High-order Implicit relaxation schemes for hyperbolic models

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Outline

Physical and mathematical context

Relaxation methods

Discretization

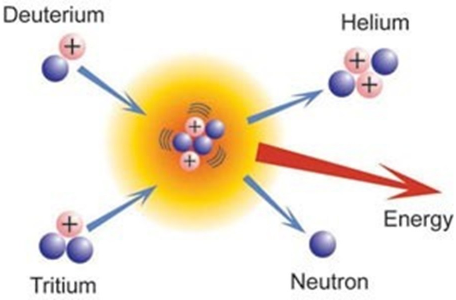
Numerical results

Conclusion and perspectives

Physical and mathematical context

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroidal room where the plasma is confined using powerful magnetic fields.
- The simulation of these instabilities is an important subject for ITER.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.



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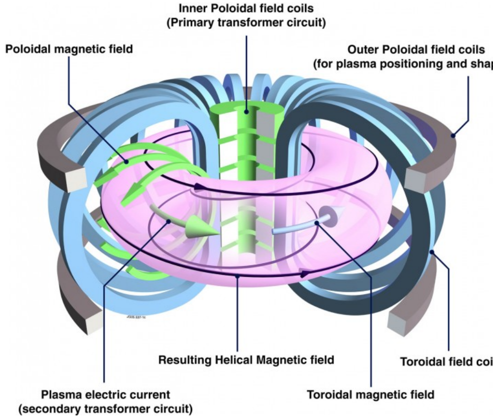

Figure: Tokamak

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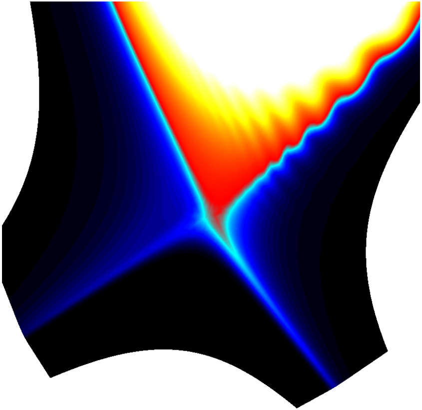

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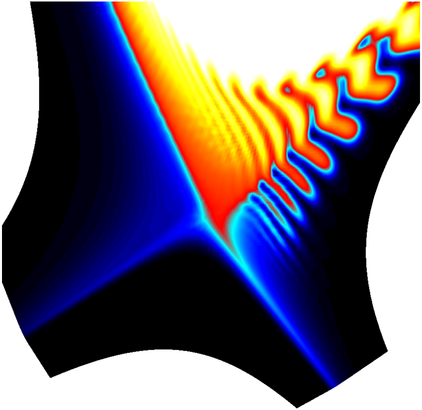

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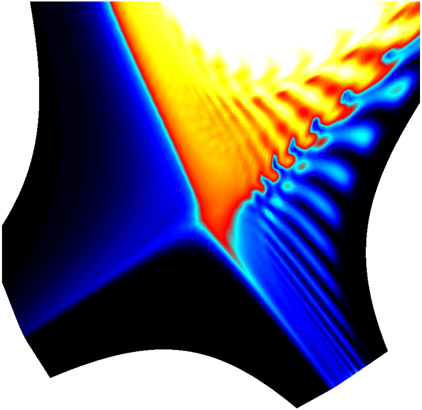

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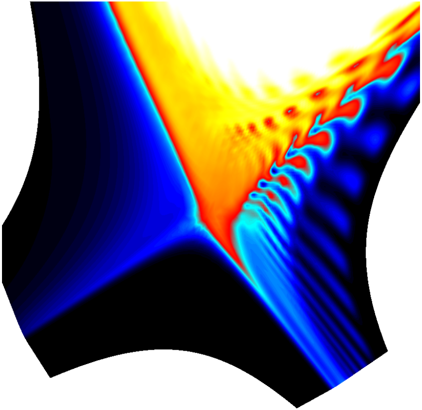

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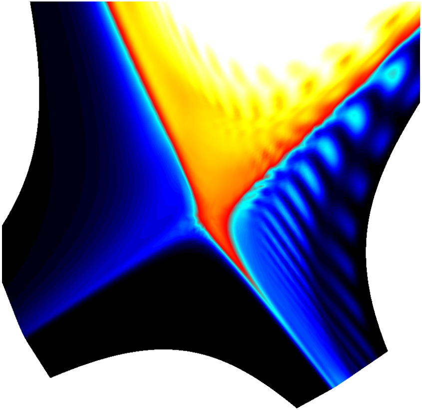


MHD in Tokamak I

Simplify Extended MHD

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p + \nabla \cdot \boldsymbol{\Pi} = \boldsymbol{J} \times \boldsymbol{B}, \\ \partial_t p + \boldsymbol{u} \cdot \nabla p + p \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q} = \eta \mid \boldsymbol{J} \mid^2 \\ \partial_t \boldsymbol{B} = -\nabla \times (-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J}), \\ \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, p the pressure, $\mathbf{u}$ the velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the current, $\boldsymbol{\Pi}$ stress tensor and $\mathbf{q}$ the heat flux.

MHD specificities in Tokamak

- **Strong anisotropic flows** (direction on the magnetic field) ===> complex geometries and aligned meshes (flux surface or magnetic field lines).
- **MHD scaling**:
  - $\boldsymbol{B}_{\parallel}$ **direction**: compressible flow and large diffusion.
  - $\boldsymbol{B}_{\perp}$ **direction**: quasi incompressible flow and small diffusion.
- **MHD Scaling** ===> compressible code with no discontinuities + fast waves.
- **Quasi stationary flows + fast waves** ===> implicit or semi implicit schemes.

Hyperbolic systems and implicit schemes

- We consider the general problem

$$\partial_t \boldsymbol{U} + \partial_x(\boldsymbol{F}(\boldsymbol{U})) = \nu \partial_x (D(\boldsymbol{U})\partial_x \boldsymbol{U})$$

- with $\boldsymbol{U} : R^n \longrightarrow R^n$ (idem for $\boldsymbol{F}(\boldsymbol{U})$) and D a matrix.
- In the following we consider the limit $\nu << 1$.

Implicit schemes

- **Implicit scheme**: allows to avoid the CFL condition filtering the fast phenomena.
- **Problem**: Direct solvers are not useful in 3D (too large matrices), we need iterative solvers.
- Conditioning of the implicit matrix: given by the ratio of the maximal and minimal eigenvalues.

- Implicit scheme :

$$\boldsymbol{U} + \Delta t \partial_x(\boldsymbol{F}(\boldsymbol{U})) - \Delta t \nu \partial_x(D(\boldsymbol{U})\partial_x \boldsymbol{U}) = \boldsymbol{U}^n$$

- At the limit $\nu << 1$ and $\Delta t >> 1$ (large time step) we solve $\partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$.

Issues of implicit schemes

- **Conclusion**: for $\nu << 1$ and $\Delta t >> 1$ the condition number of the full system closed to conditioning number of the steady hyperbolic model (the ratio of the speed waves).

Relaxation methods

General principle

- We consider the following nonlinear system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \nu \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + \boldsymbol{G}(\boldsymbol{U})$$

- with $\boldsymbol{U}$ a vector of N functions.
- **Aim**: Find a way to approximate this system with a sequence of simple systems.
- **Idea**: Xin-Jin relaxation method (very popular in the hyperbolic and finite volume community).

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = \boldsymbol{G}(\boldsymbol{U}) \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

Limit of the hyperbolic relaxation scheme

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \varepsilon \partial_x ((\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + \varepsilon \partial_x \boldsymbol{G}(\boldsymbol{U}) + o(\varepsilon^2)$$

- with $A(\boldsymbol{U})$ the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$.

- **Conclusion**: the relaxation system is an approximation of the hyperbolic original system (error in ε).
- **Stability**: the limit system is dissipative if $(\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) > 0$.

General principle II

Generalization

- The generalized relaxation is given by

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = \boldsymbol{G}(\boldsymbol{U}) \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = \dfrac{R(\boldsymbol{U})}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) + \boldsymbol{H}(\boldsymbol{U}) \end{cases}$$

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U})$$
$$+ \varepsilon \partial_x (R(\boldsymbol{U})^{-1}(\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + \varepsilon \partial_x (A(\boldsymbol{U}) \boldsymbol{G}(\boldsymbol{U}) - \boldsymbol{H}(\boldsymbol{U})) + o(\varepsilon^2)$$

Treatment of small diffusion

- Taking $R(\boldsymbol{U}) = (\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) D(\boldsymbol{U})^{-1}$, $\varepsilon = \nu$ and $\boldsymbol{H}(\boldsymbol{U}) = A(\boldsymbol{U})\boldsymbol{G}(\boldsymbol{U})$: we obtain the following limit system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \nu \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + o(\nu^2)$$

- **Limitation of the method**: the relaxation model cannot approach PDE with high diffusion.

Kinetic relaxation scheme

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \alpha^2 \partial_x u = \dfrac{1}{\varepsilon}(F(u) - v) \end{cases}$$

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}$ and note f_+ and f_- the new variables. We obtain

$$\begin{cases} \partial_t f_- - \alpha \partial_x f_- = \dfrac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \\ \partial_t f_+ + \alpha \partial_x f_+ = \dfrac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{cases}$$

- with $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\alpha}$.

First Generalization

- **Main property**: the transport is diagonal which can easily solved.

Remark

- in the Lattice Boltzmann community the discretization of this model is called D1Q2.

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0, \qquad \partial_t \eta(\boldsymbol{U}) + \partial_x \zeta(\boldsymbol{U}) \leq 0$$

- **Lattice**: $W = \{\lambda_1 ....\lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $U \in R^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- Equilibrium vector operator $\boldsymbol{f}^{eq} : R^{n_c} \rightarrow R^{n_v}$ such that $P\boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{U}$.

- Consistance with the initial PDE:

$$\mathcal{C} \left\{ \begin{aligned} P\boldsymbol{f}^{eq}(\boldsymbol{U}) &= \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) &= F(\boldsymbol{U}) \end{aligned} \right.$$

- For source terms and small diffusion terms, it is the same that the first relaxation method.
- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial \boldsymbol{f}_{eq} - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right)$$

Different kinetic models

- **1D models**:
  - The $[D1Q2]^n$ model. Two transport equations (velocities $\lambda_\pm$) by physical variable.
  - The $[D1Q3]^n$ model. Three transport equations (velocities $\lambda_\pm$ and λ_0) by physical variable. Based on flux-splitting method (Van-Leer etc).
- **Stability**:
  - Kinetic entropy

$$H(\boldsymbol{f}) = \sum_{i=1}^{q} h_i(f_i^1, ..., f_i^N)$$

    is dissipate if

$$\sum_{i=1}^{q} h_i^*(\boldsymbol{\phi}) = \eta^*(\boldsymbol{\phi}), \quad \sum_{i=1}^{q} v_i h_i^*(\boldsymbol{\phi}) = \boldsymbol{\zeta}^*(\boldsymbol{\phi})$$

    with η the macroscopic entropy, $\boldsymbol{\zeta}$ the entropy flux and $\boldsymbol{\phi} = \eta^{'}(\boldsymbol{U})$.
  - Entropy stability for the $[D1Q2]^n$ model.
  - Entropy stability for the $[D1Q3]^n$ model if the flux splitting is entropy). Dissipative stability if not.
- **Models in d-dimension**:
  - The $[DdQ(2d)]^n$ model. Generalization of $[D1Q2]^n$ model with the same properties.
  - The $[DdQ(2d+1)]^n$ model. Generalization of $[D1Q3]^n$. Same properties ?
  - The $[DdQ(d^2)]^n$ model. Additional direction to have better accuracy for isotropic problem.

Discretization methods

Time discretization

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Main idea**: splitting scheme between transport and the relaxation.
- **Key point**: the macroscopic variables are conserved during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ explicit.

First order scheme

- We define the two operators for each step :

$$T_{\Delta t} = I_d + \Delta t \Lambda \partial_x I_d$$

$$R_{\Delta t} = I_d - \Delta t \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - I_d)$$

- **Asymptotic limit**: Chapman-Enskog expansion.
- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{\Delta t}{2}\partial_x(P\Lambda^2 \partial_x \boldsymbol{f}) + \left(\frac{\Delta t}{2} + \varepsilon\right)\partial_x\left(\left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq} - A(\boldsymbol{U})^2\right)\partial_x \boldsymbol{U}\right)$$
$$+ O(\varepsilon \Delta t + \Delta t^2 + \varepsilon^2)$$

High-Order time schemes

Second-order scheme

- Scheme for transport step $T(\Delta t)$: Crank Nicolson or exact time scheme.
- Scheme for relaxation step $R(\Delta t)$: Crank Nicolson.
- Classical full second order scheme:

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t) \circ T\left(\frac{\Delta t}{2}\right).$$

- AP full second order scheme:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{2}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{4}\right).$$

- Ψ and Ψ_{ap} symmetric in time. $\Psi_{ap}(0) = I_d$.

High order scheme

- Using composition method

$$M_p(\Delta t) = \Psi_{ap}(\gamma_1 \Delta t) \circ \Psi_{ap}(\gamma_2 \Delta t) ..... \circ \Psi_{ap}(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

Space discretization - transport scheme

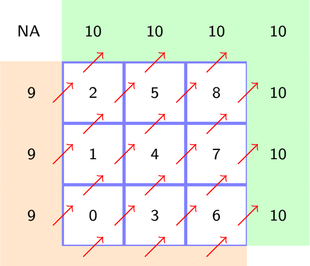
Whishlist

- Complex geometry, curved meshes (if possible),
- Flexibility, *hp* refinement,
- CFL-free,
- Matrix-free.

Candidates for transport discretization

- Implicit FV-DG schemes,
- Semi-Lagrangian schemes,
- Stochastic schemes (Glimm or particle methods).

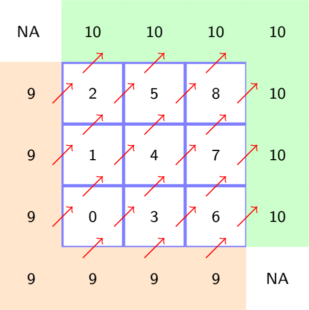
- Choice on Cartesian meshes: SL-scheme.
- Choice on Complex geometry: Implicit DG schemes.
  - Implicit Cranck-Nicholson scheme
  - Block -Triangular matrix (Upwind scheme) solved avoiding storage of the matrix.

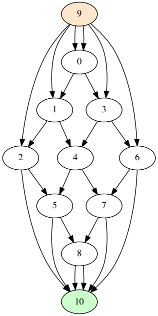


Parallel transport solver

Complex algorithm

- **Velocities**: the transport equations are independent. Possible parallelism.
- **Transport step**: partial parallelism given the implicit upwind scheme.
- **Two parallelism**: complex to manage.





Solution: StarPu

- Task-based scheduling library developed at INRIA Bordeaux.
- User submits tasks in a sequential order. StarPU schedules the tasks in parallel if possible.
- Possible MPI extension

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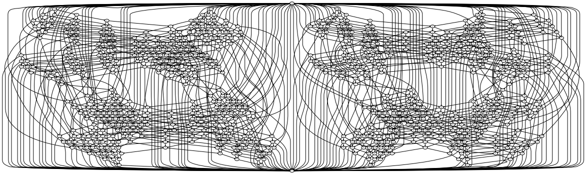

Figure: task graph for a 3D Torus.

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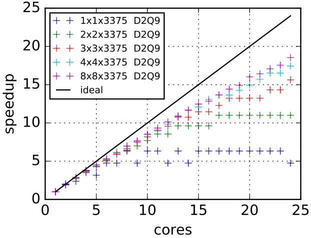
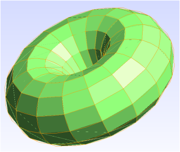
Parallelism results

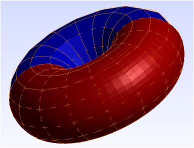
- Full D2Q9 scheme on square grids. Constant dof number per macrocell. Number N of macrocells N from 1 to $64 = 8x8$.
  - for 1 macrocell : saturation at $n_{core} = n_v$. This is expected.
  - efficiency grows with N due to topological parallelism.

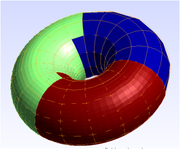


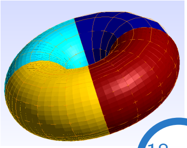
- MPI scaling
  - Toroidal mesh : 720 macroelements x 3335 dof

Nthreads/Nmpi	1	2	3	4
14	6862	2772	1491	1014



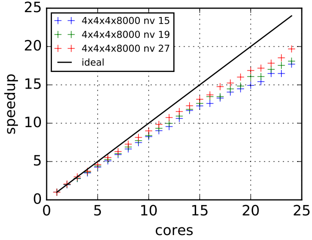






Parallelism results

- D3Q15, D3Q19, D3Q27 models on a cube with 4x4x4 elements and 8000 dof per elements with eager scheduler.
  - for 1 macrocell : saturation at $n_{core} = n_v$. This is expected.
  - efficiency grows with N due to topological parallelism.



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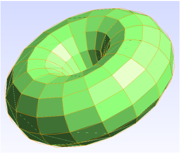

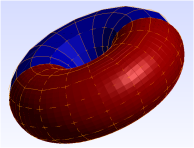

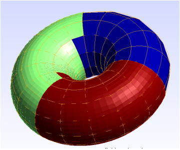

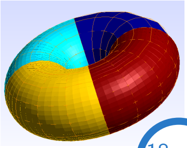

Numerical results

Burgers : quantitatives results

- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho}{2} \right) = 0$$

- **Kinetic model**: $(D1Q2)$ or $D1Q3$.
- Spatial discretization: SL-scheme, 5000 cells, order 7 space, order 2 time.
- **Test 1**: $\rho(t=0,x) = sin(2\pi x)$, viscosity$= 10^{-4}$.

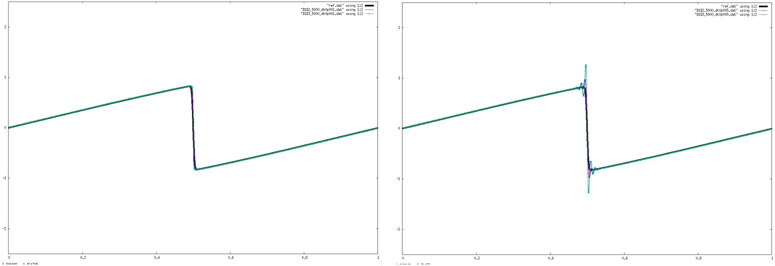

Figure: Left: comparison between different D1Q2 and D1Q3 for different time step (second order in time). Left up: $\Delta t = 0.001$ (CFL 5-10), Right up: $\Delta t = 0.005$ (CFL 20-50)

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- **Test 2**: rarefaction wave, no viscosity.

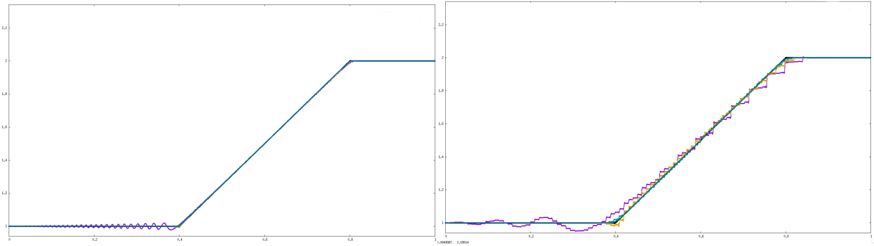

Figure: Left: comparison between different velocity set and time step . Left up: $\Delta t = 0.001$ (CFL 1-5). $V = \{-2.1, 2.1\}$ (violet) $V = \{0.9, 2.1\}$ (green) , $V = \{-2.1, 0, 2.1\}$ (yellow) and $V = \{0.9, 1.5, 2.1\}$ (blue), Right up: same for $\Delta t = 0.01$ (CFL 10-50).

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$$\partial_t \rho + \partial_x \left(\frac{\rho}{2}\right) = 0$$

- **Kinetic model**: $(D1Q2)$ or $D1Q3$.

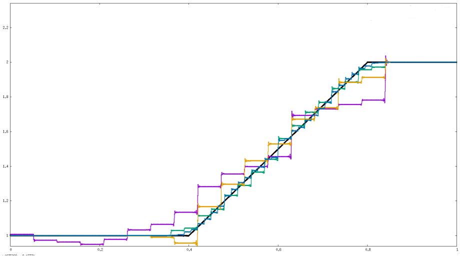

Figure: Left: comparison between different velocity set. $V = \{-2.1, 2.1\}$ (violet) $V = \{0.9, 2.1\}$ (green) , $V = \{-2.1, 0, 2.1\}$ (yellow) and $V = \{0.9, 1.5, 2.1\}$ (blue). $\Delta t = 0.05$ (CFL 50-200)

- **Remark**: Choice of kinetic model important to minimize time numerical dispersion.
- **Remark**: Time numerical dispersion comes from to second order relaxation scheme $\boldsymbol{f}^* = 2\boldsymbol{f}^{eq} - \boldsymbol{f}^n$. More the wave structure is close to the original one more $\mid \boldsymbol{f}^{eq} - \boldsymbol{f}^n \mid$ is small. Reduce the oscillations around f^{eq}.

1D isothermal Euler : Convergence

- **Model**: isothermal Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2 \rho) = 0 \end{cases}$$

- **Lattice**: $(D1 - Q2)^n$ Lattice scheme.
- For the transport (and relaxations step) we use 6-order DG scheme in space.
- **Time step**: $\Delta t = \beta \frac{\Delta x}{\lambda}$ with λ the lattice velocity. $\beta = 1$ explicit time step.
- **First test**: acoustic wave with $\beta = 50$ and $T_f = 0.4$, **Second test**: smooth contact wave with $\beta = 100$ and $T_f = 20$.

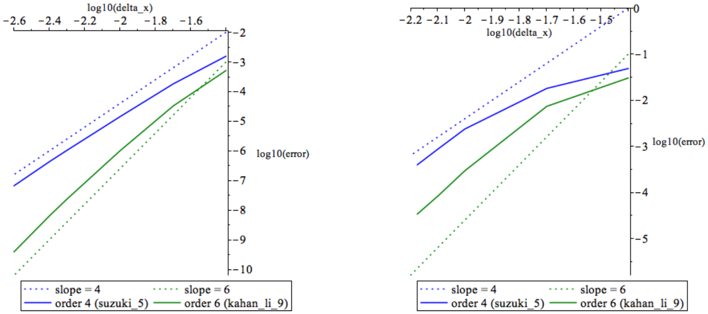

Figure: convergence rates for the first test (left) and for the second test (right).

1D Euler equations: quantitatives results

- **Model**: Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x(\rho E u + p u) = 0 \end{cases}$$

- **Kinetic model**: $(D1Q2)$ or $D1Q3$.
- For the transport (and relaxations step) we use 11-order SL scheme in space.

$$u(t=0,x) = -\sqrt{\gamma}\,\mathrm{sign}(x) M (1.0 - \cos(2\pi x/L))$$

$$p(t=0,x) = \frac{1}{M^2}(1.0 + M\gamma(1.0 - \cos(2\pi x/L))) \quad M = \frac{1}{11}$$

- **Discretization**: 4000 cells (for a domain $L = [-20, 20]$) and order 11.

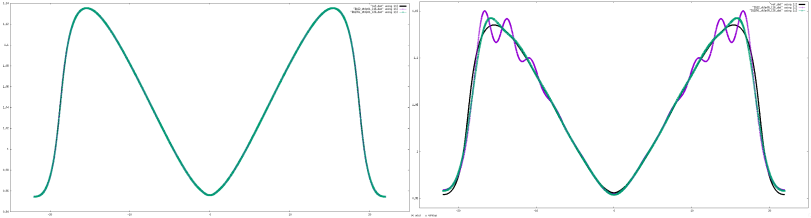

Figure: Density for the different scheme and order 2 time scheme: D1Q2 with $\lambda = 16$ (violet), D1Q3 with $\lambda = 26$ (green) and reference (black). Left : $\Delta t = 0.01$ (CFL 1-5). Right: $\Delta t = 0.05$ (CFL 5-20).

1D Euler equations: quantitatives results

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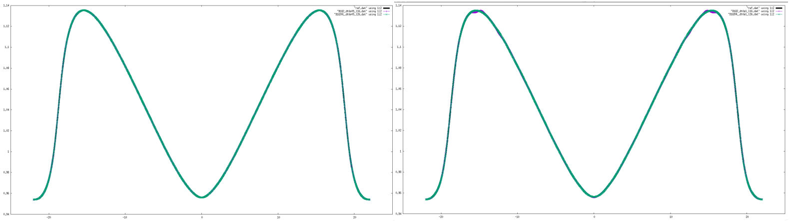
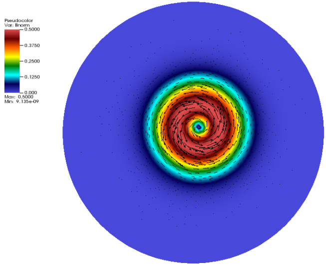


Figure: Density for the different scheme and order 4 time scheme: D1Q2 with $\lambda = 16$ (violet), D1Q3 with $\lambda = 26$ (green) and reference (black). Left : $\Delta t = 0.05$ (CFL 5-20). Right: $\Delta t = 0.1$ (CFL 10-50).

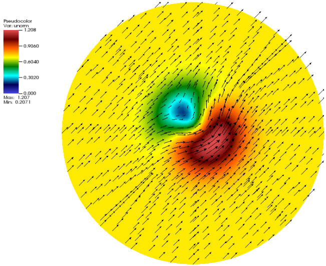
Numerical results: 2D MHD drifting vortex

- **Model** : compressible ideal MHD.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 4th order ins space. CFL around 20.
- **Test case** : advection of the vortex (steady state without drift).
- Parameters : $\rho = 1.0$, $p_0 = 1$, $u_0 = b_0 = 0.5$, $\mathbf{u}_{drift} = [1, 1]^t$, $h(r) = exp[(1 - r^2)/2]$

Magnetic field



Velocity



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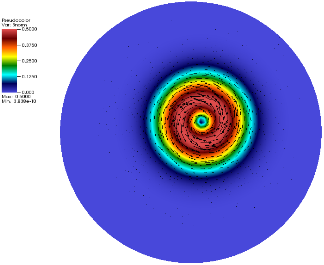
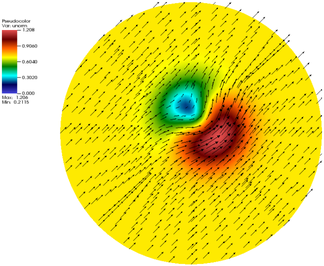


Velocity



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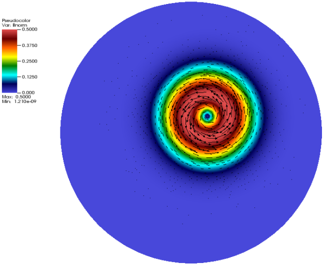
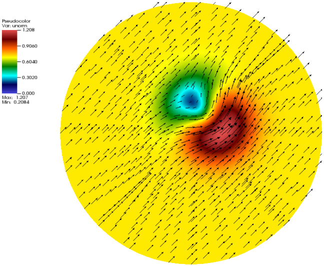


Velocity



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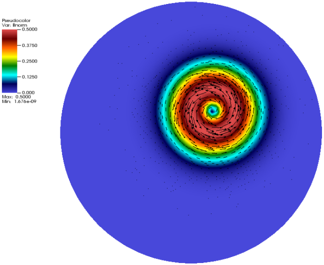
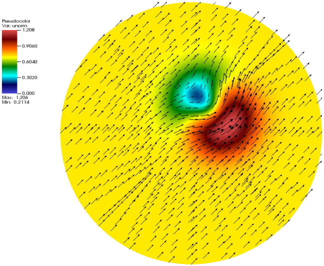


Velocity



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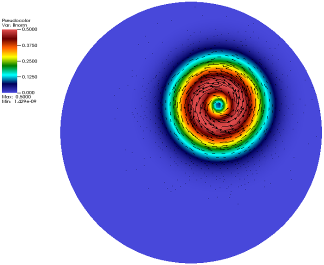
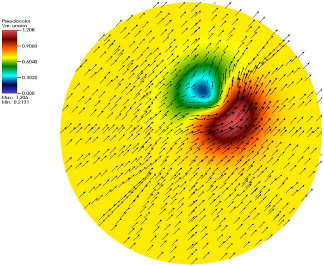


Velocity



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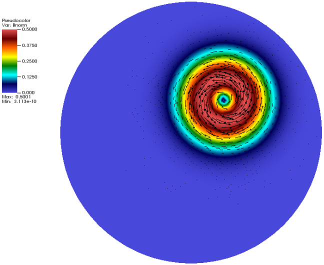
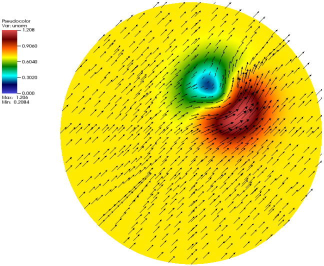


Velocity



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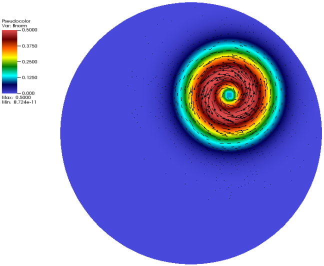
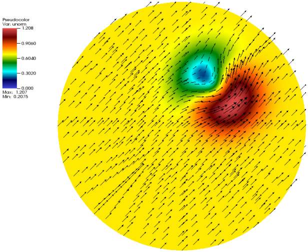


Velocity



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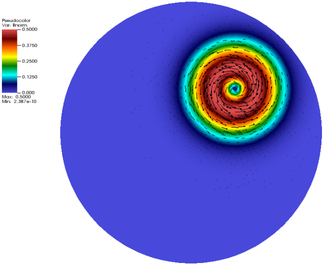

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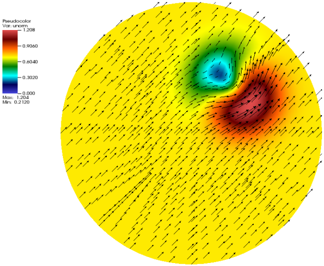

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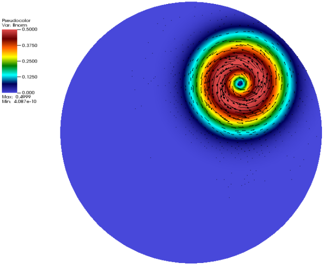
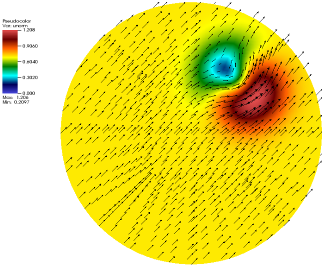


Velocity



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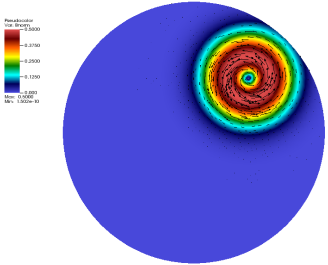

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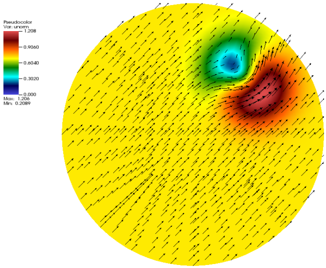

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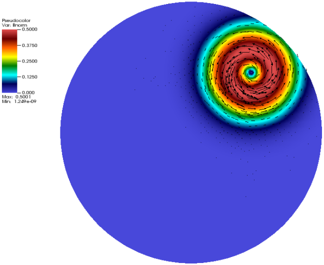

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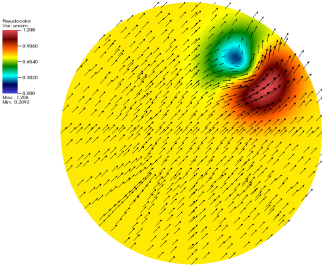

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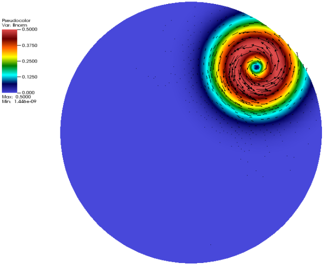

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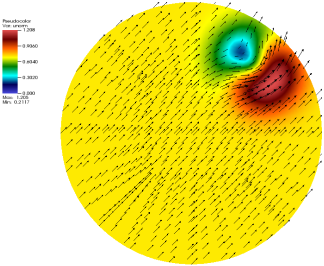

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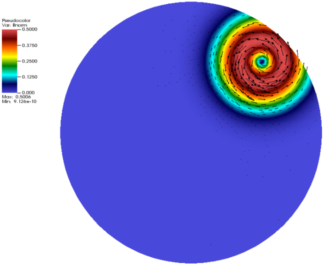

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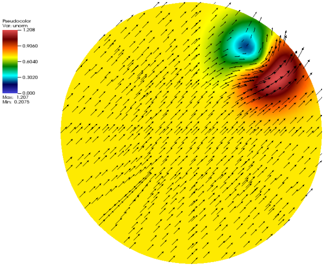

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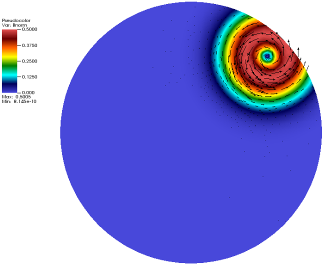

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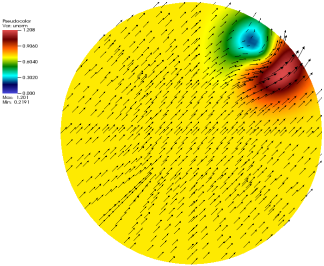

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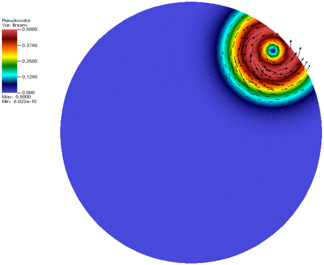

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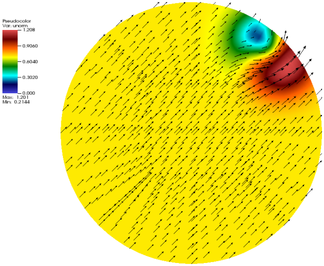

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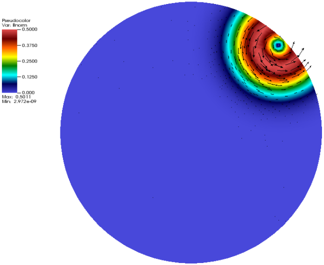

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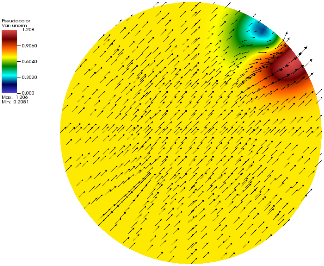

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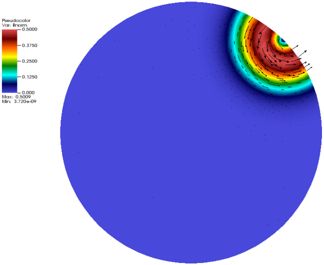

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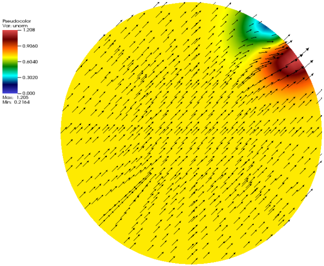

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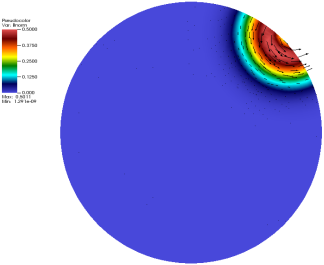
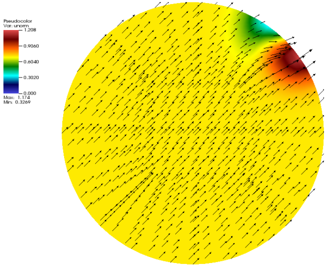
Magnetic field



Velocity



Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

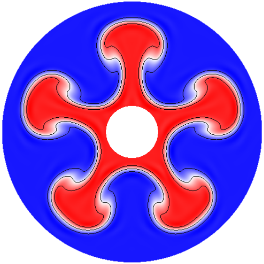


Figure: Plot of the mass fraction of gas

3D case in cylinder

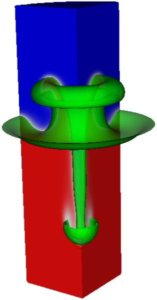


Figure: Plot of the mass fraction of gas

Numerical results: 2D-3D fluid models

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2D case in annulus

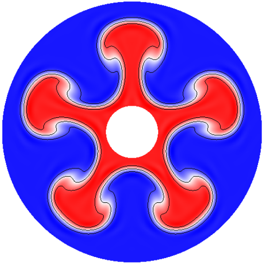

Figure: Plot of the mass fraction of gas

2D cut of the 3D case

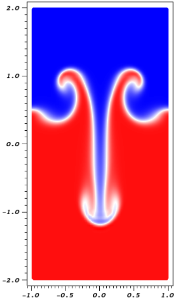

Figure: Plot of the mass fraction of gas

Conclusion and perspectives

Conclusion

Idea

- **Aim**: solve a strongly coupled nonlinear problem N.
- We construct linear problem with source term L_ε such that $\| N - L_\varepsilon \| = O(\nu^2)$.
- We solve L_ε with the discretization L_ε^h simple to solve such that $\| L_\varepsilon - L_\varepsilon^h \| = O(h^p)$.
- We obtain an error homogeneous with $O(h^p + \varepsilon)$.

Computing

- The linear problem L_ε^h can be split between simple problems.
- **Best implicit solver**: SL, Glimm scheme or DG for transport (IKR) ?
- **Parallelism**: additional parallelism with the uncoupled simple models.

Advantages/defaults

- **Advantages**: CFL- free method, High order method, Matrix - free method, complex geometry compatible method.
- **Defaults**: Large diffusion/dispersion at the order one/more. No limiting for discontinuities.

Perspectives

Numerical dispersion in time

- **Main problem**: for large time step (aim for implicit scheme) high dispersion for fast scales (acoustic for example).
- Solution:
  - Higher order scheme (4th and 6th).
  - Construct more efficient kinetic representation (additional zero velocity etc): current work.
  - Limiting or entropy dissipation methods for the relaxation step: future work.

Equilibrium

- This method not able to preserve steady states.
- **Future work**: modify the kinetic representation to preserve steady states.

Other future works

- Treatment of Complex boundary condition.
- AP method for low-mach regime.
- Convergence in the linear case.