B-Splines compatible finite element spaces. Application to plasma physics

E.Franck[1 2], M. Gaja[3], A. Ratnani[3], E. Sonnendrücker[3]

Mini Symposium: Finite element and compatible discretization,
Canum 2018, 28 may 2018.

[1]Inria Nancy Grand Est, France
[2]IRMA, university of Strasbourg, France
[3]IPP, Garching bei Munchen, Germany

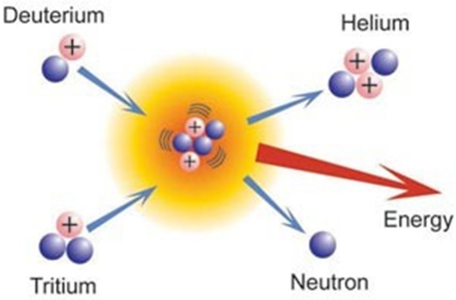
Outline

Introduction

Physical and mathematical context

Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.

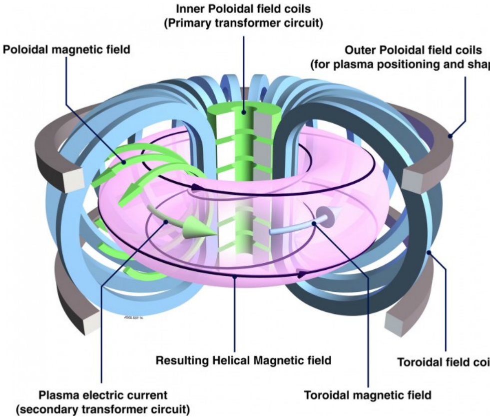
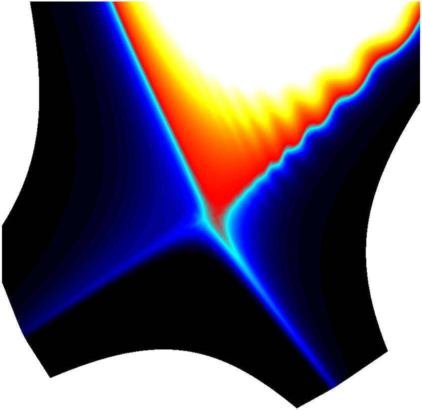


Figure: Tokamak

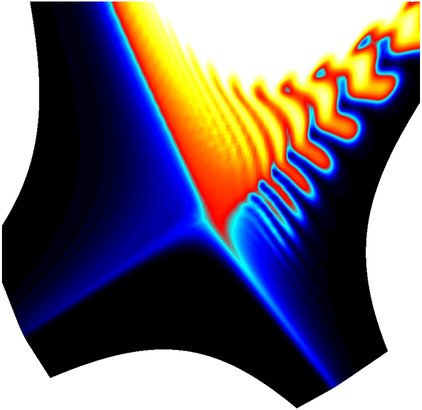
Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



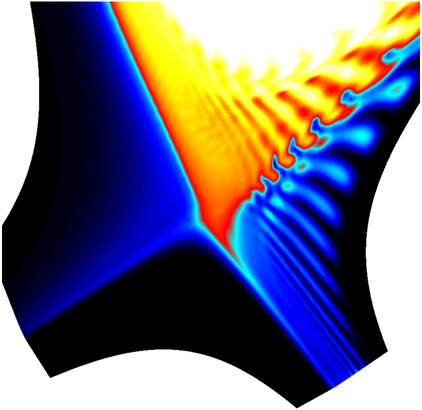
Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



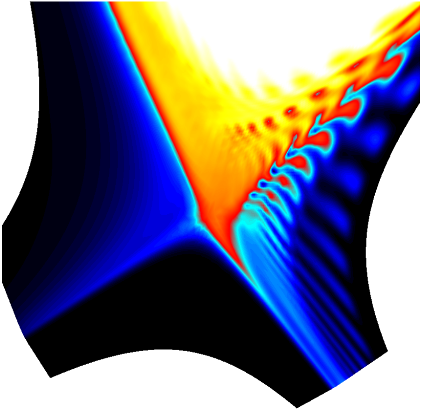
Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



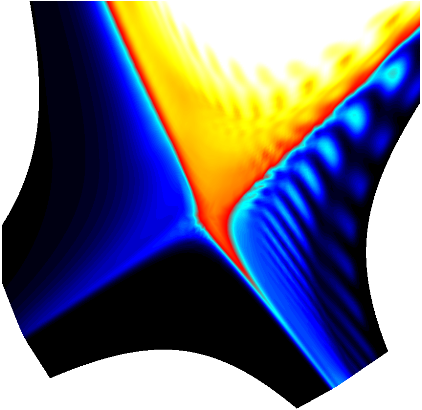
Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



Iter Project

- **Fusion DT**: At sufficiently high energies, deuterium and tritium can fuse to Helium. Free energy is released. At those energies, the atoms are ionized forming a plasma (which can be controlled by magnetic fields).
- **Tokamak**: toroïdal chamber where the plasma is confined using powerful magnetic fields.
- **Difficulty**: plasma instabilities.
  - **Disruptions**: Violent instabilities which can critically damage the Tokamak.
  - **Edge Localized Modes (ELM)**: Periodic edge instabilities which can damage the Tokamak.
- The simulation of these instabilities is an important topic for ITER.



MHD in a Tokamak

Visco-resistive MHD

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) = 0, \\ \rho\partial_t\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nu\nabla\cdot\boldsymbol{\Pi} \\ \partial_t p + \nabla\cdot(p\boldsymbol{u}) + (\gamma-1)p\nabla\cdot\boldsymbol{u} = \nabla\cdot\mathbf{q} + \eta\mid\nabla\times\boldsymbol{B}\mid^2 + \nu\boldsymbol{\Pi}:\nabla\boldsymbol{u} \\ \partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \eta\nabla\times(\nabla\times\boldsymbol{B}) \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, p the pressure, $\mathbf{u}$ the velocity, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{J}$ the current, $\boldsymbol{\Pi}$ stress tensor and $\mathbf{q}$ the heat flux.

MHD specificities in Tokamak

- **Strong anisotropic flows** (direction of the magnetic field) ===> complex geometries and aligned meshes (flux surface or magnetic field lines).
- **MHD scaling**:
  - **Diffusion**: Large Reynolds and magnetic Reynolds number.
  - $\boldsymbol{B}_{\parallel}$ **direction**: compressible flow and small Prandlt number.
  - $\boldsymbol{B}_{\perp}$ **direction**: quasi incompressible flow and large Prandlt number.
- **MHD Scaling** ===> compressible code with no discontinuities + fast waves.
- **Quasi stationary flows + fast waves** ===> implicit or semi implicit schemes.

Problem of implicit discretization

Spatial discretization

- No shocks + diffusion ==> Finite Element method.
- Strong anisotropy ==> Aligned meshes + high-order ==> Isogeometry analysis.
- Divergence constrains, stability ==> Compatible discretization.

- Solution for implicit schemes:
  - Direct solver. CPU cost and consumption memory too large in 3D.
  - Iterative solver. Problem of conditioning.

Problem of classical implicit schemes

- Huge ratio between the physical wave speeds (low Mach regime) ==> huge ratio between discrete eigenvalues.
- Transport problem: anisotropic problem ==> huge ratio between discrete eigenvalues.
- High order scheme: small/high frequencies ==> huge ratio between discrete eigenvalues.

Long term aim

- Propose High-order and stable finite element scheme.
- Propose an implicit formulation with small problems to solve (splitting).

Finite element and B-Splines

Finite element and B-Splines

Finit element method

- Equation: $-\Delta u = f$
- We define a mesh and compact basis functions $\phi_j(\mathbf{x})$ for $j \in 1,..,N$ associated to the degree of freedom j (node mesh for example).
- We write the equation on the weak form:

$$\int (\nabla u, \nabla v) = \int fv$$

- We expand the field on the basis function: $u = \sum_j u_j \phi_j(\mathbf{x})$.

- Lot of possibilities for the basis functions: P_k an Q_k Lagrange, Hermite etc.
- **Isogeometry idea**: use functions used also for the geometry description in CAO. B-Splines, Nurbs etc.

B-Splines

- **Choice**: B-Splines. Important property:
  - Arbitrary order p.
  - Regularity can be also chosen: between C^0 and C^{p-1}.
  - For high regularity Splines we add a small number of DOF to increase the degree.

B-Splines Properties

- For same degree, the Low-regular Splines are more accurate that the high-regular B-Splines (better constant).
- Conditioning better for high-regular B-Splines.

.

Construction

- In 2D/3D on cartesian grids the 2D/3D Splines are obtained by tensor product. Can be useful also for solving linear system associated.

Non cartesian grids

- We obtain non Cartesian geometries mapping the square with your physical geometry.
- Multi-patch version. Each part is mapped with a part of the physical geometries.
- **Drawback**: sometimes the mapping is singular.

mapping_2.pdf

Solvers

- As all the finite element solver we need to invert matrix. Specific solver can be used.
- Example: Laplacian

$$-\Delta u = f$$

- After discretization we obtain

$$S_{h,p,k}\boldsymbol{U} = \boldsymbol{f}$$

with $S_{h,p,k}$ the stiffness matrix for h a step mesh, p the polynomial order and regularity.

Spectral property

- We can prove that at the spectral level (GLT theory, S. Serra-Capizzano):

$$S_{h,p,k} \approx M_{h,p,k} D_h$$

with $M_{h,p,k}$ the mass matrix and D_h the finite difference matrix of the Laplacian.
- **Remark**: mass contained high-order effects and mapping.
- **Conditioning problem**: Low frequencies for D_h and high-frequencies for $p >> 1$ for $M_{h,p,k}$

Preconditioning

- Multi-Grids for low frequencies.
- $M_{h,p,k}^{-1}$ (or approximation) for high-frequencies.
- $M_{h,p,k} \approx M_{1D} \otimes M_{1D}$ for smooth mapping. Using to invert the mass.
- Ref: S. Serra-Capizzano, M. Mazza, G. Sangalli, M. Tani etc.

- Equation:

$$-\Delta u = f$$

- Square domain. Dirichlet BC.
- Convergence and efficiency of B-Splines.

B_ordre_cvg.pdf

- CPU and memory cost compare to the regularity of B-Splines:

	number of d.o.f		number of nnz		cpu-SuperLU		cpu-CG	
	$\mathcal{C}^{p-1}$	$\mathcal{C}^0$	$\mathcal{C}^{p-1}$	$\mathcal{C}^0$	$\mathcal{C}^{p-1}$	$\mathcal{C}^0$	$\mathcal{C}^{p-1}$	$\mathcal{C}^0$
p=2	4'096	16'129	98'596	253'009	0.23	0.35	$7\ 10^{-4}$	$4.1\ 10^{-3}$
p=3	4'225	36'481	196'249	896'809	0.61	1.64	$1.1\ 10^{-2}$	2 10[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Numerical results: Anisotropic diffusion

- Coupling anisotropic diffusion + equilibrium.

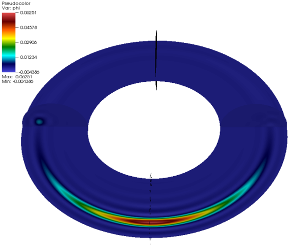
$$\partial_t T - \nabla \cdot ((\boldsymbol{B} \otimes \boldsymbol{B}) \nabla T + \varepsilon \nabla T) = 0$$

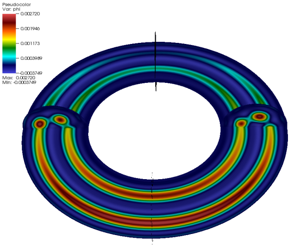
- with the magnetic field given by

$$\boldsymbol{B} = \frac{F(\psi)}{R} + \frac{1}{R} \nabla \psi \times e_{\boldsymbol{\phi}}$$

- Poloidal flux solution ψ solution of the equilibrium code:

$$\Delta^* \psi = -R^2 \frac{dp(\psi)}{d\psi} - \frac{dF(\psi)}{d\psi} F(\psi)$$





- Initial solution in left. Final solution in right.
- Solve with an Implicit in time third-order B-Splines code.

Compatible isogeometric analysis

Compatible space I: general and properties

- **Compatible space**: DeRham sequence

3D Vector fields

$$H^1(\Omega) \overset{\boldsymbol{grad}}{\longrightarrow} H(\mathbf{curl},\Omega) \overset{\boldsymbol{curl}}{\longrightarrow} H(div,\Omega) \overset{div}{\longrightarrow} L^2(\Omega)$$

$$H^1(\mathcal{P}) \overset{\widetilde{\boldsymbol{grad}}^*}{\longleftarrow} H(\mathbf{curl},\mathcal{P}) \overset{\widetilde{\boldsymbol{curl}}^*}{\longleftarrow} H(div,\mathcal{P}) \overset{\widetilde{div}^*}{\longleftarrow} L^2(\mathcal{P})$$

Compatible space I: general and properties

- **Compatible space**: DeRham sequence

3D Vector fields approximations

$$
\begin{array}{ccccccc}
H^1(\mathcal{P}) & \overset{\textbf{grad}}{\longrightarrow} & H(\textbf{curl},\mathcal{P}) & \overset{\textbf{curl}}{\longrightarrow} & H(div,\mathcal{P}) & \overset{div}{\longrightarrow} & L^2(\mathcal{P}) \\
\widetilde{\Pi}^h_{h1}\downarrow & & \widetilde{\Pi}^h_{curl}\downarrow & & \widetilde{\Pi}^h_{div}\downarrow & & \widetilde{\Pi}_{L2}\downarrow \\
V^h & \overset{\textbf{grad}^h}{\longrightarrow} & V^h_{curl} & \overset{\textbf{curl}^h}{\longrightarrow} & V^h_{div} & \overset{div^h}{\longrightarrow} & X^h \\
\mathcal{S}^{p,p,p} & & \begin{pmatrix} \mathcal{S}^{p-1,p,p} \\ \mathcal{S}^{p,p-1,p} \\ \mathcal{S}^{p,p,p-1} \end{pmatrix} & & \begin{pmatrix} \mathcal{S}^{p,p-1,p-1} \\ \mathcal{S}^{p-1,p,p-1} \\ \mathcal{S}^{p-1,p-1,p} \end{pmatrix} & & \mathcal{S}^{p-1,p-1,p-1}
\end{array}
$$

- Preservation of the operator properties:

$$div_h(\boldsymbol{Curl}_h) = 0, \quad \boldsymbol{Curl_h}(\boldsymbol{grad}_h) = 0$$

and

$$\boldsymbol{Curl}_h^* = \boldsymbol{Curl}_h, \quad \boldsymbol{grad}_h^* = div_h$$

- Dual properties useful for energy conservation, kernel properties for constraints and avoid spurious modes.

Compatible space I: general and properties

- **Compatible space**: DeRham sequence

2D Vector fields 1

$$H^1(\Omega) \overset{\boldsymbol{grad}}{\longrightarrow} H(\mathbf{curl},\Omega) \overset{\boldsymbol{rot}}{\longrightarrow} L^2(\Omega)$$

$$H^1(\mathcal{P}) \overset{\widetilde{\boldsymbol{grad}}^*}{\longleftarrow} H(\mathbf{curl},\mathcal{P}) \overset{\widetilde{\boldsymbol{rot}}^*}{\longleftarrow} L^2(\mathcal{P})$$

2D Vector fields 2

$$H^1(\Omega) \overset{\boldsymbol{curl}}{\longrightarrow} H(div,\Omega) \overset{\mathbf{div}}{\longrightarrow} L^2(\Omega)$$

$$H^1(\mathcal{P}) \overset{\widetilde{\boldsymbol{curl}}^*}{\longleftarrow} H(div,\mathcal{P}) \overset{\widetilde{\mathbf{div}}^*}{\longleftarrow} L^2(\mathcal{P})$$

- with $rot\boldsymbol{u} = \partial_x u_2 - \partial_y u_1$ and $\nabla \times f = \begin{pmatrix} \partial_y f \\ -\partial_x f \end{pmatrix}$.
- As in 3d, we have the preservation of the operator properties:

$$div_h(\boldsymbol{Curl}_h) = 0, \quad \boldsymbol{rot_h}(\boldsymbol{grad}_h) = 0$$

Why compatible spaces ?

- Example: Low-Mach Euler equation (no viscosity)

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + u\partial_x u + \frac{1}{M}\partial_x p = 0 \\ \partial_t p + u\partial_x p + p\partial_x u = 0 \end{cases}$$

- Limit when M tends to zero:

$$\partial_t rho + u\partial_x \rho = O(M), \quad \partial_x p = O(M), \quad \partial_x u = O(M).$$

- Discretization: classical finite element P_1. Gradient gives by

$$(dxu)_j \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

- Kernel of gradient: constant functions. Kernel of discret gradient P_1: constant and checkerboard modes.
- **Result**: we can compute a wrong limit.

Possible solution

- Add viscosity on the all the equations to kill unphysical modes.

$$\eta \partial_{xx} u \approx \eta \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$

- It is Stabilization.

Why compatible spaces ?

- Example: Low-Mach Euler equation (no viscosity)

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + u\partial_x u + \frac{1}{M}\partial_x p = 0 \\ \partial_t p + u\partial_x p + p\partial_x u = 0 \end{cases}$$

- Limit when M tends to zero:

$$\partial_t rho + u\partial_x \rho = O(M), \quad \partial_x p = O(M), \quad \partial_x u = O(M).$$

- Discretization: classical finite element P_1. Gradient gives by

$$(dxu)_j \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

- Kernel of gradient: constant functions. Kernel of discret gradient P_1: constant and checkerboard modes.
- **Result**: we can compute a wrong limit.

Possible solution

- Use another discretization. For example P_0 FE:

$$(dxu)_j \approx \frac{u_j - u_{j-1}}{\Delta x}$$

- No unphysical modes, but less order.
- Compatible FE: keep the order and the good kernel for classical operators in 2D/3D. Not directly valid for advection.

Example of Maxwell and properties

- **Advantage**: strong-weak form. Example: Explicit Maxwell.

$$\left\{\begin{array}{l} \boldsymbol{E}^{n+1}=\boldsymbol{E}^n+\Delta t\nabla\times\boldsymbol{B}^n=0 \\ \boldsymbol{B}^{n+1}=\boldsymbol{B}^n-\Delta t\nabla\times\boldsymbol{E}^n=0 \\ \nabla\cdot\boldsymbol{B}^{n+1}=0,\nabla\cdot\boldsymbol{E}^{n+1}=\rho \end{array}\right.$$

- We take the $\boldsymbol{B}$ equation, choose $\boldsymbol{E}\in H(curl)$ and consequently $\boldsymbol{B}\in H(div)$, multiply by test function and integrate (no ipp) to obtain

$$M_{div}\boldsymbol{B}_h^{n+1}=M_{div}\boldsymbol{B}_h^n+\Delta tC\boldsymbol{E}_h^n$$

- M_{div} the mass matrix for h(Div) space and C the weak curl matrix.
- Property of the space: $C=M_{div}\boldsymbol{Curl}_h$ with $\boldsymbol{Curl}_h$ a "finite difference curl". We obtain

$$\boldsymbol{B}_h^{n+1}=\boldsymbol{B}_h^n+\Delta t\boldsymbol{Curl}_h\boldsymbol{E}_h^n$$

- Applying div_h we obtain $div_h\boldsymbol{B}_h^{n+1}=0$.
- $\boldsymbol{B}\in H(div) ==>$ no compatibility with the first equation. So ipp on the first equation (weak form)

$$\int(\boldsymbol{E}^{n+1},\boldsymbol{C})=\int(\boldsymbol{E}^n,\boldsymbol{C})+\Delta t\int(\boldsymbol{B}^n,\nabla\times\boldsymbol{C})$$

- Taking $\boldsymbol{C}\in H(curl)$ we obtain a consistent equation.

$$M_{curl}\boldsymbol{E}^{n+1}=M_{curl}\boldsymbol{E}^n+\Delta tCurl_h^TM_{div}\boldsymbol{B}^n$$

- Taking $\boldsymbol{C}\in H(curl)$ we obtain a consistent equation.

Projector

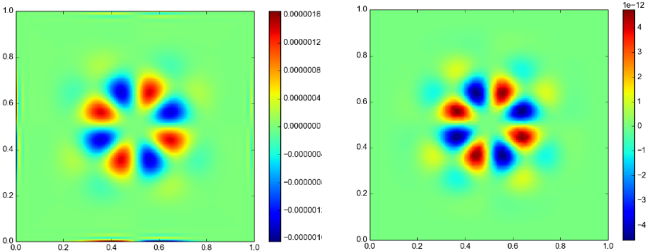
- Additionally, we need the commutative projection.
- The 3D projectors are defined by:

$$\widetilde{\Pi}^h_{h1} := \left\{ \begin{array}{l} \widetilde{\Pi}^h_{h1} f = f^0_p \in V^h \\ f^0_p(\mathbf{x}_k) = \mathbf{x}_k, \quad \forall \mathbf{x}_k \in N_h \end{array} \right. \qquad \widetilde{\Pi}^h_{L2} := \left\{ \begin{array}{l} \widetilde{\Pi}^h_{L2} \mathbf{f} = \mathbf{f}^3_p \in X^h \\ \displaystyle\int_{v_k} \mathbf{f}^3_p = \int_{s_k} \mathbf{f}, \quad \forall v_k \in \Omega_h \end{array} \right.$$

- with N_h the nodes of the mesh. Ω_h the cells of the mesh.

$$\widetilde{\Pi}^h_{curl} := \left\{ \begin{array}{l} \widetilde{\Pi}^h_{curl} \mathbf{f} = \mathbf{f}^1_p \in V^h_{curl} \\ \displaystyle\int_{e_k} \mathbf{f}^1_p \cdot \mathbf{t} = \int_{e_k} \mathbf{f} \cdot \mathbf{t}, \quad \forall e_k \in E_h \end{array} \right. \qquad \widetilde{\Pi}^h_{div} := \left\{ \begin{array}{l} \widetilde{\Pi}^h_{div} \mathbf{f} = \mathbf{f}^2_p \in V^h_{div} \\ \displaystyle\int_{f_k} \mathbf{f}^2_p \cdot \mathbf{n} = \int_{f_k} \mathbf{f} \cdot \mathbf{n}, \quad \forall f_k \in F_h \end{array} \right.$$

- with E_h the edges of the mesh. Ω_h the faces of the mesh.
- Exemple: $\rho_2 = \nabla \times (2x(1-x)y(1-y))$. Comparison between L^2 and commutative projection in $H(div)$:



Results of 3D Maxwell

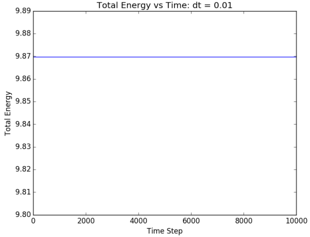
Remarks

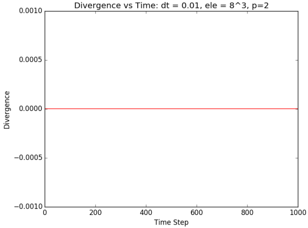
- Matrices of first and second order can by write using mass and "DF" matrices:

$$Matrix_{Grad} = M_{Curl}\,\mathbf{grad}_h, \quad Matrix_{laplacian} = \mathbf{grad}_h^T M_{Curl}\,\mathbf{grad}_h$$

- Mapping and high-order polynomial contains in the mass matrices.

- Analytic solution fof Maxwell equations. Implicit code.





- Left: Energy evolution. Right: magnetic field divergence evolution.

Problem of classical implicit schemes

- Good conservation properties.
- Need to be verified with complex mapping.

Compatible space V: practical example

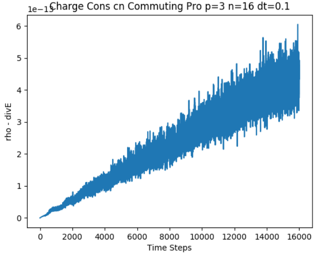
- Numerical example: 2D Maxwell model:

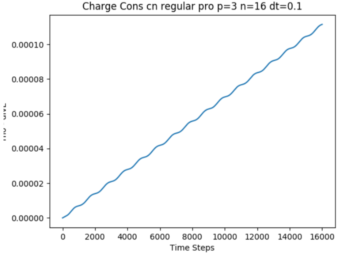
$$\begin{cases} \boldsymbol{E}^{n+1} = \boldsymbol{E}^n + \Delta t Curl(B^n) - \mu_0 \boldsymbol{J} \\ B^{n+1} = B^n - \Delta t rot(\boldsymbol{E}^n) \\ \nabla \cdot \boldsymbol{B}^{n+1} = 0, \nabla \cdot \boldsymbol{E}^{n+1} = \rho \end{cases}$$

- with $CurlB = \begin{pmatrix} \partial_y B \\ -\partial_x B \end{pmatrix}$ and $Rot(\boldsymbol{E}) = \partial_x E_y - \partial_y E_x$.
- Property to preserve

$$\nabla \cdot \partial_t \boldsymbol{E} = \partial_t \rho, \quad \text{since} \quad \partial_t \rho + \nabla \cdot \boldsymbol{J} = 0.$$

- Charge conservation for Implicit scheme with 16*16 cells. Order 3





- Left: Compatible space with commutative projection. Right: Compatible space without commutative projection.

Splitting and nonlinear solver: Full MHD

Model

- Resistive MHD model for Tokamak:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla \cdot \boldsymbol{\Pi} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + \gamma p \nabla \cdot \boldsymbol{u} = \nabla \cdot ((k_{\parallel}(\boldsymbol{B} \otimes \boldsymbol{B}) + k_{\perp} I_d) \nabla T) + \eta(T) \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \eta(T) \nabla \times (\nabla \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, $\boldsymbol{u}$ the velocity , p and T the pressure and temperature, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{\Pi} = \boldsymbol{\Pi}(\nabla \boldsymbol{u}, \boldsymbol{B})$ the stress tensor.
- with ν the viscosity, $k_{\parallel}$, $k_{\perp}$ the thermal conductivities and η the resistivity.

Important Properties

- Conservation in time: $\nabla \cdot \boldsymbol{B} = 0$ and

$$\frac{d}{dt} \int \left(\rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{\mid \boldsymbol{B} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

Possible simplification

- $\nabla \cdot \boldsymbol{\Pi} \approx \Delta \boldsymbol{u}$.
- Ohmic ($\eta \mid \nabla \times \boldsymbol{B} \mid^2$) and viscous heating $\nu \boldsymbol{\Pi} : \nabla \boldsymbol{u}$ neglected.

Three stage Energy conserving Splitting

- Convection - diffusion step:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u} \\ \partial_t p = \nabla \cdot \boldsymbol{q} + \eta(T) \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} = \eta(T) \nabla \times (\nabla \times \boldsymbol{B}) \end{cases}$$

  - Energy balance

$$\partial_t \int \left(\frac{\mid \boldsymbol{B} \mid^2}{2} + \rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

- Acoustic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} + \nabla p = 0 \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \\ \partial_t \boldsymbol{B} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

  - Energy balance

$$\partial_t \int \left(\frac{\mid \boldsymbol{B} \mid^2}{2} + \rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

- Magnetic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \partial_t p = 0 \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

  - Energy balance

$$\partial_t \int \left(\frac{\mid \boldsymbol{B} \mid^2}{2} + \rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

- **Splitting and Equilibrium**: the balance is not preserved.

Example of Compatible space for MHD: Acoustic part

- Algorithm for acoustic step

$$\begin{cases} \partial_t p_h + \nabla \cdot (p_h \boldsymbol{u}_h) + (\gamma - 1) p_h \nabla \cdot \boldsymbol{u}_h = 0 \\ \rho_h \partial_t \boldsymbol{u}_h + \nabla p_h = 0 \end{cases}$$

- $\boldsymbol{u}_h \in H(Curl)$ and $p \in H^1$. The second is exactly true using the DehRham sequence. .
- First equation needs to take on the week form.
- We solve

$$\begin{cases} A(\partial_t p_h, q) - B(p_h \boldsymbol{u}_h, q) - (\gamma - 1) B(\boldsymbol{u}_h, p_h q) = 0 \\ \rho_h \partial_t \boldsymbol{u}_h + \nabla_h p_h = 0 \end{cases}$$

with

$$A(p, q) = \int pq, \quad B(\boldsymbol{u}, p) = \int (\boldsymbol{u}, \nabla p)$$

Example of Compatible space for MHD: Acoustic part

- Algorithm for acoustic step

$$\begin{cases} \partial_t p_h + \nabla \cdot (p_h \boldsymbol{u}_h) + (\gamma - 1) p_h \nabla \cdot \boldsymbol{u}_h = 0 \\ \rho_h \partial_t \boldsymbol{u}_h + \nabla p_h = 0 \end{cases}$$

- $\boldsymbol{u}_h \in H(Curl)$ and $p \in H^1$. The second is exactly true using the DehRham sequence. .
- After time discretization

$$\begin{cases} A(p_h^{n+1}, q) - c_i \left(B(p_h^{n+1} \boldsymbol{u}_h^{n+1}, q) + (\gamma - 1) B(\boldsymbol{u}_h^{n+1}, p_h^{n+1} q) \right) \\ = A(p_h^n, q) + c_e \left(B(p_h^n \boldsymbol{u}_h^n, q) + (\gamma - 1) B(\boldsymbol{u}_h^n, p_h^n q) \right) \\ \rho_h \boldsymbol{u}_h^{n+1} + c_i \nabla_h p_h^{n+1} = \rho_h \boldsymbol{u}_h^n - c_e \nabla_h p_h^n \end{cases}$$

- with $c_i = \theta \Delta t$ and $c_e = (1 - \theta) \Delta t$.

Example of Compatible space for MHD: Acoustic part

- Algorithm for acoustic step

$$\left\{ \begin{array}{l} \partial_t p_h + \nabla \cdot (p_h \boldsymbol{u}_h) + (\gamma - 1) p_h \nabla \cdot \boldsymbol{u}_h = 0 \\ \rho_h \partial_t \boldsymbol{u}_h + \nabla p_h = 0 \end{array} \right.$$

- $\boldsymbol{u}_h \in H(Curl)$ and $p \in H^1$. The second is exactly true using the DehRham sequence. .
- We solve

$$\left\{ \begin{array}{l} A(p_h^{n+1}, q) + M_1(p_h^{n+1}, q) + (\gamma - 1) M_2(p_h^{n+1}, q) = R(q) \\ \rho_h \boldsymbol{u}_h^{n+1} + c_i \nabla_h p_h^{n+1} = \rho_h \boldsymbol{u}_h^n - c_e \nabla_h p_h^n \end{array} \right.$$

with

$$M_1(p_h^{n+1}, q) = c_i^2 \int \frac{p_h^{n+1}}{\rho_h} (\nabla p_h^{n+1}, \nabla q) - c_i \int (p_h^{n+1} \boldsymbol{u}_h^n, \nabla q) + c_i c_e \int \frac{p_h^{n+1}}{\rho_h} (\nabla p_h^n, \nabla q)$$

$$M_2(p_h^{n+1}, q) = c_i^2 \int \frac{p_h^{n+1}}{\rho_h} (\nabla p_h^{n+1}, \nabla q) + c_i^2 \int \frac{q}{\rho_h} \mid \nabla p_h^{n+1} \mid^2 - c_i \int (\boldsymbol{u}_h^n, p_h^{n+1} \nabla q)$$
$$- c_i \int (\boldsymbol{u}_h^n, \nabla p_h^{n+1}) q + c_i c_e \int \frac{p_h^{n+1}}{\rho_h} (\nabla p_h^n, \nabla q) + c_i c_e \int \frac{q}{\rho_h} (\nabla p_h^{n+1}, \nabla p_h^n)$$

and

$$R(q) = A(p_h^n, q) + c_e \left(B(p_h^n \boldsymbol{u}_h^n, q) + (\gamma - 1) B(\boldsymbol{u}_h^n, p_h^n q) \right)$$

Example of Compatible space for MHD: Acoustic part

- Algorithm for acoustic step

$$\begin{cases} \partial_t p_h + \nabla\cdot(p_h \boldsymbol{u}_h) + (\gamma-1)p_h\nabla\cdot\boldsymbol{u}_h = 0 \\ \rho_h\partial_t\boldsymbol{u}_h + \nabla p_h = 0 \end{cases}$$

- $\boldsymbol{u}_h \in H(Curl)$ and $p \in H^1$. The second is exactly true using the DehRham sequence. .

- Final Algorithm with Picard: for each k the following system

$$A(p_h^*, q) + M_1(p_h^*, q) + (\gamma-1)M_2(p_h^*, q) = R(q)$$

with $p_h^{k+1} = wp_h^* + (1-w)p_h^k$ and

$$M_1(p_h^*, q) = c_i^2\int \frac{p_h^k}{\rho_h}(\nabla p_h^*, \nabla q) - c_i\int(p_h^*\boldsymbol{u}_h^n, \nabla q) + c_i c_e\int\frac{p_h^*}{\rho_h}(\nabla p_h^n, \nabla q)$$

$$M_2(p_h^*, q) = c_i^2\int\frac{p_h^k}{\rho_h}(\nabla p_h^*, \nabla q) + c_i^2\int\frac{q}{\rho_h}(\nabla p_h^k, \nabla p_h^*) - c_i\int(\boldsymbol{u}_h^n, p_h^*\nabla q)$$

$$-c_i\int(\boldsymbol{u}_h^n, \nabla p_h^*)q + c_i c_e\int\frac{p_h^*}{\rho_h}(\nabla p_h^n, \nabla q) + c_i c_e\int\frac{q}{\rho_h}(\nabla p_h^*, \nabla p_h^n)$$

- When $\mid p_h^{k+1} - p_k \mid < \varepsilon$ we take $p_h^{n+1} = p_h^{k+1}$ and we compute the velocity

$$\rho_h\boldsymbol{u}_h^{n+1} + c_i\nabla_h p_h^{n+1} = \rho_h\boldsymbol{u}_h^n - c_e\nabla_h p_h^n$$

Substep and solvers

- Splitting allows to obtain more simple systems to solve. How ?
- Example: acoustic.
- At each nonlinear step we must solve an equation **like**:

$$-a\Delta p + \boldsymbol{u}\cdot\nabla p + cp = f$$

- and just a matrix-vector product for the velocity update.
- Case $|\ \boldsymbol{u}\ | << 1$: classical multigrid method + GLT smoother for high-order B-Splines.
- Other case. More complex. Stabilization helps probably.
- **Magnetic or Magneto-acoustic step**. Problem **like**:

$$-a(\nabla\cdot\boldsymbol{u}) + b\nabla\times(\nabla\times(\boldsymbol{u}\times\boldsymbol{b}))\times\boldsymbol{b} + c\boldsymbol{u} = \mathbf{f}$$

- More complex since for $d=0$ the kernel can be non zero. For $d<<1$ ill-conditioned system.
- Works of A. Ratnani and M. Mazza. PC for

$$-a\nabla(\nabla\cdot\boldsymbol{u}) + b\nabla\times(\nabla\times\boldsymbol{u}) = \mathbf{f}$$

- Work well for $10^{-3}\leq\frac{a}{b}\leq 10^{3}$. On going work: Larger ratio and introduction of the magnetic field.

Conclusion

Compatible spaces:

- Energy preserving time scheme + compatible spaces allows:
  - Preserve energy at the discrete level in the ideal case. More stability ?
  - Preserve strongly $\nabla \cdot \boldsymbol{B} = 0$.
  - In each step we solve simple problems (convection-diffusion-reaction problems) + matrix vector product.
  - High-order and High-regularity. Possible to align poloidal mesh to magnetic surfaces.
  - Have a simple way to assembly/store the matrices (product of mass with "DF" matrices).
  - **Needs**: stabilization for advection and preconditioning for elliptic solvers.

Following works for MHD

- Validate the acoustic step and after the magnetic step.
- Write and validate the convection diffusion step. Stabilization for convection (Holger's talk ?)
- Preconditioning for vectorial elliptic problems and anisotropic diffusion.
- Add the mapping to the circle and after the Tokamak. Realistic test cases.