High-order Implicit relaxation schemes. Application to low-mach viscous problems

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Outline

Physical and mathematical context

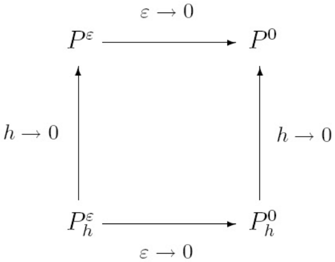
Asymptotic Preserving scheme

- We consider PDE depending of a small parameter with an asymptotic limit.
- Exemple: hyperbolic heat equation

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\partial_x u = 0, \\ \partial_t u + \frac{1}{\varepsilon}\partial_x p = -\frac{1}{\varepsilon}g - \frac{\sigma}{\varepsilon^2}u, \end{cases} \longrightarrow \partial_t p - \partial_x\left(\frac{1}{\sigma}(\partial_x p + g)\right) = 0.$$

Asymptotic preserving scheme

- AP scheme: a consistent scheme for the initial PDE which gives at the limit a consistent scheme of the limit PDE.
- Uniform AP scheme: convergence and stability independent of ε.



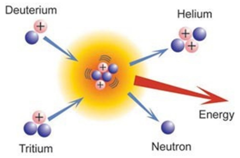
- Application: simulate problem with varying physical parameter and regime. Example: radiative transfer (strong varying σ).
- Other application: use AP scheme to create a new scheme for the limit model. Example: relaxation scheme for Euler equation.

Applications

Applications considered

- Steady or quasi-steady flows (long time limit).
- Multi-scale model: capture the slow scale and filter the fast one (low mach limit).

- **Fusion DT**: At sufficiently high energies, deuterium and tritium (plasma) can fuse to Helium. Free energy is released.
- **Tokamak**: toroïdal chamber where the plasma is confined using magnetic fields.
- **Difficulty**: plasma instabilities. Important topic for ITER.



Simulation of MHD instabilities

- Simulation: slow flow around plasma equilibrium (in green):

$$\left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0,\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\nu\nabla\cdot\boldsymbol{\Pi}\\ \partial_t p+\nabla\cdot(p\boldsymbol{u})+(\gamma-1)p\nabla\cdot\boldsymbol{u}=\nabla\cdot\mathbf{q}+\eta\mid\nabla\times\boldsymbol{B}\mid^2+\nu\boldsymbol{\Pi}:\nabla\boldsymbol{u}\\ \partial_t\boldsymbol{B}-\nabla\times(\boldsymbol{u}\times\boldsymbol{B})=\eta\nabla\times(\nabla\times\boldsymbol{B})\\ \nabla\cdot\boldsymbol{B}=0\end{array}\right.$$

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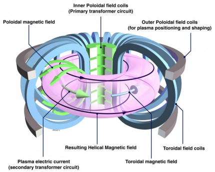

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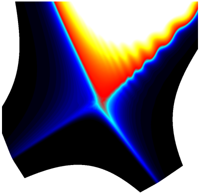

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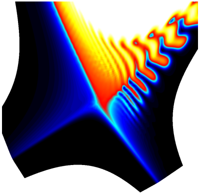

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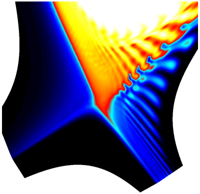

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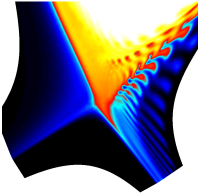

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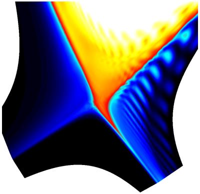

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Implicit method

Classical solution

- Explicit scheme: CFL given by the highest frequency discretized of the waves.
- **Solution**: implicit scheme to filter the frequencies not considered.

- Solution for implicit schemes:
  - Direct solver. CPU cost and consumption memory too large in 3D.
  - Iterative solver. Problem of conditioning.

Problem of conditioning

- Multi-scale PDE (low Mach regime) ==> huge ratio between discrete eigenvalues.
- High order scheme for transport: small/high discrete frequencies and anisotropy ==> huge ratio between discrete eigenvalues.
- Possible solution: preconditioning (often based on splitting and reformulation).
- Storage the matrix and perhaps the preconditioning: large memory consumption.

Main idea

- **Step 1**: Write a larger and simple system, depending of a small parameter with the initial system as a limit.
- **Step 2**: Design an implicit AP scheme for the new larger system and use it.

- **Aim**: Avoid conditioning and storage problem.

Implicit Relaxation method and results

Relaxation scheme

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \alpha^2 \partial_x u = \frac{1}{\varepsilon}(F(u) - v) \end{cases}$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_x((\lambda^2 - \mid \partial F(u) \mid^2)\partial_x u) + O(\varepsilon^2)$$

- **Stability**: the limit system is dissipative if $(\lambda^2 - \mid \partial F(u) \mid^2) > 0$.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\begin{cases} \partial_t f_- - \lambda \partial_x f_- = \frac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \partial_t f_+ + \lambda \partial_x f_+ = \frac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{cases}$$

- with $u = f_- + f_+$ and $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\lambda}$.

First Generalization

- **Main property**: the transport is diagonal (D1Q2 model) which can be easily solved.
- **Generalization**: one Xin-Jin or D1Q2 model by macroscopic variable.

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $U \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- We define the macroscopic variable by $P\boldsymbol{f} = \boldsymbol{U}$.

- Consistence conditon (R. Natalini, D. Aregba-Driollet, F. Bouchut) :

$$\mathcal{C} \begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{F}(\boldsymbol{U}) \end{cases}$$

- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial \boldsymbol{f} eq - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

Space discretization - transport scheme

Whishlist

- Complex geometry, curved meshes or unstructured meshes,
- CFL-free,
- Matrix-free.
- High-Order in space

Candidates for transport discretization

- **LBM-like**: exact transport solver,
- Implicit FV-DG schemes,
- Semi-Lagrangian schemes,

LBM-like method: exact transport

- **Advantages**:
  - Exact transport at the velocity $\lambda = \frac{v\Delta t}{\Delta x}$. Very very cheap cost.
- **Drawbacks**:
  - Link time step and mesh: complex to manage large time step, unstructured grids and multiply kinetic velocities.

Space discretization

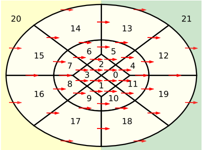
Semi Lagrangian methods

- Forward or Backward methods. Mass or nodes interpolation/projection.
- **Advantages**:
  - Possible on unstructured meshes. High order in space.
  - **Exact in time** and Matrix-free.
- **Drawbacks**:
  - No dissipation and difficult on very unstructured grids.

Implicit FV- DG methods

- Implicit Crank Nicolson scheme + FV DG scheme
- **Advantages**:
  - Very general meshes. High order in space. Dissipation to stabilize.
  - Upwind fluxes ==> triangular block matrices.
- **Drawbacks**:
  - Second order in time: numerical time dispersion.

- Current choice 1D: SL-scheme.
- Current choice in 2D-3D: DG schemes.
  - Block - triangular matrix solved avoiding storage.
  - Solve the problem in the topological order given by connectivity graph.



Time discretization

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Main idea**: splitting scheme between transport and the relaxation (P. J. Dellar, 13).
- **Key point**: the macroscopic variables are conserved during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ explicit.

First order scheme (first order transport)

- We define the two operators for each step :

$$T_{\Delta t}:(I_d+\Delta t\Lambda\partial_x I_d)\boldsymbol{f}^{n+1}=\boldsymbol{f}^n$$

$$R_{\Delta t}:\boldsymbol{f}^{n+1}+\theta\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U})-\boldsymbol{f}^{n+1})=\boldsymbol{f}^n-(1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U})-\boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t}\circ R_{\Delta t}$ is consistent with

$$\partial_t\boldsymbol{U}+\partial_x\boldsymbol{F}(\boldsymbol{U})=\frac{\Delta t}{2}\partial_x(P\Lambda^2\partial_x\boldsymbol{f})+\left(\frac{(2-\omega)\Delta t}{2\omega}\right)\partial_x\left(D(\boldsymbol{U})\partial_x\boldsymbol{U}\right)+O(\Delta t^2)$$

- with $\omega=\frac{\Delta t}{\varepsilon+\theta\Delta t}$ and $D(\boldsymbol{U})=\left(P\Lambda^2\partial_{\boldsymbol{U}}\boldsymbol{f}^{eq}-\mid\partial\boldsymbol{F}(\boldsymbol{U})\mid^2\right)$.

Time discretization

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First order scheme (exact transport)

- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

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Drawback

- For $[D1Q2]^2$ scheme we have a large error: $D(\boldsymbol{U}) = \left(\lambda^2 I_d - \partial \boldsymbol{F}(\boldsymbol{U})^2\right)$

High-Order time schemes

Second-order scheme

- **Order of convergence**: one excepted for $\omega = 2$ and exact transport. In this case: second order.
- **Remark**: same results for Strang splitting. Probably true only for macro variables.
- Classical full second order scheme:

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t, \omega = 2) \circ T\left(\frac{\Delta t}{2}\right).$$

- with T exact or second order time scheme (Crank-Nicolson).
- Since $R(\Delta t = 0) \neq I_d$ We cannot prove convergence for all variables. Second order scheme:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{2}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{4}\right).$$

High order scheme

- Using composition method

$$M_p(\Delta t) = \Psi_{ap}(\gamma_1 \Delta t) \circ \Psi_{ap}(\gamma_2 \Delta t) ..... \circ \Psi_{ap}(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

Burgers: convergence results

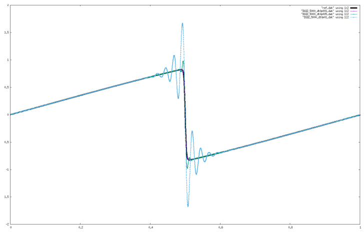
- **Model**: Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 2000 cells, degree 11.
- **Test**: $\rho(t = 0, x) = sin(2\pi x)$. $T_f = 0.14$ (before the shock) and no viscosity.
- Scheme: splitting schemes and Suzuki composition + splitting.

	SPL 1, $\theta = 1$		SPL 1, $\theta = 0.5$		SPL 2, $\theta = 0.5$		Suzuki	
Δt	Error	order	Error	order	Error	order	Error	order
0.005	$2.6E-2$	-	$1.3E-3$	-	$7.6E-4$	-	$4.0E-4$	-
0.0025	$1.4E-2$	0.91	$3.4E-4$	1.90	$1.9E-4$	2.0	$3.3E-5$	3.61
0.00125	$7.1E-3$	0.93	$8.7E-5$	1.96	$4.7E-5$	2.0	$2.4E-6$	3.77
0.000625	$3.7E-3$	0.95	$2.2E-5$	1.99	$1.2E-5$	2.0	$1.6E-7$	3.89

- Scheme: second order splitting scheme.
- Same test after the shock:



1D isothermal Euler : Convergence

- **Model**: isothermal Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2 \rho) = 0 \end{cases}$$

- **Lattice**: $(D1 - Q2)^n$ Lattice scheme.
- For the transport (and relaxations step) we use 6-order DG scheme in space.
- **Time step**: $\Delta t = \beta \frac{\Delta x}{\lambda}$ with λ the lattice velocity. $\beta = 1$ explicit time step.
- **First test**: acoustic wave with $\beta = 50$ and $T_f = 0.4$, **Second test**: smooth contact wave with $\beta = 100$ and $T_f = 20$.

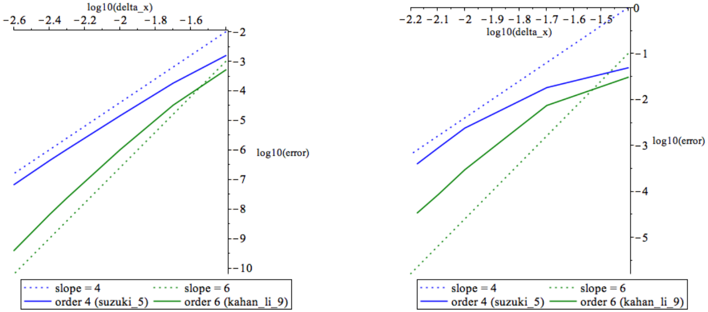
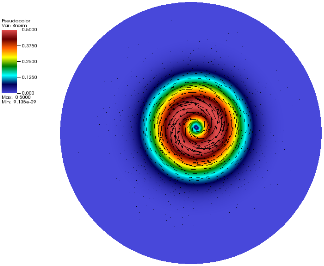


Figure: convergence rates for the first test (left) and for the second test (right).

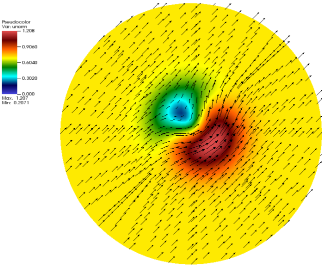
Numerical results: 2D MHD drifting vortex

- **Model** : compressible ideal MHD.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2^{nd} order Implicit DG scheme. 4th order ins space. CFL around 20.
- **Test case** : advection of the vortex (steady state without drift).
- Parameters : $\rho = 1.0$, $p_0 = 1$, $u_0 = b_0 = 0.5$, $\mathbf{u}_{drift} = [1,1]^t$, $h(r) = exp[(1-r^2)/2]$

Magnetic field



Velocity



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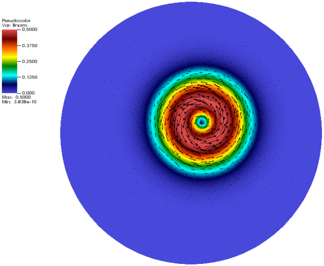

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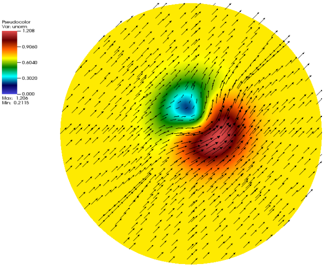

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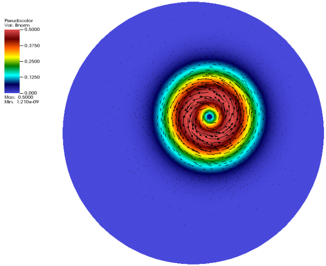
Magnetic field



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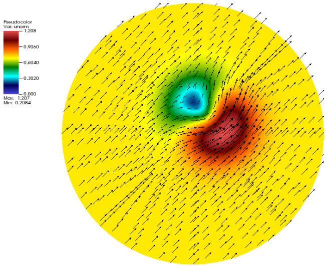

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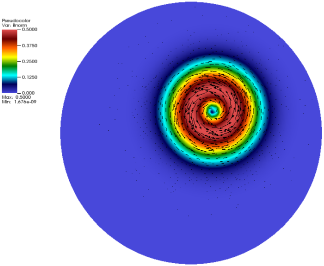

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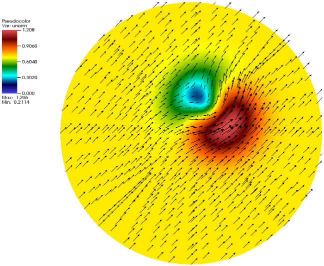

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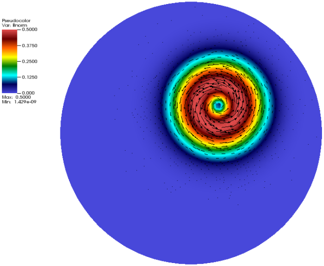

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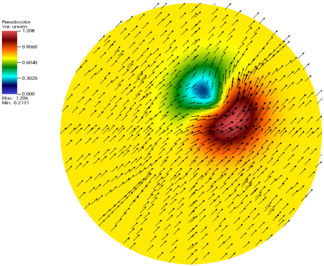

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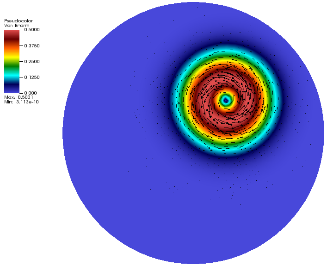
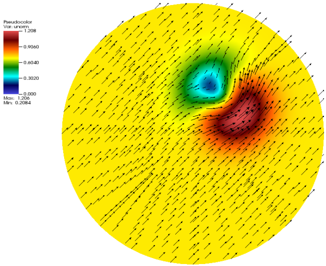


Velocity



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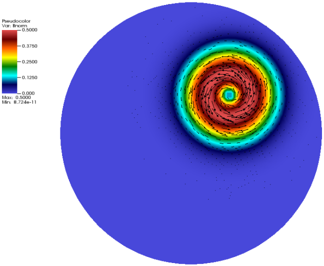

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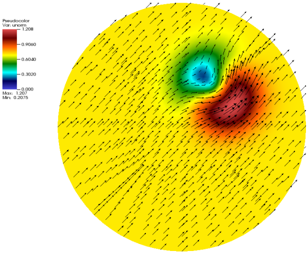

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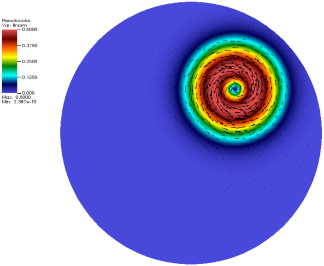

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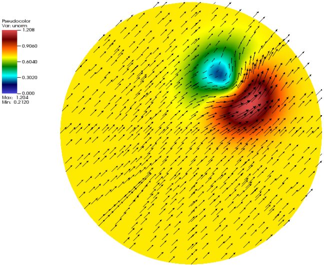

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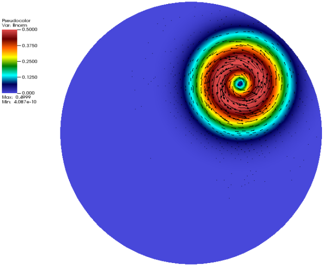

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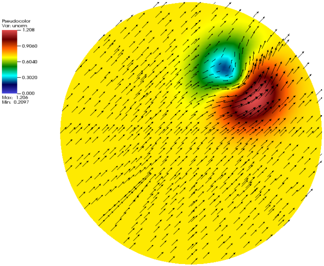

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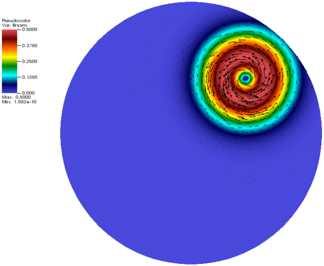

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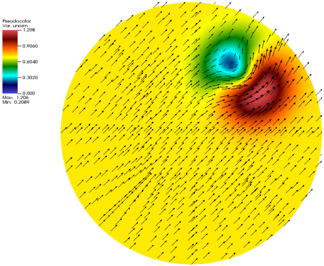

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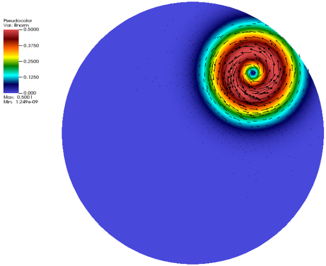

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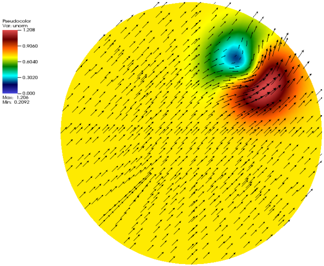

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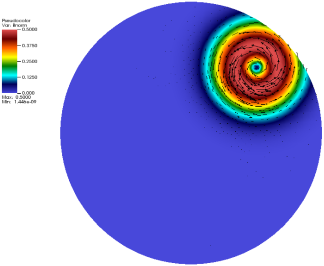

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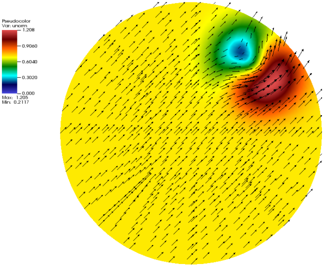

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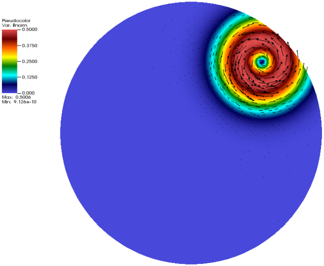

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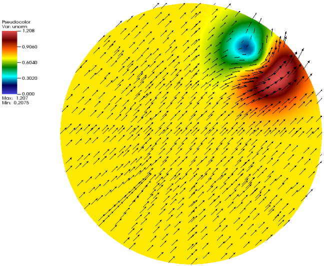

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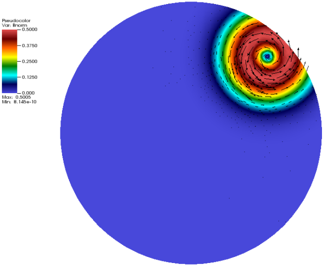

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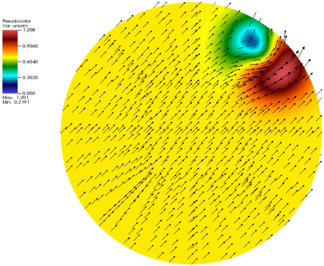

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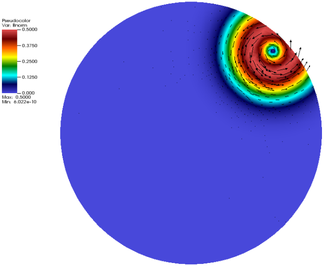

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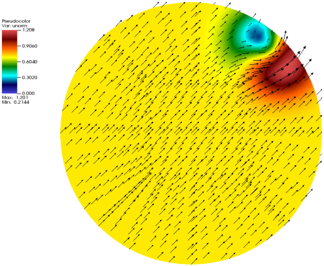

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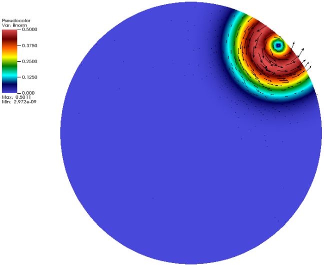

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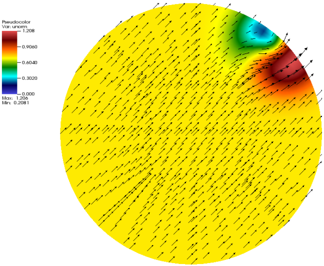

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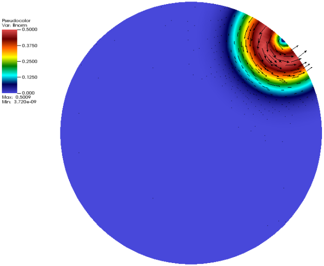

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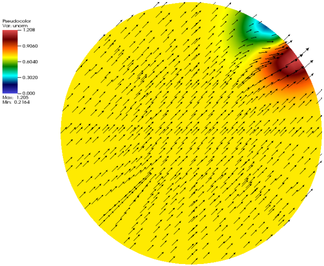

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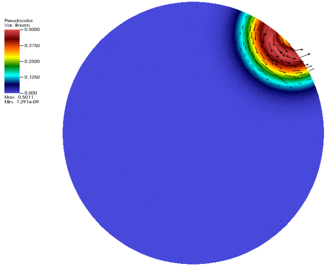

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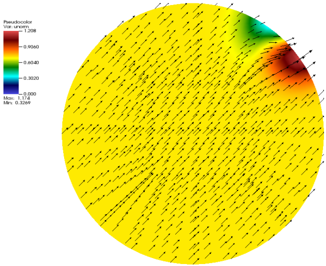

Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
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- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

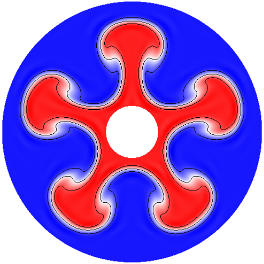

Figure: Plot of the mass fraction of gas

3D case in cylinder

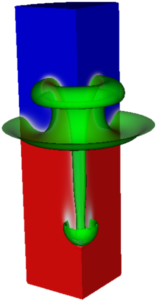

Figure: Plot of the mass fraction of gas

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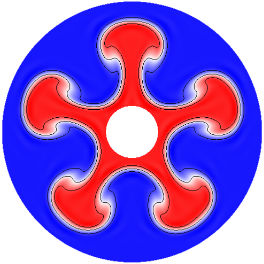


Figure: Plot of the mass fraction of gas

2D cut of the 3D case

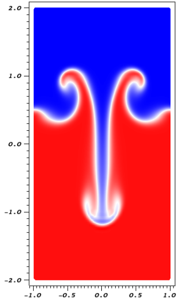


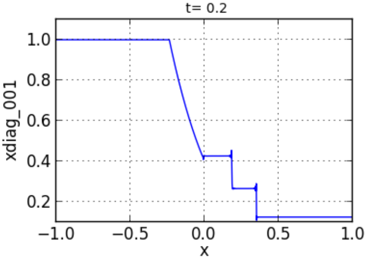
Figure: Plot of the mass fraction of gas

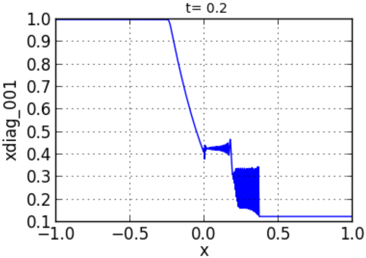
Classical kinetic representation

Limitation

- High-order extension allows to correct the main default of relaxation: large error.
- In two situations the High-order extension is not sufficient:
  - For discontinuous solutions like shocks.
  - For strongly multi-scale problem like low-Mach problem.

- **Euler equation**: Sod problem.
- Second order time scheme + SL scheme:





- Left: density $\Delta t = 1.0^{-4}$. Right: density $\Delta t = 4.0^{-4}$
- **Conclusion**: shock and high order time scheme needs limiting methods.

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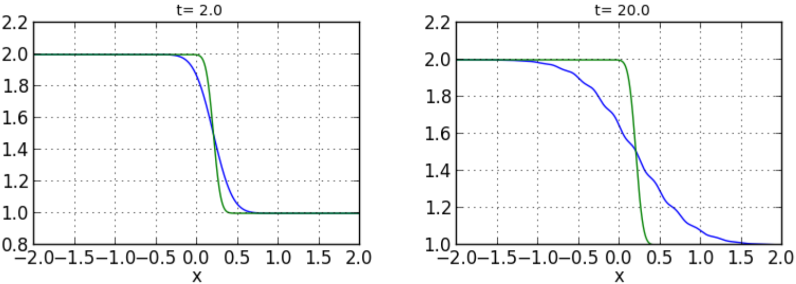
- **Euler equation**: smooth contact (u =cts, p=cts).
- First/Second order time scheme + SL scheme. $T_f = \frac{2}{M}$ and 100 time step.



- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: First order method too much dissipative for low Mach flow (dissipation with acoustic coefficient).

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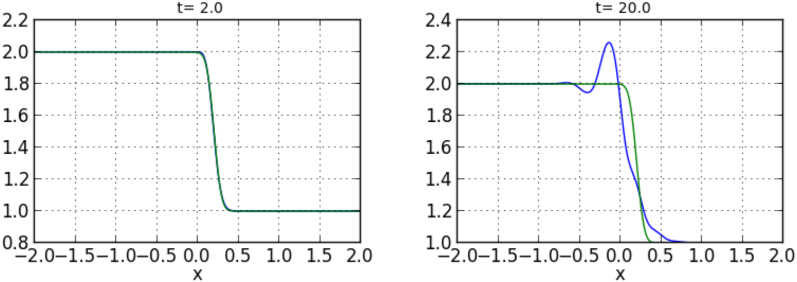


- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: Second order method too much dispersive for low Mach flow (dispersion with acoustic coefficient).

Kinetic representation for multi-scale problems

Classical kinetic representation

"Physic" kinetic representations

- Kinetic model mimics the moment model of Boltzmann equation. Euler isothermal

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2\rho) = 0 \end{cases}$$

- D1Q3 model: three velocities $\{-\lambda, 0, \lambda\}$. Equilibrium: quadrature of Maxwellian.

$$\rho = f_- + f_0 + f_+, \quad q = \rho u = -\lambda * f_- + 0 * f_0 + \lambda * f_+, \quad \boldsymbol{f}_{eq} = \begin{pmatrix} \frac{1}{2}(\rho u(u-\lambda) + c^2\rho) \\ \rho(\lambda^2 - u^2 - c^2) \\ \frac{1}{2}(\rho u(u+\lambda) + c^2\rho) \end{pmatrix}$$

- **Limit model** :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2\rho) = \varepsilon\left(\partial_{xx} u + u^3 \partial_{xx}\rho\right) \end{cases}$$

- **Good point**: no diffusion on ρ equation. **Bad point**: stable only for low mach. No natural extension for more complex pde.

Vectorial kinetic representations

- Vectorial kinetic model (B. Graille 14): $[D1Q2]^2$ one relaxation model $\{-\lambda, \lambda\}$.
- **Good point**: stable on sub-characteristic condition $\lambda > \lambda_{max}$.
- **Bad point**: Wave structure approximated by transport at maximal velocity. The idea of D1Q2 equivalent to Rusanov scheme idea. Very bad accuracy for equilibrium or multi-scale problems (low mach).

Generic vectorial D1Q3

Idea

- Keep the vectorial structure: more stable since we can diffuse on all the variables.
- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- Consistency condition:

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, & \forall k \in \{1..N_c\} \\ \lambda_- f_-^k + \lambda_0 f_0^k + \lambda_+ f_+^k = F^k(\boldsymbol{U}), & \forall k \in \{1..N_c\} \end{cases}$$

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, & \forall k \in \{1..N_c\} \\ (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k = F^k(\boldsymbol{U}) - \lambda_0 f_0^k, & \forall k \in \{1..N_c\} \end{cases}$$

- We assume a decomposition of the flux (Bouchut 03)

$$F^k(\boldsymbol{U}) = F_0^{k,-}(\boldsymbol{U}) + F_0^{k,+}(\boldsymbol{U}) + \lambda_0 I_d$$

- We obtain the following equation for the equilibrium

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, & \forall k \in \{1..N_c\} \\ (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k = F_0^{k,-}(\boldsymbol{U}) + F_0^{k,+}(\boldsymbol{U}), & \forall k \in \{1..N_c\} \end{cases}$$

- By analogy of the kinetic theory and kinetic flux splitting scheme we propose the following decomposition $\sum_{v>0} v f^k = F_0^{k,+}(\boldsymbol{U})$ and $\sum_{v<0} v f^k = F_0^{k,-}(\boldsymbol{U})$.

Generic vectorial D1Q3

Idea

- Keep the vectorial structure: more stable since we can diffuse on all the variables.
- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- The lattice $[D1Q3]^N$ is defined by the velocity set $V = [\lambda_-, \lambda_0, \lambda_+]$ and

$$\begin{cases} \boldsymbol{f}_-^{eq}(\boldsymbol{U}) = -\dfrac{1}{(\lambda_0 - \lambda_-)}\boldsymbol{F}_0^-(\boldsymbol{U}) \\ \boldsymbol{f}_0^{eq}(\boldsymbol{U}) = \left(\boldsymbol{U} - \left(\dfrac{\boldsymbol{F}_0^+(\boldsymbol{U})}{(\lambda_+ - \lambda_0)} - \dfrac{\boldsymbol{F}_0^-(\boldsymbol{U})}{(\lambda_0 - \lambda_-)}\right)\right) \\ \boldsymbol{f}_+^{eq}(\boldsymbol{U}) = \dfrac{1}{(\lambda_+ - \lambda_0)}\boldsymbol{F}_0^+(\boldsymbol{U}) \end{cases}$$

Stability

- Sufficient condition for L^2 stability: $\partial \boldsymbol{F}_0^+$, $-\partial \boldsymbol{F}_0^-$ and $1 - \frac{\partial \boldsymbol{F}_0^+ - \partial \boldsymbol{F}_0^-}{\lambda}$ are positive.
- Condition too restrictive (Gerschgorin disc).
- Condition for entropy stability: $\boldsymbol{F}_0^+$ and $\boldsymbol{F}_0^-$ is an entropy decomposition of the flux + same condition that for L^2.

D1Q3 for scalar case

- First choice: D1Q3 Rusanov ($\lambda_0 = 0$)

$$F_0^-(\rho) = -\lambda_- \frac{(F(\rho) - \lambda_+\rho)}{\lambda_+ - \lambda_-}, \quad F_0^+(\rho) = \lambda_+ \frac{(F(\rho) - \lambda_-\rho)}{\lambda_+ - \lambda_-}$$

- Consistency (for $\lambda_- = -\lambda_+$): $\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left(\lambda^2 - \mid \partial F(\rho) \mid^2\right) \partial_x \rho + O(\Delta t^2)$

- Second choice: D1Q3 Upwind

$$F_0^-(\rho) = \chi_{\{\partial F(\rho) < \lambda_0\}} (F(\rho) - \lambda_0 \rho) \quad F_0^+(\rho) = \chi_{\{\partial F(\rho) > \lambda_0\}} (F(\rho) - \lambda_0 \rho)$$

- with χ the indicatrice function.
- Consistency: $\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left(\lambda \mid \partial F(\rho) \mid - \mid \partial F(\rho) \mid^2\right) \partial_x \rho + O(\Delta t^2)$

- Third choice: D1Q3 Lax-Wendroff ($\lambda_0 = 0$)

$$F_0^-(\rho) = \frac{1}{2}\left(F(\rho) + \frac{\alpha}{\lambda} \int^{\rho} (\partial F(u))^2\right) \quad F_0^+(\rho) = \frac{1}{2}\left(F(\rho) + \frac{\alpha}{\lambda} \int^{\rho} (\partial F(u))^2\right)$$

- with $\lambda_0 = 0$ and $\lambda_- = -\lambda_+$ and $\alpha \geq 1$.
- Consistency: $\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left((\alpha - 1) \mid \partial F(\rho) \mid^2\right) \partial_x \rho + O(\Delta t^2)$.

- The last one is not entropy stable and does not satisfy the sufficient L^2 stability condition.

D1Q3 for Euler equation I

- **Euler equation**. Two regimes where the classical method is not optimal.
  - High-Mach regime: we use a negative and positive transport for purely positive or negative flows.
  - Low-Mach regime: λ is closed to the sound speed so we have viscosity too large for density equation for example.
- First possibility: use classical flux vector splitting for Euler equation.
  - **Stegel-Warming**: $\boldsymbol{F}^{\pm} = A^{\pm}(\boldsymbol{U})\boldsymbol{U}$ with $A^{\pm}$ positive/negative part of the Jacobian.
  - **Van-Leer**:

$$\boldsymbol{F}^{\pm}(\boldsymbol{U}) = \pm\frac{1}{4}\rho c(M\pm 1)^2 \begin{pmatrix} 1 \\ \frac{(\gamma-1)u\pm 2c}{\gamma} \\ \frac{((\gamma-1)u\pm 2c)^2}{2(\gamma+1)(\gamma-1)} \end{pmatrix}$$

  - **AUSM method**: convection of ρ, q and H as Van-Leer and separated reconstruction of the pressure.
  - **Approximate Osher-Solomon**: $\boldsymbol{F}^{\pm}(\boldsymbol{U}) = \boldsymbol{F}(\boldsymbol{U}) \pm \mid \boldsymbol{F}(\boldsymbol{U}) \mid$

$$\mid \boldsymbol{F}(\boldsymbol{U}) \mid \approx \int_{\boldsymbol{U}_0}^{\boldsymbol{U}} \mid A(\boldsymbol{U}) \mid = \int_0^1 \mid A(\boldsymbol{U}_0 + t(\boldsymbol{U}-\boldsymbol{U}_0)) \mid (\boldsymbol{U}-\boldsymbol{U}_0)dt$$

  - Integral is approximated by a quadrature formula along the path (E. Toro , M Dumbser)
  - Approximate of $\mid A \mid$ using Halley approximation (M. J . Castro) and $\boldsymbol{U}_0$ is the average flow.

D1Q3 for Euler equation II

- Low Mach case:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t\rho u+\partial_x\left(\rho u^2+\frac{p}{M}\right)=0\\ \partial_t E+\partial_x(Eu+pu)=0\end{array}\right.$$

- We want to preserve as possible the limit:

$$p=cts,\quad u=cts,\quad \partial_t\rho+u\partial_x\rho=0$$

- **Idea**: Splitting of the flux (E. Toro):

$$\boldsymbol{F}(\boldsymbol{U})=\begin{pmatrix}(\rho)u\\ (\rho u)u+p\\ (\frac{1}{2}\rho u^2)u+\frac{\gamma}{\gamma-1}pu\end{pmatrix}$$

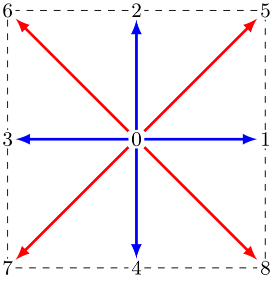
- Idea: Lax-Wendroff Flux splitting for convection and AUSM-type for the pressure term.
- Use only u, p and λ ($\approx c$) to reconstruct pressure. Important to preserve the low mach limit.
- We obtain

$$\boldsymbol{F}^{\pm}(\boldsymbol{U})=\frac{1}{2}\begin{pmatrix}(\rho u\pm\frac{u^2}{\lambda}\rho)+p\\ (\rho u^2\pm\frac{u^2}{\lambda}q)+p(1\pm\gamma\frac{u}{\lambda})\\ (\frac{1}{2}\rho u^2\pm\frac{u^2}{\lambda}\frac{1}{2}\rho u^2)+\frac{\gamma}{\gamma-1}\frac{1}{2\lambda}(p\lambda^2\pm 2\lambda pu+\lambda^2u^2)\end{pmatrix}$$

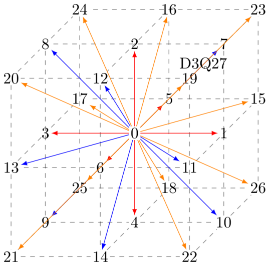
- Preserve contact. Diffusion error for ρ in $O(u^2)$.

Multi-D extension and relative velocity

- Extension of the vectorial scheme in 2D and 3D
- **2D extension**: $D2q(4 * k)$ or $D2Qq(4 * k + 1)$ with $k = 1$ or $k = 2$.
- **3D extension**: $D3q(6 * k)$, $D2Qq(6 * k + 1)$ with $k = 1$, $k = 2$ ore more.



D2Q9



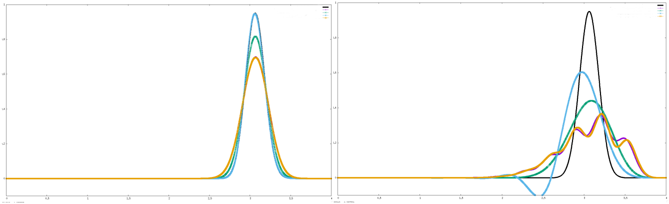
- Increase $k ==>$ increase the isotropic property of the kinetic model.
- The vectorial models with 0 velocity are not currently extended to 2D.

Advection equation

- Equation
$$\partial_t \rho + \partial_x (a(x)\rho) = 0$$
- with $a(x) > 0$ and $\partial_x a(x) > 0$. Dissipative equation.
- **Test case 1**: $a(x) = x$. 10000 cells. Order 17. $\theta = 1$ (first order).

	Rusanov		Upwind		Lax Wendroff	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.05$	$6.4E^{-2}$	-	$2.7E^{-2}$	-	$2.7E^{-2}$	-
$\Delta t = 0.025$	$3.8E^{-2}$	0.75	$1.2E^{-2}$	1.17	$5.7E^{-3}$	2.24
$\Delta t = 0.0125$	$1.9E^{-2}$	1.0	$4.2E^{-3}$	1.5	$5.5E^{-4}$	3.37
$\Delta t = 0.00625$	$7.9E^{-3}$	1.25	$1.3E^{-3}$	1.7	$5.3E^{-5}$	3.38

- **Test case 2**: $a(x) = 1 + 0.01(x - x_0)^2$. 10000 cells. Order 17. Second order time scheme.



- Left $\Delta t = 0.01$. Right $\Delta t = 0.1$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 1$ (blue), Lax-Wendroff $\alpha = 2$ (Yellow).

Advection equation

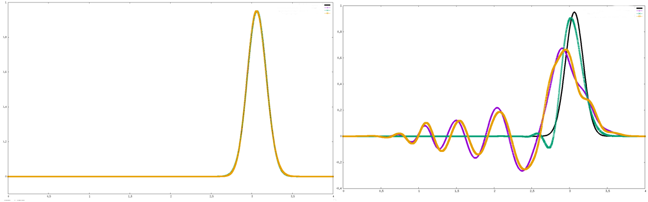
- Equation

$$\partial_t \rho + \partial_x (a(x)\rho) = 0$$

- with $a(x) > 0$ and $\partial_x a(x) > 0$. Dissipative equation.
- **Test case 1**: $a(x) = x$. 10000 cells. Order 17. $\theta = 0.5$ (second order).

	Rusanov		Upwind		Lax Wendroff	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.05$	$3.8E^{-2}$	-	$1.2E^{-4}$	-	$1.2E^{-0}$	-
$\Delta t = 0.025$	$5.3E^{-3}$	2.84	$8.1E^{-6}$	3.8	$4.1E^{-1}$	1.55
$\Delta t = 0.0125$	$3.7E^{-4}$	3.84	$5.3E^{-7}$	3.84	$1.1E^{-4}$	11.5
$\Delta t = 0.00625$	$2.3E^{-5}$	3.88	$3.3E^{-8}$	4	$6.2E^{-6}$	4.15

- **Test case 2**: $a(x) = 1 + 0.01(x - x_0)^2$. 10000 cells. Order 17. Second order time scheme.



- Left $\Delta t = 0.01$. Right $\Delta t = 0.1$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 2$ (Yellow) $= 1$ unstable.

Burgers

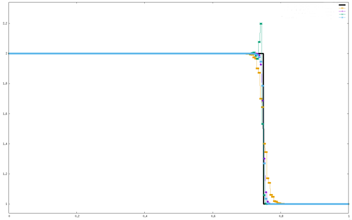
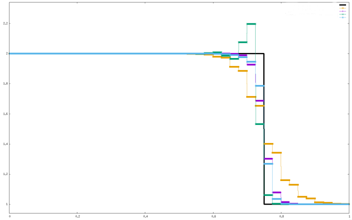
- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2}\right) = 0$$

- **Test case 1**: $\rho(t=0,x) = sin(2\pi x)$. 10000 cells. Order 17. First order time scheme.

	Rusanov		Upwind		Lax Wendroff $\alpha = 1$	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.01$	$3.9E^{-2}$	-	$1.1E^{-2}$	-	$2.3E^{-3}$	-
$\Delta t = 0.005$	$2.1E^{-2}$	0.89	$6.4E^{-3}$	0.78	$6.0E^{-4}$	1.94
$\Delta t = 0.0025$	$1.1E^{-2}$	0.93	$3.5E^{-3}$	0.87	$1.5E^{-4}$	2.00
$\Delta t = 0.00125$	$5.4E^{-3}$	1.03	$1.8E^{-3}$	0.96	$3.9E^{-5}$	1.95

- Shock wave. First order scheme in time.

- Left $\Delta t = 0.002$. Right $\Delta t = 0.01$. Reference (black), Rusanov (yellow), Upwind (violet), Lax-Wendroff (green), Lax-Wendroff $\alpha = 1.5$ (blue).

Burgers

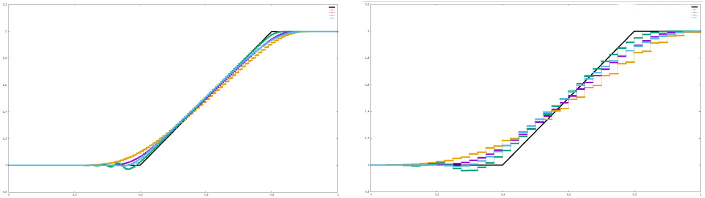
- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- **Test case 1**: $\rho(t=0,x) = sin(2\pi x)$. 10000 cells. Order 17. First order time scheme.

	Rusanov		Upwind		Lax Wendroff $\alpha = 1$	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.01$	$3.9E^{-2}$	-	$1.1E^{-2}$	-	$2.3E^{-3}$	-
$\Delta t = 0.005$	$2.1E^{-2}$	0.89	$6.4E^{-3}$	0.78	$6.0E^{-4}$	1.94
$\Delta t = 0.0025$	$1.1E^{-2}$	0.93	$3.5E^{-3}$	0.87	$1.5E^{-4}$	2.00
$\Delta t = 0.00125$	$5.4E^{-3}$	1.03	$1.8E^{-3}$	0.96	$3.9E^{-5}$	1.95

- Rarefaction wave. First order scheme in time.



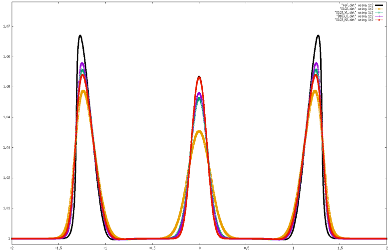
- Left $\Delta t = 0.002$. Right $\Delta t = 0.01$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 1$ (blue), Lax-Wendroff $\alpha = 2$ (Yellow).

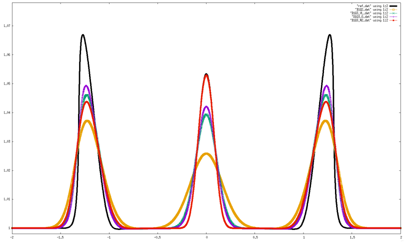
1D Euler equations

- **Model**: Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{cases}$$

- **Test case**: acoustic wave. $\rho = 1 + 0.1e^{-\frac{x^2}{\sigma}}$, $u = 0$ and $p = \rho$.
- The domain is $\Omega = [-2, 2]$. 4000 cells and 11-order SL. $\theta = 1$ (relaxation).





- Left $\Delta t = 0.002$. Right $\Delta t = 0.005$. Reference (black), Rusanov (yellow), Van-Leer (green), Osher (violet), Low-Mach (red).

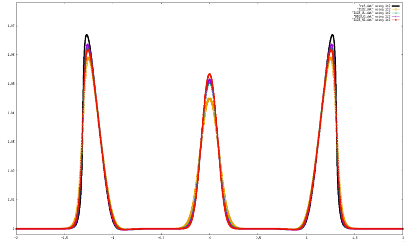
- **Conclusion**: Osher and Van-Leer more accurate that Rusanov. Low-Mach less accurate for acoustic that the two other but very accurate on the material wave.

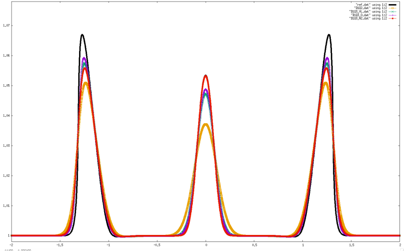
1D Euler equations

- **Model**: Euler equation

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{array}\right.$$

- **Test case**: acoustic wave. $\rho = 1 + 0.1e^{-\frac{x^2}{\sigma}}$, $u = 0$ and $p = \rho$.
- The domain is $\Omega = [-2, 2]$. 4000 cells and 11-order SL. $\theta = 0.666$ (relaxation).





- Left $\Delta t = 0.002$. Right $\Delta t = 0.005$. Reference (black), Rusanov (yellow), Van-Leer (green), Osher (violet), Low-Mach (red).

- **Conclusion**: Osher and Van-Leer more accurate that Rusanov. Low-Mach less accurate for acoustic that the two other but very accurate on the material wave.

1D Euler equations II

- **Test case**: Smooth contact. We take $p = 1$ and u is also constant.
- **Final aim**: take $\Delta t = O(\frac{1}{u})$ when u decrease to have the same error.
- We choose $\Delta t = 0.02$ and $T_f = 2$. 4000 cells. We choose $\omega = 1$:

	Schemes	Rusanov	VL	Osher	LM
$u = 10^{-2}$	$\rho(t,x)$	0.35	$1.2E^{-1}$	$9.9E^{-2}$	$1.5E^{-3}$
	$u(t,x)$	0	$5.3E^{-3}$	$1.1E^{-6}$	0
	$p(t,x)$	0	$3.6E^{-3}$	$6.1E^{-7}$	0
$u = 10^{-4}$	$\rho(t,x)$	0.35	$1.2E^{-1}$	$9.9E^{-2}$	$1.5E^{-5}$
	$u(t,x)$	0	$5.3E^{-3}$	$1.1E^{-6}$	0
	$p(t,x)$	0	$3.6E^{-3}$	$6.1E^{-7}$	0
$u = 0$	$\rho(t,x)$	0.35	$1.2E^{-1}$	$9.9E^{-2}$	0.0
	$u(t,x)$	0	$5.3E^{-3}$	$1.1E^{-6}$	0
	$p(t,x)$	0	$3.6E^{-3}$	$6.1E^{-7}$	0

- **Drawback**: When the time step is too large we have dispersive effect.
- **Possible explanation**: the error would be homogeneous to

$$| \rho^n(x) - \rho(t,x) | \approx \left[O(\Delta t u^2) + O(\Delta t^2 u \lambda^q) \right].$$

- with λ closed to the sound speed.
- **Problem**: At the second order we recover partially the problem since λ is closed to the sound speed.

1D Euler equations III

- **Possible solution**: decrease λ for the density equation.
- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$\underbrace{V=[-\lambda_f,-\lambda_s,0,\lambda_s,\lambda_f]}_{\text{slow scale}}$$

- The convective part at the slow scale. The acoustic part at the fast scale.
- **Smooth contact**: We take 200 time step and $\Delta t=\frac{0.001}{u}$:

	Error	$\mid\lambda_s\mid$	$\mid\lambda_f\mid$
$u=10^{-1}$	$2.5E^{-3}$	2	2
$u=10^{-2}$	$2.5E^{-3}$	0.2	9
$u=10^{-3}$	$2.5E^{-3}$	0.02	90

Conclusion

- **Conclusion**: the error would be homogeneous to

$$\mid \rho^n(x)-\rho(t,x)\mid \approx \left[O(\Delta t u^2)+O(\Delta t^2 u \lambda_s^q)\right].$$

- with λ_s which can be take small.
- **Drawback**: For the stability it seems necessary to have

$$\lambda_s\lambda_f \geq C\max_x(u+c)$$

1D Euler equations III

- **Possible solution**: decrease λ for the density equation.
- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$\underbrace{V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]}_{\text{fast scale}}$$

- The convective part at the slow scale. The acoustic part at the fast scale.
- **Smooth contact**: We take 200 time step and $\Delta t = \frac{0.001}{u}$:

	Error	$\mid \lambda_s \mid$	$\mid \lambda_f \mid$
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Conclusion

- **Conclusion**: the error would be homogeneous to

$$\mid \rho^n(x) - \rho(t,x) \mid \approx \left[O(\Delta t u^2) + O(\Delta t^2 u \lambda_s^q) \right].$$

- with λ_s which can be take small.
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1D Euler equations III

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- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$\underbrace{V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]}_{\text{coupling}}$$

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Conclusion

- **Conclusion**: the error would be homogeneous to

$$\mid \rho^n(x) - \rho(t,x) \mid \approx \left[O(\Delta t u^2) + O(\Delta t^2 u \lambda_s^q) \right].$$

- with λ_s which can be take small.
- **Drawback**: For the stability it seems necessary to have

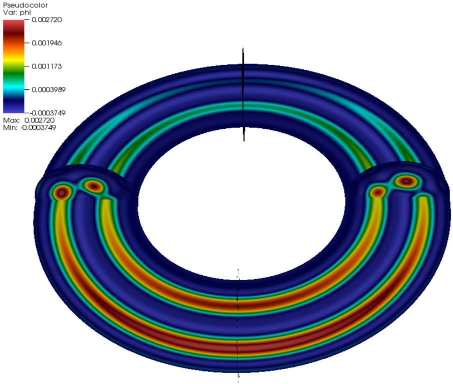
$$\lambda_s \lambda_f \geq C \max_x (u + c)$$

Kinetic relaxation method for Diffusion problem

Main parabolic problem

- Coupling anisotropic diffusion + resistivity.

$$\left\{\begin{array}{l}\partial_t T - \nabla\cdot((\mathbf{B}\otimes\mathbf{B})\nabla T + \varepsilon\nabla T) = 0\\ \partial_t \mathbf{B} - \eta\nabla\times(T^{-\frac{5}{2}}\nabla\times\mathbf{B}) = 0\\ \nabla\cdot\mathbf{B} = 0\end{array}\right.$$



- The temperature T for the case $\eta = 0$ and $\mathbf{B}$ given by magnetic equilibrium.

Kinetic model and scheme for diffusion I

- We want solve the equation: $\partial_t \rho + \partial_x (u\rho) = D\partial_{xx}\rho$
- D1Q2 Kinetic system proposed (S. Jin, F. Bouchut):

$$\begin{cases} \partial_t f_- - \dfrac{\lambda}{\varepsilon}\partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^- - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon}\partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+ - f_+) \end{cases}$$

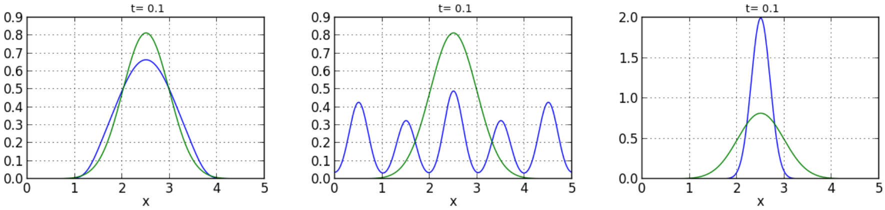
- with $f_{eq}^{\pm} = \frac{\rho}{2} \pm \frac{\varepsilon(u\rho)}{2\lambda}$. The limit is given by:

$$\partial_t \rho + \partial_x(u\rho) = \partial_x((\lambda^2 - \varepsilon^2 \mid u \mid^2)\partial_x \rho) + \lambda^2\varepsilon^2\partial_x(\partial_{xx}(u\rho) + u\partial_{xx}\rho) - \lambda^2\varepsilon^2\partial_{xxxx}\rho$$

- We introduce $\alpha >\mid u \mid$. Choosing $D = \lambda^2 - \varepsilon^2\alpha^2$ we obtain

$$\partial_t \rho + \partial_x(u\rho) = \partial_x(D\partial_x \rho) + O(\varepsilon^2)$$

- Results ($\Delta t >> \Delta_{exp}$) (Order 1. Left: $\frac{\Delta t}{\varepsilon} = 0.1$, Middle: $\frac{\Delta t}{\varepsilon} = 1$, Right: $\frac{\Delta t}{\varepsilon} = 10$):



Kinetic model and scheme for diffusion I

- We want solve the equation: $\partial_t \rho + \partial_x (u\rho) = D\partial_{xx}\rho$
- D1Q2 Kinetic system proposed (S. Jin, F. Bouchut):

$$\begin{cases} \partial_t f_- - \dfrac{\lambda}{\varepsilon}\partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^- - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon}\partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+ - f_+) \end{cases}$$

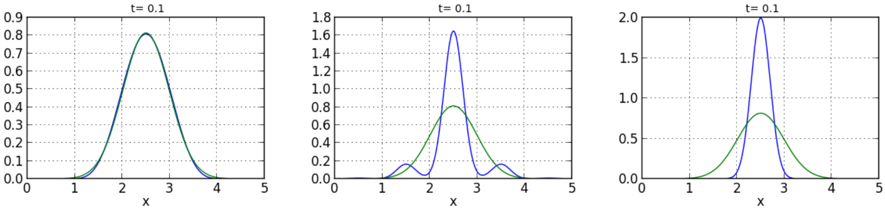
- with $f_{eq}^{\pm} = \frac{\rho}{2} \pm \frac{\varepsilon(u\rho)}{2\lambda}$. The limit is given by:

$$\partial_t \rho + \partial_x (u\rho) = \partial_x((\lambda^2 - \varepsilon^2 \mid u \mid^2)\partial_x \rho) + \lambda^2\varepsilon^2\partial_x(\partial_{xx}(u\rho) + u\partial_{xx}\rho) - \lambda^2\varepsilon^2\partial_{xxxx}\rho$$

- We introduce $\alpha >\mid u \mid$. Choosing $D = \lambda^2 - \varepsilon^2\alpha^2$ we obtain

$$\partial_t \rho + \partial_x (u\rho) = \partial_x(D\partial_x \rho) + O(\varepsilon^2)$$

- Results (Order 2. Left: $\frac{\Delta t}{\varepsilon} = 0.1$, Middle: $\frac{\Delta t}{\varepsilon} = 1$, Right: $\frac{\Delta t}{\varepsilon} = 10$):



Kinetic model and scheme for diffusion I

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$$\partial_t \rho + \partial_x(u\rho) = \partial_x(D\partial_x \rho) + O(\varepsilon^2)$$

- We can choose $\varepsilon = \Delta t^{\gamma}$ and $\omega = 2$.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
	Error	order	Error	order	Error	order
$\Delta t = 0.1$	$6.4E-2$	-	0.28	-	0.47	-
$\Delta t = 0.02$	$3.9E-3$	1.74	0.27	0	0.48	0
$\Delta t = 0.01$	$4.5E-4$	3.1	0.27	0	0.48	0
$\Delta t = 0.005$	$8.7E-5$	2.37	0.27	0	0.48	0

- The splitting scheme is not AP.

Kinetic model and scheme for diffusion II

Consistency analysis

- We consider $\partial_t \rho = D\partial_{xx}$.
- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \frac{\Lambda}{\varepsilon}\partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \rho = \Delta t \partial_x \left(\left(\frac{1-\omega}{\omega} + \frac{1}{2} \right) \frac{\lambda^2}{\varepsilon^2} \partial_x \rho \right) + O(\Delta t^2)$$

- Taking $D = \lambda^2$, $\theta = 0.5$ and $\varepsilon = \sqrt{\Delta t}$ we obtain the diffusion equation.
- **Question**: what is the error term is this case ?
- **First results** (for these choices of parameters):
  - Second order at the numerical level.
  - At the minimum the first order theoretically.
- **Question**: what is the error term ? Can we optimize the constant of convergence ?

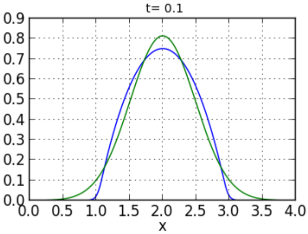
Kinetic model and scheme for diffusion II

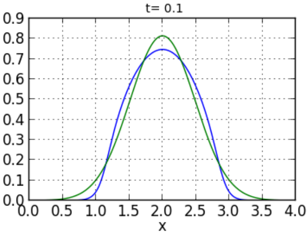
- We want solve the equation: $\partial_t \rho + \partial_x F(\rho) = \partial_{xx} D(\rho)$
- D1Q4 Kinetic system proposed (R. Natalini, F. Bouchut):

$$\partial_t \boldsymbol{f} + \Gamma \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}_{eq} - \boldsymbol{f})$$

with $\Gamma = \Lambda + \frac{1}{\sqrt{\varepsilon}}\Theta$. Consistency verified: if $\Theta^2 \boldsymbol{f}^{eq} = D(\rho)$ and $\Gamma \boldsymbol{f}^{eq} = F(\rho)$.

- Results: We choose $F(\rho) = 0$ and $D(\rho) = \left(\frac{\rho^p}{p}\right)$ with 2000 cells, order 11.
- $p = 1$ (green) $p = 2$ (violet). Left $\Delta t = 0.001$. Right $\Delta t = 0.005$





- The second kinetic scheme allows to treat also nonlinear diffusion.
- **Future work**: consistency for nonlinear case and stability for different schemes.
- Extension for other diffusion models: $\nabla(\nabla \cdot I_d)$ or $\nabla \cdot (A \nabla I_d)$.

Kinetic model and scheme for diffusion II

- We want solve the equation: $\partial_t \rho + \partial_x F(\rho) = \partial_{xx} D(\rho)$
- D1Q4 Kinetic system proposed (R. Natalini, F. Bouchut):

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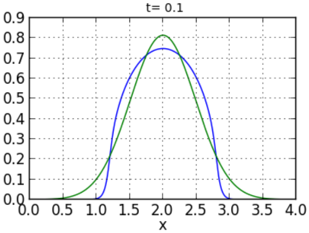
- Results: We choose $F(\rho) = 0$ and $D(\rho) = \left(\frac{\rho^p}{p}\right)$ with 2000 cells, order 11.
- $p = 3$. Left $\Delta t = 0.001$. Right $\Delta t = 0.005$



- The second kinetic scheme allows to treat also nonlinear diffusion.
- **Future work**: consistency for nonlinear case and stability for different schemes.
- Extension for other diffusion models: $\nabla(\nabla \cdot I_d)$ or $\nabla \cdot (A\nabla I_d)$.

Conclusion

Application with too restrictive CFL

- **Multi-scale models**: different physical speeds. Low-Mach Euler low β MHD.
- **Diffusion + other**: CFL is given by the diffusion. Fine grids given by another problem.
- **Varying parameters**: waves/diffusion with strongly varying coefficient. Acoustic, Maxwell, elasticity, neutronic.
- **Meshes with local refinement**: strong CFL in some area. Sismology for example.

Main idea

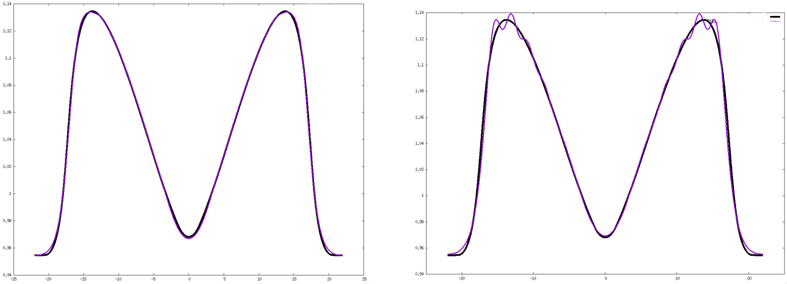
- Target: Nonlinear problem N.
- **First**: we construct the kinetic problem K_ε such that $\| K_\varepsilon - N \| \leq C_\varepsilon \varepsilon$
- **Second**: we discretize K_ε such that $\| K_\varepsilon - K_\varepsilon^{h,\Delta t} \| \leq C_{\Delta t} \Delta t^p + C_h h^q$
- We obtain a consistent method by triangular inequality.

Advantages

- **Initial problem**: invert a nonlinear conservation law is very difficult. High CPU cost (storage and assembly of problem. Slow convergence of iterative solvers).
- **Advantages**: no matrices storage and inversion. High parallelism/optimization.
- **Drawbacks**: large error in some cases. Complex for boundary condition.
- **Future**: 2D/3D NS and MHD, BC, convergence/stability, kinetic models in plasma.

Conclusion II

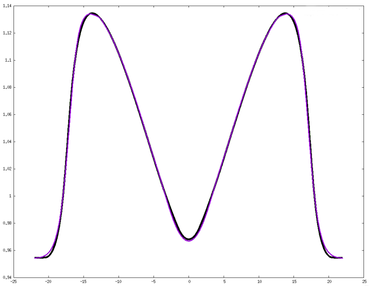
- **Test**: low-mach case. 8800 cells $h = 0.005$, Degree of polynomial: 3.
- $\Delta t = 0.04$: CFL FV ≈ 100, CFL HO ≈ 300.
- Comparison: implicit Crank-Nicolson and D1Q2 implicit.

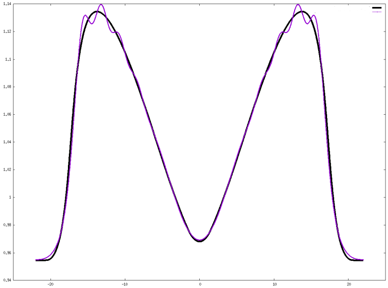


- Left: scheme (1). Right: scheme (2), Black: reference solution.

Conclusion II

- **Test**: low-mach case. 8800 cells $h = 0.005$, Degree of polynomial: 3.
- $\Delta t = 0.04$: CFL FV $\approx$ 100, CFL HO $\approx$ 300.
- Comparison: implicit Crank-Nicolson and D1Q2 SL.

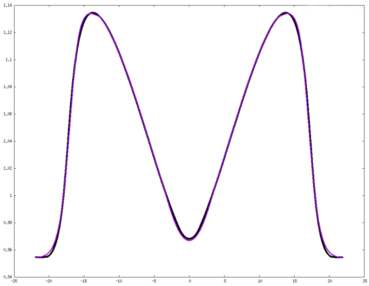


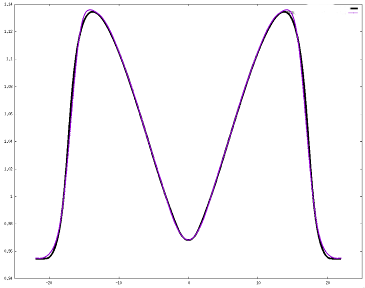


- Left: scheme (1). Right: scheme (3), Black: reference solution.

Conclusion II

- **Test**: low-mach case. 8800 cells $h = 0.005$, Degree of polynomial: 3.
- $\Delta t = 0.04$: CFL FV ≈ 100, CFL HO ≈ 300.
- Comparison: implicit Crank-Nicolson and D1Q3 implicit.

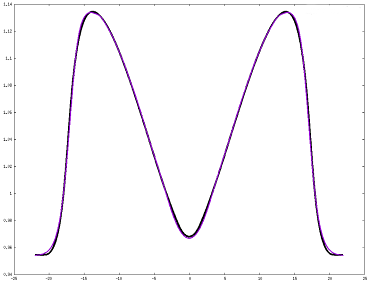


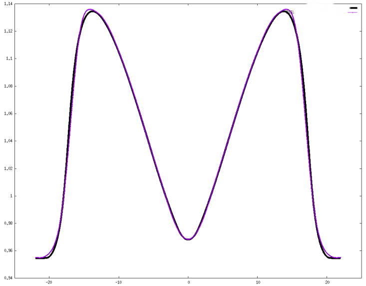


- Left: scheme (1). Right: scheme (4), Black: reference solution.

Conclusion II

- **Test**: low-mach case. 8800 cells $h = 0.005$, Degree of polynomial: 3.
- $\Delta t = 0.04$: CFL FV $\approx$ 100, CFL HO $\approx$ 300.





- Left: scheme (1). Right: scheme (4), Black: reference solution.

Conclusion

- Conclusion: as expected D1Q3 (Van-Leer) SL closed to the CN implicit scheme.
- CPU time difficult to compare since the code are different.
- But: 170 sec for (1), 110 sec for (2), 1.6 sec for (3), 1.7 sec for (4)