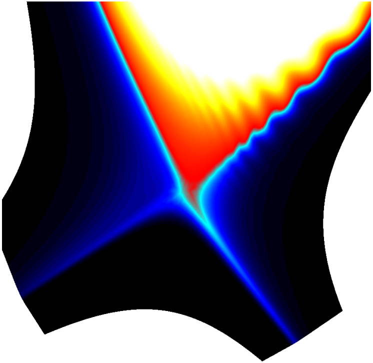

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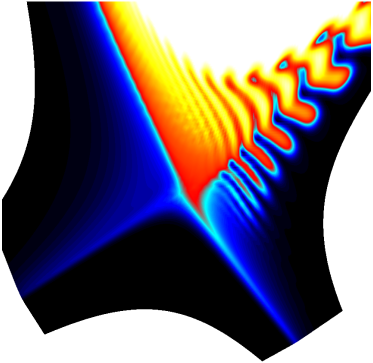

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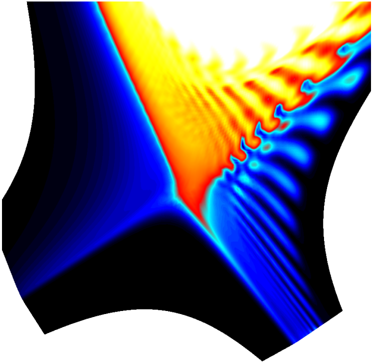

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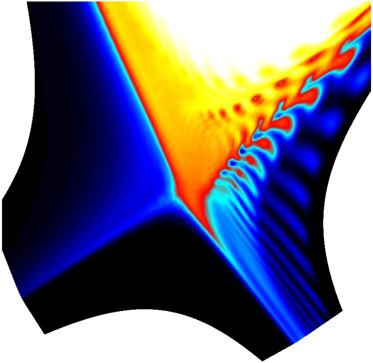

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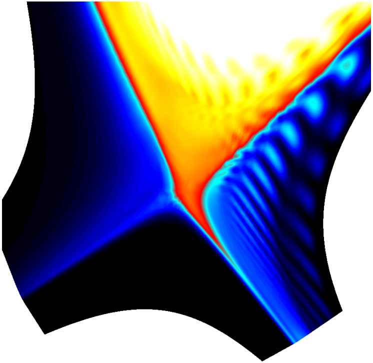


Derivation and study of Fluid models

Vlasov equations and equilibrium

- First model to describe a plasma : **Two species Vlasov-Maxwell** kinetic equation.
- We define $f_s(t,\mathbf{x},\mathbf{v})$ the distribution function associated with the species s. $\mathbf{x}\in D_{\mathbf{x}}$ and $\mathbf{v}\in R^3$.

Two-species Vlasov equation

$$\begin{cases} \partial_t f_s + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \dfrac{q_s}{m_s}(\boldsymbol{E}+\mathbf{v}\times\boldsymbol{B})\cdot\nabla_{\mathbf{v}} f_s = C_s = \sum_t C_{st}, \\ \frac{1}{c^2}\partial_t \boldsymbol{E} - \nabla\times\boldsymbol{B} = -\mu_0\boldsymbol{J}, \\ \partial_t\boldsymbol{B} = -\nabla\times\boldsymbol{E}, \\ \nabla\cdot\boldsymbol{B}=0, \quad \nabla\cdot\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}. \end{cases}$$

Invariants (no collisional case)

- Mass and momentum:

$$\frac{d}{dt}\left(\frac{1}{2}\sum_s\int m_s f_s d\mathbf{x}d\mathbf{v}\right)=0, \quad \frac{d}{dt}\left(\frac{1}{2}\sum_s\int m_s f_s \mathbf{v} d\mathbf{x}d\mathbf{v}\right)=0.$$

- Total energy:

$$\frac{d}{dt}\left(\frac{1}{2}\sum_s\int m_s f_s \mid \mathbf{v}\mid^2 d\mathbf{x}d\mathbf{v} + \frac{1}{2\mu_0 c^2}\int \mid\boldsymbol{E}\mid^2 d\mathbf{x} + \frac{1}{2\mu_0}\int\mid\boldsymbol{B}\mid^2 d\mathbf{x}\right)=0.$$

Collisional operator and invariants

- Derivation of fluid model:
  - Collisional regime: $w << \nu$ with w and ν the plasma and collision frequencies.

$$\partial_t f_s + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s} (\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) \cdot \nabla_{\mathbf{v}} f_s = \frac{1}{\varepsilon} C_s(f, f)$$

  - We define the equilibrium distribution: the Maxwellian $M_s(\mathbf{v})$ defined by

$$M_s(t, \mathbf{x}, \mathbf{v}) = \frac{n_s}{(2\pi T_s / m_s)^{\frac{3}{2}}} e^{-\frac{m_s}{2T}(\mathbf{v} - \mathbf{u}_s)^2}$$

  with n_s the number of particles, T_s the temperature and $\mathbf{u}_s$ the average velocity.
- Properties of the collision operator
  - For each species: $\int_{R^3} m_s \mathbf{v} C_{ss} d\mathbf{v} = 0$, $\int_{R^3} \frac{1}{2} m_s \mid \mathbf{v} \mid^2 C_{ss} d\mathbf{v} = 0$,
  - No conversion of particles: $\int_{R^3} m_s \mathbf{v} C_{s_1 s_2} d\mathbf{v} = 0$
  - Global momentum and energy conservation: $\int_{R^3} g(\mathbf{v})_s C_{st} d\mathbf{v} + \int_{R^3} g(\mathbf{v})_t C_{ts} d\mathbf{v} = 0$ with $g(\mathbf{v}) = m_s \mathbf{v}$ or $g(\mathbf{v}) = m_s \frac{1}{2} \mid \mathbf{v} \mid^2$

Asymptotic Study

- We plug the Chapman-Enskog expansion $f_s(t, \mathbf{x}, \mathbf{v}) = f_s^0 + \varepsilon f_s^1 + \varepsilon^2 f_s^2$ to obtain that

$$f_s(t, \mathbf{x}, \mathbf{v}) = M_s(\mathbf{v}) + \varepsilon f_s^1 + O(\varepsilon^2)$$

- Pugging this expansion and taking the moment we obtain fluid models.

Two-fluid model

- Computing the moments of the Vlasov equation we obtain the following two fluid model

Two fluid moments

$$
\begin{cases}
\partial_t n_s + \nabla_{\mathbf{x}} \cdot (m_s n_s \boldsymbol{u}_s) = 0, \\
\partial_t (m_s n_s \boldsymbol{u}_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \boldsymbol{u}_s \otimes \boldsymbol{u}_s) + \nabla_{\mathbf{x}} p_s + \nabla_{\mathbf{x}} \cdot \overline{\overline{\mathbf{\Pi}}}_s = \sigma_s \boldsymbol{E} + \boldsymbol{J}_s \times \boldsymbol{B} + \mathbf{R}_s, \\
\partial_t (m_s n_s \epsilon_s) + \nabla_{\mathbf{x}} \cdot (m_s n_s \boldsymbol{u}_s \epsilon_s + p_s \boldsymbol{u}_s) + \nabla_{\mathbf{x}} \cdot \left(\overline{\overline{\mathbf{\Pi}}}_s \cdot \boldsymbol{u}_s + \mathbf{q}_s \right) \\
= \sigma_s \boldsymbol{E} \cdot \boldsymbol{u}_s + Q_s + \mathbf{R}_s \cdot \boldsymbol{u}_s, \\
\\
\frac{1}{c^2} \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J}, \\
\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \\
\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \cdot \boldsymbol{E} = \frac{\sigma}{\varepsilon_0}.
\end{cases}
$$

- $n_s = \int_{R^3} f_s d\mathbf{v}$ the particle number , $m_s n_s \boldsymbol{u}_s = \int_{R^3} m_s \mathbf{v} f_s d\mathbf{v}$ the momentum, ϵ_s the total energy and $\rho_s = m_s n_s$ the density.
- The isotropic pressures are p_s, the stress tensors $\overline{\overline{\mathbf{\Pi}}}_s$ and the heat fluxes $\mathbf{q}_s$.
- $\mathbf{R}_s$ and Q_s are associated with the interspecies collision (force and energy transfer).
- The current is given by $\boldsymbol{J} = \sum_s \boldsymbol{J}_s = \sum_s \sigma_s \boldsymbol{u}_s$ with $\sigma_s = q_s n_s$.

Energy conservation

$$
\frac{d}{dt} \left(\int_{D_x} (\rho_e \varepsilon_e + \rho_i \varepsilon_i) + \frac{1}{2\mu_0 c^2} \int_{D_x} \mid \boldsymbol{E} \mid^2 d\mathbf{x} + \frac{1}{2\mu_0} \int_{D_x} \mid \boldsymbol{B} \mid^2 d\mathbf{x} \right) = 0
$$

MHD: assumptions and generalized Ohm's law

MHD: assumptions

- **quasi neutrality assumption**: $n_i = n_e \Longrightarrow \rho \approx m_i n_i + O(\frac{m_e}{m_i})$, $\mathbf{u} \approx \boldsymbol{u}_i + O(\frac{m_e}{m_i})$
- **Magneto-static assumption** : $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + O(\frac{V_0}{c})$.

- We define $\rho = \rho_i + \rho_e$ and $\mathbf{u} = \frac{\rho_i \boldsymbol{u}_i + \rho_e \boldsymbol{u}_e}{\rho}$.

Velocity relation

- Consequence of the quasi-neutrality:

$$\boldsymbol{u}_e = \boldsymbol{u} - \frac{m_i}{e\rho}\boldsymbol{J} + O\left(\frac{m_e}{m_i}\right)$$

- Summing the mass and moment equation for the two species we obtain:

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}} + O\left(\frac{m_e}{m_i}\right)$$

- For the pressure equation, we replace the electronic velocity by full velocity using the previous relation.

MHD: derivation

- **Ohm law**: Relation between the electric field and the other variables.
- Taking the electron density and momentum equations we obtain

$$m_e\left(\partial_t(n_e\boldsymbol{u}_e)+\nabla\cdot(n_e\boldsymbol{u}_e\otimes\boldsymbol{u}_e)\right)+\nabla p_e=-en_e\boldsymbol{E}+\boldsymbol{J}_e\times\boldsymbol{B}-\nabla\cdot\overline{\overline{\Pi}}_e+\mathbf{R}_e,$$

- We multiply the previous equation by $-e$ and we define $\boldsymbol{J}_e=-en_e\boldsymbol{u}_e$, we obtain

$$\frac{m_e}{e^2n_e}\left(\partial_t\boldsymbol{J}_e+\nabla\cdot(\boldsymbol{J}_e\otimes\boldsymbol{u}_e)\right)=\boldsymbol{E}+\boldsymbol{u}_e\times\boldsymbol{B}+\frac{1}{en_e}\nabla p_e+\frac{1}{en_e}\nabla\cdot\overline{\overline{\Pi}}_e-\frac{1}{en_e}\mathbf{R}_e,$$

- Using the quasi neutrality $\mathbf{R}_e=\eta\frac{e}{m_i}\rho\boldsymbol{J}$ and $\boldsymbol{u}_e=\boldsymbol{u}-\frac{m_i}{e\rho}\boldsymbol{J}$ we obtain

Generalized Ohm's law

$$\boldsymbol{E}+\underbrace{\boldsymbol{u}\times\boldsymbol{B}}_{\text{drift velocity}}=\underbrace{\eta\boldsymbol{J}}_{\text{resistivity}}\underbrace{+\frac{m_i}{\rho e}\boldsymbol{J}\times\boldsymbol{B}}_{\text{hall term}}\underbrace{-\frac{m_i}{\rho e}\nabla p_e-\frac{m_i}{\rho e}\nabla\cdot\overline{\overline{\Pi}}_e}_{\text{pressure term}}+O\left(\frac{m_e}{m_i}\right).$$

Final simplification

$$\left(\frac{m_e}{m_i}\right)<<1\qquad\left(\frac{V_0}{c}\right)<<1$$

Extended MHD: model

Extended MHD

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\
\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}, \\
\dfrac{1}{\gamma - 1} \partial_t p_i + \dfrac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_i + \dfrac{\gamma}{\gamma - 1} p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i = -\overline{\overline{\boldsymbol{\Pi}}}_i : \nabla \boldsymbol{u}, \\
\dfrac{1}{\gamma - 1} \partial_t p_e + \dfrac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_e + \dfrac{\gamma}{\gamma - 1} p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e = \dfrac{1}{\gamma - 1} \dfrac{m_i}{e\rho} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \dfrac{\nabla \rho}{\rho} \right) \\
-\overline{\overline{\boldsymbol{\Pi}}}_e : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_e : \nabla \left(\frac{m_i}{e\rho} \boldsymbol{J} \right) + \eta |\boldsymbol{J}|^2, \\
\partial_t \boldsymbol{B} = -\nabla \times \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \dfrac{m_i}{\rho e} \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_e - \dfrac{m_i}{\rho e} \nabla p_e + \dfrac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right), \\
\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \boldsymbol{J}.
\end{cases}
$$

- **Remark**: We can write easily the equation on the total pressure $p_e + p_i$. Possible simplification $p_e = \frac{p}{2}$.

- In Black: ideal MHD. In Black and blue: Viscous-resistive MHD. All the term: Extended MHD.

MHD Invariants

- The mass ρ the momentum $\rho\boldsymbol{u}$ and the total energy $E=\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{|\boldsymbol{B}|^2}{2}+\frac{1}{\gamma-1}p$ with $p=\rho T$ are conserved in time.

Sketch of proof

- Mass and momentum conservation: divergence form + the flux-divergence theorem + null BC.
- Total energy: multiply the first equation by $\frac{|\boldsymbol{u}|^2}{2}$ the second by $\boldsymbol{u}$ and the last one by $\boldsymbol{B}$ we obtain

$$\partial_t E+\nabla\cdot\left[\boldsymbol{u}\left(\rho\frac{|\boldsymbol{u}|^2}{2}+\frac{\gamma}{\gamma-1}p\right)-(\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}\right]+\nabla\cdot\mathbf{q}+\nabla\cdot(\overline{\overline{\boldsymbol{\Pi}}}\cdot\boldsymbol{u})+\eta\nabla\cdot(\boldsymbol{J}\times\boldsymbol{B})$$
$$+\nabla\cdot\left[\frac{m_i}{\rho e}\left((\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}-\nabla p_e\times\boldsymbol{B}-\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e\times\boldsymbol{B}-\frac{\gamma}{\gamma-1}p_e\boldsymbol{J}-\boldsymbol{J}\cdot\overline{\overline{\boldsymbol{\Pi}}}_e\right)\right]=0$$

  with $\overline{\overline{\boldsymbol{\Pi}}}=\overline{\overline{\boldsymbol{\Pi}}}_i+\overline{\overline{\boldsymbol{\Pi}}}_e$ and $\mathbf{q}=\mathbf{q}_i+\mathbf{q}_e$.
- We conclude with the divergence-flux theorem + BC null.

Tokamak Ordering

- We define ρ_i ion Larmor radius, V_i thermal velocity, w_i gyro frequency.
- **Tokamak Ordering**: Small flow $\frac{V_0}{V_i}=O(\delta)$ and very low frequency $\frac{w_0}{w_i}=O(\delta^2)$ with $\delta=\frac{\rho_i}{L}$

Extended MHD with ordering

$$
\left\{
\begin{array}{l}
\delta^2\partial_t\rho+\delta^2\nabla\cdot(\rho\boldsymbol{u})=0,\\
\delta^3\rho\partial_t\boldsymbol{u}+\delta^3\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\delta\nabla p=\delta\boldsymbol{J}\times\boldsymbol{B}-\delta\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}},\\
\delta^2\left(\dfrac{1}{\gamma-1}\partial_t p+\dfrac{1}{\gamma-1}\boldsymbol{u}\cdot\nabla p+\dfrac{\gamma}{\gamma-1}p\nabla\cdot\boldsymbol{u}+\nabla\cdot\mathbf{q}\right)=\delta^2\left(\frac{1}{\gamma-1}\frac{m_i}{e\rho}\boldsymbol{J}\cdot\left(\nabla p_e-\gamma p_e\frac{\nabla\rho}{\rho}\right)\right.\\
\left.-\overline{\overline{\boldsymbol{\Pi}}}:\nabla\boldsymbol{u}+\overline{\overline{\boldsymbol{\Pi}}}_e:\nabla\left(\frac{m_i}{e\rho}\boldsymbol{J}\right)+\eta|\boldsymbol{J}|^2\right),\\
\delta^2\partial_t\boldsymbol{B}=-\delta^2\nabla\times\left(-\boldsymbol{u}\times\boldsymbol{B}+\eta\boldsymbol{J}-\dfrac{m_i}{\rho e}\nabla\cdot\overline{\overline{\boldsymbol{\Pi}}}_e-\dfrac{m_i}{\rho e}\nabla p_e+\dfrac{m_i}{\rho e}(\boldsymbol{J}\times\boldsymbol{B})\right),\\
\nabla\cdot\boldsymbol{B}=0,\quad \delta\nabla\times\boldsymbol{B}=\boldsymbol{J}.
\end{array}
\right.
$$

Closure I

- **Closure**: Write the dependency of $\overline{\overline{\boldsymbol{\Pi}}}$ and $\mathbf{q}$ with the variables p, $\mathbf{u}$ and ρ.
- Taking $f_s = M_s(\mathbf{v}) + O(\varepsilon)$ and neglecting the ε terms, we obtain

$$\overline{\overline{\boldsymbol{\Pi}}} = 0 \quad \text{and } \mathbf{q} = 0$$

- This approximation gives ideal MHD or Euler equation (in the dynamic gas context).
- Taking $f_s = M_s(\mathbf{v}) + \varepsilon f_1 + O(\varepsilon^2)$ and neglecting the ε^2 terms, we obtain

$$\overline{\overline{\boldsymbol{\Pi}}} = \overline{\overline{\boldsymbol{\Pi}}}(\mathbf{W}, \mathbf{b}, p) \quad \mathbf{q} = \mathbf{q}(T, \mathbf{b})$$

with $\mathbf{W} = \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T - \frac{2}{3}\nabla \cdot \boldsymbol{u}$ and $\mathbf{b} = \frac{\mathbf{B}}{|\boldsymbol{B}|}$.

- This approximation gives viscous MHD and Navier-Stokes equation (in the gas dynamic context).

Heat flux and anisotropic diffusion

- $\mathbf{q}_{i,e} = -n_{i,e}\left(\chi_{\parallel}^{i,e}(\mathbf{b}\cdot\nabla T_{i,e})\mathbf{b} + \chi_c^{i,e}\mathbf{b}\times\nabla T_{i,e} + \chi_{\perp}^{i,e}\mathbf{b}\times(\mathbf{b}\times\nabla T_{i,e})\right)$
- **Ordering**: $\chi_{\parallel}^{i,e} = O\left(\frac{\lambda_{i,e}}{L}\right)^2$, $\chi_c^{i,e} = O\left(\frac{\rho_{i,e}}{L_\perp}\right)$, $\chi_{\perp}^{i,e} = O\left(\frac{\rho_{i,e}}{L_\perp}\right)^2$ and $\frac{\lambda_{i,e}}{L} >> \frac{\rho_{i,e}}{L_\perp}$.

Closure and simplification

Stress tensor

- Total Stress tensor $\overline{\overline{\Pi}} = \overline{\overline{\Pi}}_i + \overline{\overline{\Pi}}_e \approx \overline{\overline{\Pi}}_i$ since $\frac{|\overline{\overline{\Pi}}_i|}{|\overline{\overline{\Pi}}_i|} = O(\frac{m_e}{m_i})$.
- **Tokamak Ordering**: Small flow $\frac{V_0}{V_i} = O(\delta)$ and very low frequency $\frac{w_0}{w_i} = O(\delta^2)$.
- Stress tensor expansion $\overline{\overline{\Pi}} = \overline{\overline{\Pi}}_\parallel + \delta^2 \overline{\overline{\Pi}}_{gv} + \delta^4 \overline{\overline{\Pi}}_\perp \Longrightarrow \overline{\overline{\Pi}} \approx \overline{\overline{\Pi}}_\parallel + \delta^2 \overline{\overline{\Pi}}_{gv}$.
- The term $\nabla \cdot \overline{\overline{\Pi}}_\parallel$ dissipate energy (compensated by the viscous parallel heating $\overline{\overline{\Pi}} : \nabla \boldsymbol{u}$).
- The term $\nabla \cdot \overline{\overline{\Pi}}_{gv}$ does not dissipate energy ($\overline{\overline{\Pi}}_{gv} : \nabla \boldsymbol{u} = 0$)

Simplification of the velocity

- Velocity expansion

$$\parallel \boldsymbol{B} \parallel^2 \mathbf{u} = \underbrace{(\mathbf{u}, \boldsymbol{B})\boldsymbol{B}}_{\mathbf{u}_\parallel} + \underbrace{(\boldsymbol{E} \times \boldsymbol{B})}_{\mathbf{u}_E} + \frac{m_i}{\rho e} \underbrace{(\boldsymbol{B} \times \nabla p_i)}_{\mathbf{u}_i} + O(\delta)$$

- This approximation is used in JOREK.

Wave and stability

Linearization of the MHD

Linearization of ideal MHD

- We consider a flow $\boldsymbol{u} = \boldsymbol{u}_0 + \delta \boldsymbol{u}$ (with $\boldsymbol{u}_0$ constant), $\boldsymbol{B} = \boldsymbol{B}_0 + \delta \boldsymbol{B}$, $p = p_0 + \delta p$ and $\rho = \rho_0 + \delta\rho$.

- We obtain

$$\partial_t \delta\rho = -\nabla \cdot (\rho \delta \boldsymbol{u}), \quad \partial_t \delta p = -\delta \boldsymbol{u} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \delta \boldsymbol{u}, \quad \partial_t \delta \boldsymbol{B} = \nabla \times (\delta \boldsymbol{u} \times \boldsymbol{B}_0)$$

and

$$\rho_0 \partial_t \delta \boldsymbol{u} + \rho_0 \boldsymbol{u}_0 \cdot \nabla \delta \boldsymbol{u} + \nabla \delta p = \delta \boldsymbol{J} \times \boldsymbol{B}_0 + \boldsymbol{J}_0 \times \delta \boldsymbol{B}$$

- We define the Lagrangian displacement $\partial_t \boldsymbol{\xi} = \delta \boldsymbol{u}$. Using this definition and taking all term together we obtain

Linearized force operator

- Lagrangian displacement

$$\rho \partial_{tt} \boldsymbol{\xi} = \rho_0 \boldsymbol{u}_0 \cdot \nabla (\partial_t \boldsymbol{\xi}) + F_a(\boldsymbol{B}_0)\boldsymbol{\xi} + F_p(p_0)\boldsymbol{\xi}$$

with

$$F_a(\boldsymbol{B}_0)\boldsymbol{\xi} = \frac{1}{\mu_0}\left[\nabla \times (\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0))) \times \boldsymbol{B}_0 + \boldsymbol{J}_0 \times (\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0))\right]$$

$$F_p(p_0)\boldsymbol{\xi} = \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p_0 \nabla \cdot \boldsymbol{\xi})$$

Plane wave study of MHD

- **Plane wave analysis** : we consider a solution

$$\xi(t,\boldsymbol{x}) = \xi^0 f\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right),$$

with ξ^0 a vector independent of t and $\boldsymbol{x}$. Plugging the Plane wave in the model we obtain $A(\boldsymbol{k},\omega)\xi^0 = 0$.
- To have a non-trivial solution, the kernel of $A(\boldsymbol{k},\omega)$ must be non-trivial. The dispersion relation generate a non-trivial Kernel.

- We consider $\boldsymbol{B}_0$, ρ_0 and p_0 constant. We define $\boldsymbol{k} = \boldsymbol{k}_{\parallel} + \boldsymbol{k}_{\perp}$, $c^2 = \gamma\frac{p_0}{\rho_0}$ and $V_a^2 = \frac{|\boldsymbol{B}_0|^2}{\mu_0\rho_0}$.
- Dispersion matrix $A(\boldsymbol{k},\omega)$:

$$-\rho_0\left[(w^2 + w(\boldsymbol{k}\cdot\boldsymbol{u}_0))\xi - c^2(\boldsymbol{k}\cdot\xi)\boldsymbol{k} + \frac{1}{\rho_0\mu_0}(\boldsymbol{k}\times((\boldsymbol{k}\cdot\boldsymbol{B}_0)\xi - (\boldsymbol{k}\cdot\xi)\boldsymbol{B}_0)\times\boldsymbol{B}_0)\right] = 0$$

$$-\rho_0\left[(w^2 + w(\boldsymbol{k}\cdot\boldsymbol{u}_0))\xi - (c^2 + V_a^2)(\boldsymbol{k}\cdot\xi)\boldsymbol{k} + \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)}{\rho_0\mu_0}((\boldsymbol{k},\boldsymbol{B}_0)\xi - (\boldsymbol{B}_0\cdot\xi)\boldsymbol{k} - (\boldsymbol{k}\cdot\xi)\boldsymbol{B}_0)\right] =$$

Remark

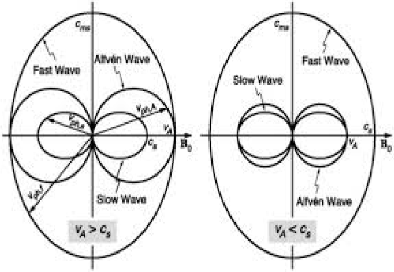
- **MHD specific**: the speed wave depend strongly of the magnetic field direction.

Final wave structure of MHD

Wave Structure of the MHD

- **Alfvén velocity** and **Sound velocity** :
  $$V_a = \sqrt{\frac{\mathbf{B}_0^2}{\rho_0}} \text{ and } c = \sqrt{\frac{\gamma p_0}{\rho_0}}$$
- Four types of waves in plasma:
  - The matter wave $\lambda_0 = (\boldsymbol{u}_0, \boldsymbol{n})$,
  - The Alfven wave $\lambda_a = (\boldsymbol{u}_0, \boldsymbol{n}) \pm V_a$
  - The slow wave
    $$\lambda_s = (\boldsymbol{u}_0, \boldsymbol{n}) \pm \left(\frac{1}{2}(V_a^2 + c^2) - V_{ac}\right)^{\frac{1}{2}}$$
  - The fast wave
    $$\lambda_s = (\boldsymbol{u}_0, \boldsymbol{n}) \pm \left(\frac{1}{2}(V_a^2 + c^2) + V_{ac}\right)^{\frac{1}{2}}$$

  with $\boldsymbol{n} = \frac{\boldsymbol{k}}{\|\boldsymbol{k}\|}$ and
  $$V_{ac} = ((V_a^2 + c^2)^2 - 4v_a^2c^2cos^2\theta)^{\frac{1}{2}}$$



Tokamak

- Classical regime: $V_a >> c >> \| \mathbf{u} \|$.
- Close to X-point: $V_a >> c$ and $c \approx \| \mathbf{u} \|$.
- Extended MHD: two additional dispersive waves.

Stability I

Stability of Linearized ideal MHD

- We consider a flow with a velocity $\boldsymbol{u}_0 = 0$. We consider $\boldsymbol{\xi}(\mathbf{x}, t) = \xi_0(\mathbf{x})e^{-iwt}$.
- As written before, the Lagrangian displacement satisfies

$$\rho_0 \partial_{tt}\boldsymbol{\xi} = F(p_0, \boldsymbol{B}_0)\boldsymbol{\xi} \Longrightarrow -\rho_0 w^2 \boldsymbol{\xi} = F(p_0, \boldsymbol{B}_0)\boldsymbol{\xi}$$

- The operator F is self-adjoint (energy conservation). Therefore the w^2 are purely real (w purely real or imaginary)
- **Stability** : depends on the sign of the imaginary part since
  - $w^2 > 0$ stable oscillations.
  - $w^2 < 0$ exponential instability

Energy conservation

- **Energy conservation**: $\partial_t(\delta K + \delta W) = cts$ with

$$\delta K = \frac{1}{2}\int \rho \mid \partial_t \boldsymbol{\xi} \mid^2, \quad \delta W = -\frac{1}{2}\int \overline{\boldsymbol{\xi}} \cdot F(\boldsymbol{\xi})$$

Stability II

Results of stability

- Using the conservation energy and some property we obtain that

$$w^2 = \frac{\delta W(\overline{\xi}, \xi)}{K(\overline{\xi}, \xi)}$$

- Therefore the instability depends on the sign of the potential energy $\delta W(\overline{\xi}, \xi)$.

Potential energy

$$\delta W = \frac{1}{2}\int \left(\mid \nabla \times (\xi \times \boldsymbol{B}_0) \mid^2 + \mid \boldsymbol{B}_0(\nabla \cdot \xi_\perp + 2\xi_\perp \cdot (\mathbf{b} \cdot \nabla \mathbf{b})) \mid^2 + \gamma p_0 \mid \nabla \cdot \xi \mid^2 \right)$$

$$- \int \left(2(\xi_\perp \cdot \nabla p_0)(\mathbf{b}_0 \cdot \nabla \mathbf{b}_0 \cdot \xi_\perp) + \mathbf{J}_{\parallel,0}(\xi_\perp \times \mathbf{b}_0) \cdot (\nabla \times (\xi \times \boldsymbol{B}_0)) \right)$$

- Red term magnetic field line bending (Alfvén wave) $\Longrightarrow$ stabilizing
- Blue term magnetic field compression (fast wave) $\Longrightarrow$ stabilizing
- Green term compression (slow wave) $\Longrightarrow$ stabilizing
- Violet term pressure gradient $\Longrightarrow$ destabilizing
- Orange term parallel current $\Longrightarrow$ destabilizing

Equilibrium and reduced models

JOREK code

JOREK

- **Models**: reduced MHD models (reduction of the solution space) using potential formulation of the fields.
- **Physics in models**: two fluid and neoclassical effect, coupling with neutral ...
- Typical run of JOREK:
  - Computation of the equilibrium on a grid aligned to the magnetic surfaces.
  - Computation of the MHD instabilities perturbing the axisymmetric equilibrium.

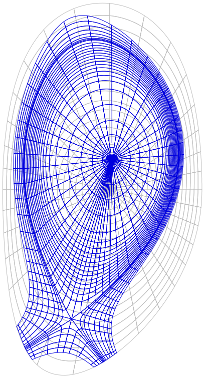


Figure: Aligned grid

Numerical methods

- **Spatial Discretization**: 2D Cubic Bezier finite elements + Fourier expansion.
- **Temporal discretization**: Implicit scheme + Gmres + Toroidal modes Block Jacobi preconditioning

Equilibrium I

- We consider the resistive MHD with a uniform flow $\boldsymbol{u}=0$. We obtain the following equilibrium

$$\begin{cases} \boldsymbol{u}=0 \\ \boldsymbol{J}\times\boldsymbol{B}=\nabla p \\ \partial_t \boldsymbol{B}=\frac{\eta}{\nu_0}\Delta\boldsymbol{B} \end{cases}$$

- $\tau << \tau_{diff}$ with τ the characteristic time and $\tau_{diff}=\frac{\mu_0 L^2}{\eta}$ the characteristic time of the diffusion.

MHD equilibrium

- The equilibrium is mainly defined by the force balanced

$$\boldsymbol{J}\times\boldsymbol{B}=\nabla p$$

- The equilibrium induces that $\boldsymbol{B}\cdot\nabla p=0$, $\nabla\cdot\boldsymbol{J}=0$ and we assume that $\nabla p\cdot\mathbf{e}_\phi=0$.
- In a Tokamak we assume that

$$\boldsymbol{B}=\mu_0\frac{F(\psi,Z)}{R}\mathbf{e}_\phi+\frac{1}{R}(\nabla\psi\times\mathbf{e}_\phi)$$

- with ψ the poloidal magnetic flux.
- By definition of the magnetic field, we have:

$$\mu_0\boldsymbol{J}=\frac{1}{R}\nabla\left(\mu_0 F(\psi,Z)\right)\times\mathbf{e}_\phi-\frac{1}{R}\Delta^*\psi\mathbf{e}_\phi \text{ with } \Delta^*=R^2\nabla\cdot\left(\frac{1}{R}\nabla\cdot\right)$$

Equilibrium II

- Plugging the previous results in the equilibrium and taking this in the toroidal direction

$$\mu_0\partial_R\left(F(\psi)\right)\partial_Z\psi-\mu_0\partial_Z(F(\psi))\partial_R\psi=\nabla P\cdot\mathbf{e}_\phi=0$$

- Since $\boldsymbol{B}\cdot\nabla p=0$ we have $\frac{1}{R}\left[\psi,p\right]=0$ which gives $p=p(\psi)$.
- Using the fact that P and F depend only of ψ, plugging the definition $\boldsymbol{B}$ and $\boldsymbol{J}$ in the equilibrium and taking this in the direction we obtain the equilibrium.

Grad-Shafranov equation

$$\Delta^*\psi=-\mu_0R^2\frac{dp(\psi)}{d\psi}-\mu_0^2F(\psi)\frac{dF(\psi)}{d\psi}$$

- **Equilibrium**: given by a nonlinear second order elliptic equation (Picard or Newton solver).

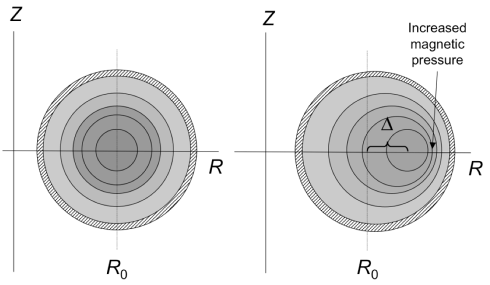
Aligned grid

- Computation of the equilibrium on polar grid
- Computation of new grid aligned on the iso-surface of ψ.
- Computation of the equilibrium of the new grid.

Grad-Shafranov Shift and β plasma

Shift

- **Property of GS operator**: induce a shift of the magnetic surface
- **Shift estimation**: $\frac{\Delta}{r} \approx \beta_p \frac{r}{R_0}$
- with r and R_0 the minor and major radius.
- $\beta_p = \frac{2\mu_0 |p|}{|\mathbf{B}_p|}$ the ratio of the pressure and poloidal magnetic pressure.



Limit

- No reasonable physics equilibrium when the shift is equal to minor radius. Consequently

$$\beta_p = \frac{| \mathbf{B}_\phi |^2}{| \mathbf{B}_p |^2} \beta < \frac{R_0}{r}$$

- At the end we can deduce a maximum value of β. A typical example:

$$| \mathbf{B}_\phi |^2 \approx 10 \, | \mathbf{B}_p |^2, \quad R_0 = 3r \Longrightarrow \beta_{max} = 0.03$$

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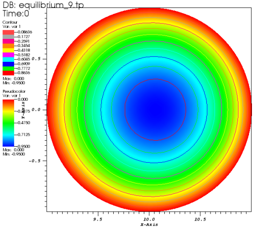

Figure: GS solution for β fixed, $r = 1$ and $R0 = 10$

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Grad-Shafranov Shift and β plasma

Shift

- **Property of GS operator**: induce a shift of the magnetic surface
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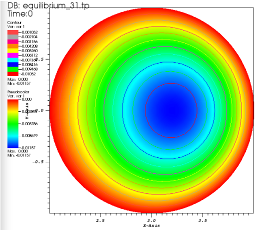


Figure: GS solution for β fixed, $r = 1$ and $R0 = 3$

Limit

- No reasonable physics equilibrium when the shift is equal to minor radius. Consequently

$$\beta_p = \frac{\mid \boldsymbol{B}_\phi \mid^2}{\mid \boldsymbol{B}_p \mid^2} \beta < \frac{R_0}{r}$$

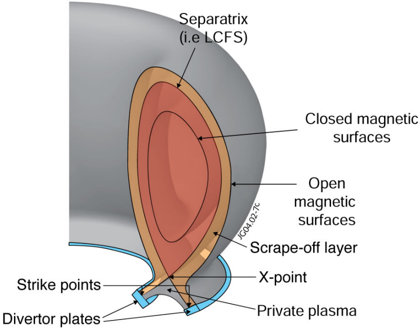
- At the end we can deduce a maximum value of β. A typical example:

$$\mid \boldsymbol{B}_\phi \mid^2 \approx 10 \mid \boldsymbol{B}_p \mid^2, \quad R_0 = 3r \Longrightarrow \beta_{max} = 0.03$$

Geometry of the poloidal equilibrium

Definition

- **Divertor**: device to evacuate impurities and excess heat.
- **X-Point** : saddle point of the poloidal magnetic flux (no poloidal magnetic field at this point).
- **Separatrix**: last closed magnetic surface of the magnetic field.
- **Scrape-off layer**: (plasma region characterized by the open field lines).



Numerical difficulties

- **Singularity**: The X-point generates a singularity in the mapping between the logical and physical mesh.
- **Boundary condition**: no trivial Bohm BC condition at the x-point (mach number closed to one).

Reduced MHD: assumptions and principle of derivation

- **Aim**: Reduce the number of variables and eliminate the fast waves in the reduced MHD model.
- We consider the cylindrical coordinate $(R, Z, \phi) \in \Omega \times [0, 2\pi]$.

Reduced MHD: Assumption

$$\boldsymbol{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi, \quad \boldsymbol{u} = -\underbrace{R\nabla u \times \mathbf{e}_\phi}_{=\frac{\boldsymbol{E}\times\boldsymbol{B}_\phi}{|\boldsymbol{B}_\phi|^2}} + v_{||}\boldsymbol{B} + \underbrace{\tau_{IC}\frac{R}{\rho}\left(\mathbf{e}_\phi \times \nabla p\right)}_{=\frac{\boldsymbol{B}_\phi\times\nabla p}{|\boldsymbol{B}_\phi|^2}}$$

with u the electrical potential, ψ the magnetic poloidal flux, $v_{||}$ the parallel velocity.

- To avoid high order operators, we introduce the vorticity $w = \Delta_{pol} u$ and the toroidal current $j = \triangle^*\psi = R^2\nabla\cdot\left(\frac{1}{R^2}\nabla_{pol}\psi\right)$.
- Derivation: we plug $\boldsymbol{B}$ and $\boldsymbol{u}$ in the equations + some computations. For the equations on u and $v_{||}$ we use the following projections

$$\mathbf{e}_\phi \cdot \nabla \times R^2\left(\rho\partial_t\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{J}\times\boldsymbol{B} + \nu\Delta\boldsymbol{u}\right)$$

and

$$\boldsymbol{B}\cdot\left(\rho\partial_t\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{J}\times\boldsymbol{B} + \nu\Delta\boldsymbol{u}\right).$$

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EUROfusion

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