Implicit relaxation schemes for compressible fluid models

D. Coulette[2], E. Franck[12], P. Helluy[12]
A. Ratnani[3], E. Sonnendrücker[3]

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[1] Inria Nancy Grand Est, France
[2] IRMA, university of Strasbourg, France
[3] NMPP, IPP, Garching bei München, Germany

Outline

Mathematical and physical problems

Relaxation methods

Mathematical and physical problems

Hyperbolic systems and implicit scheme

We consider the general problem

$$\partial_t \boldsymbol{U} + \partial_x(\boldsymbol{F}(\boldsymbol{U})) = \nu \partial_x (D(\boldsymbol{U})\partial_x \boldsymbol{U})$$

with $\boldsymbol{U} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (idem for $\boldsymbol{F}(\boldsymbol{U})$) and D a matrix.

In the following we consider the limit $\nu << 1$.

Implicit scheme

- **Implicit scheme**: allows to avoid the CFL condition filtering the fast phenomena.
- **Problem**: Direct solver, not useful in 3D (too large matrices), we need iterative solvers.
- Conditioning of the implicit matrix: given by the ratio of the maximal and minimal eigenvalues.

- Implicit scheme :

$$\boldsymbol{U} + \Delta t \partial_x(\boldsymbol{F}(\boldsymbol{U})) - \Delta t \nu \partial_x (D(\boldsymbol{U})\partial_x \boldsymbol{U}) = \boldsymbol{U}^n$$

- At the limit $\nu << 1$ and $\Delta t >> 1$ (large time step) we solve $\partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$.

Problem of the implicit scheme

- **Conclusion**: for $\nu << 1$ and $\Delta t >> 1$ the conditioning number of the full system closed to conditioning number of the steady model (the ratio of the speed waves).
- **Exemples**: low-Mach Euler equation, low-Mach and low-β MHD.

Idea

Limit of the classical method

- High memory consumption to store Jacobian and perhaps preconditioning.
- CPU time does not increase linearly with respect to the problem size (effect of the ill-conditioning linked to the physic).

Future of scientific computing

- Machines able to make lots of parallel computing.
- Small memory by node.

Idea: Divide and Conquer

- Propose algorithm which approximates the full problem by a collection of simpler ones.
- Perform the resolution of the simple problems.
- Limit memory consumption using matrix-free method.

Outline of the session

Aim

- Present implicit methods for compressible full (no potential formulation) models based on Divide and Conquer with small memory consumption.

Relaxation methods

- Classical relaxation method (my talk)
  - Presentation of the generalized **Xin-Jin relaxation** method: approximation of the classical model by simpler and linear larger model.
  - Time schemes. Application in the FE/IGA context and results.
- Kinetic relaxation method (D. Coulette talk's)
  - Alternative version of relaxation method based on kinetic formalism.
  - DG context and task-based parallelization (key point).

Splitting method and Compatible FE

- M. Gaja talk's
  - Presentation of splitting method + compatible space to separate the time scale in the matrices.
  - Efficients solver for simple (elliptic) models in the IGA context.

Relaxation methods

General principle

- We consider the following nonlinear system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \nu \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + \boldsymbol{G}(\boldsymbol{U})$$

- with $\boldsymbol{U}$ a vector of N functions.
- **Aim**: Find a way to approximate this system with a sequence of simple systems.
- **Idea**: Xin-Jin relaxation method (very popular in the hyperbolic and finite volume community).

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = \boldsymbol{G}(\boldsymbol{U}) \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

Limit of the hyperbolic relaxation scheme

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \varepsilon \partial_x ((\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + \varepsilon \partial_x \boldsymbol{G}(\boldsymbol{U}) + o(\varepsilon^2)$$

- with $A(\boldsymbol{U})$ the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$.

- **Conclusion**: the relaxation system is an approximation of the hyperbolic original system (error in ε).
- **Stability**: the limit system is dissipative if $(\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) > 0$.

General principle II

Generalization

- The generalized relaxation is given by

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = \boldsymbol{G}(\boldsymbol{U}) \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = \frac{R}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) + \boldsymbol{H}(\boldsymbol{U}) \end{cases}$$

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \varepsilon \partial_x (R^{-1}(\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + \varepsilon(\partial_x \boldsymbol{G}(\boldsymbol{U}) - \partial_x \boldsymbol{H}(\boldsymbol{U})) + o(\varepsilon^2)$$

Treatment of small diffusion

- Taking $R = (\alpha^2 - \mid A(\boldsymbol{U}) \mid^2) D(\boldsymbol{U})^{-1}$, $\varepsilon = \nu$ and $\boldsymbol{H}(\boldsymbol{U}) = A(\boldsymbol{U})\boldsymbol{G}(\boldsymbol{U})$: we obtain the following limit system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \nu \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + o(\nu^2)$$

- **Limit of the method**: the relaxation model cannot approach pde with high diffusion.

Time discretization

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Main idea**: splitting scheme between transport and the relaxation.

First order scheme

- We define the three operator for each steps :

$$T_{\Delta t} = I_d + \Delta t \begin{pmatrix} 0 & \partial_x \\ \alpha^2 \partial_x & 0 \end{pmatrix}$$

$$S_{\Delta t} = I_d + \Delta t \begin{pmatrix} \boldsymbol{G}(I_d) & 0 \\ 0 & 0 \end{pmatrix}$$

$$R_{\Delta t} = I_d + \Delta t \begin{pmatrix} 0 & 0 \\ -\frac{R}{\varepsilon} \boldsymbol{F}(I_d) & \frac{R}{\varepsilon} I_d - \boldsymbol{H}(I_d) \end{pmatrix}$$

- The final scheme $T_{\Delta t} \circ S_{\Delta t} \circ R_{\Delta t}$ is consistant with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U}) + \frac{\Delta t}{2} \partial_x (\alpha^2 \partial_x \boldsymbol{U}) + \left(\frac{\Delta t}{2} + \varepsilon \right) \partial_x \left(R^{-1} \left(\alpha^2 I_d - A(\boldsymbol{U})^2 \right) \partial_x \boldsymbol{U} \right)$$

$$+ O(\varepsilon \Delta t + \Delta t^2 + \varepsilon^2)$$

- **Remark**: the viscosity induced by the splitting have the same form that the viscosity induced by the relaxation.

Discretization of the transport step

Main property

- Transport part:

$$\left\{\begin{array}{l} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \alpha^2 \partial_x \boldsymbol{U} = 0 \end{array}\right.$$

- Can be rewritten as N independent acoustic wave problems.

- We propose a efficient way to solve a single wave equation in the FE/IGA context.

$$\begin{pmatrix} I_d & \theta \Delta t \partial_x \\ \alpha^2 \theta \Delta t \partial_x & I_d \end{pmatrix} \begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} I_d & -(1-\theta)\Delta t \partial_x \\ -\alpha^2 (1-\theta) \Delta t \partial_x & I_d \end{pmatrix} \begin{pmatrix} u^n \\ v^n \end{pmatrix}$$

- Now we propose to apply a Schur decomposition to the implicit matrix.

Final algorithm problem

$$\left\{\begin{array}{ll} \text{Predictor :} & v^* = v^n - (1-\theta)\Delta t u \\ \text{Update :} & (I_d - \alpha^2 \theta^2 \Delta t^2 \partial_{xx}) u^{n+1} = -\theta \Delta t \partial_x v^* + (u^n - (1-\theta)\Delta t v^n) \\ \text{Corrector :} & v^{n+1} = v^* - \alpha^2 \theta \Delta t \partial_x v^{n+1} \end{array}\right.$$

- **Systems to solve**: 2 mass matrices and on laplacian by wave equations.
- **Parallelization** (simple BC): N independent mass matrices, N independent stiffness matrices, N independent mass matrices.
- **Parallelization** (complex BC): N independent mass matrices, one linear matrix of the size N (N laplacian weakly coupled by the boundary), N independent mass matrices.

2D- Extension

Generalization

- The generalized relaxation is given by

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V}_x + \partial_y \boldsymbol{V}_y = 0 \\ \partial_t \boldsymbol{V}_x + \alpha^2 B_{xx}\partial_x \boldsymbol{U} + \alpha^2 B_{xy}\partial_y \boldsymbol{U} = \frac{\Omega_{xx}}{\varepsilon}(\boldsymbol{F}_x(\boldsymbol{U}) - \boldsymbol{V}_x) + \frac{\Omega_{xy}}{\varepsilon}(\boldsymbol{F}_y(\boldsymbol{U}) - \boldsymbol{V}_y) \\ \partial_t \boldsymbol{V}_y + \alpha^2 B_{yx}\partial_x \boldsymbol{U} + \alpha^2 B_{yy}\partial_y \boldsymbol{U} = \frac{\Omega_{yx}}{\varepsilon}(\boldsymbol{F}_x(\boldsymbol{U}) - \boldsymbol{V}_x) + \frac{\Omega_{yy}}{\varepsilon}(\boldsymbol{F}_y(\boldsymbol{U}) - \boldsymbol{V}_y) \end{cases}$$

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}_x(\boldsymbol{U}) + \partial_y \boldsymbol{F}_y(\boldsymbol{U}) = \varepsilon \nabla \cdot (\Omega^{-1}(\alpha^2 B - A^q)\nabla \boldsymbol{U}) + o(\varepsilon^2)$$

- **Remark**: classical choice for B is $B_{xx} = B_{yy} = I_d$ and $B_{yx} = B_{xy} = 0$
- B can be a way to reduce the diffusion adding null wave in the linear system.
- **Discretization**: same space, time discretization and algorithm that in 1D.

Parallelization of the models

- **Transport step** (simple BC): $d * N$ independent mass matrices, N independent stiffness matrices, N independent mass matrices.
- **Transport step** (complex BC): $d * N$ independent mass matrices, one linear matrix of the size N (structure depend of B), $d * N$ independent.
- **Relaxation step**: $d * N$ independent mass matrices.

High-Order time schemes

Second-order scheme

- Scheme for transport step $T(\Delta t)$: Semi Lagrangian for (KRS) or Cranck-Nicholson (KRS with DG or HRS).
- Scheme for relaxation step $R(\Delta t)$: Cranck-Nicholson (KRS and HRS).
- Classical full second order scheme:

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t) \circ T\left(\frac{\Delta t}{2}\right).$$

- AP full second order scheme:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{2}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{4}\right).$$

- Ψ and Ψ_{ap} symmetric in time. $\Psi_{ap}(0) = I_d$.

High order scheme

- Using composition method

$$M_p(\Delta t) = \Psi_{ap}(\gamma_1 \Delta t)\Psi_{ap}(\gamma_2 \Delta t).....\Psi_{ap}(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

Results Burgers I

- **Model** : Viscous - Burgers model.
- Spatial discretization: $N_{cell} = 10000$, $order = 3$. Initial condition : Gaussian.
- **Explicit time step** : stable for $\Delta t < 1.0E^{-5}$.
- **Implicit time step** : $\Delta t = 1.0E^{-3}$

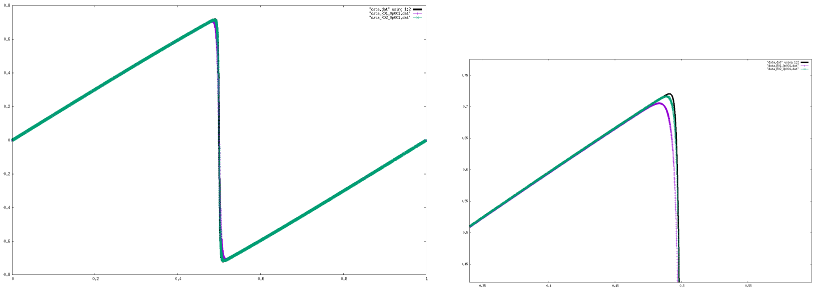

Figure: Left: numerical solution for the first order and the second order schemes for $\Delta t = 0.001$, Right: Zoom

- **Remark**: for discontinuous solutions (or strong gradient solutions) the scheme admits high numerical dispersion and instabilities.
- **Instability**: oscillations $\longrightarrow \alpha$ increase and α increase $\longrightarrow$ oscillations increase.

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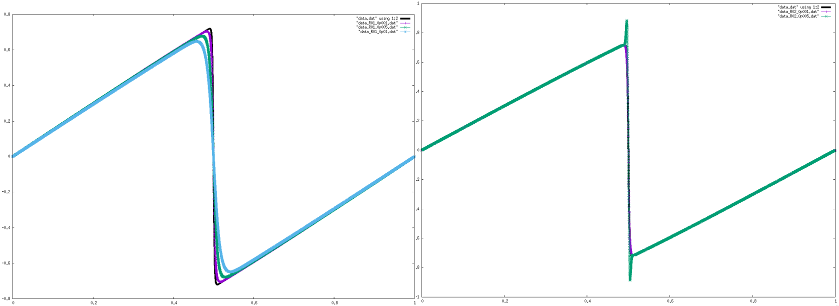

Figure: Left: numerical solution for the first order scheme, Right: numerical solution for the second order scheme. $\nu = 10^{-3}$

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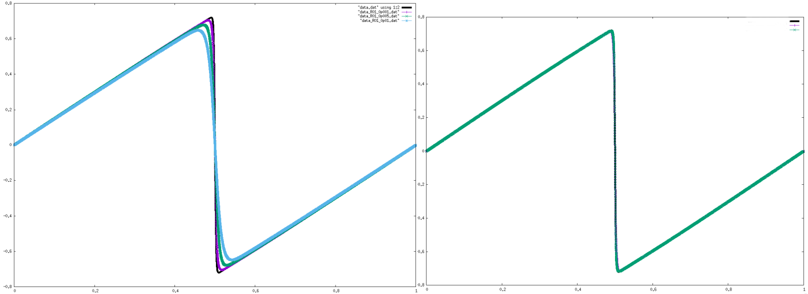

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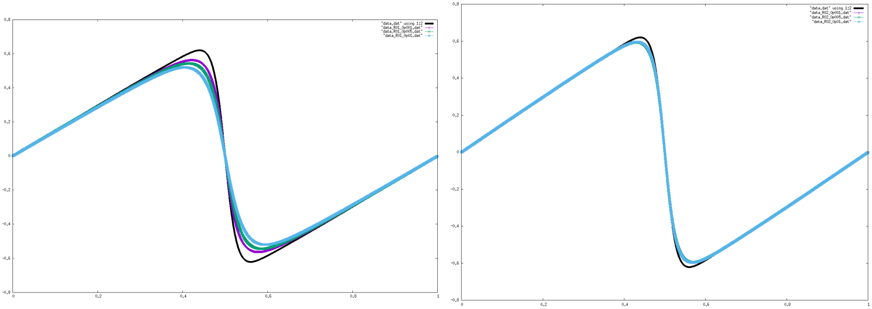

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Results Navier-Stokes I

- **Model**: compressible Navier-Stokes equation

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \\ \partial_t\rho u + \partial_x(\rho u^2 + p) = \partial_x(\nu(\rho)\partial_x u) - \rho g \\ \partial_t E + \partial_x(Eu + pu) = \partial_x(\nu(\rho)\partial_x \frac{u^2}{2}) + \partial_x(\eta\partial_x T) - \rho v g \end{cases}$$

- **Test**: Propagation of acoustic wave (no viscosity, no gravity).
- **CPU Time** for initial $Mach = 0$:

	CN method			Relaxation method		
Δt / cells	5.10^3	10^4	2.10^4	5.10^3	10^4	2.10^4
$\Delta t = 0.005$	160	540	2350	135	430	1920
$\Delta t = 0.01$	90	315	1550	70	220	1000
$\Delta t = 0.02$	55	175	765	40	125	530
$\Delta t = 0.05$	30	100	420	20	65	270

- **CPU Time** for initial $Mach = 0.5$:

	CN method			Relaxation method		
Δt / cells	5.10^3	10^4	2.10^4	5.10^3	10^4	2.10^4
$\Delta t = 0.01$	145	480	2150	100	320	1470
$\Delta t = 0.02$	80	290	1200	60	200	970

Conclusion:

- In this case the Relaxation method is competitive with the classical scheme without important optimization (no parallelization of the problem, etc).

Results Navier-Stokes II

- Simple test case: $\rho(t,x) = 1 + G(x - ut)$, $u(t,x) = 2$ and $T(t,x) = 0$.

Scheme Δt	$\Delta t = 1.0E^{-2}$	$\Delta t = 5.0E^{-3}$	$\Delta t = 2.5E^{-3}$	$\Delta t = 1.25E^{-3}$
CN scheme	$8.8E^{-3}$	$2.25E^{-3}$	$5.7E^{-3}$	$1.4E^{-3}$
Relaxation scheme	$2.25E^{-3}$	$5.7E^{-4}$	$1.4E^{-4}$	$3.6E^{-5}$

- **Conclusion**: the relaxation scheme converges with the second order as expected.
- Spatial discretization: $N_{cell} = 10000$, $order = 3$. Initial condition : Gaussian.

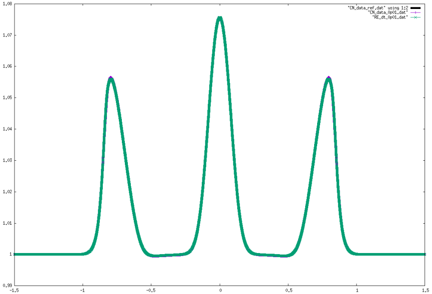

Figure: Fine solution (black). CN solution (violet) and Relaxation solution(green), $M = 0$, $\Delta t = 0.01$

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Scheme Δt	$\Delta t = 1.0E^{-2}$	$\Delta t = 5.0E^{-3}$	$\Delta t = 2.5E^{-3}$	$\Delta t = 1.25E^{-3}$
CN scheme	$8.8E^{-3}$	$2.25E^{-3}$	$5.7E^{-3}$	$1.4E^{-3}$
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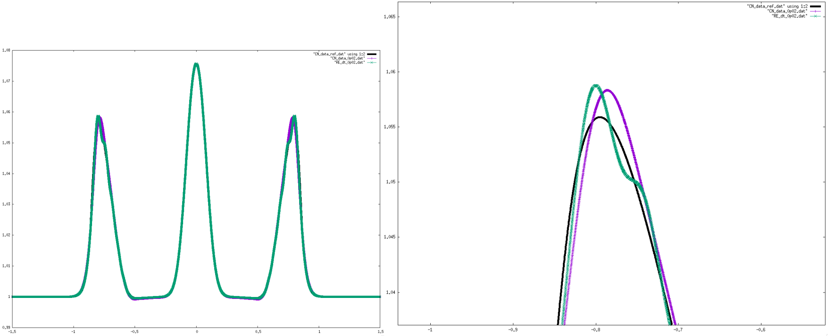

Figure: Fine solution (black). CN solution (violet) and Relaxation solution(green), $M = 0$, $\Delta t = 0.02$

Results Navier-Stokes II

- Simple test case: $\rho(t,x) = 1 + G(x - ut)$, $u(t,x) = 2$ and $T(t,x) = 0$.

Scheme Δt	$\Delta t = 1.0E^{-2}$	$\Delta t = 5.0E^{-3}$	$\Delta t = 2.5E^{-3}$	$\Delta t = 1.25E^{-3}$
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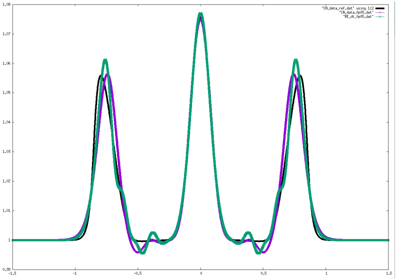

Figure: Fine solution (black). CN solution (violet) and Relaxation solution (green), $M = 0$, $\Delta t = 0.05$

- The two methods (CN and relaxation) capture well the fine solution.

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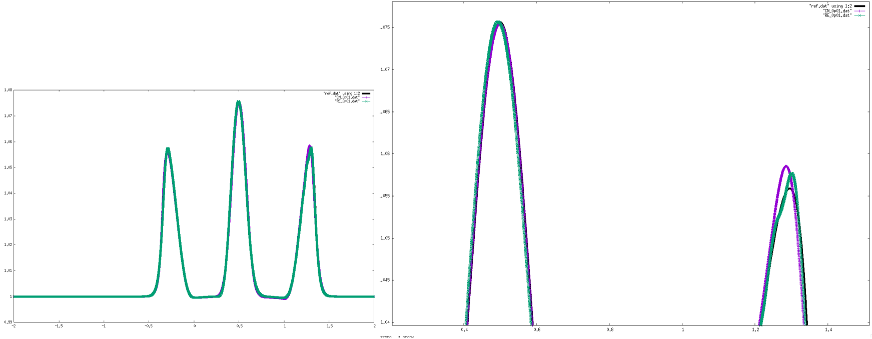

Figure: Fine solution (black). CN solution (violet) and Relaxation solution(green), $M = 0.5$, $\Delta t = 0.01$

Results Navier-Stokes II

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CN scheme	$8.8E^{-3}$	$2.25E^{-3}$	$5.7E^{-3}$	$1.4E^{-3}$
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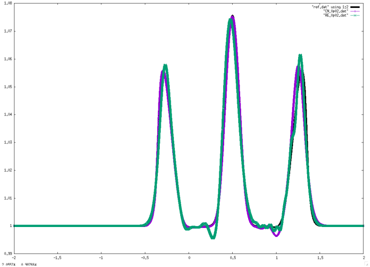

Figure: Fine solution (black). CN solution (violet) and Relaxation solution (green), $M = 0.5$, $\Delta t = 0.02$

- The two methods (CN and relaxation) capture well the fine solution.

Results 2D I

- **Model**: 2D compressible isothermal Navier-Stokes equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = S_r \rho \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + c^2 \rho I_d) = \mu \Delta \boldsymbol{u} + (\mu + \lambda) \nabla (\nabla \cdot \boldsymbol{u}) + S_r \boldsymbol{u} \end{cases}$$

- **Test I**: Steady state between source and spatial part. **Order of convergence**:

	Error	Order
$\Delta t = 0.025$	$1.6E^{-2}$	x
$\Delta t = 0.0125$	$3.8E^{-3}$	x
$\Delta t = 0.00625$	$9.3E^{-4}$	x
$\Delta t = 0.003125$	$2.3E^{-4}$	x

- **Test II**: Propagation of acoustic wave (no viscosity, no gravity).
- **CPU Time** for initial $Mach = 0$:

	CN method			CN Newton			Relaxation method		
Δt / cells	100^2	200^2	400^2	100^2	200^2	400^2	100^2	200^2	400^2
$\Delta t = 0.01$	340	1320	5650	610	2410	9800	330	1260	5040
$\Delta t = 0.02$	170	670	3060	310	1250	6850	165	650	2555
$\Delta t = 0.05$	75	300	1290	140	555	3080	70	275	1115
$\Delta t = 0.1$	45	170	760	100	380	2190	40	155	625

Conclusion:

- The Relaxation method is competitive with the classical schemes (linearized of Newton) without important optimization (no parallelization of the problem, etc).

- **Test I**: Acoustic wave for isothermal Euler equation.

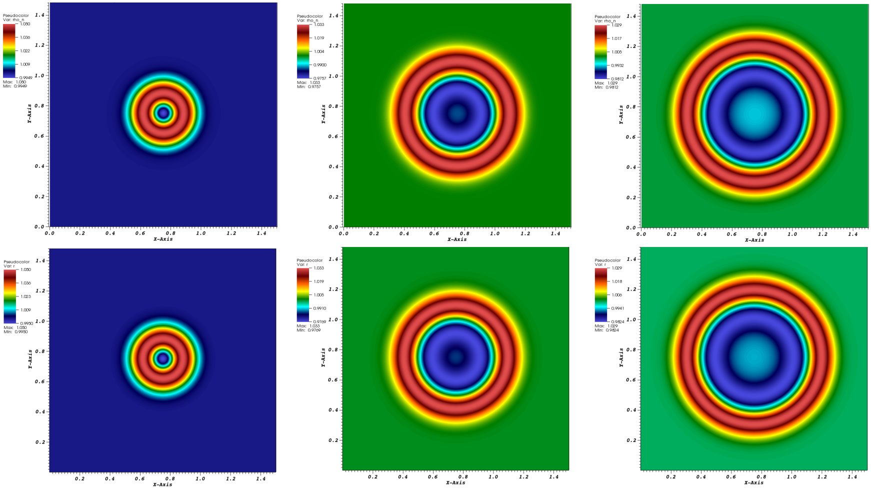

Figure: Comparison between the CN coupled with Newton method (top) and the relaxation (bottom) for a time step $\Delta t = 0.01$.

Results 2D II

- **Test I**: Acoustic wave for isothermal Euler equation.

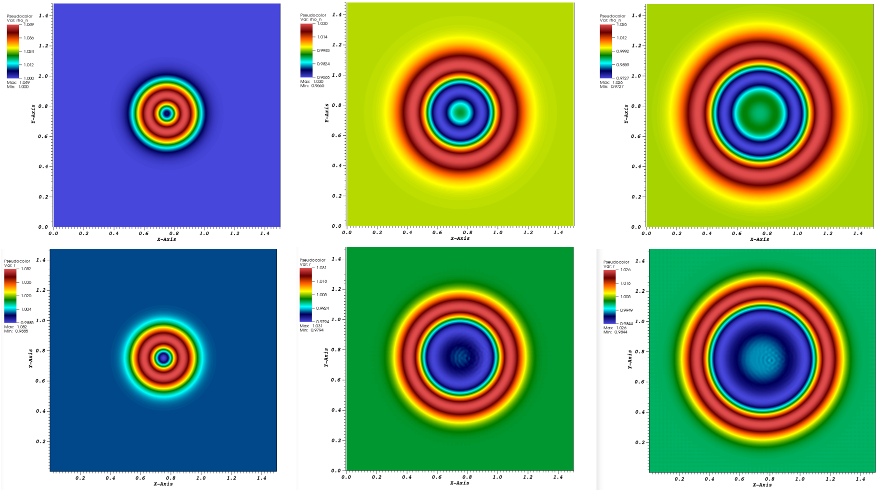

Figure: Comparison between the CN coupled with Newton method (top) and the relaxation (bottom) for a time step $\Delta t = 0.05$.

Results 2D II

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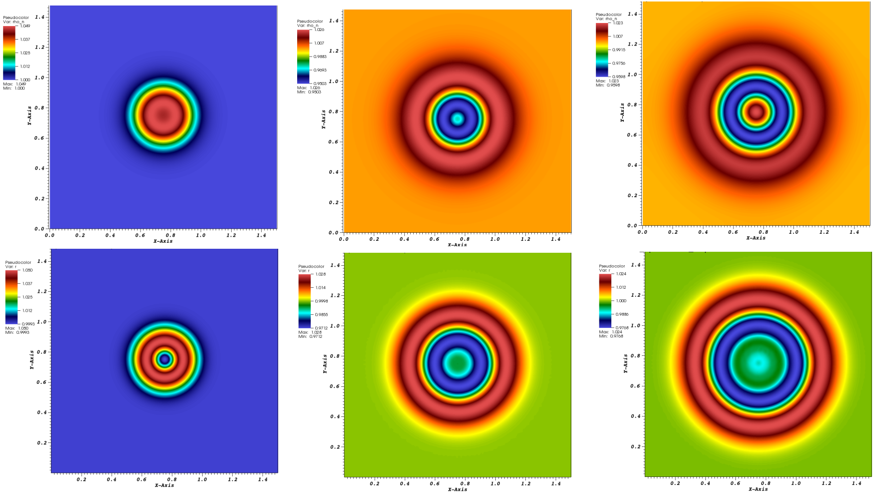

Figure: Comparison between the CN coupled with Newton method (top) and the relaxation (bottom) for a time step $\Delta t = 0.1$.

Results 2D II

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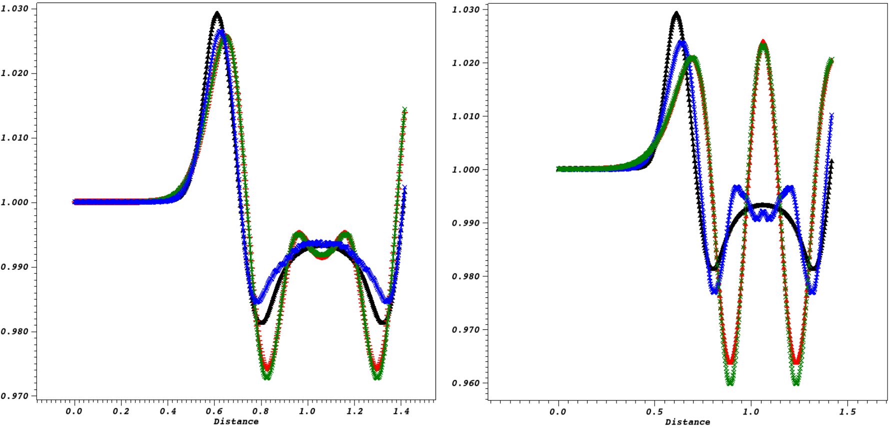

Figure: 1D cut. Fine solution (black), CN method (red), Newton (green) and relaxation (blue). $\Delta t = 0.05$ (left) and $\Delta t = 0.1$ (right)

Future works on relaxation methods

Diffusion

- Propose relaxation for diffusion equation (main point the nonlinearitly must be local).
- Model:

$$\partial_t\rho - \partial_x(D(\rho)\nabla\rho) = f$$

Baby MHD model

- Propose relaxation for a baby model with the additional difficulties linked to the MHD
- Model:

$$\begin{cases} \partial_t \boldsymbol{B} + \nabla\times\left(\boldsymbol{u}\times\boldsymbol{B} + \frac{1}{\rho_0}\nabla T\right) = \eta\nabla\times(\nabla\times\boldsymbol{B}) \\ \partial_t T - \nabla\cdot\left((k_\parallel - k_\perp)(\boldsymbol{b}\otimes\boldsymbol{b})\nabla T + k_\perp\nabla T\right) = 0 \end{cases}$$

- **Difficulties**: anisotropic diffusion and divergence free constrains
- **Div free constrains**: Powell method + classical relaxation or specific relaxation for curl and compatible FE space.

Equilibrium

- Since we over transport all the quantities decoupling the variables we create additional numerical diffusion in time not compatible with equilibrium conservation.
- **Aim** : find method to preserve with a better accuracy the equilibrium.