Compatible isogeometric discretization. Application to linear MHD

E.Franck[1,2], M. Gaja[3], M. Mazza[3], A. Ratnani[3], E. Sonnendrücker[3]

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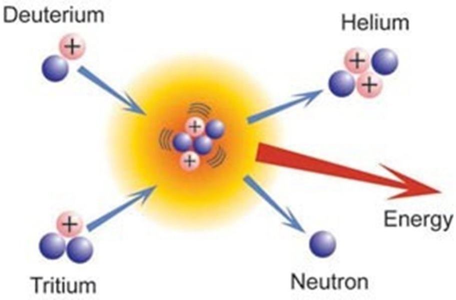
[1]Inria Nancy Grand Est, France
[2]IRMA, university of Strasbourg, France
[3]IPP, Garching bei Munchen, Germany

Outline

Physical and mathematical context

Fusion and Tokamak Instabilities

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- The simulation of these instabilities is an important topic for ITER.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.



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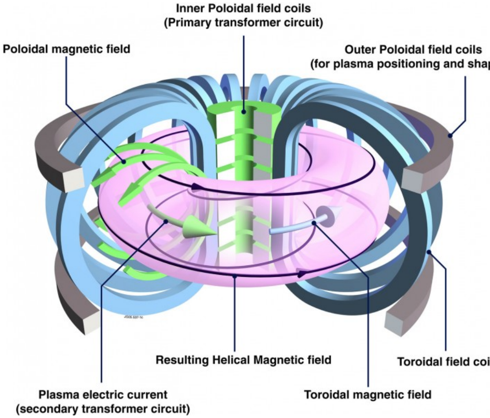

Figure: Tokamak

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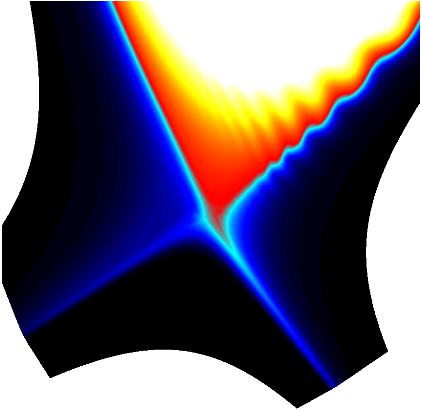

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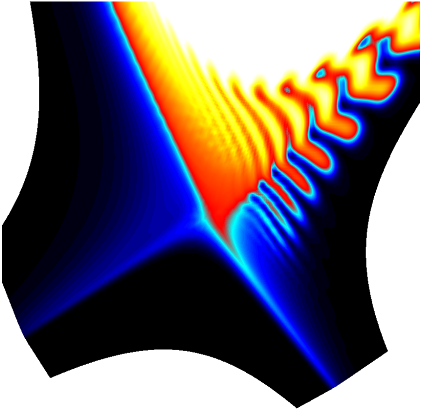

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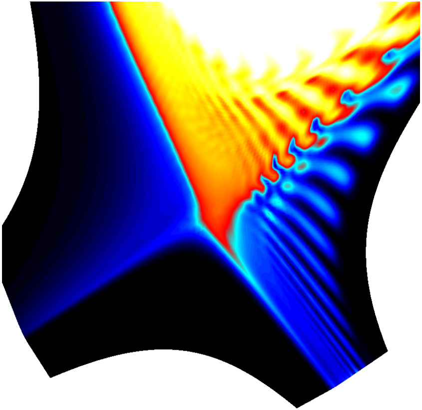

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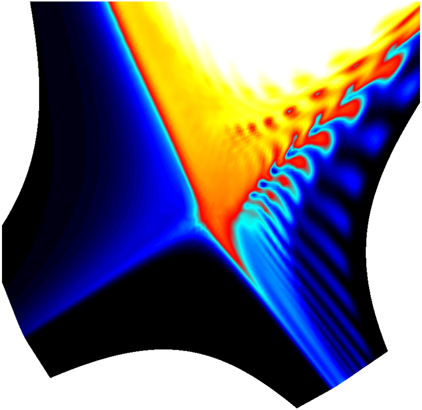

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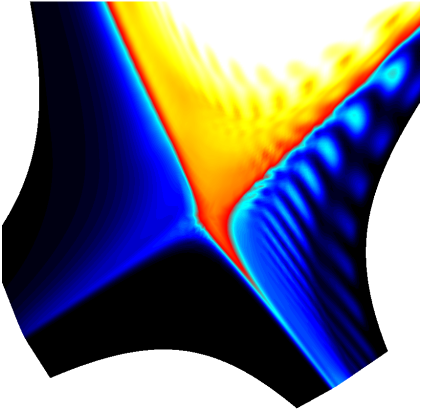

Model

- Resistive MHD model for Tokamak:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla \cdot \boldsymbol{\Pi} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = \nabla \cdot ((k_{\parallel} (\boldsymbol{B} \otimes \boldsymbol{B}) + k_{\perp} I_d) \nabla T) + \eta \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \eta \nabla \times (\nabla \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, $\boldsymbol{u}$ the velocity , p and T the pressure and temperature, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{\Pi} = \boldsymbol{\Pi}(\nabla \boldsymbol{u}, \boldsymbol{B})$ the stress tensor.
- with ν the viscosity, $k_{\parallel}$, $k_{\perp}$ the thermal conductivities and η the resistivity.

Important Properties

- Conservation in time: $\nabla \cdot \boldsymbol{B} = 0$ and

$$\frac{d}{dt} \int \left(\rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{\mid \boldsymbol{B} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

Possible simplification

- $\nabla \cdot \boldsymbol{\Pi} \approx \Delta \boldsymbol{u}$.
- Ohmic ($\eta \mid \nabla \times \boldsymbol{B} \mid^2$) and viscous heating $\nu \boldsymbol{\Pi} : \nabla \boldsymbol{u}$ neglected.

Three stage Energy conserving Splitting

- **MHD for tokamak**: strongly anisotropic, quasi stationary flows (during linear phase).
- **Quasi stationary flows + fast waves** ===> implicit or semi implicit schemes.
- **Choice**: Implicit CN scheme + Splitting.
- Convection step:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u} \\ \partial_t p = \nabla \cdot ((k_{\parallel} (\boldsymbol{B} \otimes \boldsymbol{B} + k_{\perp} I_d) \nabla T) + \eta \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} = \eta \nabla \times (\nabla \times \boldsymbol{B}) \end{cases}$$

- Acoustic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} + \nabla p = 0 \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \\ \partial_t \boldsymbol{B} = 0 \end{cases}$$

- Magnetic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \partial_t p = 0 \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \end{cases}$$

- Each step preserves the total energy.

Two stage Energy conserving Splitting

- Convection step:

$$\left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0,\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}=\nu\Delta\boldsymbol{u}\\ \partial_t p=\nabla\cdot((k_{\parallel}(\boldsymbol{B}\otimes\boldsymbol{B})+k_{\perp}I_d)\nabla T)+\eta\mid\nabla\times\boldsymbol{B}\mid^2+\nu\boldsymbol{\Pi}:\nabla\boldsymbol{u}\\ \partial_t\boldsymbol{B}=\eta\nabla\times(\nabla\times\boldsymbol{B})\end{array}\right.$$

- Magneto-Acoustic step:

$$\left\{\begin{array}{l}\partial_t\rho=0,\\ \rho\partial_t\boldsymbol{u}+\nabla p=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\\ \partial_t p+\nabla\cdot(p\boldsymbol{u})+(\gamma-1)p\nabla\cdot\boldsymbol{u}=0\\ \partial_t\boldsymbol{B}-\nabla\times(\boldsymbol{u}\times\boldsymbol{B})=0\\ \nabla\cdot\boldsymbol{B}=0\end{array}\right.$$

- Each step preserves the total energy.
- The equilibrium relation $\nabla p=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$ is not split. More adapted to equilibrium preservation.
- **Future work**: test of these splitting schemes for different regimes.

Dimensionless linearized model

- We define: the Mach Number $M = \frac{u_0}{c}$ (with c sound speed), the Reynolds number $R_e = \frac{L\rho_0 u_0}{\nu}$, the magnetic Reynolds number $R_m = \frac{LV\mu_0}{\eta}$, the Prandlt number $P_r = \frac{\nu c_p}{\eta}$, the β-number $\beta = \frac{c^2}{V_A^2}$ with V_A the Alfven velocity defined by $V_A^2 = \frac{B_0^2}{\rho_0 \mu_0}$.
- We linearize the previous model with
  - $\boldsymbol{u} = \boldsymbol{a} + \delta \boldsymbol{u}$ with $\mid \boldsymbol{a} \mid = 1$ constant
  - $\rho = \rho_0 + \delta\rho$ with $\rho_0 = 1$
  - $T = T_0 + \delta T$ with $T_0 = 1$
  - $\boldsymbol{B} = \boldsymbol{B}_0 + \delta \boldsymbol{B}$ with $\boldsymbol{B}_0 = \boldsymbol{b}$ a non constant magnetic field.
- We choose the characteristic velocity $V = \frac{u_0}{M^p \beta^q}$. If $p = q = 0$ we have $V = u_0$. If $p = 1$ and $q = 0$ we have $V = c$. If $p = 1$ and $q = 0.5$ we have $V = V_A$.
- We obtain

$$\begin{cases} \partial_t p + M^p \beta^q \boldsymbol{a} \cdot \nabla p + \gamma M^p \beta^q \nabla \cdot \boldsymbol{u} = \frac{(\gamma - 1) M^p \beta^q}{R_e P_r} \nabla \cdot ((k_{\parallel} (\boldsymbol{b} \times \boldsymbol{b}) + k_{\perp} I_d) \nabla T) \\ \partial_t \boldsymbol{u} + M^p \beta^q \boldsymbol{a} \cdot \nabla \boldsymbol{u} + \frac{\beta^q}{\gamma M^{2-p}} \nabla p = \frac{1}{M^{2-p} \beta^{1-q}} ((\nabla \times \boldsymbol{B}) \times \boldsymbol{b}_0 + \boldsymbol{j}_0 \times \boldsymbol{B}) + \frac{M^p \beta^q}{R_e} \Delta \boldsymbol{u} \\ \partial_t \boldsymbol{B} + M^p \beta^q \boldsymbol{a} \cdot \nabla \boldsymbol{B} - M^p \beta^q \nabla \times (\boldsymbol{u} \times \boldsymbol{b}_0) = -\frac{1}{R_m} \nabla \times \nabla \times \boldsymbol{B} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

Three stage Energy conserving Splitting

- Convection step:

$$\begin{cases} \partial_t p + M^p\beta^q \boldsymbol{a}\cdot\nabla p = \dfrac{(\gamma-1)M^p\beta^q}{R_e P_r}\nabla\cdot((k_\parallel(\boldsymbol{b}\times\boldsymbol{b})+k_\perp I_d)\nabla T), \\ \partial_t \boldsymbol{u} + M^p\beta^q \boldsymbol{a}\cdot\nabla \boldsymbol{u} = \dfrac{M^p\beta^q}{R_e}\Delta\boldsymbol{u} \\ \partial_t \boldsymbol{B} + M^p\beta^q \boldsymbol{a}\cdot\nabla \boldsymbol{B} = -\dfrac{1}{R_m}\nabla\times\nabla\times\boldsymbol{B} \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

- Acoustic step:

$$\begin{cases} \partial_t \boldsymbol{u} + \dfrac{\beta^q}{\gamma M^{2-p}}\nabla p = 0 \\ \partial_t p + \gamma M^p\beta^q\nabla\cdot\boldsymbol{u} = 0 \\ \partial_t \boldsymbol{B} = 0 \end{cases}$$

- Magnetic step:

$$\begin{cases} \partial_t \boldsymbol{u} = \dfrac{1}{M^{2-p}\beta^{1-q}}\left((\nabla\times\boldsymbol{B})\times\boldsymbol{b}_0 + \boldsymbol{j}_0\times\boldsymbol{B}\right) \\ \partial_t p = 0 \\ \partial_t \boldsymbol{B} - M^p\beta^q\nabla\times(\boldsymbol{u}\times\boldsymbol{b}_0) = 0 \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

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- The equilibrium relation $\frac{\beta^q}{\gamma M^{2-p}}\nabla p = \frac{1}{M^{2-p}\beta^{1-q}}\left((\nabla\times\boldsymbol{B})\times\boldsymbol{b} + \boldsymbol{j}\times\boldsymbol{B}\right)$ is not split. Better to preserve equilibrium.
- **Future work**: test of these splitting schemes for different regimes.

Compatible isogeometric analysis

Compatible space I

- **Isogeometric analysis**: use the same basis functions to represent the geometry and physical unknowns.
- **B-Splines**: functions of arbitrary degree p and regularity between C^0 and C^{p-1}.
- **B-Splines**: by 1D tensor product. Complex geometries obtained by global mapping.
- **Compatible space**: DeRham sequence

3D Vector fields

$$H^1(\Omega) \overset{\boldsymbol{grad}}{\longrightarrow} H(\mathbf{curl},\Omega) \overset{\boldsymbol{curl}}{\longrightarrow} H(div,\Omega) \overset{div}{\longrightarrow} L^2(\Omega)$$

$$H^1(\mathcal{P}) \overset{\widetilde{\boldsymbol{grad}}^*}{\longleftarrow} H(\mathbf{curl},\mathcal{P}) \overset{\widetilde{\boldsymbol{curl}}^*}{\longleftarrow} H(div,\mathcal{P}) \overset{\widetilde{div}^*}{\longleftarrow} L^2(\mathcal{P})$$

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3D Vector fields approximations

$$
\begin{array}{ccccccc}
H^1(\mathcal{P}) & \overset{\textbf{grad}}{\longrightarrow} & H(\textbf{curl},\mathcal{P}) & \overset{\textbf{curl}}{\longrightarrow} & H(div,\mathcal{P}) & \overset{div}{\longrightarrow} & L^2(\mathcal{P}) \\
\widetilde{\Pi}^h_{h1}\downarrow & & \widetilde{\Pi}^h_{curl}\downarrow & & \widetilde{\Pi}^h_{div}\downarrow & & \widetilde{\Pi}_{L2}\downarrow \\
V^h & \overset{\textbf{grad}^h}{\longrightarrow} & V^h_{curl} & \overset{\textbf{curl}^h}{\longrightarrow} & V^h_{div} & \overset{div^h}{\longrightarrow} & X^h \\
\mathcal{S}^{p,p,p} & & \begin{pmatrix} \mathcal{S}^{p-1,p,p} \\ \mathcal{S}^{p,p-1,p} \\ \mathcal{S}^{p,p,p-1} \end{pmatrix} & & \begin{pmatrix} \mathcal{S}^{p,p-1,p-1} \\ \mathcal{S}^{p-1,p,p-1} \\ \mathcal{S}^{p-1,p-1,p} \end{pmatrix} & & \mathcal{S}^{p-1,p-1,p-1}
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2D Vector fields 1

$$H^1(\Omega) \overset{\boldsymbol{grad}}{\longrightarrow} H(\mathbf{curl},\Omega) \overset{\boldsymbol{rot}}{\longrightarrow} L^2(\Omega)$$

$$H^1(\mathcal{P}) \overset{\widetilde{\boldsymbol{grad}}^*}{\longleftarrow} H(\mathbf{curl},\mathcal{P}) \overset{\widetilde{\boldsymbol{rot}}^*}{\longleftarrow} L^2(\mathcal{P})$$

2D Vector fields 2

$$H^1(\Omega) \overset{\boldsymbol{curl}}{\longrightarrow} H(div,\Omega) \overset{\mathbf{div}}{\longrightarrow} L^2(\Omega)$$

$$H^1(\mathcal{P}) \overset{\widetilde{\boldsymbol{curl}}^*}{\longleftarrow} H(div,\mathcal{P}) \overset{\widetilde{\mathbf{div}}^*}{\longleftarrow} L^2(\mathcal{P})$$

- We can, as in 3D, construct a Discrete DeRham sequence.

Compatible space II

- Advantage of Compatible B-Splines space:
  - High degree, high regularity.
  - Preservation of the properties (3D case here)

$$div_h(\boldsymbol{Curl}_h) = 0, \quad \boldsymbol{Curl_h}(\boldsymbol{grad}_h) = 0$$

and

$$\boldsymbol{Curl}_h^* = \boldsymbol{Curl}_h, \quad \boldsymbol{grad}_h^* = div_h$$

  - Dual properties useful for energy conservation, kernel properties for constraints and avoid spurious modes.
- **Other point**: strong form (equation verified at the coefficient level). Example: Explicit Maxwell.

$$\begin{cases} \boldsymbol{E}^{n+1} = \boldsymbol{E}^n + \Delta t \nabla \times \boldsymbol{B}^n = 0 \\ \boldsymbol{B}^{n+1} = \boldsymbol{B}^n - \Delta t \nabla \times \boldsymbol{E}^n = 0 \\ \nabla \cdot \boldsymbol{B}^{n+1} = 0, \nabla \cdot \boldsymbol{E}^{n+1} = \rho \end{cases}$$

- We take the $\boldsymbol{B}$ equation, choose $\boldsymbol{E} \in H(curl)$ and consequently $\boldsymbol{B} \in H(div)$, multiply by test function and integrate to obtain

$$M\boldsymbol{B}_h^{n+1} = M\boldsymbol{B}_h^n + \Delta t C \boldsymbol{E}_h^n$$

- with M the mass matrix and C the weak curl matrix.

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- Property of the space: $C = M\boldsymbol{Curl}_h$ therefore we have the following strong form

$$\boldsymbol{B}_h^{n+1} = \boldsymbol{B}_h^n + \Delta t \boldsymbol{Curl}_h \boldsymbol{E}_h^n$$

- Applying div_h we obtain $div_h \boldsymbol{B}_h^{n+1} = 0$.

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- Taking $\boldsymbol{B} \in H(div)$ we don't have compatibility with the first equation since we have $\nabla \times \boldsymbol{B}$. Idea: integrate by part the first equation (weak form)

$$\int (\boldsymbol{E}^{n+1}, \boldsymbol{C}) = \int (\boldsymbol{E}^n, \boldsymbol{C}) + \Delta t \int (\boldsymbol{B}^n, \nabla \times \boldsymbol{C})$$

- Taking $\boldsymbol{C} \in H(curl)$ we obtain a consistent equation.

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- At the matrix level, we obtain

$$M_{curl}\boldsymbol{E}^{n+1} = M_{curl}\boldsymbol{E}^n + \Delta t Curl_h^T M_{div}\boldsymbol{B}^n$$

- Taking $\boldsymbol{C} \in H(curl)$ we obtain a consistent equation.

Compatible space III

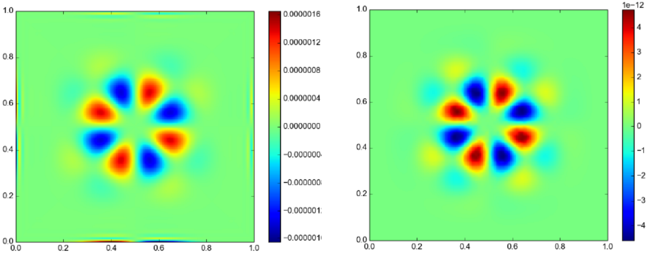
- Additionally we need the commutative projection.
- The 3D projectors are defined by:

$$\widetilde{\Pi}^h_{h1} := \begin{cases} \widetilde{\Pi}^h_{h1} f = f^0_p \in V^h \\ f^0_p(\mathbf{x}_k) = \mathbf{x}_k, \quad \forall \mathbf{x}_k \in N_h \end{cases} \qquad \widetilde{\Pi}^h_{L2} := \begin{cases} \widetilde{\Pi}^h_{L2} \mathbf{f} = \mathbf{f}^3_p \in X^h \\ \int_{v_k} \mathbf{f}^3_p = \int_{s_k} \mathbf{f}, \quad \forall v_k \in \Omega_h \end{cases}$$

- with N_h the nodes of the mesh. Ω_h the cells of the mesh.

$$\widetilde{\Pi}^h_{curl} := \begin{cases} \widetilde{\Pi}^h_{curl} \mathbf{f} = \mathbf{f}^1_p \in V^h_{curl} \\ \int_{e_k} \mathbf{f}^1_p \cdot \mathbf{t} = \int_{e_k} \mathbf{f} \cdot \mathbf{t}, \quad \forall e_k \in E_h \end{cases} \qquad \widetilde{\Pi}^h_{div} := \begin{cases} \widetilde{\Pi}^h_{div} \mathbf{f} = \mathbf{f}^2_p \in V^h_{div} \\ \int_{f_k} \mathbf{f}^2_p \cdot \mathbf{n} = \int_{f_k} \mathbf{f} \cdot \mathbf{n}, \quad \forall f_k \in F_h \end{cases}$$

- with E_h the edges of the mesh. Ω_h the faces of the mesh.
- Exemple: $\rho_2 = \nabla \times (2x(1-x)y(1-y))$. Comparison between L^2 and commutative projection in $H(div)$:



Discretization of sub-models

Advection diffusion model

- Model:

$$\begin{cases} \partial_t p + M^p\beta^q \boldsymbol{a}\cdot\nabla p = \dfrac{(\gamma-1)M^p\beta^q}{R_e P_r}\nabla\cdot\left((k_{\parallel}(\boldsymbol{b}\times\boldsymbol{b}) + k_{\perp} I_d)\nabla T\right), \\ \partial_t \boldsymbol{u} + M^p\beta^q \boldsymbol{a}\cdot\nabla \boldsymbol{u} = \dfrac{M^p\beta^q}{R_e}\Delta \boldsymbol{u} \\ \partial_t \boldsymbol{B} + M^p\beta^q \boldsymbol{a}\cdot\nabla \boldsymbol{B} = -\dfrac{1}{R_m}\nabla\times\nabla\times \boldsymbol{B} \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

- **Classical problem**: classical FE space.
- Low Mach case (and $\beta < 1$):
  - **Dominant term**: anisotropic diffusion ($\eta << k_{\parallel}$, $\nu << k_{\parallel}$)
  - Equation on p: Need robust solver for anisotropic diffusion (not so violent case).
  - Equation on $\boldsymbol{u}$ and $\boldsymbol{B}$: Need robust solver for the mass matrix (like GLT PC or C. Manni PC based on $M_{2D} \approx M_{1D} \otimes M_{1D}$ sufficient).
- Sonic case (and $\beta < 1$):
  - Equation on $\boldsymbol{u}$ and $\boldsymbol{B}$: robust solver for large Pecklet number and stabilization (M. Campos-Pinto and E. Sonnendrücker work's).
  - Way to assure $\nabla\cdot\boldsymbol{B} = 0$.

2D Acoustic model

- Model:

$$\begin{cases} \partial_t \boldsymbol{u} + \dfrac{\beta^q}{\gamma M^{2-p}} \nabla p = 0 \\ \\ \partial_t p + \gamma M^p \beta^q \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

  with the energy balance

$$dt \int \left(M^2 \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{p}{2\gamma} \right) = 0$$

  and the new equation on the vorticity

$$\partial_t w = \partial_t (rot \boldsymbol{u}) = 0$$

- Time scheme : Theta-scheme ($\theta = 0.5$ for Crank-Nicolson scheme). We obtain

$$\begin{cases} \boldsymbol{u}^{n+1} + \dfrac{\beta^q \theta \Delta t}{\gamma M^{2-p}} \nabla p^{n+1} = \boldsymbol{u}^n - \dfrac{\beta^q (1-\theta) \Delta t}{\gamma M^{2-p}} \nabla p^n \\ \\ p^{n+1} + \theta \Delta t \gamma M^p \beta^q \nabla \cdot \boldsymbol{u}^{n+1} = p^n - (1-\theta) \Delta t \gamma M^p \beta^q \nabla \cdot \boldsymbol{u}^n = 0 \end{cases}$$

- For $\theta = 0.5$ the scheme is a second order scheme symmetric in time and preserve energy.

Acoustic model: different formulations

- Different space-time discretizations:
  - **Full Weak H^1 Formulation**: full system in the weak form (non compatible case).
  - **Full Weak H^1-$H(Curl)$**: full system in the weak form, $p \in H^1$ and $\boldsymbol{u} \in H(curl)$.
  - **Full Weak L^2-$H(div)$**: full system in the weak form, $p \in L^2$ and $\boldsymbol{u} \in H(div)$.
  - **Strong-weak H^1-$H(Curl)$**:
    We consider $p \in H^1$ and $\boldsymbol{u} \in H(curl)$. The second equation is solved strongly

$$\boldsymbol{u}^{n+1} + \theta\Delta t \left[\frac{1}{\gamma M^{2-p}}\nabla\right] p^{n+1} = \boldsymbol{u}^n - (1-\theta)\Delta t \left[\frac{1}{\gamma M^{2-p}}\nabla\right] p^n$$

The first equation weakly. We introduce a test function $q \in H^1$, multiply the first equation by q and integrate by parts to obtain:

$$\int p^{n+1} q - \theta\Delta t M^p \gamma \int (\boldsymbol{u}^{n+1}, \nabla q) = \int p^n q + (1-\theta)\Delta t M^p \int (\boldsymbol{u}^n, \nabla q)$$

Since the equation on $\boldsymbol{u}$ is strong we can plug it in the last equation and we obtain

$$A_1(p,q) = \int p^{n+1} q + \theta^2 \Delta t^2 \frac{1}{M^{2(1-p)}} \int (\nabla p^{n+1}, \nabla q) = b$$

with

$$b_1(q) = \int p^n q + \Delta t M^p \gamma \int (\boldsymbol{u}^n, \nabla q) - \theta(1-\theta)\Delta t \frac{1}{M^{2(1-p)}} \int (\nabla p^n, \nabla q)$$

Acoustic model: different formulations

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  - **Full Weak L^2-$H(div)$**: full system in the weak form, $p \in L^2$ and $\boldsymbol{u} \in H(div)$.
  - **Strong-weak H^1-$H(Curl)$**:

Final Algorithm

- We solve

$$A_1(p^{n+1}, q) = b_1(q)$$

with A_1 the weak form of a scalar elliptic problem.

- We compute strongly

$$\boldsymbol{u}^{n+1} = -\theta \Delta t \left[\frac{\beta^q}{\gamma M^{2-p}} \nabla \right] p^{n+1} + \boldsymbol{u}^n - (1-\theta)\Delta t^2 \left[\frac{\beta^q}{\gamma M^{2-p}} \nabla \right] p^n$$

Acoustic model: different formulations

- Different space-time discretizations:
  - **Full Weak H^1 Formulation**: full system in the weak form (non compatible case).
  - **Full Weak H^1-$H(Curl)$**: full system in the weak form, $p \in H^1$ and $\boldsymbol{u} \in H(curl)$.
  - **Full Weak L^2-$H(div)$**: full system in the weak form, $p \in L^2$ and $\boldsymbol{u} \in H(div)$.

  - **Strong-weak L^2-$H(div)$**:
    We consider $p \in L^2$ and $\boldsymbol{u} \in H(Div)$.The second equation is solved strongly

$$p^{n+1} + \theta\Delta t \left[M^p \gamma \nabla \cdot\right] \boldsymbol{u}^{n+1} = p^n - (1-\theta)\Delta t \left[M^p \gamma \nabla \cdot\right] \boldsymbol{u}^n$$

The first is take weakly. We take a test function $\boldsymbol{v} \in H(div)$, multiply the first equation by $\boldsymbol{v}$ and integrate by parts we obtain

$$\int (\boldsymbol{u}^{n+1}, \boldsymbol{v}) - \theta\Delta t \frac{1}{\gamma M^{2-p}} \int (p^{n+1} \nabla \cdot \boldsymbol{v}) = \int (\boldsymbol{u}^n, \boldsymbol{v}) + (1-\theta)\Delta t \frac{1}{\gamma M^{2-p}} \int (p^n, \nabla \cdot \boldsymbol{v})$$

Since the equation on $\boldsymbol{u}$ is strong we can plug it in the last equation and we obtain

$$A_2(\boldsymbol{u}, \boldsymbol{v}) = \int (\boldsymbol{u}^{n+1}, \boldsymbol{v}) + \theta^2 \Delta t^2 \frac{1}{M^{2(1-p)}} \int (\nabla \cdot \boldsymbol{u}^{n+1}, \nabla \cdot \boldsymbol{v}) = b(\boldsymbol{v})$$

with

$$b(\boldsymbol{v}) = \int (\boldsymbol{u}^n, \boldsymbol{v}) + \Delta t \frac{1}{\gamma M^{2-p}} \int (p^n, \nabla \cdot \boldsymbol{v}) - \theta(1-\theta)\Delta t^2 \frac{1}{M^{2(1-p)}} \int (\nabla \cdot \boldsymbol{u}^n, \nabla \cdot \boldsymbol{v})$$

Acoustic model: different formulations

- Different space-time discretizations:
  - **Full Weak H^1 Formulation**: full system in the weak form (non compatible case).
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  - **Full Weak L^2-$H(div)$**: full system in the weak form, $p \in L^2$ and $\boldsymbol{u} \in H(div)$.
  - **Strong-weak L^2-$H(div)$**:

Final Algorithm

- We solve

$$A_2(\boldsymbol{u}^{n+1}, \boldsymbol{v}) = b_2(\boldsymbol{v})$$

with A_2 the weak form of a vectorial elliptic problem.

- We compute strongly

$$p^{n+1} = -\theta\Delta t \left[M^p \gamma \nabla\cdot\right] \boldsymbol{u}^{n+1} + p^n - (1-\theta)\Delta t \left[M^p \gamma \nabla\cdot\right] \boldsymbol{u}^n$$

Acoustic model: properties, solver and PC

- **Full weak formulation**: preserve total energy.
- **Strong-weak formulation**: preserve total energy ?

Strong-weak $H^1 - H(curl)$

- Preserve vorticity equation since $\boldsymbol{u}$ given by $\nabla_h p$ and $rot_h(\nabla_h) = 0$.
- Elliptic equation to invert

$$w^2 p - \Delta p = f$$

- **Efficient solver**: CG + MG-GLT preconditioning. MG for low frequencies. GLT for high frequencies.

Strong-weak $L^2 - H(div)$

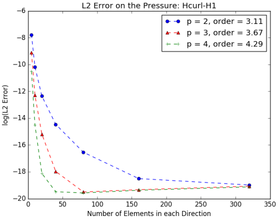
- Probable equilibrium conservation if source term.
- Elliptic equation to invert

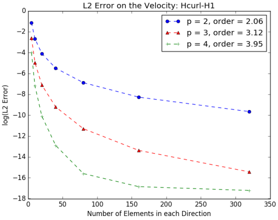
$$w^2 \boldsymbol{u} - \nabla(\nabla \cdot \boldsymbol{u}) = \mathbf{f}$$

- More complex elliptic problem when w^2 ($\approx \Delta t^{-2}$) tends to zero.
- **Possible Efficient solver**: CG + Auxiliary-space Hiptmair preconditioning coupled with GLT.

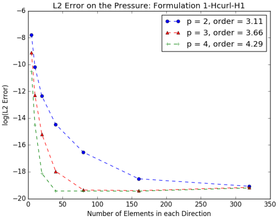
Convergence Results and CPU Time in 2D, dt = 0.1s

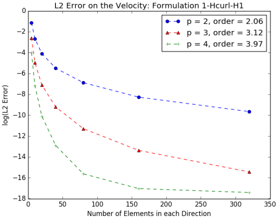
Full Model Conv.:



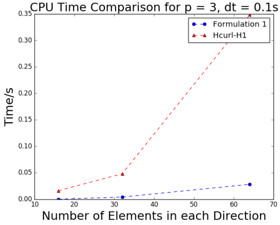


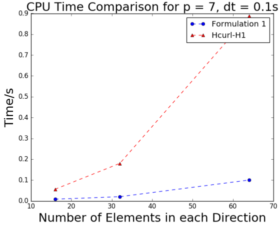
Formulation Conv.:



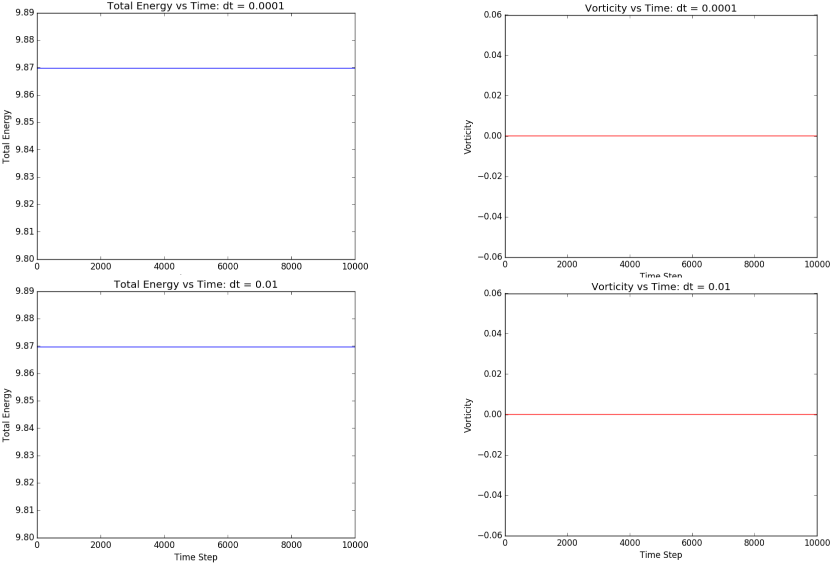


CPU Time:





Energy and Vorticity: 16×16, $p = 3$



3D Maxwell-like model

- Model:

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \times \boldsymbol{B} = 0 \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{u} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- **Time scheme**: second order Crank-Nicolson time scheme.
- **Compatible space**. We choose $\boldsymbol{u} \in H(curl)$ and $\boldsymbol{B} \in H(div)$.
- **Two possibilities**: full weak formulation or strong-weak $H(div) - H(curl)$.

Strong weak form algorithm

- We put the strong equation on $\boldsymbol{B}$ in the weak equation on $\boldsymbol{u}$ and integrate by part.
- We solve

$$A(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) = b(\boldsymbol{v})$$

  with

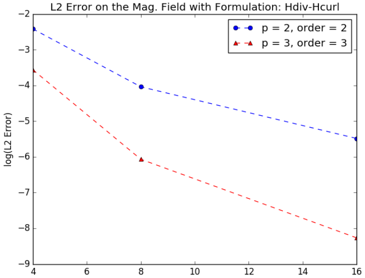
$$A(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) = \int (\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + \theta^2 \Delta t^2 \int (\nabla \times (\boldsymbol{u}_h^{n+1}), \nabla \times \boldsymbol{v}) = b(\boldsymbol{v})$$

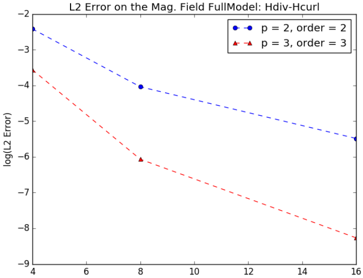
- We compute strongly

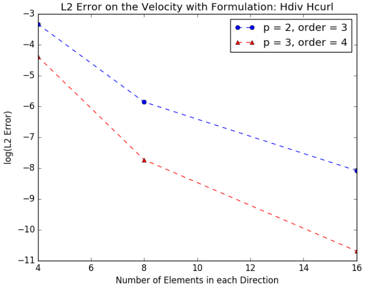
$$\boldsymbol{B}_h^{n+1} = \Delta t \boldsymbol{Curl}_h(\boldsymbol{u}_h^{n+1}) + \boldsymbol{B}_h^n + (1-\theta)\Delta t \boldsymbol{Curl}_h(\boldsymbol{u}_h^n)$$

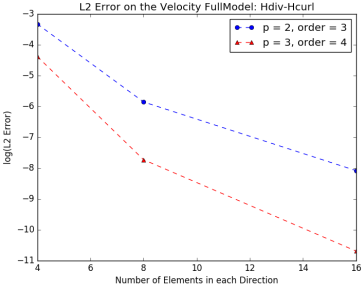
- $\nabla \cdot_h \boldsymbol{B}_h = 0$ is preserved in time.

Convergence Maxwell: Formulation, HdivHcurl

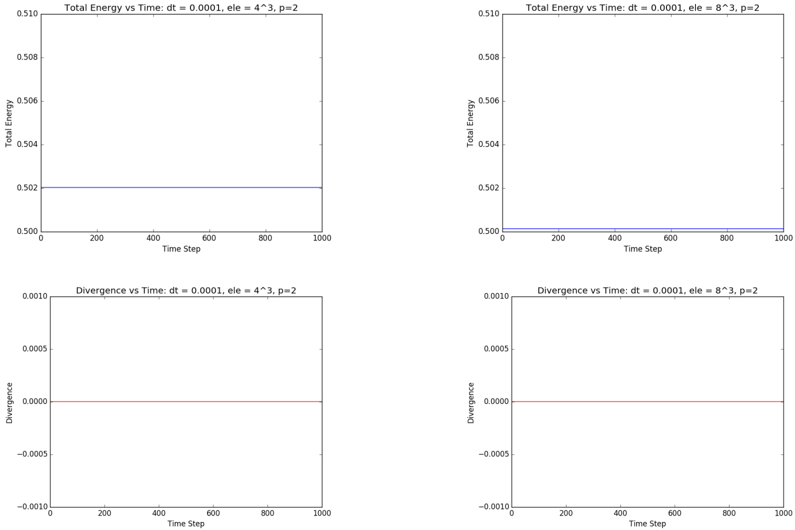








Energy and Divergence



- For 16 elements, degree 3, 40 steps and dt= 0.0025, the model with formulation was 4 times faster than the full model.

Magnetic step

- Magnetic step model:

$$\begin{cases} \partial_t \boldsymbol{u} = \dfrac{1}{M^{2-p}\beta^{1-q}}\left((\nabla \times \boldsymbol{B}) \times \boldsymbol{b} + \boldsymbol{j} \times \boldsymbol{B}\right) \\ \partial_t \boldsymbol{B} - M^p \beta^q \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- **Time scheme**: second order Crank-Nicolson time scheme.
- **Compatible space**. We choose $\boldsymbol{u} \in H(div)$ and $\boldsymbol{B} \in H(div)$.
- **Two possibilities**: full weak formulation or strong-weak $H(div)$-$H(div)$.

Strong weak form algorithm

- We put the strong equation on $\boldsymbol{B}$ in the weak equation on $\boldsymbol{u}$ and integrate by part.
- We solve

$$A(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) = b(\boldsymbol{v})$$

$$A(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) = \int (\boldsymbol{u}_h^{n+1}, \boldsymbol{v}) + \theta^2 \Delta t^2 \int (\nabla \times (\boldsymbol{u}_h^{n+1} \times \boldsymbol{b}), \nabla \times (\boldsymbol{v} \times \boldsymbol{b})) = b(\boldsymbol{v})$$

- We compute the projection step

$$(\boldsymbol{u}_h^n \times \boldsymbol{b})_{h(curl)} = \Pi(\boldsymbol{u}^n \times \boldsymbol{b}), \qquad (\boldsymbol{u}_h^{n+1} \times \boldsymbol{b})_{h(curl)} = \Pi(\boldsymbol{u}^{n+1} \times \boldsymbol{b})$$

- We compute strongly

$$\boldsymbol{B}^{n+1} = M^p \beta^q \theta \Delta t \nabla \times (\boldsymbol{u}^{n+1} \times \boldsymbol{b})_{h(curl)} + \boldsymbol{B}^n + (1-\theta)\Delta t M^p \beta^q \nabla \times (\boldsymbol{u}^n \times \boldsymbol{b})_{h(curl)}$$

- $\nabla \cdot_h \boldsymbol{B}_h = 0$ is preserved in time.

Magnetic step: remark and solver

- The elliptic operator invert in the previous algorithm is

$$\omega^2 \boldsymbol{u} + \frac{1}{M^{2-p}\beta^{1-q}} \left(\nabla \times \left(\nabla \times \left(\boldsymbol{u} \times \boldsymbol{b} \right) \right) \right) \times \boldsymbol{b} = \boldsymbol{f}$$

- This operator can be decomposed like

$$\omega^2 \boldsymbol{u} - \frac{1}{M^{2-2p}\beta^{1-2q}} \left(\nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{b} \cdot \nabla(\boldsymbol{b} \cdot \nabla \boldsymbol{u}) - \nabla(\boldsymbol{b}, \boldsymbol{b} \cdot \nabla \boldsymbol{u}) - \boldsymbol{b} \cdot \nabla(\boldsymbol{b} \nabla \cdot \boldsymbol{u}) \right) = \boldsymbol{f}$$

- Fast wave: first term, Alfven wave: second term.
- **First idea of physic PC**: Previous operator equivalent to

$$\begin{cases} \boldsymbol{u} + \frac{1}{\omega} \nabla(\boldsymbol{b}, \boldsymbol{C}) - \frac{1}{\omega} \boldsymbol{b} \cdot \nabla \boldsymbol{C} = \boldsymbol{f} \\ \boldsymbol{C} + \frac{1}{\omega} \boldsymbol{b} \nabla \cdot \boldsymbol{u} - \frac{1}{\omega} \boldsymbol{b} \cdot \nabla \boldsymbol{u} = 0 \end{cases}$$

- The model can be written as $I_d + \frac{1}{\omega} F + \frac{1}{\omega} A = \left(I_d + \frac{1}{\omega} F \right) \left(I_d + \frac{1}{\omega} A \right) + O(\frac{1}{\omega^2})$. If ω is large (small Δt) we solve the two models one after one.
- Solver for each model: Schur decomposition.

Algorithm

- Solve $\omega^2 \boldsymbol{u}^* - \nabla(\nabla \cdot \boldsymbol{u}^*) = \boldsymbol{f}$
- Compute $\boldsymbol{C}^* = -\frac{1}{\omega} \boldsymbol{b} \nabla \cdot \boldsymbol{u}^*$
- Solve $\omega^2 \boldsymbol{C} - \boldsymbol{b} \cdot \nabla(\boldsymbol{b} \cdot \nabla \boldsymbol{C}) = \boldsymbol{C}^* + \boldsymbol{b} \cdot \nabla \boldsymbol{u}^*$
- Compute $\boldsymbol{u} = -\frac{1}{\omega} \boldsymbol{b} \cdot \nabla \boldsymbol{C}$

Equilibrium and init

Initialization

- **Equilibrium magnetic field**:

$$\boldsymbol{b} = \frac{F(\psi)}{R} + \frac{1}{R}\nabla\psi \times e_{\phi}$$

- Poloidal flux solution of ψ:

$$\Delta^*\psi = -R^2\frac{dp(\psi)}{d\psi} - \frac{dF(\psi)}{d\psi}F(\psi)$$

- We choose $\psi \in H_1$ and solve the previous equation.
- We define $\boldsymbol{A} \in H(Curl)$ with

$$\boldsymbol{A} = \Pi_{curl}\left(\frac{1}{R}\psi e_{\phi} + ...\right)$$

- We define $\boldsymbol{b} = Curl_h\boldsymbol{A}$ to assure initially $Div_h\boldsymbol{b} = 0$.

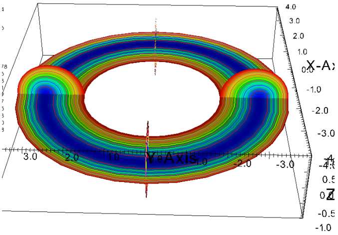


Figure: 3D equilibrium

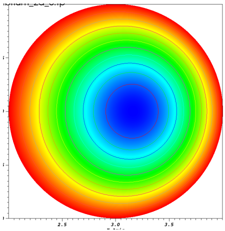


Figure: poloidal cut of equilibrium

Numerical issues

- **High-order discretization**: between third and fifth polynomial degree in space, second order in time.
- **Important drawback**: numerical dispersion ==> less accurate simulation, loss of positivity, worse conditioning.

Numerical dispersion in time

- **Solution**: stabilization like TG-stabilization. Add a dissipative process on the target wave.
- **Drawback**: loose of energy conservation. Other solutions ? stabilization ?

Numerical dispersion in time

- **Stabilization**: same remark as for time dispersion.
- **B-Splines property**: more regularity.
- **Results of the iso-geometric analysis community**: less numerical dispersion increasing the regularity.
- High regular B-Splines admit probably less numerical dispersion than classical FE or Bezier Elements.

Current work for JOREK

Splitting in JOREK

- Splitting similar to the previous one, tested in JOREK.
- Splitting: two steps for advection diffusion, one for magneto-acoustic.
- Non optimized version tested current in JOREK for the model 199 (later 303)
- Coupling with Petsc for the sub-steps.

Program

- Benchmarking to improve the splitting (current work).
- Implementation: Optimized version.
- Coupling with Jacobian free Matrix and Parallelization.
- Extension to the diamagnetic and neoclassical terms.

Other possible works in JOREK

- **Other idea**: Second order Xin-Jin/kinetic Relaxation scheme.
- **Advantages**:
  - **Implicit step**: invert only simple diffusion equations and mass matrices.
  - Perhaps unify all the reduced and full MHD model (+ basic stabilization).
- **Defaults**:
  - Full MHD formalism: complex BC
  - Slightly more dispersive method.

Other application

Galbrun equation

- Equation describing the Lagrangian motion ζ given by

$$\rho_0 D_{tt}\zeta = \mathbf{F}(\zeta)$$

- with

$$\mathbf{F}(\zeta) = \nabla\left(p_0 \nabla\cdot\zeta\right) + \nabla(\zeta\cdot\nabla p_0) - \boldsymbol{B}_0\times(\nabla\times(\nabla\times(\zeta\times\boldsymbol{B}_0))) + \boldsymbol{J}_0\times\nabla\times(\zeta\times\boldsymbol{B}_0)$$

- and $D_t = \partial_t + \boldsymbol{u}_0\cdot\nabla$ the material derivative.

Heliosismology

- **Data**: "Lagrangian motion" at the surface of the sun in the telescope direction.
- Inverse problem to recover the background p_0, $\boldsymbol{u}_0$ etc
- Solve harmonic problem:

$$(-\omega i + \boldsymbol{u}_0\cdot\nabla)^2\zeta = \mathbf{F}(\zeta).$$

- Some numerical issues are similar to ours.

Conclusion

Conclusion

- **B-Splines Compatible space**: allows to preserve some properties (divergence free, energy)) with high-order on complex geometries.
- **B-Splines**: High-regularity more adapted to wave problems since less numerical dispersion (need to be verified).
- **Splitting**: Allows to separate the convection-diffusion from the waves and use the Strong-weak algorithm for the wave part.

Perspectives

- **Characteristic velocity**: sound speed/ Alfven speed ? Improve splitting for this limit with large Reynolds and low Prandt number.
- Improve properties preserving for the magneto-acoustic in Tokamak geometry.
- Couple with an equilibrium code, use for $\boldsymbol{b}$ the equilibrium magnetic field, improve the equilibrium conservation.
- Stabilization and $\nabla \cdot \boldsymbol{B} = 0$ for all steps.
- GLT or Hiptmair PC for acoustic. PC for the magnetic step.
- Extention in the nonlinear case (commutation between linearization and splitting ??).