Numerical methods for stiff hyperbolic systems

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Outline

Mathematical context

Stiff hyperbolic systems

Problem

- We consider the general stiff problem:

$$\partial_t \boldsymbol{U} + \frac{1}{\varepsilon^a}\partial_x \boldsymbol{F}(\boldsymbol{U}) + \frac{1}{\varepsilon^b}\partial_x \boldsymbol{G}(\boldsymbol{U}) = \frac{1}{\varepsilon^c}\boldsymbol{R}(\boldsymbol{U}) - \frac{\sigma}{\varepsilon^d}\boldsymbol{D}(\boldsymbol{U})$$

Limit

- First case: $a = b = c = 1$ and $\sigma = 0$. long time limit:

$$\partial_x \boldsymbol{F}(\boldsymbol{U}) + \partial_x \boldsymbol{G}(\boldsymbol{U}) = \boldsymbol{R}(\boldsymbol{U})$$

- Second case: $a = b = 0$, $c = 1$ and $\sigma = 0$. relaxation limit:

$$\partial_t \boldsymbol{V} + \partial_x \boldsymbol{K}_1(\boldsymbol{V}) = 0$$

- Third case: $a = b = c = 1$, $d = 2$ $\sigma = 1$. diffusion limit:

$$\partial_t \boldsymbol{V} + \partial_x \boldsymbol{K}_1(\boldsymbol{V}) - \partial_x(\boldsymbol{K}_2(\boldsymbol{V})\partial_x \boldsymbol{V}) = 0$$

- 4th: $a = c = 0$, $b = 1$ and $\sigma = 0$. fast wave limit:

$$\partial_t \boldsymbol{U} + \partial_x \tilde{\boldsymbol{G}}(\boldsymbol{U}) = 0, \quad \partial_x \tilde{\boldsymbol{F}}(\boldsymbol{U}) = 0$$

Diffusion limit: damped wave equation

Damped wave equation

$$\left\{\begin{array}{l}\partial_t p + \frac{1}{\varepsilon}\partial_x u = 0 \\ \partial_t u + \frac{1}{\varepsilon}\partial_x p = -\frac{\sigma}{\varepsilon^2} u\end{array}\right. , \quad \longrightarrow \partial_t p - \partial_x\left(\frac{1}{\sigma}\partial_x p\right) = 0$$

- Ref: Jin-Levermore 96, Gosse-Toscani 01.
- We plug $u = -\frac{\varepsilon}{\sigma}\partial_x p + O(\varepsilon^2)$ in first equation.

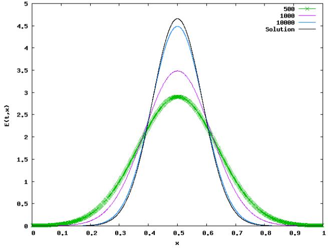
Godunov scheme

$$\left\{\begin{array}{l}\frac{p_j^{n+1} - p_j}{\Delta t} + \frac{1}{\varepsilon}\frac{u_{j+1} - u_{j-1}}{\Delta x} - \frac{\Delta x}{2\varepsilon}\frac{p_{j+1} - 2p_j + p_{j-1}}{\Delta x^2} = 0 \\ \frac{u_j^{n+1} - u_j}{\Delta t} + \frac{1}{\varepsilon}\frac{p_{j+1} - p_{j-1}}{\Delta x} - \frac{\Delta x}{2\varepsilon}\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = -\frac{\sigma}{\varepsilon^2} u_j\end{array}\right.$$

- **Limit scheme**:

$$\frac{p_j^{n+1} - p_j}{\Delta t} - \left(\frac{1}{\sigma} + \frac{\Delta x}{2\varepsilon}\right)\frac{p_{j+1} - 2p_j + p_{j-1}}{\Delta x^2} = O(\varepsilon)$$

- **CFL condition**: $\Delta t \leq f(\varepsilon) h$



- Diffusion and numerical solutions for $\varepsilon = 0.001$.

Long time limit: Euler gravity

Euler gravity

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \frac{1}{\varepsilon}\partial_x(\rho u^2) + \frac{1}{\varepsilon}\partial_x p = -\frac{1}{\varepsilon}\rho\partial_x\phi \\ \partial_t E + \frac{1}{\varepsilon}\partial_x(Eu + pu) = -\frac{1}{\varepsilon}\rho u \partial_x \phi \end{cases}$$

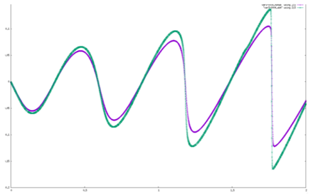
- Class of steady solutions: for $u = 0$ and $\partial_x p = -\rho\partial_x\phi$ the system does not move.
- C. Berthon, C. Klingenberg (and al) 15-16-17.

Rusanov scheme

- **Example**: $\rho = e^{-x\partial_x\phi}$, $p = e^{-x\partial_x\phi}$ and $\phi = gx$.

$$\begin{cases} \rho^{n+1} = \rho^n + \frac{\Delta x}{\lambda}\partial_{xx}\rho + O(\Delta x^2) \\ (\rho u)^{n+1} = (\rho u)^n + \frac{\Delta x}{\lambda}\partial_{xx}(\rho u) + O(\Delta x^2) \\ E^{n+1} = E^n + \frac{\Delta x}{\lambda}\partial_{xx}E + O(\Delta x^2) \end{cases}$$

- with $\lambda > max_x(\mid u \mid + c)$ with c the sound speed.
- **Conclusion**: the equilibriums are not preserved.



- Pertubated equilibrium.

Relaxation limit: HRM model

HRM model

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho Y + \partial_x(\rho Y u) = \frac{1}{\varepsilon}(\rho Y^{eq}(\rho) - \rho Y) \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \end{cases}$$

- with Y the mass fraction and $p = p(\rho, Y)$ (Ambrosso 09 etc).
- **Relaxation limit**: the mass fraction is close to given equilibrium.

Splitting scheme

- Only write for the mass fraction part

and
$$(\rho Y)^* = (\rho Y)^n + \frac{\Delta t}{\varepsilon}(\rho^n Y^{eq}(\rho^n) - \rho^n Y^n)$$

$$\frac{(\rho Y)^{n+1} - (\rho Y)^*}{\Delta t} + \frac{(\rho Y u)^*_{j+1} - (\rho Y u)^*_{j-1}}{\Delta x} - \lambda \frac{(\rho Y)^*_{j+1} - 2(\rho Y)^*_j + (\rho Y)^*_{j-1}}{\Delta x} = 0$$

- **Stability** we must take $\Delta t < C\varepsilon \Delta x$.

Fast wave limit: Low-Mach Euler equation

Euler low-mach

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \frac{1}{M}\partial_x p = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{cases}$$

- S. Dellacherie, C. Chalons, C. Klingenberg (and al) 14-15-17.
- **Limit for M small**: $u = cts + O(M)$, $p = cts + O(M)$ and $\partial_t \rho + u\partial_x \rho = O(M)$.

Rusanov scheme

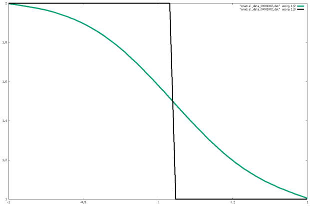
- At the limit: density advection. Advection scheme:

$$\partial_t \rho_j + \frac{(\rho u)_{j+1} - (\rho u)_{j-1}}{\Delta x} - \mid u \mid \frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{\Delta x} = 0$$

- Limit scheme of Rusanov scheme for Euler:

$$\partial_t \rho_j + \frac{(\rho u)_{j+1} - (\rho u)_{j-1}}{\Delta x} - \frac{\lambda}{M} \frac{\rho_{j+1} - 2\rho_j + \rho_{j-1}}{\Delta x} = 0$$

- The scheme for Euler dissipate too much.
- **Stability**: $\Delta t \leq CM\Delta x$.
- CFL constrains by "fast velocity / small amplitude" acoustic waves. Filter in time/space these waves.



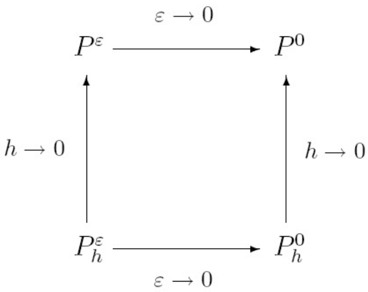
- Contact with $u = 0.01$. $T_f = 10$.
- Black curve: exact sol.
- Green curve: numerical sol with 100 cells.

Important notion: AP and Well-Balanced schemes

- We consider PDE depending of a small parameter ε with an asymptotic limit.

Asymptotic preserving scheme

- AP scheme: a consistent scheme for the initial PDE which gives at the limit a consistent scheme of the limit PDE.
- Uniform AP scheme: convergence and stability independent of ε.



- Application: simulate problem with varying physical parameter and regime. Example: radiative transfer.
- Other application: use AP scheme to create a new scheme for the limit model. Example: relaxation scheme for Euler equation.

Well Balanced scheme

- A scheme which preserve exact (or with high accuracy ?) a steady state of the continuous PDE.

AP/WB schemes for hyperbolic PDE with source terms

Damped wave equation: Godunov scheme

Damped wave equation:

$$\left\{\begin{array}{l}\partial_t p + \frac{1}{\varepsilon}\partial_x u = 0\\ \partial_t u + \frac{1}{\varepsilon}\partial_x p = -\frac{\sigma}{\varepsilon^2}u\end{array}\right.$$

- **Riemann Invariant**: $u + p$ (eigenvalue 1) and $u - p$ (eigenvalue -1) .
- Important relation to obtain the limit: $\partial_x p = -\frac{\sigma}{\varepsilon}u$.

- Upwind scheme for $\partial_t u + \partial_x(au) = 0$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}}{\Delta x_j} = 0$$

with $x_j =\mid x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \mid$ and $u_{j+\frac{1}{2}} = u_j^n$ for $a > 0$ and $u_{j+\frac{1}{2}} = u_{j+1}^n$ for $a < 0$.
- Godunov acoustic scheme: Upwind scheme on the Riemann invariant. We obtain

$$\left\{\begin{array}{l}\frac{p_j^{n+1} - p_j^n}{\Delta t} + \frac{u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}}{\varepsilon\Delta x_j} = 0\\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{p^n_{j+\frac{1}{2}} - p^n_{j-\frac{1}{2}}}{\varepsilon\Delta x_j} = 0,\end{array}\right. \qquad \left\{\begin{array}{l}u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} = u_j^n + p_j^n\\ u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} = u_{j+1}^n - p_{j+1}^n.\end{array}\right.$$

- **Main drawback**: the fluxes ignore the balance between the pressure gradient and the source.

Damped wave equation: Jin-Levermore AP scheme

Jin-Levermore scheme:

- Plug the balance law $\partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2)$ in the fluxes (Jin-Levermore 96).
- Scheme write on irregular grids.

- We write

$$p(x_j) = p(x_{j+\frac{1}{2}}) + (x_j - x_{j+\frac{1}{2}})\partial_x p(x_{j+\frac{1}{2}})$$

- Coupling the previous relation (and the same for x_{j+1}) with the fluxes

$$\begin{cases} u_j + p_j = u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}\Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}, \\ u_{j+1} - p_{j+1} = u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}}\Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}. \end{cases}$$

Jin-Levermore scheme:

$$\begin{cases} \frac{p_j^{n+1} - p_j^n}{\Delta t} + \frac{M_{j+\frac{1}{2}} u^n_{j+\frac{1}{2}} - M_{j-\frac{1}{2}} u^n_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} \\ \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{p^n_{j+\frac{1}{2}} - p^n_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \frac{\sigma}{\varepsilon^2} u_j^n = 0, \end{cases}, \quad \begin{cases} u_{j+\frac{1}{2}} = \frac{u_j + u_{j+1}}{2} + \frac{p_j - p_{j+1}}{2} \\ p_{j+\frac{1}{2}} = \frac{p_j + p_{j+1}}{2} + \frac{u_j - u_{j+1}}{2} \end{cases}$$

with $\Delta x_{j+\frac{1}{2}} = \mid x_{j+1} - x_j \mid$ and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}$.

Damped wave equation: Jin-Levermore AP scheme

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- Plug the balance law $\partial_x p = -\frac{\sigma}{\varepsilon} u + O(\varepsilon^2)$ in the fluxes (Jin-Levermore 96).
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- Scheme write on irregular grids.

- We write

$$p(x_j) = p(x_{j+\frac{1}{2}}) - \frac{\Delta x_j}{2} \frac{\sigma}{\varepsilon} u(x_{j+\frac{1}{2}})$$

- Coupling the previous relation (and the same for x_{j+1}) with the fluxes

$$\begin{cases} u_j + p_j = u_{j+\frac{1}{2}} + p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_j}{2\varepsilon} u_{j+\frac{1}{2}}, \\ u_{j+1} - p_{j+1} = u_{j+\frac{1}{2}} - p_{j+\frac{1}{2}} + \frac{\sigma_{j+\frac{1}{2}} \Delta x_{j+1}}{2\varepsilon} u_{j+\frac{1}{2}}. \end{cases}$$

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with $\Delta x_{j+\frac{1}{2}} = \mid x_{j+1} - x_j \mid$ and $M_{j+\frac{1}{2}} = \frac{2\varepsilon}{2\varepsilon + \sigma_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}}}$.

Gosse-Toscani scheme

- **Other scheme**: Gosse - Toscani scheme.
- **Derivation of the scheme**: Localization of the source on the interface and the Riemann problem associated.
- **Other solution**: we use the following source term $\frac{1}{2}(u_{j+\frac{1}{2}}+u_{j-\frac{1}{2}})$ with the Jin-Levermore scheme.

Gosse-Toscani scheme:

$$\begin{cases} \frac{p_j^{n+1}-p_j^n}{\Delta t}+\frac{M_{j+\frac{1}{2}}u_{j+\frac{1}{2}}-M_{j-\frac{1}{2}}u_{j-\frac{1}{2}}}{\varepsilon\Delta x_j} \\ \frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{M_{j+\frac{1}{2}}p_{j+\frac{1}{2}}-M_{j-\frac{1}{2}}p_{j-\frac{1}{2}}}{\varepsilon\Delta x_j}-\frac{M_{j+\frac{1}{2}}-M_{j-\frac{1}{2}}}{\Delta x_j\varepsilon}p_j^n+\left(\frac{\sigma_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}{2\varepsilon^2\Delta x_j}+\frac{\sigma_{j-\frac{1}{2}}\Delta x_{j-\frac{1}{2}}}{2\varepsilon^2\Delta x_j}\right)u_j^n=0 \end{cases}$$

with

$$u_{j+\frac{1}{2}}=\frac{u_j^n+u_{j+1}^n}{2}+\frac{p_j^n-p_{j+1}^n}{2},\qquad p_{j+\frac{1}{2}}=\frac{p_j^n+p_{j+1}^n}{2}+\frac{u_j^n-u_{j+1}^n}{2}$$

and $M_{j+\frac{1}{2}}=\frac{2\varepsilon}{2\varepsilon+\sigma_{j+\frac{1}{2}}\Delta x_{j+\frac{1}{2}}}$.

Analysis of the Godunov scheme

- **Consistency error**:
  - First equation: $\left(\frac{\Delta x}{\varepsilon}+\Delta t\right)$. Second equation: $\left(\frac{\Delta x^2}{\varepsilon}+\Delta t\right)$
- **Time discretization**:
  - Explicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon+\varepsilon^2}\right)\leq 1$. Semi-implicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon}\right)\leq 1$.

Analysis of the Jin-Levermore scheme

- **Consistency error**:
  - First equation: $(\Delta x+\Delta t)$. Second equation: $\left(\frac{\Delta x^2}{\varepsilon}+\Delta t\right)$
- **Time discretization**:
  - Explicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon+\varepsilon^2}\right)\leq 1$. Semi-implicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon}\right)\leq 1$.

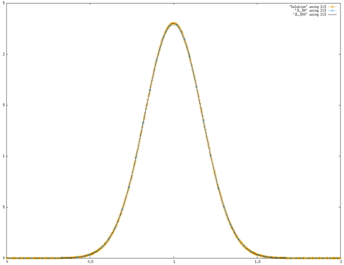
Analysis of the Gosse-Toscani scheme

- **Consistency error**:
  - First and second equation: $(\Delta x+\Delta t)$.
- **Time discretization**:
  - Explicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon}\right)\leq 1$. Semi-implicit CFL: $\Delta t\left(\frac{1}{\Delta x\varepsilon+\Delta x^2}\right)\leq 1$.

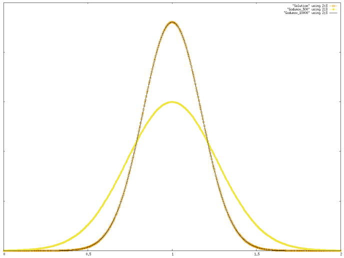
Numerical example

- **Validation test for the AP scheme**: the data are $p(0,x) = G(x)$ with $G(x)$ a Gaussian $u(0,x) = 0$ and $\sigma = 1$, $\varepsilon = 0.001$.

Jin-Levermore scheme



Godunov scheme



Scheme	L^2 error	CPU time
Godunov, 10000 cells	0.0376	505 sec
Godunov, 500 cells	0.42	5.31 sec
AP-JL, 500 cells	4.3E-3	5.42 sec
AP-JL, 50 cells	0.012	0.46 sec
AP-GT, 500 cells	1.3E-4	2.38 sec
AP-GT, 50 cells	0.012	0.013 sec

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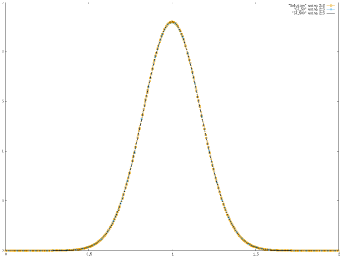
Gosse-Toscani scheme



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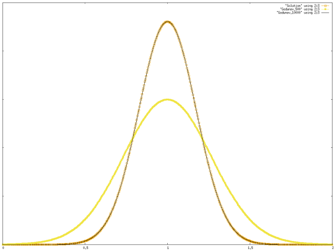

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Test for Well-Balanced property

- We propose to study also the Well-Balanced property for the family of steady state:

$$\begin{cases} u(t,x) = C_1 \\ p(t,x) = -\frac{\sigma}{\varepsilon} C_1 x + C_2 \end{cases}$$

- This steady-state generate also the affine steady state of the limit equation.
- For this, we initialize the different schemes with a steady state and simulate with a large final time (T_f=20).

- Results for different scheme and meshes.

Scheme/mesh	Uniform Mesh	Random Mesh
Godunov, 100 cells	0.0	2.83E-3
Godunov, 1000 cells	5.0E-17	2.7E-4
AP-JL, 100 cells	0.0	3.3E-3
AP-JL, 1000 cells	6.3E-17	3.9E-4
AP-GT, 100 cells	3.1E-16	3.1E-16
AP-GT, 1000 cells	3.0E-16	2.8E-15

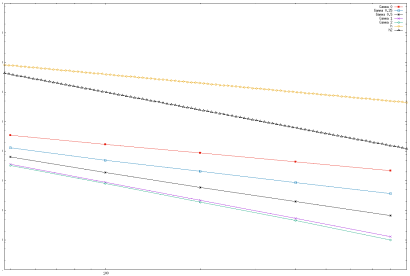
Conclusion

- Only the Gosse-Toscani scheme is WB for all meshes.

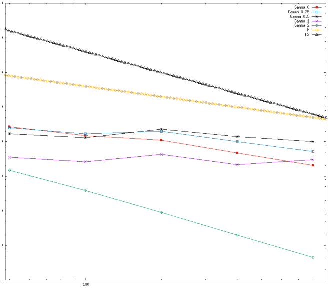
Test for uniform convergence in 1D

- We solve the damped wave equation for different values of ε.
- $p(t,x) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x), \quad u(t,x) = (-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x))$
- **Convergence uniform**: convergence independent of ε.
- **Test**: $\varepsilon = h^{\gamma}$ on uniform and random meshes.

JL scheme on uniform mesh



JL scheme on random mesh

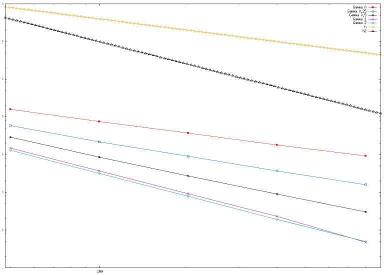


- The GT scheme and the JL scheme (only on uniform mesh) are uniform AP with the error homogeneous to $O(h\varepsilon + h^2)$.
- On Random mesh the JL scheme is not an uniform AP scheme.

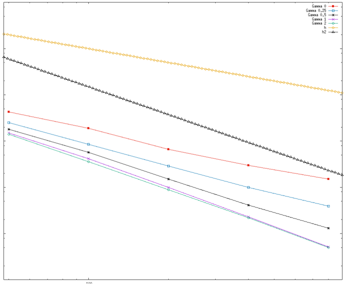
Test for uniform convergence in 1D

- We solve the damped wave equation for different values of ε.
- $p(t,x) = (\alpha(t) + \frac{\varepsilon^2}{\sigma}\alpha'(t))\cos(\pi x), \quad u(t,x) = \left(-\frac{\varepsilon}{\sigma}\alpha(t)\sin(\pi x)\right)$
- **Convergence uniform**: convergence independent of ε.
- **Test**: $\varepsilon = h^{\gamma}$ on uniform and random meshes.

GT scheme on uniform mesh



GT scheme on random mesh



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Analysis of AP schemes: modified equations

- The modified equation associated with the Upwind scheme is

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\partial_x u - \frac{\Delta x}{2\varepsilon}\partial_{xx} p = 0, \\ \partial_t u + \frac{1}{\varepsilon}\partial_x p - \frac{\Delta x}{2\varepsilon}\partial_{xx} u = -\frac{\sigma}{\varepsilon^2} u. \end{cases}$$

- Plugging $\varepsilon\partial_x p + O(\varepsilon^2) = -\sigma u$ in the first equation, we obtain

$$\partial_t p - \frac{1}{\sigma}\partial_{xx} p - \frac{\Delta x}{2\varepsilon}\partial_{xx} p = 0.$$

- **Conclusion**: the regime is captured only on fine grids.

- The modified equation associated to the Gosse-Toscani scheme is

$$\begin{cases} \partial_t p + M\frac{1}{\varepsilon}\partial_x u - M\frac{\Delta x}{2\varepsilon}\partial_{xx} p = 0, \\ \partial_t u + M\frac{1}{\varepsilon}\partial_x p - M\frac{\Delta x}{2\varepsilon}\partial_{xx} u = -M\frac{\sigma}{\varepsilon^2} u. \end{cases}$$

- Plugging $M\varepsilon\partial_x p + O(\varepsilon^2) = -M\sigma u$ in the first equation, we obtain

$$\partial_t p - \frac{M}{\sigma}\partial_{xx} p - \frac{1-M}{\sigma}\partial_{xx} p = 0.$$

- **Conclusion**: the regime is captured on all grids.

AP schemes

- AP schemes modify the numerical diffusion to correct the scheme on coarse grid.
- The JL scheme does not converge in the intermediary regimes.
- **Interpretation**: since the linear steady states are not preserved the limit diffusion scheme in these regimes is not consistent.

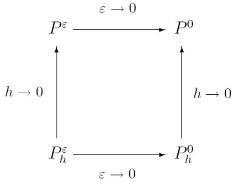
Idea

- The exact preservation of linear steady-state is necessary for uniform AP schemes ?

Uniform convergence in space

- Naive convergence estimate : $||P_h^\varepsilon - P^\varepsilon||_{naive} \leq C\varepsilon^{-b}h^c$
- **Idea**: use triangular inequalities and AP diagram (Jin-Levermore-Golse).

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq \min(||P_h^\varepsilon - P^\varepsilon||_{naive}, ||P_h^\varepsilon - P_h^0|| + ||P_h^0 - P^0|| + ||P^\varepsilon - P^0||)$$



- Intermediary estimations :
  - $||P^\varepsilon - P^0|| \leq C_a\varepsilon^a$,
  - $||P_h^0 - P^0|| \leq C_d h^d$,
  - $||P_h^\varepsilon - P_h^0|| \leq C_e\varepsilon^e$,
  - $d \geq c,\ e \geq a$.

- We using $\min(x, y+z) \leq \min(x,y) + \min(x,z)$ and $d \geq c,\ e \geq a$ to obtain

$$||P_h^\varepsilon - P^\varepsilon||_{L^2} \leq C\left(\min(\varepsilon^{-b}h^c, \varepsilon^e) + h^d + \min(\varepsilon^{-b}h^c, \varepsilon^a)\right) \leq 2C\left(h^d + \min(\varepsilon^{-b}h^c, \varepsilon^a)\right)$$

- Defining $\varepsilon_{th}^{-b}h^c = \varepsilon_{th}^a$ we obtain $\min(\varepsilon^{-b}h^c, \varepsilon^a) \leq \varepsilon_{th}^a = h^{\frac{ac}{a+b}}$.

Space result

We assume that $\|\boldsymbol{V}^\varepsilon(0) - \boldsymbol{V}_h^\varepsilon(0)\|_{L^2(\Omega)} \leq Ch \parallel p(0) \parallel_{H^2}$ and $C_1 h < \Delta x_j < C_2 h \quad \forall j$.

$$\|\boldsymbol{V}^\varepsilon - \boldsymbol{V}_h^\varepsilon\|_{L^2([0,T]\times\Omega)} \leq C\min\left(h^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}, h + 2\varepsilon\right) \parallel p_0 \parallel_{H^3(\Omega)} \leq Ch^{\frac{1}{3}} \parallel p_0 \parallel_{H^3(\Omega)}$$

Euler equation with external forces

- Euler equation with gravity and friction:

$$\left\{\begin{array}{l}\partial_t \rho + \frac{1}{\varepsilon}\partial_x(\rho u) = 0,\\ \partial_t \rho u + \frac{1}{\varepsilon}\partial_x(\rho u^2) + \frac{1}{\varepsilon}\partial_x p = -\frac{1}{\varepsilon}(\rho\partial_x\phi + \frac{\sigma}{\varepsilon}\rho u),\\ \partial_t E + \frac{1}{\varepsilon}\partial_x(Eu + pu) = -\frac{1}{\varepsilon}(\rho u\partial_x\phi + \frac{\sigma}{\varepsilon}\rho u^2).\end{array}\right.$$

- with ϕ the gravity potential, σ the friction coefficient.

Subset of solutions :

- Hydrostatic Steady-state ($\alpha = 1$, $\beta = 0$):

$$\left\{\begin{array}{l} u = 0,\\ \partial_x p = -\rho\partial_x\phi.\end{array}\right.$$

- High friction limit ($\alpha = 0$, $\beta = 1$), no gravity: $u = 0$
- Diffusion limit ($\alpha = 1$, $\beta = 1$):

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x(\rho u) = 0,\\ \partial_t E + \partial_x(Eu) + p\partial_x u = 0,\\ u = -\frac{1}{\sigma}\left(\partial_x\phi + \frac{1}{\rho}\partial_x p\right).\end{array}\right.$$

Design of AP nodal scheme I

Jin Levermore method:

Plug the relation $\partial_x p + O(\varepsilon) = -\rho\partial_x\phi - \frac{\sigma}{\varepsilon}\rho u$ in the Lagrangian fluxes

- Classical Lagrange+remap scheme (LP scheme):

$$\begin{cases} \partial_t \rho_j + \dfrac{\rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - \rho_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} = 0 \\ \partial_t (\rho u)_j + \dfrac{(\rho u)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - (\rho u)_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \dfrac{p^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} = -\frac{1}{\varepsilon}\left(\rho_j (\partial_x \phi)_j + \frac{\sigma}{\varepsilon}\rho_j u_j\right) \\ \partial_t E_j + \dfrac{E_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - E_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} + \dfrac{p^*_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - p^*_{j-\frac{1}{2}} u^*_{j-\frac{1}{2}}}{\varepsilon \Delta x_j} = -\frac{1}{\varepsilon}\left(\rho_j u_j (\partial_x \phi)_j + \frac{\sigma}{\varepsilon}\rho_j u_j^2\right) \end{cases}$$

with Lagrangian fluxes

$$\begin{cases} p^*_{j+\frac{1}{2}} + (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} = p_j + (\rho c)_{j+\frac{1}{2}} u_j \\ p^*_{j+\frac{1}{2}} - (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} = p_{j+1} - (\rho c)_{j+\frac{1}{2}} u_{j+1} \end{cases}$$

and the upwind flux

$$u^*_{j+\frac{1}{2}} f_{j+\frac{1}{2}} = \begin{cases} u^*_{j+\frac{1}{2}} f_j \\ u^*_{j+\frac{1}{2}} f_{j+1} \end{cases}$$

Design of AP nodal scheme I

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with Lagrangian fluxes   with the new Lagrangian fluxes

$$\begin{cases} p^*_{j+\frac{1}{2}} + (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} - \dfrac{\Delta x_{j+\frac{1}{2}}}{2}\left((\rho\partial_x\phi)_{j+\frac{1}{2}} + \dfrac{\sigma_{j+\frac{1}{2}}}{\varepsilon}\rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}}\right) = p_j + (\rho c)_{j+\frac{1}{2}} u_j \\ p^*_{j+\frac{1}{2}} - (\rho c)_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}} + \dfrac{\Delta x_{j+\frac{1}{2}}}{2}\left((\rho\partial_x\phi)_{j+\frac{1}{2}} + \dfrac{\sigma_{j+\frac{1}{2}}}{\varepsilon}\rho_{j+\frac{1}{2}} u^*_{j+\frac{1}{2}}\right) = p_{j+1} - (\rho c)_{j+\frac{1}{2}} u_{j+1} \end{cases}$$

with $\rho_{j+\frac{1}{2}}$ and $(\rho\partial_x\phi)_{j+\frac{1}{2}}$ averages between the interface and the upwind flux and the upwind flux

$$u^*_{j+\frac{1}{2}} f_{j+\frac{1}{2}} = \begin{cases} u^*_{j+\frac{1}{2}} f_j \\ u^*_{j+\frac{1}{2}} f_{j+1} \end{cases}$$

Design of AP nodal scheme I

Jin Levermore method:

Plug the relation $\partial_x p + O(\varepsilon) = -\rho\partial_x\phi - \frac{\sigma}{\varepsilon}\rho u$ in the Lagrangian fluxes

- New scheme (LP-AP scheme):

$$\begin{cases} \partial_t \rho_j + \dfrac{\rho_{j+\frac12} u^*_{j+\frac12} - \rho_{j-\frac12} u^*_{j-\frac12}}{\varepsilon^\alpha \Delta x_j} = 0 \\ \partial_t (\rho u)_j + \dfrac{(\rho u)_{j+\frac12} u^*_{j+\frac12} - (\rho u)_{j-\frac12} u^*_{j-\frac12}}{\varepsilon^\alpha \Delta x_j} + \dfrac{p^*_{j+\frac12} - p^*_{j-\frac12}}{\varepsilon^\alpha \Delta x_j} = -\dfrac{1}{\varepsilon^\alpha}\left((\rho\partial_x\phi)_{j+\frac12} + \dfrac{\sigma}{\varepsilon^\beta}\rho_{j+\frac12} u^*_{j+\frac12}\right) \\ \partial_t E_j + \dfrac{E_{j+\frac12} u^*_{j+\frac12} - E_{j-\frac12} u^*_{j-\frac12}}{\varepsilon^\alpha \Delta x_j} + \dfrac{p^*_{j+\frac12} u^*_{j+\frac12} - p^*_{j-\frac12} u^*_{j-\frac12}}{\varepsilon^\alpha \Delta x_j} = -\dfrac{1}{\varepsilon\alpha}\left((\rho\partial_x\phi)_{j+\frac12} u^*_{j+\frac12} + \dfrac{\sigma}{\varepsilon^\beta}\rho_{j+\frac12} {u^*_{j+\frac12}}^2\right) \end{cases}$$

with Lagrangian fluxes

$$\begin{cases} p^*_{j+\frac12} + (\rho c)_{j+\frac12} u^*_{j+\frac12} - \dfrac{\Delta x_{j+\frac12}}{2}\left((\rho\partial_x\phi)_{j+\frac12} + \dfrac{\sigma_{j+\frac12}}{\varepsilon^\beta}\rho_{j+\frac12} u^*_{j+\frac12}\right) = p_j + (\rho c)_{j+\frac12} u_j \\ p^*_{j+\frac12} - (\rho c)_{j+\frac12} u^*_{j+\frac12} + \dfrac{\Delta x_{j+\frac12}}{2}\left((\rho\partial_x\phi)_{j+\frac12} + \dfrac{\sigma_{j+\frac12}}{\varepsilon^\beta}\rho_{j+\frac12} u^*_{j+\frac12}\right) = p_{j+1} - (\rho c)_{j+\frac12} u_{j+1} \end{cases}$$

with $\rho_{j+\frac12}$ and $(\rho\partial_x\phi)_{j+\frac12}$ averages between the interface and the upwind flux

$$u^*_{j+\frac12} f_{j+\frac12} = \begin{cases} u^*_{j+\frac12} f_j \\ u^*_{j+\frac12} f_{j+1} \end{cases}$$

Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}}$ is exactly preserved.

- **Question**: How the scheme preserved the continuous steady state ?
- **First choice**:

$$(\rho \partial_x \phi)_{j+\frac{1}{2}} = \frac{1}{2} (\rho_j + \rho_{j+1}) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}$$

- Only the continuous steady state with $\rho \partial_x \phi = Cts$ are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}}(\rho\partial_x\phi)_{j+\frac{1}{2}}$ is exactly preserved.

- **Question**: How the scheme preserved the continuous steady state ?
- **Second choice**:

$$(\rho\partial_x\phi)_{j+\frac{1}{2}} = \left(\frac{\rho_{j+1} - \rho_j}{ln(\rho_{j+1}) - ln(\rho_j)}\right)\frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}$$

- Only the continuous steady state with $\rho = p = e^{-xg}$, $\phi = gx$ are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

$$\partial_x p = -\rho\partial_x\phi$$

Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}}(\rho\partial_x\phi)_{j+\frac{1}{2}}$ is exactly preserved.

- **Question**: How the scheme preserved the continuous steady state ?
- **Second choice**:

$$(\rho\partial_x\phi)_{j+\frac{1}{2}} = \left(\frac{\rho_{j+1} - \rho_j}{ln(\rho_{j+1}) - ln(\rho_j)}\right)\frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}$$

- Only the continuous steady state with $\rho = p = e^{-xg}$, $\phi = gx$ are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

$$\Delta_{j+\frac{1}{2}}\left(\frac{1}{\Delta x_{j+\frac{1}{2}}}\int_{x_j}^{x_{j+1}}\partial_x p\right) = -\Delta_{j+\frac{1}{2}}\left(\frac{1}{\Delta x_{j+\frac{1}{2}}}\int_{x_j}^{x_{j+1}}\rho\partial_x\phi\right)$$

Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state $p_{j+1} - p_j = -\Delta x_{j+\frac{1}{2}} (\rho \partial_x \phi)_{j+\frac{1}{2}}$ is exactly preserved.

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$$(\rho \partial_x \phi)_{j+\frac{1}{2}} = \left(\frac{\rho_{j+1} - \rho_j}{ln(\rho_{j+1}) - ln(\rho_j)} \right) \frac{\phi_{j+1} - \phi_j}{\Delta x_{j+\frac{1}{2}}}$$

- Only the continuous steady state with $\rho = p = e^{-xg}$, $\phi = gx$ are exactly preserved.

Idea

- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

$$\Delta_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \partial_x \overline{p}_{j+\frac{1}{2}} \right) = -\Delta_{j+\frac{1}{2}} \left(\frac{1}{\Delta x_{j+\frac{1}{2}}} \int_{x_j}^{x_{j+1}} \overline{\rho}_{j+\frac{1}{2}} \partial_x \overline{\phi}_{j+\frac{1}{2}} \right)$$

with $\overline{p}_{j+\frac{1}{2}}$ (same for ρ and ϕ) average polynomial interpolation.

Properties

Ap property

- The semi-implicit scheme is AP on general grids with a parabolic CFL condition.

WB property

- The discrete steady state $p_{j+1}-p_j=-\Delta x_{j+\frac{1}{2}}(\rho\partial_x\phi)_{j+\frac{1}{2}}$ is exactly preserved.

- **Question**: How the scheme preserved the continuous steady state ?
- **Second choice**:

$$(\rho\partial_x\phi)_{j+\frac{1}{2}}=\left(\frac{\rho_{j+1}-\rho_j}{ln(\rho_{j+1})-ln(\rho_j)}\right)\frac{\phi_{j+1}-\phi_j}{\Delta x_{j+\frac{1}{2}}}$$

- Only the continuous steady state with $\rho=p=e^{-xg}$, $\phi=gx$ are exactly preserved.

Idea

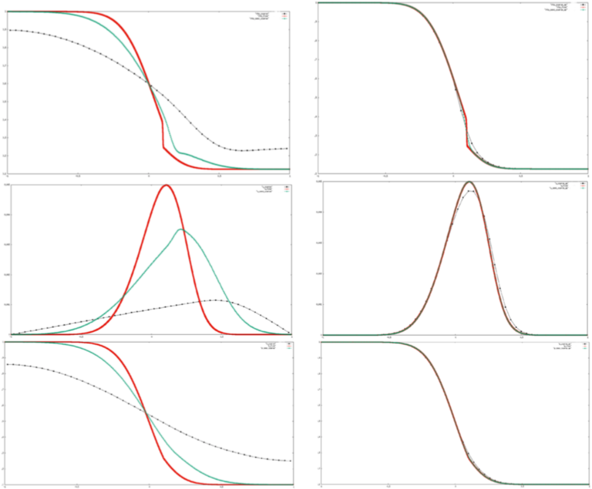
- To treat general steady-state: construct a new discrete equilibrium which is a very high order approximation to the continuous one.

the final equilibrium $p_{j+1}-p_j=-\Delta x_{j+\frac{1}{2}}(\rho\partial_x\phi)^{HO}_{j+\frac{1}{2}}$

$$(\rho\partial_x\phi)^{HO}_{j+\frac{1}{2}}=\Delta x_{j+\frac{1}{2}}\left(\frac{1}{\Delta x_{j+\frac{1}{2}}}\int_{x_j}^{x_{j+1}}\left(\partial_x\overline{p}_{j+\frac{1}{2}}+\overline{\rho}_{j+\frac{1}{2}}\partial_x\overline{\phi}_{j+\frac{1}{2}}\right)-\frac{p_{j+1}-p_j}{\Delta x_{j+\frac{1}{2}}}\right)$$

Results

- Comparison between AP and Non AP scheme for Euler equation.



- Left: non AP, Right: AP. Red: fine solution, black: coarse solution and green: middle coarse solution.

Results

- **Well-Balanced property.**
- **Test case**: $\rho(t,x) = 3 + 2\sin(2\pi x)$ and $\phi(x) = -\sin(2\pi x)$. Random mesh

Schemes	LR		LR-AP (2)		LR-AP (3)		LR-AP (4)	
cells	Err	q	Err	q	Err	q	Err	q
20	0.8335	-	0.0102	-	0.0079	-	0.0067	-
40	0.4010	1.05	0.0027	1.91	8.4E-4	3.23	1.5E-4	5.48
80	0.2065	0.96	7.0E-4	1.95	7.7E-5	3.45	4.1E-6	5.19
160	0.1014	1.02	1.7E-4	2.04	7.0E-6	3.46	1.0E-7	5.36

- **Test case**: $\rho(t,x) = e^{-gx}$, $u(t,x) = 0$, $p(t,x) = e^{-gx}$ et $\phi = gx$. Random mesh

Schemes	LR		LR-AP (2)		LR-AP (3)		LR-AP (4)	
cells	Err	q	Err	q	Err	q	Err	q
20	0.0280	-	6.5E-4	-	1.8E-5	-	8.0E-7	-
40	0.0152	0.88	1.4E-4	2.21	2.0E-6	3.17	3.8E-8	4.4
80	0.0072	1.08	3.3E-5	2.08	2.0E-7	3.32	2.0E-9	4.25
160	0.0038	0.92	8.8E-6	1.90	2.8E-8	2.84	1.1E-10	4.18

WB scheme

Not exact preservation of general steady-state, but arbitrary high order accuracy around the steady-state

Implicit relaxation method for low Mach Euler equations

Low Mach and implicit scheme

Aim: Low Mach Euler equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{M} \nabla p = 0, \\ \partial_t E + \nabla \cdot ((E + p)\mathbf{u}) = 0, \end{cases}$$

- CFL condition $\Delta t \leq hM$.
- **Aim**: choose a time step adapted to $\mathbf{u}$. Filter the fast waves.
- **Solution**: implicit scheme.

Implicit scheme

- **Direct solver**: too expensive in CPU time and memory consumption.
- **Iterative solver**: used in practice. But ofter ill-conditioning for hyperbolic models.
- **Euler equation**: ill-conditioned mainly in the low-Mach regime.

Idea

- Using relaxation model and AP schemes to obtain implicit scheme without matrices.

Relaxation scheme

- We consider the relaxation model (Jin-Xin 95) for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \alpha^2 \partial_x u = \frac{1}{\varepsilon}(F(u) - v) \end{cases}$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_x ((\lambda^2 - \mid \partial F(u) \mid^2) \partial_x u) + O(\varepsilon^2)$$

- **Stability**: the limit system is dissipative if $(\lambda^2 - \mid \partial F(u) \mid^2) > 0$.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\begin{cases} \partial_t f_- - \lambda \partial_x f_- = \frac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \partial_t f_+ + \lambda \partial_x f_+ = \frac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{cases}$$

- with $u = f_- + f_+$ and $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\lambda}$.

Remark

- **Main property**: the transport is diagonal (D1Q2 model) which can be easily solved.

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ ($n_c < n_v$) such that $\boldsymbol{U} = P\boldsymbol{f}$, with $U \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- We define the macroscopic variable by $P\boldsymbol{f} = \boldsymbol{U}$.

- Consistence conditon (R. Natalini, D. Aregba-Driollet, F. Bouchut) :

$$\mathcal{C} \begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{F}(\boldsymbol{U}) \end{cases}$$

- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial \boldsymbol{f} eq - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

First Generalization

- **Generalization** $[D1Q2]^n$: one Xin-Jin or D1Q2 model by macroscopic variable.

Time scheme

Time scheme

- **Property**: the nonlinearity is local and non-locality is linear.
- **Main idea**: time splitting scheme between transport and source.

Consistency in time

- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $\Psi(\Delta t) = T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \left(\frac{(2-\omega)\Delta t}{2\omega}\right) \partial_x \left(D(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

- with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$ and $D(\boldsymbol{U}) = \left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq} - A(\boldsymbol{U})^2\right)$.

Drawback

- For $[D1Q2]^2$ scheme we have a large error: $D(\boldsymbol{U}) = \left(\lambda^2 I_d - A(\boldsymbol{U})^2\right)$

High-order extension

High order scheme

- Second order splitting

$$\Psi(\Delta t) = T\left(\frac{1}{2}\Delta t\right) \circ R\left(\Delta t\right) \circ T\left(\frac{1}{2}\Delta t\right)$$

- Higher order scheme using composition:

$$M_p(\Delta t) = \Psi(\gamma_1 \Delta t) \circ \Psi(\gamma_2 \Delta t)..... \circ \Psi(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.
- High-order convergence only for macroscopic variables.

Space solver

- **Exact transport**: the choice of the velocities link time and space discretization.
- **Semi- Lagrangian**: Interpolation $2q + 1$ gives a consistency error $O(\frac{h^{2d+2}}{\Delta t})$.
- **implicit DG**: DG (k polynomial and Gauss-Lobatto) point gives a consistency error $O(h^k) + O(\Delta t^2)$.

Burgers: convergence results

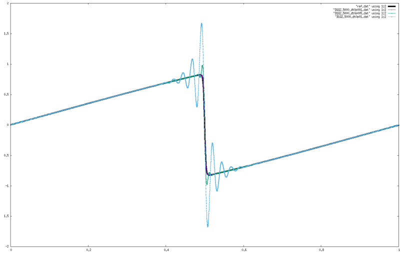
- **Model**: Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 2000 cells, degree 11.
- **Test**: $\rho(t=0,x) = sin(2\pi x)$. $T_f = 0.14$ (before the shock) and no viscosity.
- Scheme: splitting schemes and Suzuki composition + splitting.

	SPL 1, $\theta = 1$		SPL 1, $\theta = 0.5$		SPL 2, $\theta = 0.5$		Suzuki	
Δt	Error	order	Error	order	Error	order	Error	order
0.005	$2.6E^{-2}$	-	$1.3E^{-3}$	-	$7.6E^{-4}$	-	$4.0E^{-4}$	-
0.0025	$1.4E^{-2}$	0.91	$3.4E^{-4}$	1.90	$1.9E^{-4}$	2.0	$3.3E^{-5}$	3.61
0.00125	$7.1E^{-3}$	0.93	$8.7E^{-5}$	1.96	$4.7E^{-5}$	2.0	$2.4E^{-6}$	3.77
0.000625	$3.7E^{-3}$	0.95	$2.2E^{-5}$	1.99	$1.2E^{-5}$	2.0	$1.6E^{-7}$	3.89

- Scheme: second order splitting scheme.
- Same test after the shock:



Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

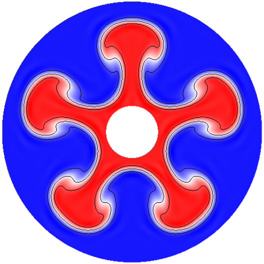
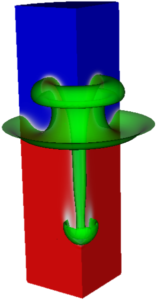


Figure: Plot of the mass fraction of gas

3D case in cylinder



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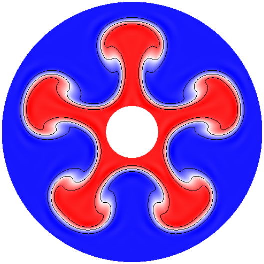


Figure: Plot of the mass fraction of gas

2D cut of the 3D case

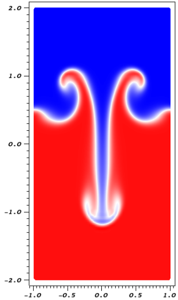


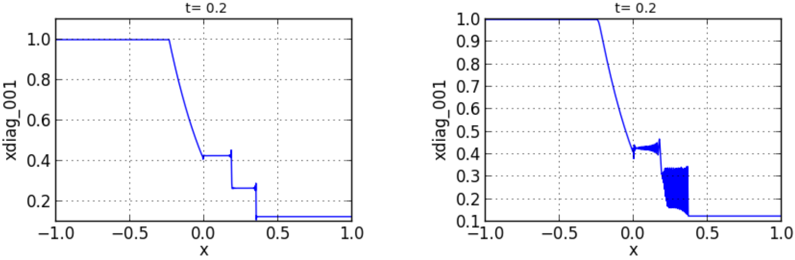
Figure: Plot of the mass fraction of gas

Classical kinetic representation

Limitation

- High-order extension allows to correct the main default of relaxation: large error.
- In two situations the High-order extension is not sufficient:
  - For discontinuous solutions like shocks.
  - For strongly multi-scale problem like low-Mach problem.

- **Euler equation**: Sod problem.
- Second order time scheme + SL scheme:



- Left: density $\Delta t = 1.0^{-4}$. Right: density $\Delta t = 4.0^{-4}$
- **Conclusion**: shock and high order time scheme needs limiting methods.

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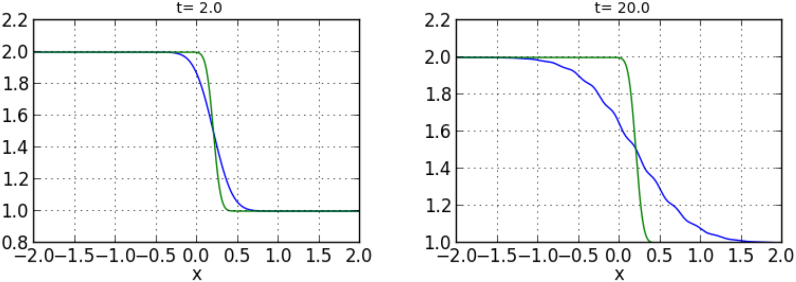
- **Euler equation**: smooth contact (u =cts, p=cts).
- First/Second order time scheme + SL scheme. $T_f = \frac{2}{M}$ and 100 time step.



- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: First order method too much dissipative for low Mach flow (dissipation with acoustic coefficient).

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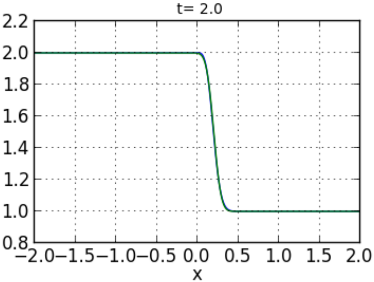
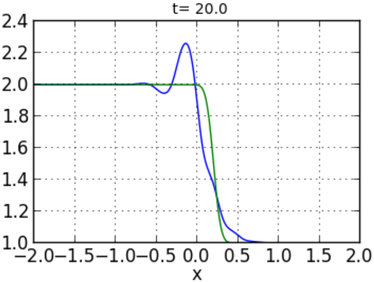




- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: Second order method too much dispersive for low Mach flow (dispersion with acoustic coefficient).

Generic vectorial D1Q3

Idea

- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- Consistency condition:

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, & \forall k \in \{1..N_c\} \\ \lambda_- f_-^k + \lambda_0 f_0^k + \lambda_+ f_+^k = F^k(\boldsymbol{U}), & \forall k \in \{1..N_c\} \end{cases}$$

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, & quad\forall k \in \{1..N_c\} \\ (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k = F^k(\boldsymbol{U}) - \lambda_0 f_0^k, & \forall k \in \{1..N_c\} \end{cases}$$

- We assume a decomposition of the flux (Bouchut 03, Natalini -Aregba 00)

$$F^k(\boldsymbol{U}) = F_0^{k,-}(\boldsymbol{U}) + F_0^{k,+}(\boldsymbol{U}) + \lambda_0 I_d$$

- We obtain the following equation for the equilibrium

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, & \forall k \in \{1..N_c\} \\ (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k = F_0^{k,-}(\boldsymbol{U}) + F_0^{k,+}(\boldsymbol{U}), & \forall k \in \{1..N_c\} \end{cases}$$

- By analogy of the kinetic theory and kinetic flux splitting scheme we propose the following decomposition $\sum_{v>0} v f^k = F_0^{k,+}(\boldsymbol{U})$ and $\sum_{v<0} v f^k = F_0^{k,-}(\boldsymbol{U})$.

Generic vectorial D1Q3

Idea

- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- The lattice $[D1Q3]^N$ is defined by the velocity set $V = [\lambda_-, \lambda_0, \lambda_+]$ and

$$\begin{cases} \boldsymbol{f}_-^{eq}(\boldsymbol{U}) = -\dfrac{1}{(\lambda_0 - \lambda_-)}\boldsymbol{F}_0^-(\boldsymbol{U}) \\ \boldsymbol{f}_0^{eq}(\boldsymbol{U}) = \left(\boldsymbol{U} - \left(\dfrac{\boldsymbol{F}_0^+(\boldsymbol{U})}{(\lambda_+ - \lambda_0)} - \dfrac{\boldsymbol{F}_0^-(\boldsymbol{U})}{(\lambda_0 - \lambda_-)}\right)\right) \\ \boldsymbol{f}_+^{eq}(\boldsymbol{U}) = \dfrac{1}{(\lambda_+ - \lambda_0)}\boldsymbol{F}_0^+(\boldsymbol{U}) \end{cases}$$

Stability

- Condition only on the macroscopic flux splitting.
- Condition for entropy stability:
  - $\boldsymbol{F}_0^+$ and $\boldsymbol{F}_0^-$ is an entropy decomposition of the flux
  - $\partial \boldsymbol{F}_0^+$, $-\partial \boldsymbol{F}_0^-$ and $1 - \frac{\partial \boldsymbol{F}_0^+ - \partial \boldsymbol{F}_0^-}{\lambda}$ are positive.

D1Q3 for scalar case

- First choice: D1Q3 Rusanov ($\lambda_0 = 0$)

$$F_0^-(\rho) = -\lambda_- \frac{(F(\rho) - \lambda_+ \rho)}{\lambda_+ - \lambda_-}, \quad F_0^+(\rho) = \lambda_+ \frac{(F(\rho) - \lambda_- \rho)}{\lambda_+ - \lambda_-}$$

- Consistency (for $\lambda_- = -\lambda_+$): $\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left(\lambda^2 - \mid \partial F(\rho) \mid^2\right) \partial_x \rho + O(\Delta t^2)$

- Second choice: D1Q3 Upwind

$$F_0^-(\rho) = \chi_{\{\partial F(\rho) < \lambda_0\}} \left(F(\rho) - \lambda_0 \rho\right) \quad F_0^+(\rho) = \chi_{\{\partial F(\rho) > \lambda_0\}} \left(F(\rho) - \lambda_0 \rho\right)$$

- with χ the indicatrice function.
- Consistency: $\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left(\lambda \mid \partial F(\rho) \mid - \mid \partial F(\rho) \mid^2\right) \partial_x \rho + O(\Delta t^2)$

- Third choice: D1Q3 Lax-Wendroff ($\lambda_0 = 0$)

$$F_0^-(\rho) = \frac{1}{2} \left(F(\rho) + \frac{\alpha}{\lambda} \int^{\rho} (\partial F(u))^2\right) \quad F_0^+(\rho) = \frac{1}{2} \left(F(\rho) + \frac{\alpha}{\lambda} \int^{\rho} (\partial F(u))^2\right)$$

- with $\lambda_0 = 0$ and $\lambda_- = -\lambda_+$ and $\alpha \geq 1$.
- Consistency: $\partial_t \rho + \partial_x F(\rho) = \sigma \Delta t \partial_x \left((\alpha - 1) \mid \partial F(\rho) \mid^2\right) \partial_x \rho + O(\Delta t^2)$.

- The last one is not entropy stable and L^2 stability in some case.

D1Q3 for Euler equation II

- Low Mach case:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t\rho u+\partial_x\left(\rho u^2+\frac{p}{M}\right)=0\\ \partial_t E+\partial_x(Eu+pu)=0\end{array}\right.$$

- We want to preserve as possible the limit:

$$p=cts,\quad u=cts,\quad \partial_t\rho+u\partial_x\rho=0$$

- **Idea**: Splitting of the flux (E. Toro 12):

$$\boldsymbol{F}(\boldsymbol{U})=\begin{pmatrix}(\rho)u\\ (\rho u)u+p\\ (E)u+pu\end{pmatrix}$$

- Idea: Lax-Wendroff Flux splitting for convection and AUSM-type (M. Liou 93) for the pressure term.
- Use only u, p and λ ($\approx c$) to reconstruct pressure. Important to preserve the low mach limit.
- We obtain

$$\boldsymbol{F}^{\pm}(\boldsymbol{U})=\frac{1}{2}\begin{pmatrix}(\rho u\pm\frac{u^2}{\lambda}\rho)+p\\ (\rho u^2\pm\frac{u^2}{\lambda}q)+p(1\pm\gamma\frac{u}{\lambda})\\ (Eu\pm\frac{u^2}{\lambda}E)+\left(pu\pm\frac{1}{\lambda}\gamma(u^2+\lambda^2)p\right)\end{pmatrix}$$

- Preserve contact. Diffusion error for ρ in $O(u^2)$.

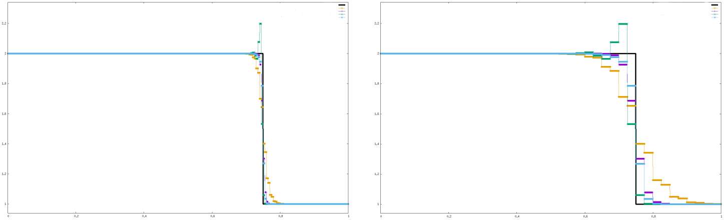
- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- **Test case 1**: $\rho(t=0,x) = sin(2\pi x)$. 10000 cells. Order 17. First order time scheme.

	Rusanov		Upwind		Lax Wendroff $\alpha = 1$	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.01$	$3.9E^{-2}$	-	$1.1E^{-2}$	-	$2.3E^{-3}$	-
$\Delta t = 0.005$	$2.1E^{-2}$	0.89	$6.4E^{-3}$	0.78	$6.0E^{-4}$	1.94
$\Delta t = 0.0025$	$1.1E^{-2}$	0.93	$3.5E^{-3}$	0.87	$1.5E^{-4}$	2.00
$\Delta t = 0.00125$	$5.4E^{-3}$	1.03	$1.8E^{-3}$	0.96	$3.9E^{-5}$	1.95

- Shock wave. First order scheme in time.



- Left $\Delta t = 0.002$. Right $\Delta t = 0.01$. Reference (black), Rusanov (yellow), Upwind (violet), Lax-Wendroff (green), Lax-Wendroff $\alpha = 1.5$ (blue).

Burgers

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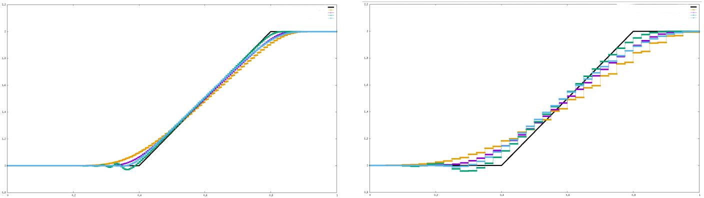

- Left $\Delta t = 0.002$. Right $\Delta t = 0.01$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 1$ (blue), Lax-Wendroff $\alpha = 2$ (Yellow).

1D Euler equations II

- **Test case**: Smooth contact. We take $p = 1$ and u is also constant.
- **Final aim**: take $\Delta t = O(\frac{1}{u})$ when u decrease to have the same error.

- We choose $\Delta t = 0.02$ and $T_f = 2$. 4000 cells. First order time scheme. We compare different D1Q3 schemes.

	Schemes	Rusanov	VL	Osher	Low Mach
$u = 10^{-2}$	$\rho(t,x)$	0.26	$1.0E^{-1}$	$8.4E^{-2}$	$1.0E^{-3}$
	$u(t,x)$	0	$3.4E^{-3}$	$6.0E^{-7}$	0
	$p(t,x)$	0	$5.0E^{-4}$	$4.3E^{-8}$	0
$u = 10^{-4}$	$\rho(t,x)$	0.26	$1.0E^{-1}$	$8.4E^{-2}$	$1.0E^{-5}$
	$u(t,x)$	0	$3.4E^{-3}$	$6.0E^{-7}$	0
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$u = 0$	$\rho(t,x)$	0.26	$1.0E^{-1}$	$4.8E^{-2}$	0.0
	$u(t,x)$	0	$3.4E^{-3}$	$6.0E^{-7}$	0
	$p(t,x)$	0	$5.0E^{-4}$	$4.3E^{-8}$	0

- **Drawback**: When the time step is too large we have dispersive effect.
- **Possible explanation**: the error would be homogeneous to

$$\mid \rho^n(x) - \rho(t,x) \mid \approx \left[O(\Delta t u^2) + O(\Delta t^2 u \lambda^q)\right].$$

- with λ closed to the sound speed.
- **Problem**: At the second order we recover partially the problem since λ is closed to the sound speed.

1D Euler equations III

- **Possible solution**: decrease λ for the density equation.
- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$\underbrace{V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]}_{\text{slow scale}}$$

- The convective part at the slow scale. The acoustic part at the fast scale.
- **Smooth contact**: We take 200 time step and $\Delta t = \frac{0.001}{u}$:

Error	$u = 10^{-1}$	$u = 10^{-2}$	$u = 10^{-3}$	$u = 10^{-4}$
$\alpha = 1$	$2.5E^{-3}$	$2.5E^{-3}$	$2.5E^{-3}$	$2.5E-3$
λ_s	2	0.2	0.02	0.002
λ_f	2	20	200	2000

Conclusion

- **Conclusion**: the error would be homogeneous to

$$\mid \rho^n(x) - \rho(t,x) \mid \approx \left[O(\Delta t u^2) + O(\Delta t^2 u \lambda_s^q) \right].$$

- with λ_s which can be take small.
- **Drawback**: For the stability it seems necessary to have

$$\lambda_s \lambda_f \geq C \max_x (u + c)$$

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- **Drawback**: For the stability it seems necessary to have

$$\lambda_s \lambda_f \geq C \max_x (u + c)$$

Conclusion

Ap schemes for diffusion limit

- **AP scheme**: plug the term source effect in the fluxes.
- **Uniform AP**: scheme: previous construction not sufficient. WB also ?
- Other Works:
  - 2D extension on unstructured meshes for damped wave equations [BDF12], [FHNG11], [BDFL16].
  - Extension on 2D unstructured meshes for Friedrich's systems [BDF14].
  - Extension on 2D unstructured meshes for nonlinear radiative problem [BDF11], [BDF12] and Euler equations [F14], [FM16].

Kinetic relaxation schemes

- **Implicit schemes**: without matrices based on kinetic relaxation schemes.
- High order time extension [CFHMN17], [CFHMN18] and parallel algorithm [Cemracs18].
- Future Works:
  - D1Q3 schemes for hyperbolic problem in 1D (in redaction). Extension in 2D/3D application to low-Mach Euler equation.
  - Implicit Kinetic schemes for anisotropic diffusion (in redaction).
  - Boundary conditions (Post doc of F. Drui).
  - Incompressibility, divergence constrains.