High-order Implicit relaxation schemes for hyperbolic models

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Seminar LBM, IHP, Paris

Thanks to: M. Boileau[2], C. Courtes[4], M. Mehrenberger[2], L. Navoret[2], L. Mendoza[2] ,H. Guillard[5], L. Thanhuser[3], C. Klingenberg [3]

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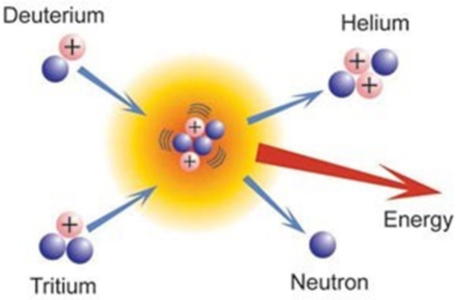
Outline

Physical and mathematical context

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



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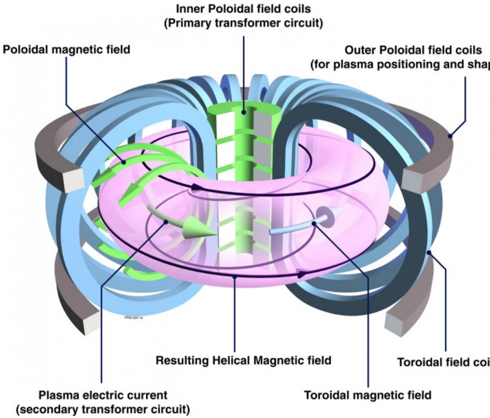

Figure: Tokamak

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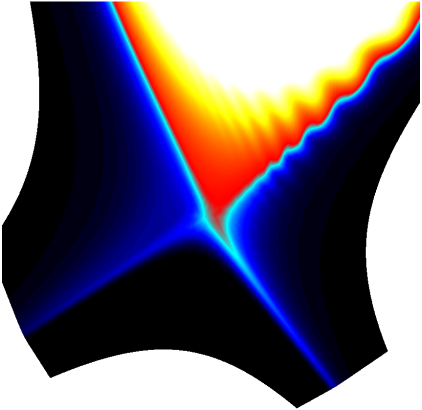

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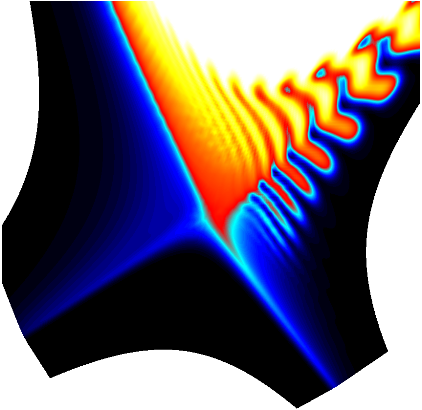

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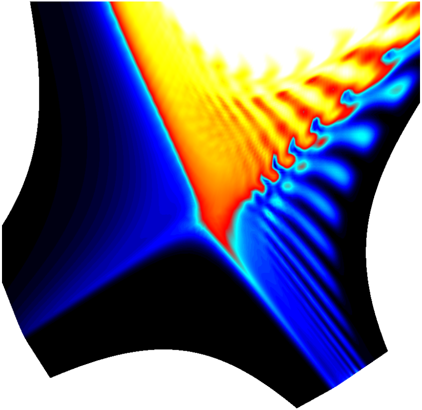

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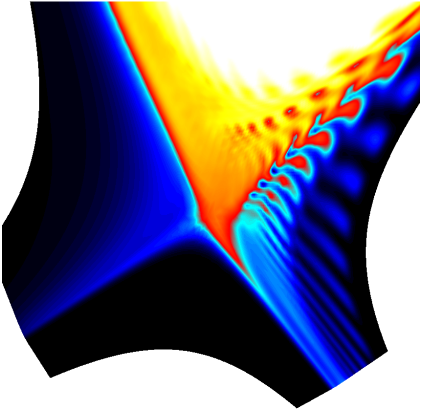

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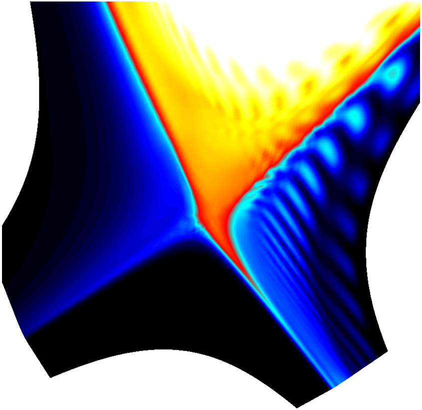


MHD in a Tokamak

Simplified Extended MHD

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla \cdot \boldsymbol{\Pi} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = \nabla \cdot \mathbf{q} + \eta \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \eta \nabla \times (\nabla \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, p the pressure, $\mathbf{u}$ the velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the current, $\boldsymbol{\Pi}$ stress tensor and $\mathbf{q}$ the heat flux.

MHD specificities in Tokamak

- **Strong anisotropic flows** (direction of the magnetic field) ===> complex geometries and aligned meshes (flux surface or magnetic field lines).
- **MHD scaling**:
  - **Diffusion**: Large Reynolds and magnetic Reynolds number.
  - $\boldsymbol{B}_{\parallel}$ **direction**: compressible flow and small Prandlt number.
  - $\boldsymbol{B}_{\perp}$ **direction**: quasi incompressible flow and large Prandlt number.
- **MHD Scaling** ===> compressible code with no discontinuities + fast waves.
- **Quasi stationary flows + fast waves** ===> implicit or semi implicit schemes.

Problem of implicit discretization

- Solution for implicit schemes:
  - Direct solver. CPU cost and consumption memory too large in 3D.
  - Iterative solver. Problem of conditioning.

Problem of conditioning

- Huge ratio between the physical wave speeds (low Mach regime) ==> huge ratio between discrete eigenvalues.
- Transport problem: anisotropic problem ==> huge ratio between discrete eigenvalues.
- High order scheme: small/high frequencies ==> huge ratio between discrete eigenvalues.
- Possible solution: preconditioning (often based on splitting and reformulation).

Storage problem

- Storage the matrix and perhaps the preconditioning: large memory consumption.
- Possibility: Jacobian free method (additional cost, but store only vectors).

Implicit Relaxation method and results

General principle

- We consider the following nonlinear system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \nu \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + \boldsymbol{G}(\boldsymbol{U})$$

- with $\boldsymbol{U}$ a vector of N functions.
- **Aim**: Find a way to approximate this system with a sequence of simple systems.
- **Idea**: Xin-Jin (95) relaxation method (very popular in the hyperbolic and finite volume community).

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = \boldsymbol{G}(\boldsymbol{U}) \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = \frac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

Limit of the hyperbolic relaxation scheme

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \varepsilon \partial_x ((\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + \varepsilon \partial_x \boldsymbol{G}(\boldsymbol{U}) + o(\varepsilon^2)$$

- with $A(\boldsymbol{U})$ the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$.

- **Conclusion**: the relaxation system is an approximation of the hyperbolic original system (error in ε).
- **Stability**: the limit system is dissipative if $(\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) > 0$.

General principle II

Generalization

- The generalized relaxation is given by

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = \boldsymbol{G}(\boldsymbol{U}) \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = \dfrac{R(\boldsymbol{U})}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) + \boldsymbol{H}(\boldsymbol{U}) \end{cases}$$

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U})$$
$$+ \varepsilon \partial_x (R(\boldsymbol{U})^{-1}(\alpha^2 - \mid A(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + \varepsilon \partial_x (A(\boldsymbol{U})\boldsymbol{G}(\boldsymbol{U}) - \boldsymbol{H}(\boldsymbol{U})) + o(\varepsilon^2)$$

Treatment of small diffusion

- Taking $R(\boldsymbol{U}) = (\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) D(\boldsymbol{U})^{-1}$, $\varepsilon = \nu$ and $\boldsymbol{H}(\boldsymbol{U}) = A(\boldsymbol{U})\boldsymbol{G}(\boldsymbol{U})$: we obtain the following limit system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \nu \partial_x (D(\boldsymbol{U})\partial_x \boldsymbol{U}) + o(\nu^2)$$

- **Limitation of the method**: the relaxation model cannot approach PDE with high diffusion.

Kinetic relaxation scheme

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \alpha^2 \partial_x u = \dfrac{1}{\varepsilon}(F(u) - v) \end{cases}$$

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}$ and note f_+ and f_- the new variables. We obtain

$$\begin{cases} \partial_t f_- - \alpha \partial_x f_- = \dfrac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \\ \partial_t f_+ + \alpha \partial_x f_+ = \dfrac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{cases}$$

- with $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\alpha}$.

First Generalization

- **Main property**: the transport is diagonal which can be easily solved.

Remark

- In the Lattice Boltzmann community the discretization of this model is called D1Q2.

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0, \qquad \partial_t \eta(\boldsymbol{U}) + \partial_x \zeta(\boldsymbol{U}) \leq 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $U \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- Equilibrium vector operator $\boldsymbol{f}^{eq} : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_v}$ such that $P\boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{U}$.

- Consistence with the initial PDE (R. Natalini 00, F. Bouchut 99-03 ...) :

$$\mathcal{C} \begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{U} \\ P\Lambda\boldsymbol{f}^{eq}(\boldsymbol{U}) = F(\boldsymbol{U}) \end{cases}$$

- For source terms and small diffusion terms, it is the same as the first relaxation method.
- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial \boldsymbol{f}_{eq} - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right)$$

Time discretization

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Main idea**: splitting scheme between transport and the relaxation (P. J. Dellar, 13).
- **Key point**: the macroscopic variables are conserved during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ explicit.

First order scheme

- We define the two operators for each step :

$$T_{\Delta t} = I_d + \Delta t \Lambda \partial_x I_d$$

$$R_{\Delta t} = I_d - \Delta t \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - I_d)$$

- **Asymptotic limit**: Chapman-Enskog expansion.
- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{\Delta t}{2}\partial_x(P\Lambda^2\partial_x \boldsymbol{f}) + \left(\frac{\Delta t}{2} + \varepsilon\right)\partial_x\left(\left(P\Lambda^2\partial_{\boldsymbol{U}}\boldsymbol{f}^{eq} - A(\boldsymbol{U})^2\right)\partial_x \boldsymbol{U}\right)$$

$$+ O(\varepsilon\Delta t + \Delta t^2 + \varepsilon^2)$$

High-Order time schemes

Second-order scheme

- Scheme for transport step $T(\Delta t)$: Crank Nicolson or exact time scheme.
- Scheme for relaxation step $R(\Delta t)$: Crank Nicolson.
- Classical full second order scheme:

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t) \circ T\left(\frac{\Delta t}{2}\right).$$

- **Numerical test**: second order but probably only for the macroscopic variables.
- AP full second order scheme:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{2}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{4}\right).$$

- Ψ and Ψ_{ap} symmetric in time. $\Psi_{ap}(0) = I_d$.

High order scheme

- Using composition method

$$M_p(\Delta t) = \Psi_{ap}(\gamma_1 \Delta t) \circ \Psi_{ap}(\gamma_2 \Delta t) ..... \circ \Psi_{ap}(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

Space discretization - transport scheme

Whishlist

- Complex geometry, curved meshes or unstructured meshes,
- CFL-free,
- Matrix-free.

Candidates for transport discretization

- **LBM-like**: exact transport solver,
- Implicit FV-DG schemes,
- Semi-Lagrangian schemes,
- Stochastic schemes (Glimm or particle methods).

LBM-like method: exact transport

- **Advantages**:
  - Exact transport at the velocity $\lambda = \frac{v\Delta t}{\Delta x}$. Very very cheap cost.
- **Drawbacks**:
  - Link time step and mesh: complex to manage large time step, unstructured grids and multiply kinetic velocities.

Space discretization

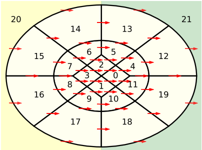
Semi Lagrangian methods

- Forward or Backward methods. Mass or nodes interpolation/projection.
- **Advantages**:
  - Possible on unstructured meshes. High order in space.
  - **Exact in time** and Matrix-free.
- **Drawbacks**:
  - No dissipation and difficult on very unstructured grids.

Implicit FV- DG methods

- Implicit Crank Nicolson scheme + FV DG scheme
- **Advantages**:
  - Very general meshes. High order in space. Dissipation to stabilize.
  - Upwind fluxes ==> triangular block matrices.
- **Drawbacks**:
  - Second order in time: numerical time dispersion.

- Current choice 1D: SL-scheme.
- Current choice in 2D-3D: DG schemes.
  - Block - triangular matrix solved avoiding storage.
  - Solve the problem in the topological order given by connectivity graph.



Burgers : quantitative results

- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 5000 cells, degree 7 in space, order 2 time.
- **Test 1**: $\rho(t=0,x) = sin(2\pi x)$, viscosity= 10^{-4}.

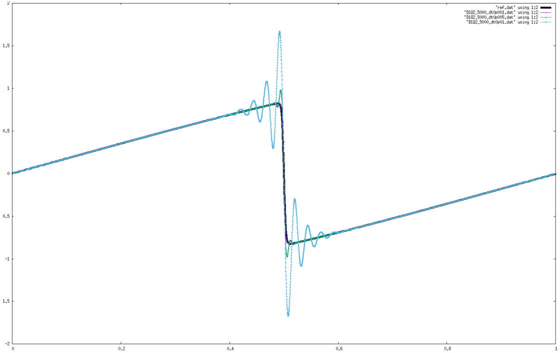


Figure: Comparison for different time step. Violet: $\Delta t = 0.001$ (CFL 5-30), Green: $\Delta t = 0.005$ (CFL 20-120), Blue $\Delta t = 0.01$ (CFL 50-300), Black : reference

1D isothermal Euler : Convergence

- **Model**: isothermal Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2 \rho) = 0 \end{cases}$$

- **Lattice**: $(D1 - Q2)^n$ Lattice scheme.
- For the transport (and relaxations step) we use 6-order DG scheme in space.
- **Time step**: $\Delta t = \beta \frac{\Delta x}{\lambda}$ with λ the lattice velocity. $\beta = 1$ explicit time step.
- **First test**: acoustic wave with $\beta = 50$ and $T_f = 0.4$, **Second test**: smooth contact wave with $\beta = 100$ and $T_f = 20$.

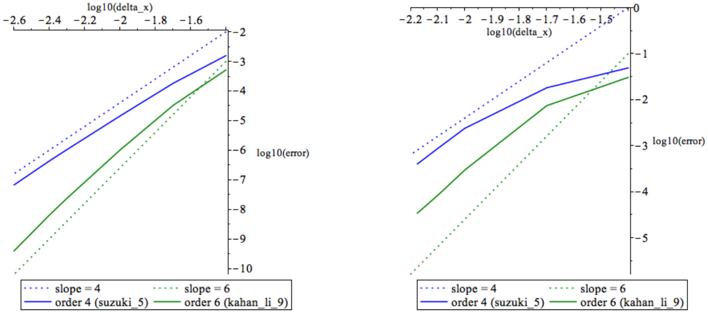


Figure: convergence rates for the first test (left) and for the second test (right).

- **Test case**: discontinuous initial data (Sod problem). No viscosity, $\beta = 3$. 6 order space-time scheme.

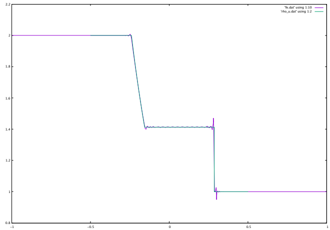
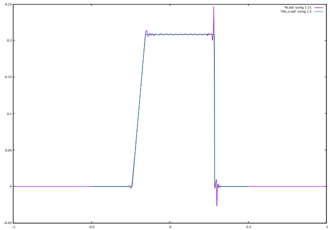
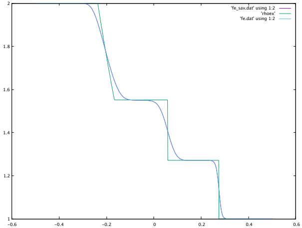



Figure: density (left) and velocity (right).

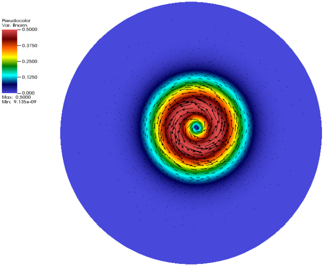
- With refinement in space we can reduce the oscillations.
- **Test case**: Sod problem. $\nu = 5.10^{-4}$, $\beta = 5$. 6 order space-time scheme.



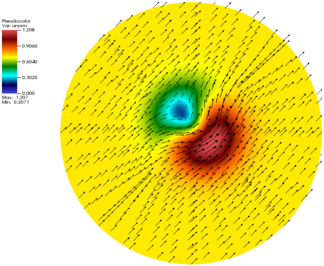
Numerical results: 2D MHD drifting vortex

- **Model** : compressible ideal MHD.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2^{nd} order Implicit DG scheme. 4th order ins space. CFL around 20.
- **Test case** : advection of the vortex (steady state without drift).
- Parameters : $\rho = 1.0$, $p_0 = 1$, $u_0 = b_0 = 0.5$, $\mathbf{u}_{drift} = [1, 1]^t$, $h(r) = exp[(1 - r^2)/2]$

Magnetic field



Velocity



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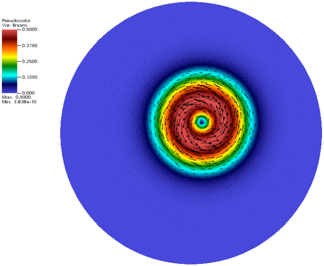

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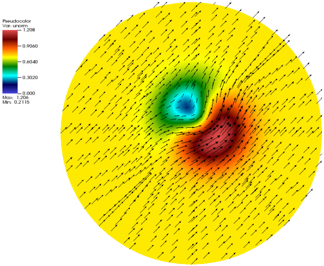

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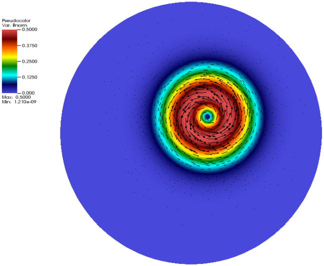

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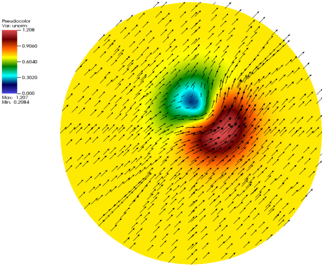

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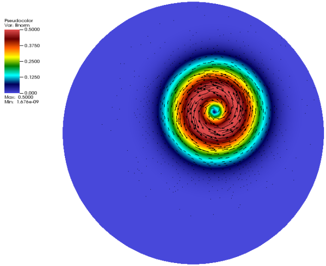

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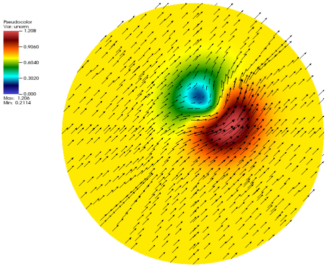

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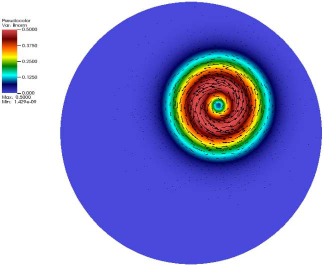
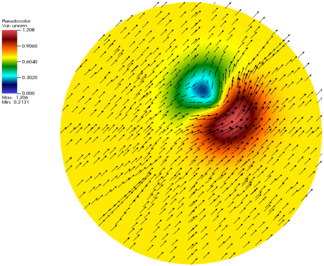


Velocity



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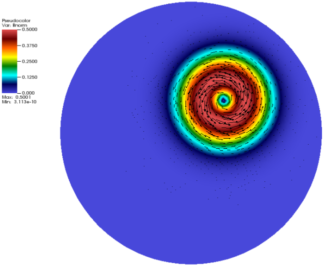

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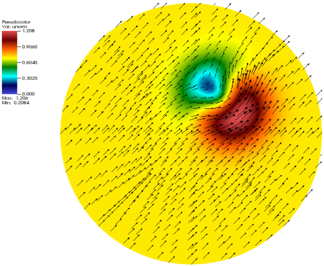

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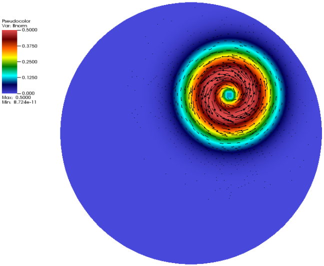

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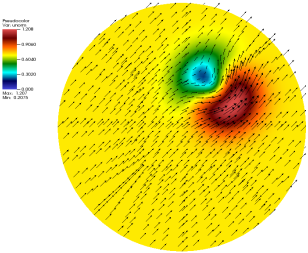

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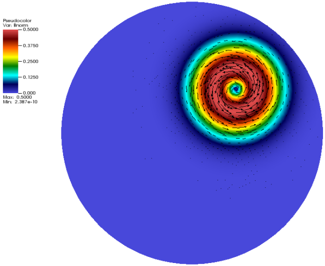
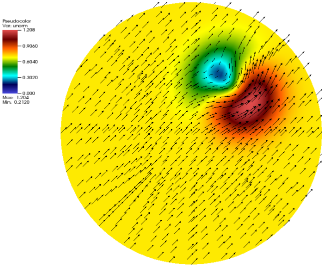


Velocity



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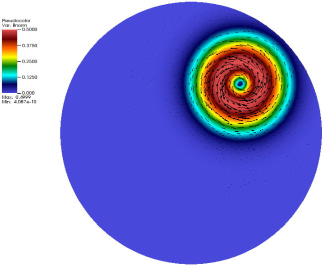

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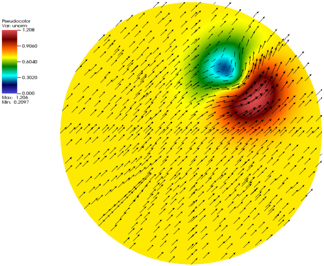

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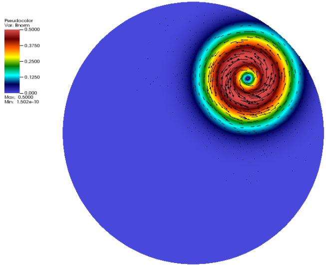

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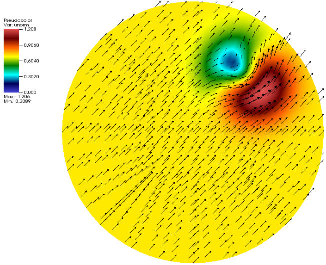

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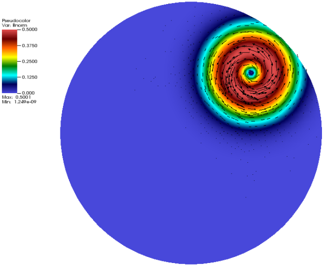

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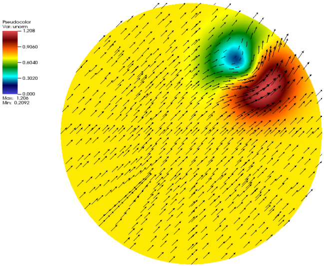

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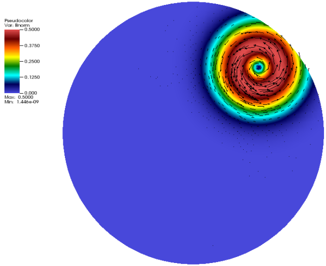

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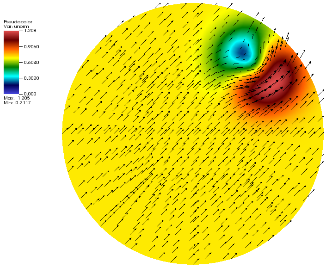

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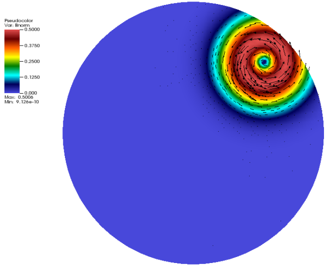

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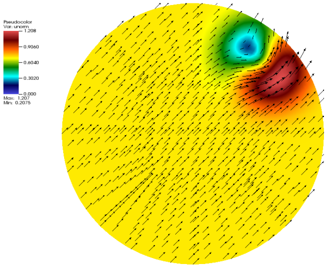

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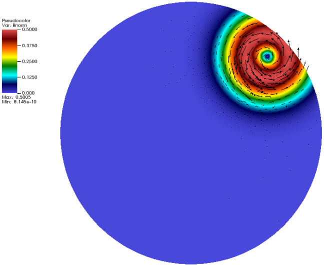

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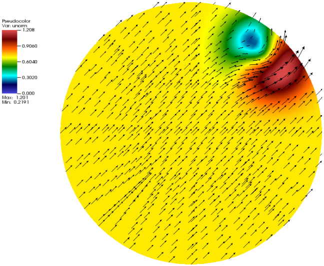

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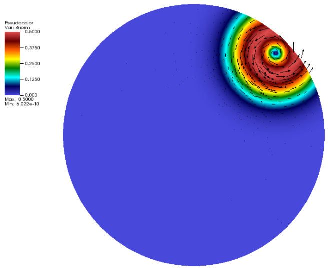

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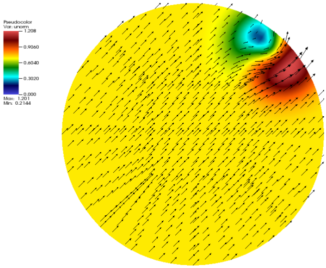

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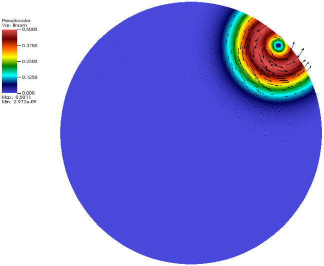

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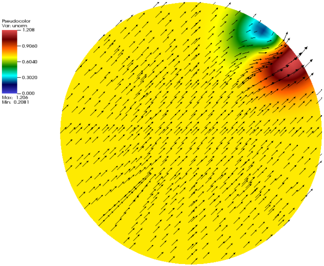

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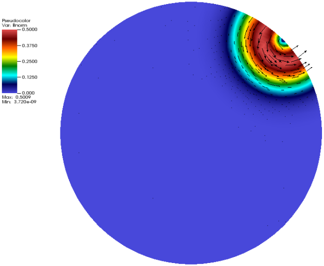

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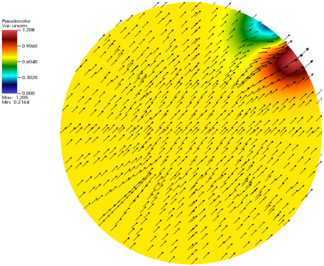

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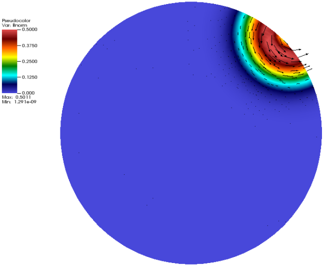

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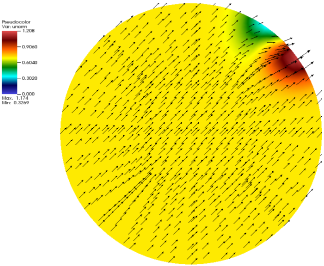

Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

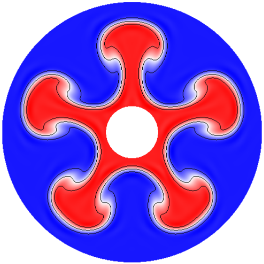

Figure: Plot of the mass fraction of gas

3D case in cylinder

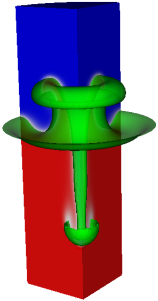

Figure: Plot of the mass fraction of gas

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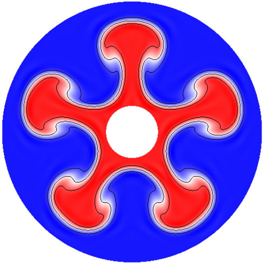


Figure: Plot of the mass fraction of gas

2D cut of the 3D case

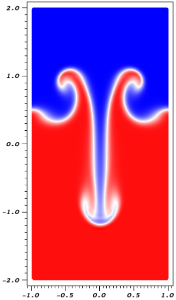


Figure: Plot of the mass fraction of gas

Kinetic representation of hyperbolic systems

Key point: design of the kinetic representation

Main idea

- Target: Nonlinear problem N.
- **First**: we construct the kinetic problem K_ε such that $\| K_\varepsilon - N \| \leq C_\varepsilon \varepsilon$
- **Second**: we discretize K_ε such that $\| K_\varepsilon - K_\varepsilon^{h,\Delta t} \| \leq C_{\Delta t} \Delta t^p + C_h h^q$
- We obtain a consistent method by triangular inequality.

First point: Analysis of the error

- **Assuming**: large time step and high order in space. **Main problem**: time error.
- The error in time comes from the transport step and relaxation step.
- If we use SL-scheme no time error in the transport step.
- **Main problem**: time error relaxation/splitting (order 1/2: diffusion/dispersion).
- This error homogeneous to $\left(P\Lambda^2 \partial \boldsymbol{f}_{eq} - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2\right)$. The closer the wave structure of K_{eps} is to the one of N, the smaller this error.

Second point: stability

- The kinetic model must be stable with the minimal sub-characteristic stability condition.

Classical kinetic representation

"Physic" kinetic representations

- Kinetic representation mimics the moment model construction of Boltzmann equation.
- Example: Euler isothermal

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t\rho u+\partial_x(\rho u^2+c^2\rho)=0\end{array}\right.$$

- D1Q3 model: three velocities $\{-\lambda,0,\lambda\}$. Equilibrium: quadrature of Maxwellian.

$$\rho=f_-+f_0+f_+,\quad q=\rho u=-\lambda*f_-+0*f_0+\lambda*f_+,\quad \boldsymbol{f}_{eq}=\begin{pmatrix}\frac{1}{2}(\rho u(u-\lambda)+c^2\rho)\\ \rho(\lambda^2-u^2-c^2)\\ \frac{1}{2}(\rho u(u+\lambda)+c^2\rho)\end{pmatrix}$$

- **Limit model** :

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t\rho u+\partial_x(\rho u^2+c^2\rho)=\varepsilon\left(\partial_{xx}u+u^3\partial_{xx}\rho\right)\end{array}\right.$$

- **Good point**: no diffusion on ρ equation. **Bad point**: stable only for low mach.

Vectorial kinetic representations

- Vectorial kinetic model (B. Graille 14): $[D1Q2]^2$ one relaxation model $\{-\lambda,\lambda\}$ (previous slide) by equation.
- **Good point**: stable on sub-characteristic condition $\lambda>\lambda_{max}$.
- **Bad point**: large error. Wave propagation approximated by transport at maximal velocity in the two directions.

New kinetic models. Scalar case I

Idea

- Design vectorial kinetic model with un-symmetric velocities and additional central velocity (typically zero).
- **Problem**: Stability not trivial. **Idea**: use entropy construction (F. Dubois 13).

- We consider $\partial_t \rho + \partial_x F(\rho)$ with the entropy equation $\partial_t \eta(\rho) + \partial_x \zeta(\rho) \leq 0$.
- We consider a model D1Q3 with $V = \{\lambda_-, \lambda_0, \lambda_+\}$. We take

$$\rho = f_- + f_0 + f_+, \quad F(\rho) = \lambda_- f_- + \lambda_0 f_0 + \lambda_+ f_+$$

- We define an entropy $H = h_-(f_-) + h_0(f_0) + h_+(f_+)$ with h_0, $h_\pm$ convex functions.
- We define $\phi = \partial_\rho \eta(\rho)$ and $\eta^*(\phi)$ the dual entropy (by the Legendre transform).

Lemma

- If the following condition are satisfied

$$\eta^*(\phi) = h_- + h_0 + h_+, \quad \zeta^*(\phi) = \lambda_- h_- + \lambda_0 h_0 + \lambda_+ h_+$$

- We have $\partial_t H(\boldsymbol{f}) \leq 0$ and this entropy admits a minimum defined by

$$(f^{eq})_i = \frac{\partial h_i^*}{\partial \phi}$$

Scalar case II

Design kinetic model

- **Method**: choose a physical entropy. Compute the atomic dual entropies and the equilibrium.
- **Stability condition**: convex condition of the atomic entropy.

- We fix arbitrary $h_0^\star(\phi)$ consequently we obtain the following solution

$$\begin{cases} h_-^\star(\phi) = -\dfrac{[\zeta^\star(\phi) - \lambda_+ \eta^\star(\phi)] + (\lambda_+ - \lambda_0) h_0^\star(\phi)}{(\lambda_+ - \lambda_-)} \\ h_+^\star(\phi) = \dfrac{[\zeta^\star(\phi) - \lambda_- \eta^\star(\phi)] + (\lambda_- - \lambda_0) h_0^\star(\phi)}{(\lambda_+ - \lambda_-)} \end{cases}$$

- The function $h_0^\star(\phi)$ which "saturate" the convex conditions on the three equations.
- Using final atomic entropies we derivate to obtain the equilibrium.

$$\begin{cases} f_-^{eq} = \frac{\lambda_0}{\lambda_+ - \lambda_-}\rho - \frac{F^-(\rho)}{\lambda_0 - \lambda_-} \\ f_0^{eq} = \left(\rho - \left(\frac{F^+(\rho)}{(\lambda_+ - \lambda_0)} - \frac{F^-(\rho)}{(\lambda_0 - \lambda_-)}\right)\right) \\ f_+^{eq} = -\frac{\lambda_0}{\lambda_+ - \lambda_-}\rho + \frac{F^+(\rho)}{\lambda_+ - \lambda_0} \end{cases}$$

with

$$F^\pm = \int [(\partial F(\rho) - \lambda_0)]^\pm + C_\pm$$

- This model D1Q3 upwind is stable on the condition $\lambda_- \leq F^{'}(\rho) \leq \lambda_+$.
- **Advantage**: adaptation of the model depending on the flow direction.

Vectorial case

- We consider the equation

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0, \quad \partial_t \eta(\boldsymbol{U}) + \partial_x \boldsymbol{\zeta}(\boldsymbol{U}) \leq 0$$

- Vectorial $[D1Q3]^N$ model (to simplify $\lambda_0 = 0$). One D1Q3 model by equation.
- Same theory with

$$H = h_-(f_-^1, .., f_-^N) + h_0(f_0^1, ..., f_0^N) + h_+(f_+^1, ..., f_+^N)$$

- **Problem**: At the end, we must integrate the positive/ negative part of the Jacobian to compute f_0^{eq}. Not possible in general (idem in the flux-splitting theory).

D1Q3 flux-splitting model

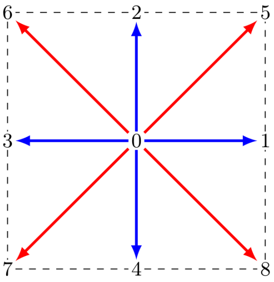
- **Idea**: we choose an entropic flux-splitting $\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{F}^-(\boldsymbol{U}) + \boldsymbol{F}^+(\boldsymbol{U})$ such as $\partial_t \eta + \partial_x \boldsymbol{\zeta}^-(\boldsymbol{U}) + \partial_x \boldsymbol{\zeta}^+(\boldsymbol{U}) \leq 0$.
- We obtain:

$$\begin{cases} f_-^{eq} = -\frac{1}{\lambda_-} \boldsymbol{F}^-(\boldsymbol{U}) \\ f_0^{eq} = \left(\boldsymbol{U} - \left(\frac{\boldsymbol{F}^+(\boldsymbol{U})}{\lambda_+} + \frac{\boldsymbol{F}^-(\boldsymbol{U})}{\lambda_-} \right) \right) \\ f_+^{eq} = \frac{1}{\lambda_+} \boldsymbol{F}^+(\boldsymbol{U}) \end{cases}$$

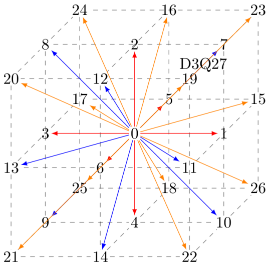
- **Stability**: $\lambda_- I_d < D < \lambda_+ I_d$ with D the eigenvalues matrix of $\partial \boldsymbol{F}_0^{\pm}(\boldsymbol{U})$.

Multi-D extension and relative velocity

- Extension of the vectorial scheme in 2D and 3D
- **2D extension**: $D2q(4 * k)$ or $D2Qq(4 * k + 1)$ with $k = 1$ or $k = 2$.
- **3D extension**: $D3q(6 * k)$, $D2Qq(6 * k + 1)$ with $k = 1$, $k = 2$ ore more.



D2Q9



- Increase k ==> increase the isotropic property of the kinetic model.
- The vectorial models with 0 velocity are not currently extended to 2D.
- **Related future work**: Extension to the relative velocity idea (T. Fevrier 15) at the vectorial models.
- **Relative velocity**: Relax the moment of the kinetic model in a repair moving at a given velocity (analogy with ALE).

Advection equation

- Equation
$$\partial_t \rho + \partial_x (a(x)\rho) = 0$$
- with $a(x) > 0$ and $\partial_x a(x) > 0$. Dissipative equation.
- **Test 1**: Velocity is given by $a(x) = 1.0 + 0.05x^2$ with the domain $[0, 5]$ and $T_f = 1$.
- We compare the numerical dispersion in time due to the models:
  - $D1Q2$ model: M_a^0 $(\lambda_\pm = \pm 1.5)$, M_b^0 $(\lambda_\pm = \{0, 1.5\})$, M_c^0 $(\lambda_\pm = \{0.75, 1.5\})$.
  - $D1Q3$ model: M_a^1 $(\lambda_{-,0,+} = \{-1.5, 0, 1.5\})$, M_b^1 $(\lambda_{-,0,+} = \{0, 0.75, 1.5\})$, M_c^1 $\{0.75, 1.1, 1.5\})$

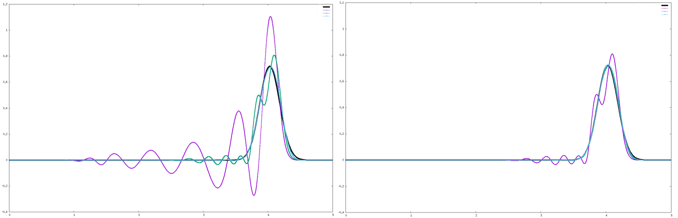


Figure: Left: comparison between different D1Q2 (violet M_a^0, green M_b^0, blue M_c^0, dark ref solution). Right: comparison between different D1Q3 (violet M_a^1, green M_b^1, blue M_c^1, dark ref solution) $\Delta t = 0.1$ (CFL $\approx 100 - 300$).

Advection equation

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$$\partial_t \rho + \partial_x (a(x)\rho) = 0$$
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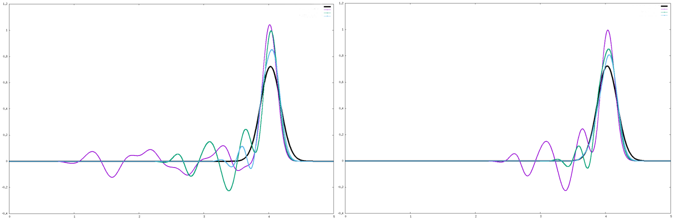

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Burgers

- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- **Kinetic model**: $(D1Q2)$ or $D1Q3$.
- Spatial discretization: SL-scheme, 1000 cells, order 7 space, order 2 time.
- **Test 2**: rarefaction wave, no viscosity.

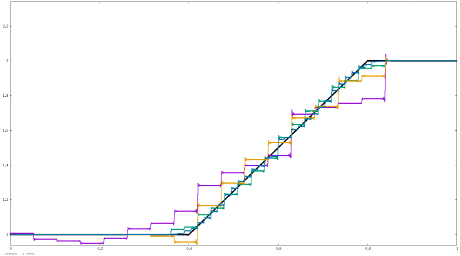

Figure: Left: comparison between different velocity set. $V = \{-2.1, 2.1\}$ (violet) $V = \{0.9, 2.1\}$ (green) , $V = \{-2.1, 0, 2.1\}$ (yellow) and $V = \{0.9, 1.5, 2.1\}$ (blue). $\Delta t = 0.05$ (CFL 50-200)

- **Remark**: Choice of kinetic model important to minimize time numerical dispersion.

1D Euler equations: quantitatives results

- **Model**: Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t \rho E + \partial_x(\rho E u + p u) = 0 \end{cases}$$

- **Kinetic model**: $(D1Q2)$ or $D1Q3$.
- For the transport (and relaxations step) we use 11-order SL scheme in space.

$$u(t=0,x) = -\sqrt{\gamma}\,\text{sign}(x) M(1.0 - \cos(2\pi x/L))$$

$$p(t=0,x) = \frac{1}{M^2}(1.0 + M\gamma(1.0 - \cos(2\pi x/L))) \quad M = \frac{1}{11}$$

- **Discretization**: 4000 cells (for a domain $L = [-20, 20]$) and order 11.

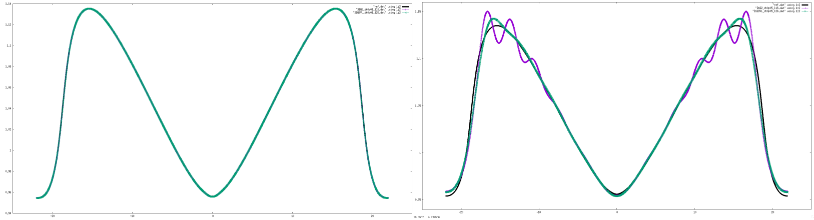

Figure: Density. Second time scheme: D1Q2 with $\lambda = 16$ (violet), D1Q3 with $\lambda = 26$ (green) and reference (black). Left : $\Delta t = 0.01$ (CFL 1-5). Right: $\Delta t = 0.05$ (CFL 5-20).

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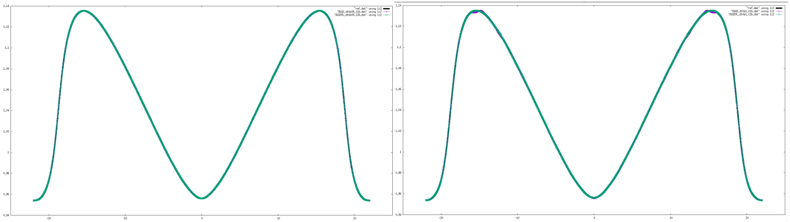


Figure: Density. Second time scheme: D1Q2 with $\lambda = 16$ (violet), D1Q3 with $\lambda = 26$ (green) and reference (black). Left : $\Delta t = 0.05$ (CFL 5-20). Right: $\Delta t = 0.1$ (CFL 10-50).

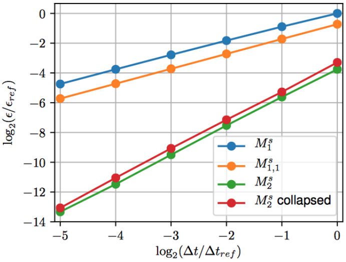
Other works

Current Work I: equilibrium

Equilibrium

- Classical problem: $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = S(\mathbf{U})$. Steady-state important to preserve: $\partial_x \mathbf{F}(U) = S(\mathbf{U})$
- **Problem**: kinetic relaxation scheme not appropriate for that.
  - **First problem**: construct kinetic source to have equilibrium in relaxation step.
  - **Main problem**: time and spatial error in the transport step.

- **Example**: Euler with gravity. Equilibrium between gradient pressure and gravity.



- **Result**: convergence with second order in time but no preservation of the steady state.

Current Work II: diffusion

- We want solve the equation: $\partial_t \rho + \partial_x (u\rho) = D\partial_{xx}\rho$
- Kinetic system proposed (S. Jin, F. Bouchut):

$$\begin{cases} \partial_t f_- - \dfrac{\lambda}{\varepsilon}\partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^- - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon}\partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+ - f_+) \end{cases}$$

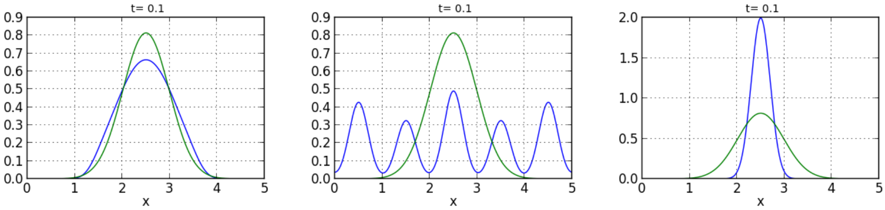
- with $f_{eq}^{\pm} = \frac{\rho}{2} \pm \frac{\varepsilon(u\rho))}{2\lambda}$. The limit is given by:

$$\partial_t \rho + \partial_x (u\rho) = \partial_x((\lambda^2 - \varepsilon^2 \mid \partial F(\rho) \mid^2)\partial_x \rho) + \lambda^2\varepsilon^2\partial_x(\partial_{xx}F(\rho) + \partial F(\rho)_{xx}\rho) - \lambda^2\varepsilon^2\partial_{xxxx}\rho$$

- We introduce $\alpha >\mid \partial F(\rho) \mid$. Choosing $D = \lambda^2 - \varepsilon^2\alpha^2$ we obtain

$$\partial_t \rho + \partial_x (u\rho) = \partial_x (D\partial_x \rho) + O(\varepsilon^2)$$

- Results ($\Delta t >> \Delta_{exp}$) (Order 1. Left: $\frac{\Delta t}{\varepsilon} = 0.1$, Middle: $\frac{\Delta t}{\varepsilon} = 1$, Right: $\frac{\Delta t}{\varepsilon} = 10$):



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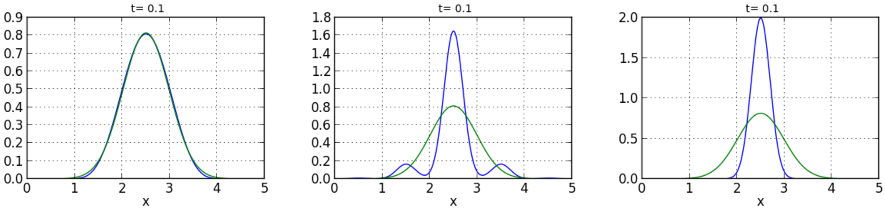
$$\partial_t\rho + \partial_x(u\rho) = \partial_x(D\partial_x\rho) + O(\varepsilon^2)$$

- Results (Order 2. Left: $\frac{\Delta t}{\varepsilon} = 0.1$, Middle: $\frac{\Delta t}{\varepsilon} = 1$, Right: $\frac{\Delta t}{\varepsilon} = 10$):



Current Work II: diffusion

- We want solve the equation: $\partial_t \rho + \partial_x(u\rho) = D\partial_{xx}\rho$
- Kinetic system proposed (S. Jin, F. Bouchut):

$$\left\{\begin{array}{l} \partial_t f_- - \dfrac{\lambda}{\varepsilon}\partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^- - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon}\partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+ - f_+) \end{array}\right.$$

- with $f_{eq}^{\pm} = \frac{\rho}{2} \pm \frac{\varepsilon(u\rho))}{2\lambda}$. The limit is given by:

$$\partial_t \rho + \partial_x(u\rho) = \partial_x((\lambda^2 - \varepsilon^2 \mid \partial F(\rho) \mid^2)\partial_x \rho) + \lambda^2\varepsilon^2\partial_x(\partial_{xx}F(\rho) + \partial F(\rho)_{xx}\rho) - \lambda^2\varepsilon^2\partial_{xxxx}\rho$$

- We introduce $\alpha >\mid \partial F(\rho) \mid$. Choosing $D = \lambda^2 - \varepsilon^2\alpha^2$ we obtain

$$\partial_t \rho + \partial_x(u\rho) = \partial_x(D\partial_x\rho) + O(\varepsilon^2)$$

- **Consistency limit condition**: $\varepsilon > \Delta t$. ε is a non physical parameter. We can choose $\varepsilon = \alpha\Delta t$ with $\alpha >> 1$

	$\alpha = 10$		$\alpha = 50$	
	Error	order	Error	order
$\Delta t = 0.02$	$1.7E^{-2}$	-	$3.5E^{-1}$	-
$\Delta t = 0.01$	$4.4E^{-4}$	5.3	$1.5E^{-1}$	1.2
$\Delta t = 0.005$	$1.4E^{-5}$	5	$3.36E^{-2}$	2.1
$\Delta t = 0.0025$	$5.6E^{-6}$	1.3	$1.78E^{-3}$	4.2

- Convergent only for $\alpha >> 1$ since spitting scheme are not AP. **Future work**: Design AP scheme.

Current Work III: Positive discretization

- **Most important error**: the error due to the relaxation.
- **Time numerical dispersion**: when ε is zero the second order relaxation scheme is $\boldsymbol{f}^* = 2\boldsymbol{f}^{eq} - \boldsymbol{f}^n$. We oscillate around the equilibrium.
- More the wave structure is close to the original one more $\| \boldsymbol{f}^{eq} - \boldsymbol{f}^n \|$ is small. Reduce the oscillations around f^{eq}.

Limiting/entropic technic for relaxation

- **Relaxation step**: $\boldsymbol{f}^{n+1} = \boldsymbol{f}^{eq} + w_1(\varepsilon)(\boldsymbol{f}^n - \boldsymbol{f}^{eq})$ with $w_1(\varepsilon) = \frac{\varepsilon-(1-\theta)\Delta t}{\varepsilon+\theta\Delta t}$
  - **Entropic correction** (I. V. Karlin 98): find ε such that $H(\boldsymbol{f}^{eq} + w_1(\varepsilon)(\boldsymbol{f}^n - \boldsymbol{f}^{eq})) = H(\boldsymbol{f}^n)$ with H the entropy.
  - **Limiting technic**: We have $w_1 = -1$ ordre 2. $w_1 = 0$ ordre 1.
  - $\boldsymbol{f}^{n+1} = \boldsymbol{f}^{eq} + \phi(w_1(\varepsilon))(\boldsymbol{f}^n - \boldsymbol{f}^{eq})$ with ϕ a limiter such that $\phi(w_1) \approx -1$ if $\| \boldsymbol{f}^n - \boldsymbol{f}^{eq} \| < tol$ and $\phi(w_1) \approx 0$ if $\| \boldsymbol{f}^n - \boldsymbol{f}^{eq} \| >> 1$.

Spatial dispersion

- Limiting technic for DG solver. **Problem**: time dispersion of transport DG solver. Open question
- SL- Scheme: SL method based on bounded polynomial (B. Després 16), positive FV-SL or DG-SL.

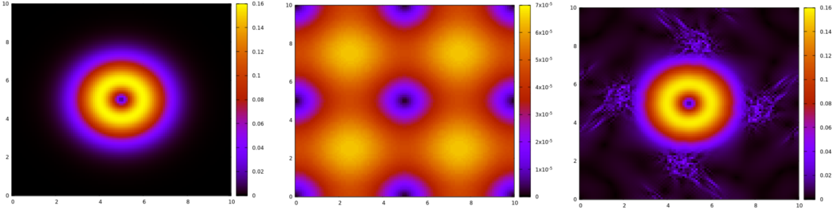
Current Work IV: Low Mach Limit

Low-Mach limit

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{M^2} \nabla p = 0 \end{cases}$$

- We need $\lambda > \frac{1}{M}$. Order one : huge diffusion, ordrr two: huge dispersion for $M << 1$.

- Similar problem: stationary MHD vortex. $\lambda = 20$



- Left: init, middle: order 1 $t = 30$, right: order 2 $t = 150$.

Solution

- Kinetic model with zero velocity + SL for transport (non error in time)
- Two scales kinetic model with order 1 only for the fast scale.

Conclusion

Advantages

- **Initial problem**: invert a nonlinear conservation law is very difficult. High CPU cost (storage and assembly of problem. Slow convergence of iterative solvers).
- **Advantage of method**: replace the complex nonlinear problem (with a huge and increasing cost) by some simple independent problems (with a small and stable cost).

Drawbacks

- High-time error (diffusion/dispersion) since we overestimate the transport. Order 1:

Euler imp	D1Q2 FV-DG
$\frac{\Delta t}{2}\partial_x(A(\boldsymbol{U})^2\partial_x\boldsymbol{U})$	$\frac{\Delta t}{2}(\partial_x(\lambda^2 I_d + \lambda^2 I_d - A(\boldsymbol{U})^2)\partial_x\boldsymbol{U})$

D1Q2 SL	D1Q3 SL
$\frac{\Delta t}{2}(\partial_x(I_d\lambda^2 - A(\boldsymbol{U})^2)\partial_x\boldsymbol{U})$	$\frac{\Delta t}{2}(\partial_x(I_d\lambda \mid A_v(\boldsymbol{U}) \mid -A(\boldsymbol{U})^2)\partial_x\boldsymbol{U})$

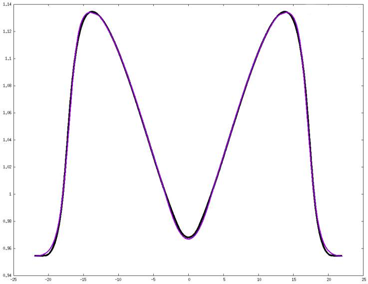
- Additional error is reduced using transport SL scheme, good kinetic representation (and limiting technic for second order).
- **Second drawback**: With this method we reformulate the equations. Some points are more complex: BC, equilibrium etc.

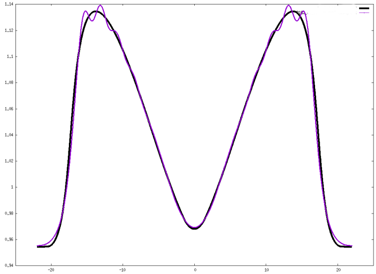
Perspectives

- BC, Equilibrium, Positivity, Diffusion, low-Mach limit, MHD, SL on general meshes.

Conclusion II

- **Test**: low-mach case. 8800 cells $h = 0.005$, Degree of polynomial: 3.
- $\Delta t = 0.04$: CFL FV $\approx$ 100, CFL HO $\approx$ 300.
- (1) Implicit CN + FE method, (2) D1Q2 CN + FE, (3) D1Q2 SL, (4) D1Q3 SL.





- Left: scheme (1). Right: scheme (2), Black: reference solution.

Conclusion II

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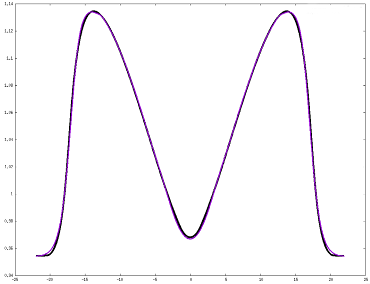

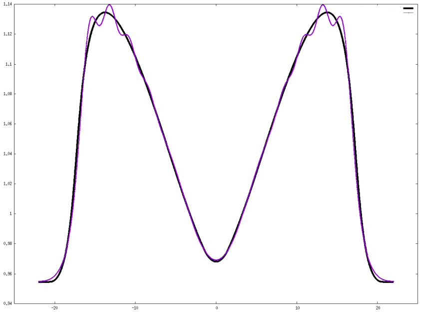

- Left: scheme (1). Right: scheme (3), Black: reference solution.

Conclusion II

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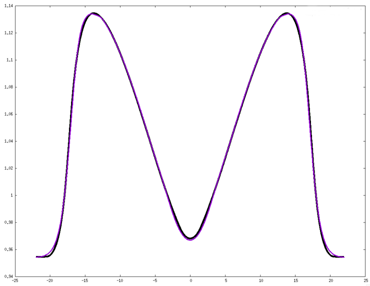

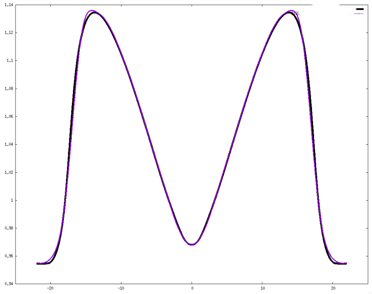

- Left: scheme (1). Right: scheme (4), Black: reference solution.

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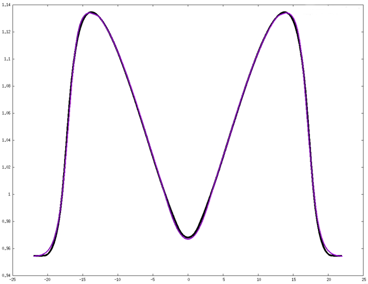

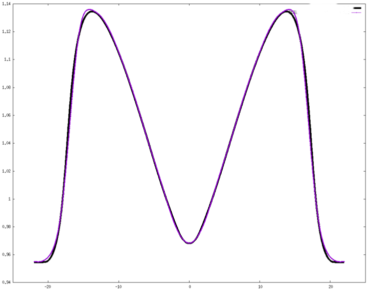

- Left: scheme (1). Right: scheme (4), Black: reference solution.

Conclusion

- Conclusion: as expected D1Q3 SL closed to CN implicit scheme.
- CPU time difficult to compare since the code are different.
- But: 170 sec for (1), 110 sec for (2), 1.6 sec for (3), 1.7 sec for (4)