LBM method as kinetic relaxation schemes. High-order methods and low-mach viscous problems

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Outline

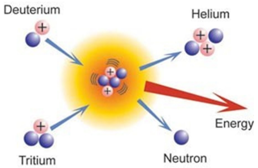
Physical and mathematical context

Application

Applications considered

- Steady or quasi-steady flows (long time limit).
- Multi-scale problem: capture the slow scale and filter the fast one (ex: low mach).

- **Fusion DT**: At sufficiently high energies, deuterium and tritium (plasmas) can fuse to Helium. Free energy is released.
- **Tokamak**: toroïdal chamber where the plasma is confined using magnetic fields.
- **Difficulty**: plasma instabilities. Important topic for ITER.



Simulation of MHD instabilities

- Simulation: slow flow around plasmas equilibrium (in green):

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\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\
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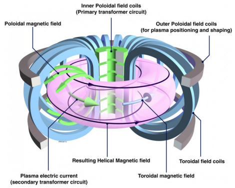

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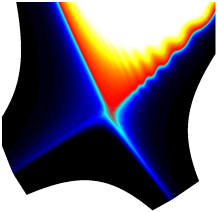

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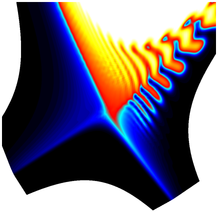

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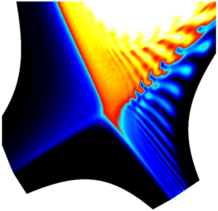

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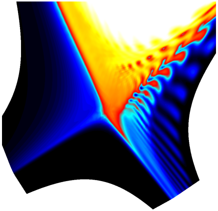

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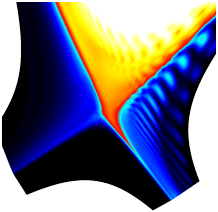

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Implicit method and general grids

Classical solution

- Explicit scheme: CFL given by the high frequency discretized of the waves.
- **Solution**: implicit scheme to filter the frequencies not considered.

- Solution for implicit schemes:
  - Direct solver. CPU cost and consumption memory too large in 3D.
  - Iterative solver. Problem of conditioning.

Problem of conditioning

- Multi-scale PDE (low Mach regime) ==> huge ratio between discrete eigenvalues.
- High order scheme for transport: small/high frequencies and anisotropy ==> huge ratio between discrete eigenvalues.
- Storage the matrix and perhaps the preconditioning: large memory consumption.

Mesh and geometry

- **Geometry**: toroidal geometry. Poloidal section: circle or D-shape.
- **Meshes**: curved meshes, unstructured meshes, Multi-Patch + mapping.

Current work

- LBM-type algorithm: CFL free and matrix-free on complex geometries.

LBM as implicit relaxation method

LBM

- We consider the following system $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$.
- We consider the new variable $M\boldsymbol{f} = \boldsymbol{U}$ and the velocities set $V = [v_1, ..., v_n]$.

Time loop

- At the time t^n, we have $\boldsymbol{f}^n$.
- We apply the transport step:

$$f_i^*(x) = f_i^n(x - v_i \Delta t) \quad \forall i \leq N$$

- Relaxation step:

$$\boldsymbol{f}^{n+1} = \boldsymbol{f}^* + \Omega(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^*)$$

with $\Omega = M^{-1}SM$ with S a diagonal matrix with $s_k \in \{0, 2\}$

- To write the first substep we choose $v_i = k\lambda \frac{\Delta t}{\Delta x}$ with k an integer (in general: 0, 1, 2).
- **Consistency** :

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \partial_x (A(\boldsymbol{U}, \lambda, S)\partial_x \boldsymbol{U}) + \Delta t^2 \partial_x B(\boldsymbol{U}, \partial_x \boldsymbol{U}, \partial_{xx} \boldsymbol{U}, \lambda, S)$$

- We can increase the order with the good parameters.
- **Advantages**: **very very** simple algorithm.
- **Drawbacks**: complicate to manage with large Δt and complex grids.

Rewritting: transport step

- Transport step:

$$f_i^*(x) = f_i^n(x - v_i \Delta t) \quad \forall i \leq N$$

- We solve $\partial_t f_i + v_i \partial_x f_i = 0$ with the characteristic method.
- Possible since we choose the velocity $v_i = k\lambda \frac{\Delta t}{\Delta x}$.
- Avoiding this constrains, $x - v_i \Delta t$ in not a mesh node but inside a cell. **Natural solution**: Backward or Forward Semi-Lagrangian method.

SL methods

- **BSL**: we compute the origin of the characteristic curve and interpolate (high-order) the value obtained.
- **FSL**: we follow of the characteristic curve and project (high-order) the value obtained.
- B-splines, Lagrange interpolation. Nodal or average projection. etc

- The transport step can be rewrite as advection equation:

$$\partial_t f_i + v_i \partial_x f_i = 0, \quad \forall i \leq N$$

solved with BSL (or FSL) with exact interpolation (projection).

Natural extension

- Relax assumption on the velocities and use full BSL solver for advection (or other solver like FV or DG).

Rewritting: relaxation step

- Relaxation step:

$$\boldsymbol{f}^{n+1} = \boldsymbol{f}^* + \Omega(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^*)$$

- We recognize an operator closed to BGK operator.
- BGK operator:

$$\partial_t \boldsymbol{f} = \frac{R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- Dicretizating the previous scheme with a θ scheme you obtain:

$$\frac{\boldsymbol{f}^{n+1} - \boldsymbol{f}^n}{\Delta t} = \frac{\theta R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^{n+1}) - \boldsymbol{f}^{n+1}) + \frac{(1-\theta)R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n)$$

- The equilibrium is construct such that $\boldsymbol{U}^{n+1} = \boldsymbol{U}^n$.
- Consequently

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Conclusion

- The relaxation can be write as a θ-scheme for a generalized BGK operator.

- **Remark**: $\Omega = I_D$ is equivalent to $\varepsilon = 0$, $R = I_d$ and $\theta = 1$ (first order scheme).
  $\Omega = 2I_D$ is equivalent to $\varepsilon = 0$, $R = I_d$ and $\theta = 0.5$ (second order scheme).

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$$\left(I_d + \frac{\theta \Delta t R}{\varepsilon}\right)\boldsymbol{f}^{n+1} = \left(I_d - \frac{(1-\theta)\Delta t R}{\varepsilon}\right)\boldsymbol{f}^n + \frac{\Delta R}{\varepsilon}\boldsymbol{f}^{eq}(\boldsymbol{U}^n)$$

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Rewritting: algorithm

- On one time step, the first step (T) is a discretization of

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = 0$$

- The second step (C) is a discretization of

$$\partial_t \boldsymbol{f} = \frac{R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- The algorithm can be view as a first order Lie splitting scheme in time:

$$\boldsymbol{f}^n = [T(\Delta t) \circ (\Delta t)]^n \boldsymbol{f}^0$$

- **Natural extension**: Second order strang splitting scheme in time:

$$\boldsymbol{f}^n = \left[T\left(\frac{1}{2}\Delta t\right) \circ (\Delta t) \circ T\left(\frac{1}{2}\Delta t\right)\right]^n \boldsymbol{f}^0.$$

- However $T\left(\frac{1}{2}\Delta t\right) \circ T\left(\frac{1}{2}\Delta t\right) = T(\Delta t)$.
- So the second order splitting is given by

$$\boldsymbol{f}^n = T\left(\frac{1}{2}\Delta t\right) \circ [T(\Delta t) \circ (\Delta t)]^n \circ T\left(\frac{1}{2}\Delta t\right) \boldsymbol{f}^0$$

Conclusion

- The algorithm can be view as a first order splitting or a second order splitting if we add a beginning and final transport step.

Rewritting: advantages and drawbacks

Conclusion

- A LBM method can be view the discretization of the model

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{R}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

which gives at the limit $\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\varepsilon)$

- obtained with
  - A time Lie splitting scheme,
  - A θ-scheme for the relaxation step (unconditionnaly stable)==> AP scheme.
  - A BSL scheme for the transport with exact interpolation (choice of velocities).

- Idea: use this other formulation to use different schemes in space and time.
- To treat complex geometries and large time steps. We propose
  - Use a high order BSL scheme (without exact interpolation) or implicit DG schemes.
  - Use another time scheme for relaxation (not studied).
  - Increase the time order of the full algorithm.
- **General model**: $[D1Q2]^n$. One D1Q2 by equation (B. Graille, S. Jin):

$$\begin{cases} \partial_t \boldsymbol{f}_+ + \lambda \partial_x \boldsymbol{f}_+ = \frac{1}{\varepsilon}(\boldsymbol{f}_+^{eq} - \boldsymbol{f}_+) \\ \partial_t \boldsymbol{f}_- - \lambda \partial_x \boldsymbol{f}_- = \frac{1}{\varepsilon}(\boldsymbol{f}_-^{eq} - \boldsymbol{f}_-) \end{cases}$$

with $\boldsymbol{f}_\pm^{eq} = \frac{1}{2}\boldsymbol{U} \pm \frac{\boldsymbol{F}(\boldsymbol{U})}{2\lambda}$.

High-order CFL free schemes and unstructured meshes

Consistency

Consistency space

- **Exact transport**: the choice of the velocities link time and space discretization.
- **Semi- Lagrangian**: Interpolation $2q+1$ gives a consistency error $O(\frac{h^{2d+2}}{\Delta t})$.

Consistency in time

- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \left(\frac{(2-\omega)\Delta t}{2\omega}\right) \partial_x \left(D(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

- with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$ and $D(\boldsymbol{U}) = \left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq} - A(\boldsymbol{U})^2\right)$.

Drawback

- For $[D1Q2]^2$ scheme we have a large error: $D(\boldsymbol{U}) = \left(\lambda^2 I_d - A(\boldsymbol{U})^2\right)$

High-Order time schemes

Second-order scheme

- Scheme for transport step $T(\Delta t)$: Crank Nicolson or exact time scheme.
- Classical full second order scheme:

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t) \circ T\left(\frac{\Delta t}{2}\right).$$

- **Numerical test**: first and second order splitting: converge at second order.
- Second order: probably only for the macroscopic variables.
- AP full second order scheme:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{2}\right) \circ R\left(\frac{\Delta t}{2}\right) \circ T\left(\frac{\Delta t}{4}\right).$$

- Ψ and Ψ_{ap} symmetric in time. $\Psi_{ap}(0) = I_d$.

High order scheme

- Using composition method

$$M_p(\Delta t) = \Psi_{ap}(\gamma_1 \Delta t) \circ \Psi_{ap}(\gamma_2 \Delta t)..... \circ \Psi_{ap}(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.
- Splitting non AP for $\varepsilon = 0$ converge with high-order for macroscopic variables.

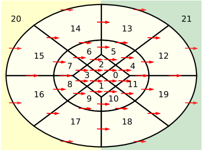
Semi Lagrangian methods

- Forward or Backward methods. Mass or nodes interpolation/projection.
- **Advantages**:
  - Possible on unstructured meshes. High order in space.
  - **Exact in time** and Matrix-free.
- **Drawbacks**:
  - No dissipation and difficult on very unstructured grids.

Implicit FV- DG methods

- Implicit Crank Nicolson scheme + FV DG scheme
- **Advantages**:
  - Very general meshes. High order in space. Dissipation to stabilize.
  - Upwind fluxes ==> triangular block matrices.
- **Drawbacks**:
  - Second order in time: numerical time dispersion.

- Current choice 1D: SL-scheme.
- Current choice in 2D-3D: DG schemes.
  - Block - triangular matrix solved avoiding storage.
  - Solve the problem in the topological order given by connectivity graph.



Burgers: convergence results

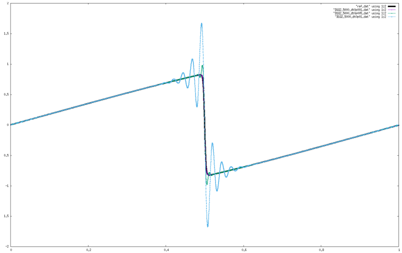
- **Model**: Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 2000 cells, degree 11.
- **Test**: $\rho(t = 0, x) = sin(2\pi x)$. $T_f = 0.14$ (before the shock) and no viscosity.
- Scheme: splitting schemes and Suzuki composition + splitting.

	SPL 1, $\theta = 1$		SPL 1, $\theta = 0.5$		SPL 2, $\theta = 0.5$		Suzuki	
Δt	Error	order	Error	order	Error	order	Error	order
0.005	$2.6E^{-2}$	-	$1.3E^{-3}$	-	$7.6E^{-4}$	-	$4.0E^{-4}$	-
0.0025	$1.4E^{-2}$	0.91	$3.4E^{-4}$	1.90	$1.9E^{-4}$	2.0	$3.3E^{-5}$	3.61
0.00125	$7.1E^{-3}$	0.93	$8.7E^{-5}$	1.96	$4.7E^{-5}$	2.0	$2.4E^{-6}$	3.77
0.000625	$3.7E^{-3}$	0.95	$2.2E^{-5}$	1.99	$1.2E^{-5}$	2.0	$1.6E^{-7}$	3.89

- Scheme: second order splitting scheme.
- Same test after the shock:



1D isothermal Euler : Convergence

- **Model**: isothermal Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2 \rho) = 0 \end{cases}$$

- **Lattice**: $(D1 - Q2)^n$ Lattice scheme.
- For the transport (and relaxations step) we use 6-order DG scheme in space.
- **Time step**: $\Delta t = \beta \frac{\Delta x}{\lambda}$ with λ the lattice velocity. $\beta = 1$ explicit time step.
- **First test**: acoustic wave with $\beta = 50$ and $T_f = 0.4$, **Second test**: smooth contact wave with $\beta = 100$ and $T_f = 20$.

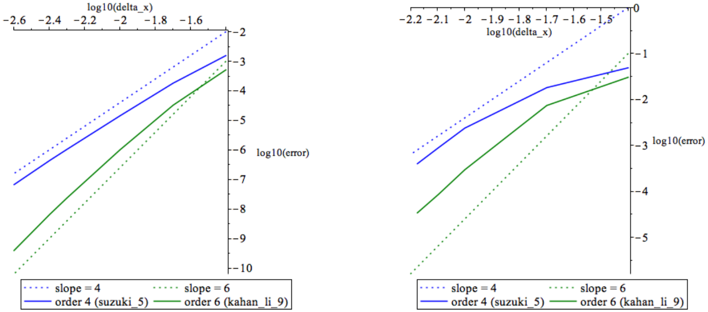


Figure: convergence rates for the first test (left) and for the second test (right).

Numerical results: 2D MHD drifting vortex

- **Model** : compressible ideal MHD.
- **Kinetic model** : $(D2-Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2^{nd} order Implicit DG scheme. 4th order ins space. CFL around 20.
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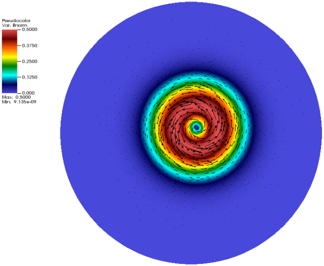
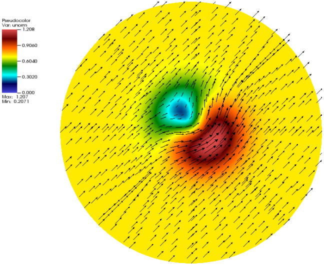
Magnetic field



Velocity



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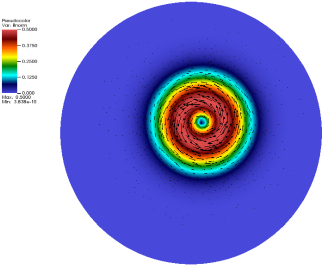

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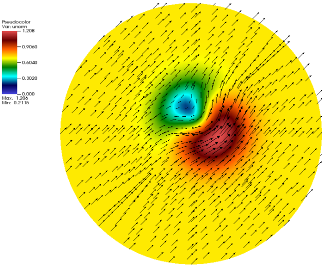

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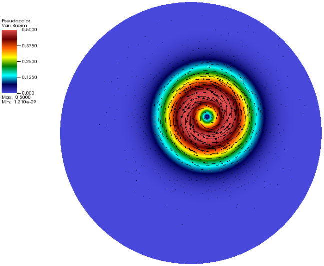

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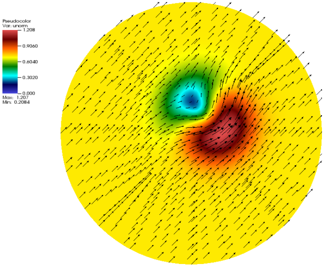

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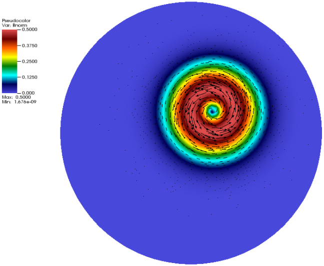

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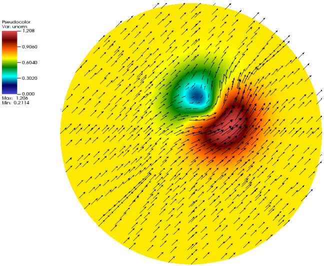

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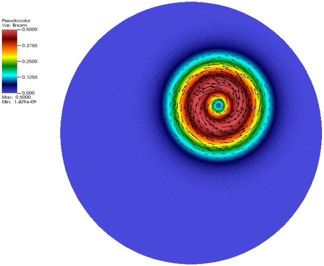

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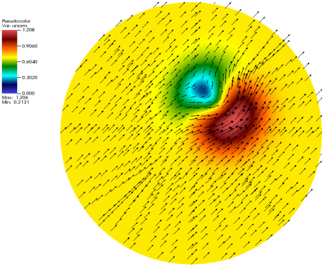

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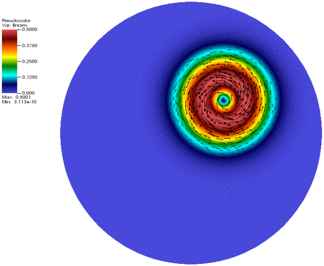

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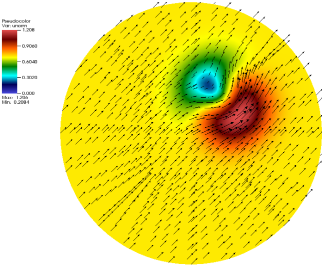

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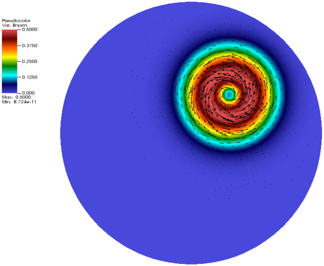

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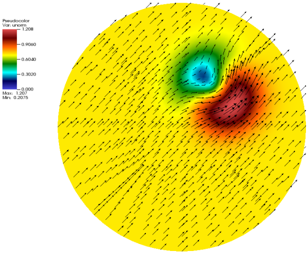

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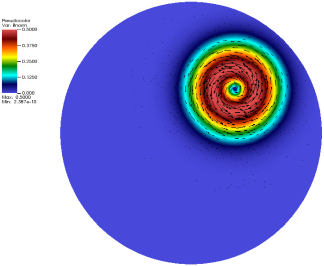

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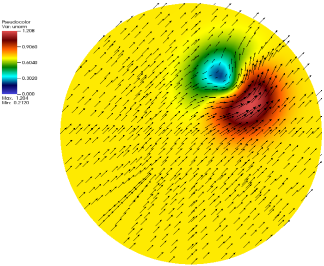

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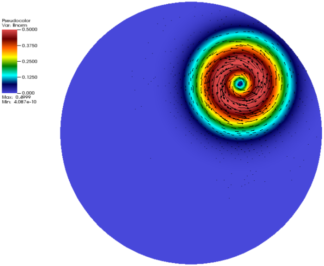

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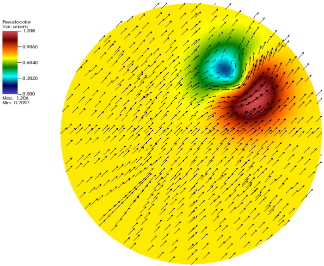

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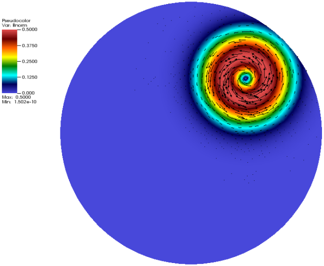

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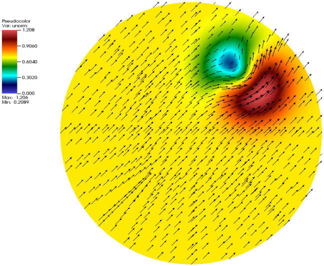

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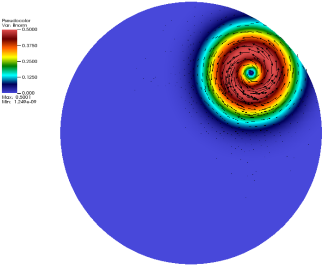

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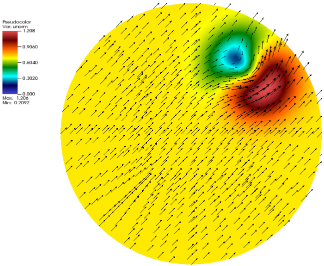

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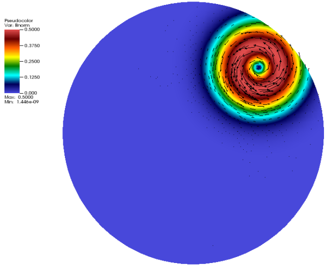

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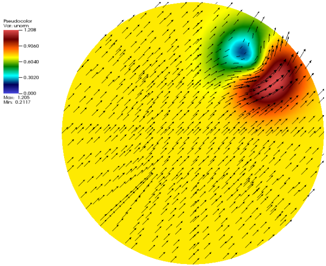

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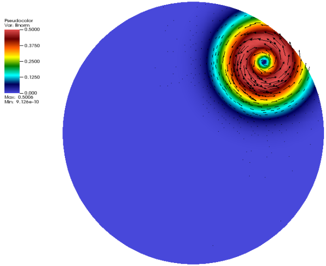

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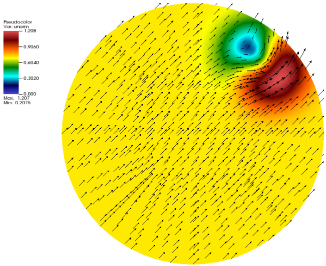

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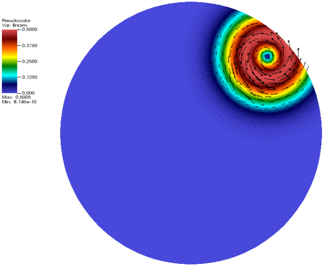

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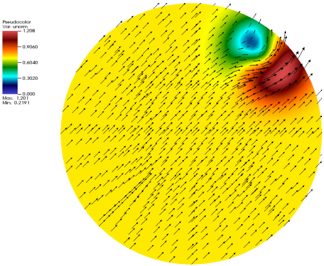

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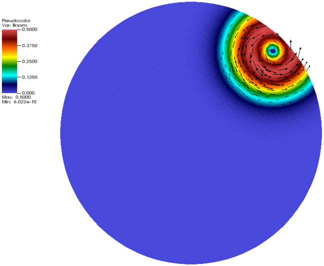

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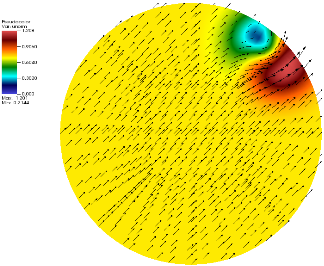

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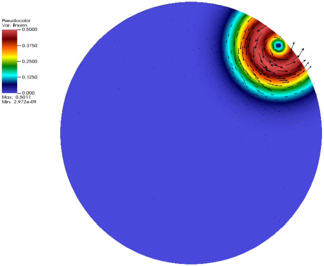

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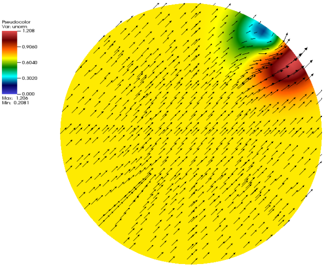

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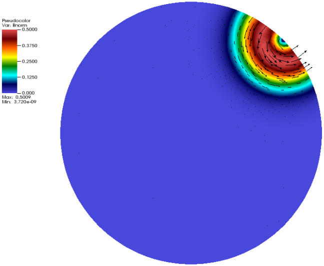

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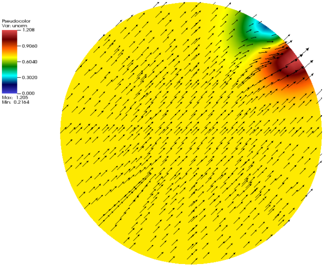

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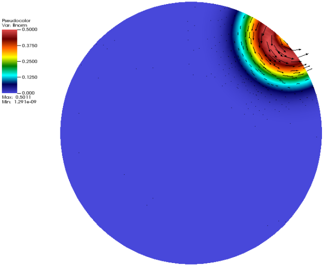

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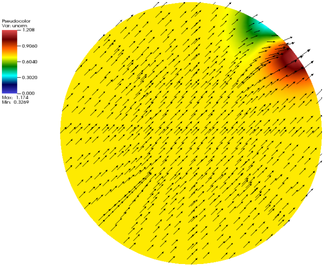

Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

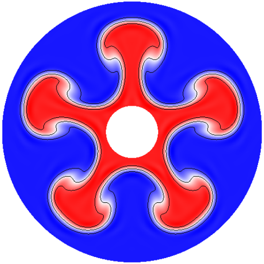

Figure: Plot of the mass fraction of gas

3D case in cylinder

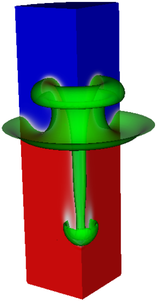

Figure: Plot of the mass fraction of gas

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- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

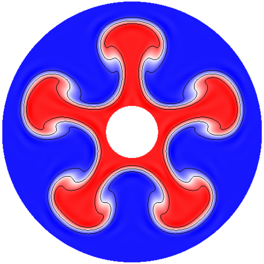


Figure: Plot of the mass fraction of gas

2D cut of the 3D case

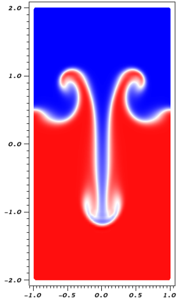


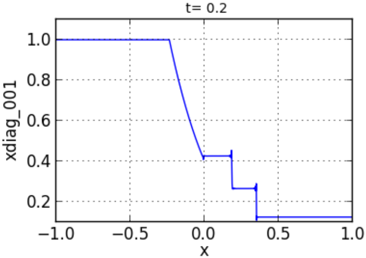
Figure: Plot of the mass fraction of gas

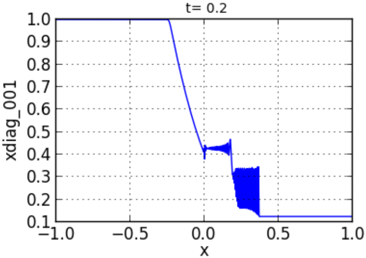
Classical kinetic representation

Limitation

- High-order extension allows to correct the main default of relaxation: large error.
- In two situations the High-order extension is not sufficient:
  - For discontinuous solutions like shocks.
  - For strongly multi-scale problem like low-Mach problem.

- **Euler equation**: Sod problem.
- Second order time scheme + SL scheme:





- Left: density $\Delta t = 1.0^{-4}$. Right: density $\Delta t = 4.0^{-4}$
- **Conclusion**: shock and high order time scheme needs limiting methods.

Classical kinetic representation

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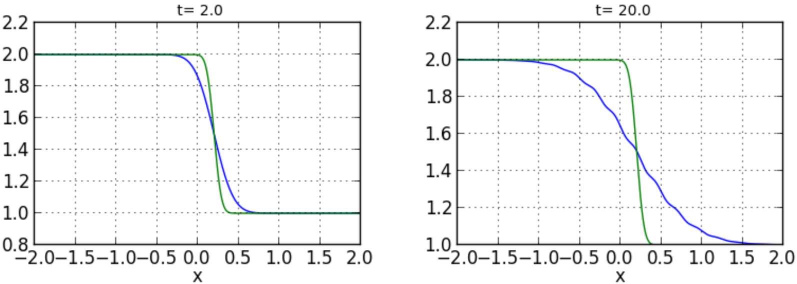
- In two situations the High-order extension is not sufficient:
  - For discontinuous solutions like shocks.
  - For strongly multi-scale problem like low-Mach problem.

- **Euler equation**: smooth contact (u =cts, p=cts).
- First/Second order time scheme + SL scheme. $T_f = \frac{2}{M}$ and 100 time step.



- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: First order method too much dissipative for low Mach flow (dissipation with acoustic coefficient).

Classical kinetic representation

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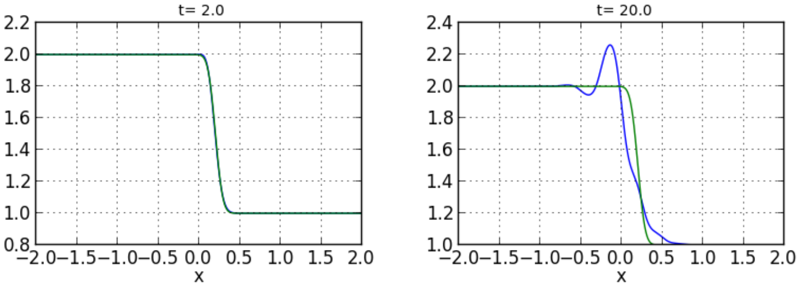


- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: Second order method too much dispersive for low Mach flow (dispersion with acoustic coefficient).

Kinetic representation for multi-scale problems

Classical kinetic representation

"Physic" kinetic representations

- Kinetic model mimics the moment model of Boltzmann equation. Euler isothermal

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2\rho) = 0 \end{cases}$$

- D1Q3 model: three velocities $\{-\lambda, 0, \lambda\}$. Equilibrium: quadrature of Maxwellian.

$$\rho = f_- + f_0 + f_+, \quad q = \rho u = -\lambda * f_- + 0 * f_0 + \lambda * f_+, \quad \boldsymbol{f}_{eq} = \begin{pmatrix} \frac{1}{2}(\rho u(u-\lambda) + c^2\rho) \\ \rho(\lambda^2 - u^2 - c^2) \\ \frac{1}{2}(\rho u(u+\lambda) + c^2\rho) \end{pmatrix}$$

- **Limit model** :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + c^2\rho) = \varepsilon\left(\partial_{xx} u + u^3 \partial_{xx}\rho\right) \end{cases}$$

- **Good point**: no diffusion on ρ equation. **Bad point**: stable only for low mach. No natural extension for more complex pde.

Vectorial kinetic representations

- Vectorial kinetic model (B. Graille 14): $[D1Q2]^2$ one relaxation model $\{-\lambda, \lambda\}$.
- **Good point**: stable on sub-characteristic condition $\lambda > \lambda_{max}$.
- **Bad point**: Wave structure approximated by transport at maximal velocity. The idea of D1Q2 equivalent to Rusanov scheme idea. Very bad accuracy for equilibrium or multi-scale problems (low mach).

Generic vectorial D1Q3

Idea

- Keep the vectorial structure: more stable since we can diffuse on all the variables.
- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- Consistency condition:

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, \quad \forall k \in \{1..N_c\} \\ \lambda_- f_-^k + \lambda_0 f_0^k + \lambda_+ f_+^k = F^k(\boldsymbol{U}), \quad \forall k \in \{1..N_c\} \end{cases}$$

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, \quad \forall k \in \{1..N_c\} \\ (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k = F^k(\boldsymbol{U}) - \lambda_0 f_0^k, \quad \forall k \in \{1..N_c\} \end{cases}$$

- We assume a decomposition of the flux (Bouchut 03)

$$F^k(\boldsymbol{U}) = F_0^{k,-}(\boldsymbol{U}) + F_0^{k,+}(\boldsymbol{U}) + \lambda_0 I_d$$

- We obtain the following equation for the equilibrium

$$\begin{cases} f_-^k + f_0^k + f_+^k = U^k, \quad \forall k \in \{1..N_c\} \\ (\lambda_- - \lambda_0) f_-^k + (\lambda_+ - \lambda_0) f_+^k = F_0^{k,-}(\boldsymbol{U}) + F_0^{k,+}(\boldsymbol{U}), \quad \forall k \in \{1..N_c\} \end{cases}$$

- By analogy of the kinetic theory and kinetic flux splitting scheme we propose the following decomposition $\sum_{v>0} v f^k = F_0^{k,+}(\boldsymbol{U})$ and $\sum_{v<0} v f^k = F_0^{k,-}(\boldsymbol{U})$.

Generic vectorial D1Q3

Idea

- Keep the vectorial structure: more stable since we can diffuse on all the variables.
- Add a central velocity (equal or close to zero) to capture the slow dynamics.

- The lattice $[D1Q3]^N$ is defined by the velocity set $V = [\lambda_-, \lambda_0, \lambda_+]$ and

$$\begin{cases} \boldsymbol{f}_-^{eq}(\boldsymbol{U}) = -\dfrac{1}{(\lambda_0 - \lambda_-)}\boldsymbol{F}_0^-(\boldsymbol{U}) \\ \boldsymbol{f}_0^{eq}(\boldsymbol{U}) = \left(\boldsymbol{U} - \left(\dfrac{\boldsymbol{F}_0^+(\boldsymbol{U})}{(\lambda_+ - \lambda_0)} - \dfrac{\boldsymbol{F}_0^-(\boldsymbol{U})}{(\lambda_0 - \lambda_-)}\right)\right) \\ \boldsymbol{f}_+^{eq}(\boldsymbol{U}) = \dfrac{1}{(\lambda_+ - \lambda_0)}\boldsymbol{F}_0^+(\boldsymbol{U}) \end{cases}$$

Stability

- Entropy stability: $\boldsymbol{F}_0^+$ and $\boldsymbol{F}_0^-$ is an entropy decomposition of the flux + $\partial \boldsymbol{F}_0^+$, $-\partial \boldsymbol{F}_0^-$ and $1 - \frac{\partial \boldsymbol{F}_0^+ - \partial \boldsymbol{F}_0^-}{\lambda}$ are positive.
- Optimal condition for L^2 stability in linear case not clear.

D1Q3 for scalar case

- First choice: D1Q3 Rusanov ($\lambda_0 = 0$)

$$F_0^-(\rho) = -\lambda_- \frac{(F(\rho) - \lambda_+\rho)}{\lambda_+ - \lambda_-}, \quad F_0^+(\rho) = \lambda_+ \frac{(F(\rho) - \lambda_-\rho)}{\lambda_+ - \lambda_-}$$

- Consistency (for $\lambda_- = -\lambda_+$): $\partial_t\rho + \partial_x F(\rho) = \sigma\Delta t\partial_x \left(\lambda^2 - \mid \partial F(\rho) \mid^2\right) \partial_x\rho + O(\Delta t^2)$

- Second choice: D1Q3 Upwind

$$F_0^-(\rho) = \chi_{\{\partial F(\rho) < \lambda_0\}} (F(\rho) - \lambda_0\rho) \quad F_0^+(\rho) = \chi_{\{\partial F(\rho) > \lambda_0\}} (F(\rho) - \lambda_0\rho)$$

- with χ the indicatrice function.
- Consistency: $\partial_t\rho + \partial_x F(\rho) = \sigma\Delta t\partial_x \left(\lambda \mid \partial F(\rho) \mid - \mid \partial F(\rho) \mid^2\right) \partial_x\rho + O(\Delta t^2)$

- Third choice: D1Q3 Lax-Wendroff ($\lambda_0 = 0$)

$$F_0^-(\rho) = \frac{1}{2}\left(F(\rho) + \frac{\alpha}{\lambda}\int^\rho (\partial F(u))^2\right) \quad F_0^+(\rho) = \frac{1}{2}\left(F(\rho) + \frac{\alpha}{\lambda}\int^\rho (\partial F(u))^2\right)$$

- with $\lambda_0 = 0$ and $\lambda_- = -\lambda_+$ and $\alpha \geq 1$.
- Consistency: $\partial_t\rho + \partial_x F(\rho) = \sigma\Delta t\partial_x \left((\alpha - 1) \mid \partial F(\rho) \mid^2\right) \partial_x\rho + O(\Delta t^2)$.

- The last one is not entropy stable and does not satisfy the sufficient L^2 stability condition.

D1Q3 for Euler equation I

- **Euler equation**. Two regimes where the classical method is not optimal.
  - High-Mach regime: we use a negative and positive transport for purely positive or negative flows.
  - Low-Mach regime: λ is closed to the sound speed so we have viscosity too large for density equation for example.
- First possibility: use classical flux vector splitting for Euler equation.
  - **Stegel-Warming**: $\boldsymbol{F}^{\pm} = A^{\pm}(\boldsymbol{U})\boldsymbol{U}$ with $A^{\pm}$ positive/negative part of the Jacobian.
  - **Van-Leer**:

$$\boldsymbol{F}^{\pm}(\boldsymbol{U}) = \pm\frac{1}{4}\rho c(M \pm 1)^2 \begin{pmatrix} 1 \\ \frac{(\gamma-1)u \pm 2c}{\gamma} \\ \frac{((\gamma-1)u \pm 2c)^2}{2(\gamma+1)(\gamma-1)} \end{pmatrix}$$

  - **AUSM method**: convection of ρ, q and H as Van-Leer and separated reconstruction of the pressure.
  - **Approximate Osher-Solomon**: $\boldsymbol{F}^{\pm}(\boldsymbol{U}) = \boldsymbol{F}(\boldsymbol{U}) \pm \mid \boldsymbol{F}(\boldsymbol{U}) \mid$

$$\mid \boldsymbol{F}(\boldsymbol{U}) \mid \approx \int_{\boldsymbol{U}_0}^{\boldsymbol{U}} \mid A(\boldsymbol{U}) \mid = \int_0^1 \mid A(\boldsymbol{U}_0 + t(\boldsymbol{U} - \boldsymbol{U}_0)) \mid (\boldsymbol{U} - \boldsymbol{U}_0)dt$$

  - Integral is approximated by a quadrature formula along the path (E. Toro , M Dumbser)
  - Approximate of $\mid A \mid$ using Halley approximation (M. J . Castro) and $\boldsymbol{U}_0$ is the average flow.

D1Q3 for Euler equation II

- Low Mach case:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x \left(\rho u^2 + \dfrac{p}{M}\right) = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{cases}$$

- We want to preserve as possible the limit:

$$p = cts, \quad u = cts, \quad \partial_t \rho + u\partial_x \rho = 0$$

- **Idea**: Splitting of the flux (Zha-Bilgen, Toro-Vasquez):

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} (\rho)u \\ (\rho u)u + p \\ (E)u + pu \end{pmatrix}$$

- Idea: Lax-Wendroff Flux splitting for convection and AUSM-type for the pressure term.
- Use only u, p and λ ($\approx c$) to reconstruct pressure. Important to preserve the low mach limit.
- We obtain

$$\boldsymbol{F}^{\pm}(\boldsymbol{U}) = \frac{1}{2}\begin{pmatrix} (\rho u \pm \alpha \frac{u^2}{\lambda}\rho) + p \\ (\rho u^2 \pm \alpha \frac{u^2}{\lambda} q) + p(1 \pm \gamma \frac{u}{\lambda}) \\ (Eu \pm \frac{u^2}{\lambda} E) + (pu \pm \alpha \frac{1}{\lambda}\gamma(u^2 + \lambda^2)p) \end{pmatrix}$$

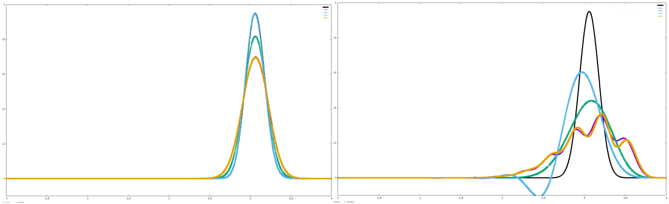
- Preserve contact.
- The scheme is construct to have diffusion error on rho homogeneous $((\alpha - 1)u^2)$ (lax wendroff scheme).

Advection equation

- Equation
$$\partial_t \rho + \partial_x (a(x)\rho) = 0$$
- with $a(x) > 0$ and $\partial_x a(x) > 0$. Dissipative equation.
- **Test case 1**: $a(x) = x$. 10000 cells. Order 17. $\theta = 1$ (first order).

	Rusanov		Upwind		Lax Wendroff	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.05$	$6.4E^{-2}$	-	$2.7E^{-2}$	-	$2.7E^{-2}$	-
$\Delta t = 0.025$	$3.8E^{-2}$	0.75	$1.2E^{-2}$	1.17	$5.7E^{-3}$	2.24
$\Delta t = 0.0125$	$1.9E^{-2}$	1.0	$4.2E^{-3}$	1.5	$5.5E^{-4}$	3.37
$\Delta t = 0.00625$	$7.9E^{-3}$	1.25	$1.3E^{-3}$	1.7	$5.3E^{-5}$	3.38

- **Test case 2**: $a(x) = 1 + 0.01(x - x_0)^2$. 10000 cells. Order 17. Second order time scheme.



- Left $\Delta t = 0.01$. Right $\Delta t = 0.1$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 1$ (blue), Lax-Wendroff $\alpha = 2$ (Yellow).

Advection equation

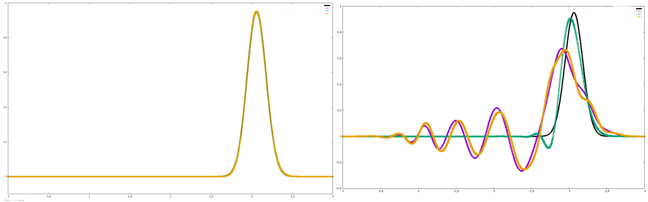
- Equation

$$\partial_t \rho + \partial_x (a(x)\rho) = 0$$

- with $a(x) > 0$ and $\partial_x a(x) > 0$. Dissipative equation.
- **Test case 1**: $a(x) = x$. 10000 cells. Order 17. $\theta = 0.5$ (second order).

	Rusanov		Upwind		Lax Wendroff	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.05$	$3.8E^{-2}$	-	$1.2E^{-4}$	-	$1.2E^{-0}$	-
$\Delta t = 0.025$	$5.3E^{-3}$	2.84	$8.1E^{-6}$	3.8	$4.1E^{-1}$	1.55
$\Delta t = 0.0125$	$3.7E^{-4}$	3.84	$5.3E^{-7}$	3.84	$1.1E^{-4}$	11.5
$\Delta t = 0.00625$	$2.3E^{-5}$	3.88	$3.3E^{-8}$	4	$6.2E^{-6}$	4.15

- **Test case 2**: $a(x) = 1 + 0.01(x - x_0)^2$. 10000 cells. Order 17. Second order time scheme.



- Left $\Delta t = 0.01$. Right $\Delta t = 0.1$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 2$ (Yellow) $= 1$ unstable.

Burgers

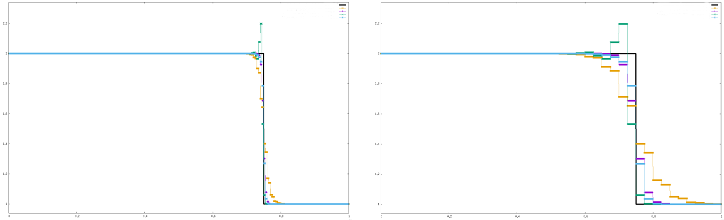
- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- **Test case 1**: $\rho(t=0,x) = sin(2\pi x)$. 10000 cells. Order 17. First order time scheme.

	Rusanov		Upwind		Lax Wendroff $\alpha = 1$	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.01$	$3.9E^{-2}$	-	$1.1E^{-2}$	-	$2.3E^{-3}$	-
$\Delta t = 0.005$	$2.1E^{-2}$	0.89	$6.4E^{-3}$	0.78	$6.0E^{-4}$	1.94
$\Delta t = 0.0025$	$1.1E^{-2}$	0.93	$3.5E^{-3}$	0.87	$1.5E^{-4}$	2.00
$\Delta t = 0.00125$	$5.4E^{-3}$	1.03	$1.8E^{-3}$	0.96	$3.9E^{-5}$	1.95

- Shock wave. First order scheme in time.



- Left $\Delta t = 0.002$. Right $\Delta t = 0.01$. Reference (black), Rusanov (yellow), Upwind (violet), Lax-Wendroff (green), Lax-Wendroff $\alpha = 1.5$ (blue).

Burgers

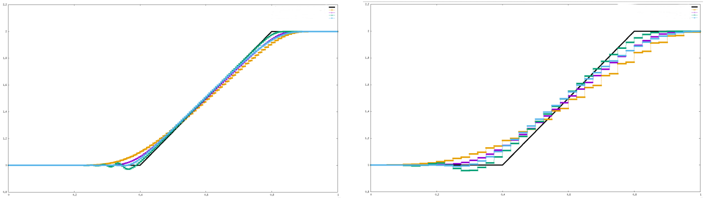
- **Model**: Viscous Burgers equations

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- **Test case 1**: $\rho(t=0,x) = sin(2\pi x)$. 10000 cells. Order 17. First order time scheme.

	Rusanov		Upwind		Lax Wendroff $\alpha = 1$	
	Error	Order	Error	Order	Error	Order
$\Delta t = 0.01$	$3.9E^{-2}$	-	$1.1E^{-2}$	-	$2.3E^{-3}$	-
$\Delta t = 0.005$	$2.1E^{-2}$	0.89	$6.4E^{-3}$	0.78	$6.0E^{-4}$	1.94
$\Delta t = 0.0025$	$1.1E^{-2}$	0.93	$3.5E^{-3}$	0.87	$1.5E^{-4}$	2.00
$\Delta t = 0.00125$	$5.4E^{-3}$	1.03	$1.8E^{-3}$	0.96	$3.9E^{-5}$	1.95

- Rarefaction wave. First order scheme in time.



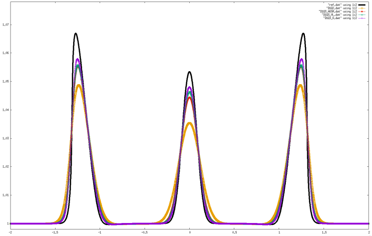
- Left $\Delta t = 0.002$. Right $\Delta t = 0.01$. Reference (black), Rusanov (violet), Upwind (green), Lax-Wendroff $\alpha = 1$ (blue), Lax-Wendroff $\alpha = 2$ (Yellow).

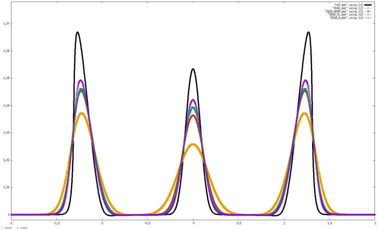
1D Euler equations

- **Model**: Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{cases}$$

- **Test case**: acoustic wave. $\rho = 1 + 0.1e^{-\frac{x^2}{\sigma}}$, $u = 0$ and $p = \rho$.
- The domain is $\Omega = [-2, 2]$. 4000 cells and 11-order SL. $\theta = 1$ (relaxation).





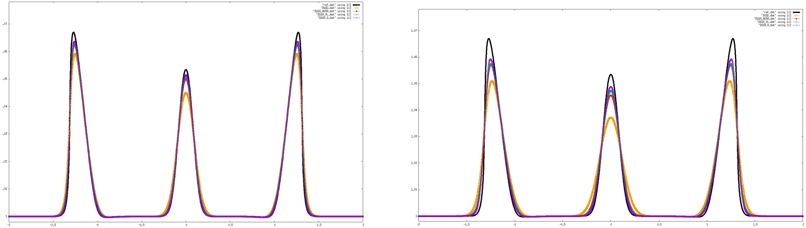
- Left $\Delta t = 0.002$. Right $\Delta t = 0.005$. Reference (black), Rusanov (yellow), Van-Leer (green), Osher (violet), AUSM (red).

- **Conclusion**: Osher and Van-Leer more accurate that Rusanov. Low-Mach less accurate for acoustic that the two other, but very accurate on the material wave.

- **Model**: Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{cases}$$

- **Test case**: acoustic wave. $\rho = 1 + 0.1 e^{-\frac{x^2}{\sigma}}$, $u = 0$ and $p = \rho$.
- The domain is $\Omega = [-2, 2]$. 4000 cells and 11-order SL. $\theta = 0.666$ (relaxation).



- Left $\Delta t = 0.002$. Right $\Delta t = 0.005$. Reference (black), Rusanov (yellow), Van-Leer (green), Osher (violet), AUSM (red).

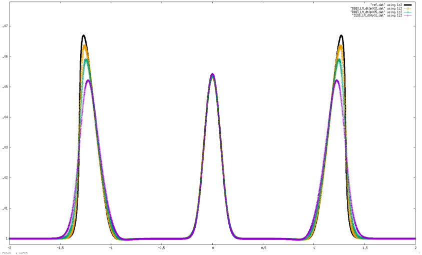
- **Conclusion**: Osher and Van-Leer more accurate that Rusanov. Low-Mach less accurate for acoustic that the two other, but very accurate on the material wave.

1D Euler equations

- **Model**: Euler equation

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = 0 \\ \partial_t E + \partial_x(Eu + pu) = 0 \end{cases}$$

- **Test case**: acoustic wave. $\rho = 1 + 0.1e^{-\frac{x^2}{\sigma}}$, $u = 0$ and $p = \rho$.
- The domain is $\Omega = [-2, 2]$. 4000 cells and 11-order SL. $\theta = 0.666$ (relaxation).



- Same test case for the low-mach scheme with $\omega = 1$. $\Delta t = 0.002$ (yellow), $\Delta t = 0.005$ (green), $\Delta t = 0.01$ (violet).

- **Conclusion**: Osher and Van-Leer more accurate that Rusanov. Low-Mach less accurate for acoustic that the two other, but very accurate on the material wave.

1D Euler equations II

- **Test case**: Smooth contact. We take $p = 1$ and u is also constant.
- **Final aim**: Where $T_f = O(\frac{1}{u})$ we want take $\Delta t = O(\frac{1}{u})$ and preserve the same error when u decrease.

- We choose $\Delta t = 0.02$ and $T_f = 2$. 4000 cells. We choose $\omega = 1$:

	Schemes	Rusanov	VL	Osher	LM
$u = 10^{-2}$	$\rho(t,x)$	0.26	$1.0E^{-1}$	$8.4E^{-2}$	$1.0E^{-3}$
	$u(t,x)$	0	$3.4E^{-3}$	$6.0E^{-7}$	0
	$p(t,x)$	0	$5.0E^{-4}$	$4.3E^{-8}$	0
$u = 10^{-4}$	$\rho(t,x)$	0.26	$1.0E^{-1}$	$8.4E^{-2}$	$1.0E^{-5}$
	$u(t,x)$	0	$3.4E^{-3}$	$6.0E^{-7}$	0
	$p(t,x)$	0	$5.0E^{-4}$	$4.3E^{-8}$	0
$u = 0$	$\rho(t,x)$	0.26	$1.0E^{-1}$	$4.8E^{-2}$	0.0
	$u(t,x)$	0	$3.4E^{-3}$	$6.0E^{-7}$	0
	$p(t,x)$	0	$5.0E^{-4}$	$4.3E^{-8}$	0

- **Drawback**: When the time step is too large we have dispersive effect.
- **Possible explanation**: the error would be homogeneous to

$$| \rho^n(x) - \rho(t,x) | \approx \left[O((\alpha - 1)\Delta t u^2) + O(\Delta t^2 u \lambda^q) \right] T_f .$$

- with λ closed to the sound speed.
- **Problem**: At the second order, we recover partially the problem since λ is closed to the sound speed.

1D Euler equations III

- **Possible solution**: decrease λ for the density equation.
- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$\underbrace{V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]}_{\text{slow scale}}$$

- The convective part associated at the slow scale. The acoustic part associated at the fast scale.
- **Smooth contact**: We take 200 time step and $\Delta t = \frac{0.001}{u}$:

Error	$u = 10^{-1}$	$u = 10^{-2}$	$u = 10^{-3}$	$u = 10^{-4}$
$\alpha = 1$	$2.5E^{-3}$	$2.5E^{-3}$	$2.5E^{-3}$	$2.5E-3$
λ_s	2	0.2	0.02	0.002
λ_f	2	20	200	2000

Conclusion

- **Conclusion**: the error would be homogeneous to

$$\mid \rho^n(x) - \rho(t,x) \mid \approx \left[O((\alpha-1)\Delta t u^2) + O(\Delta t^2 u \lambda_s^q) \right] T_f.$$

- with λ_s which can be taken as $O(u)$.
- **Drawback**: For the stability it seems necessary to have

$$\lambda_s \lambda_f \geq C \max_x (u + c)$$

1D Euler equations III

- **Possible solution**: decrease λ for the density equation.
- We propose two-scale kinetic model.
- We consider the following $[D1Q5]^3$ based on the following velocities:

$$\underbrace{V = [-\lambda_f, -\lambda_s, 0, \lambda_s, \lambda_f]}_{\text{fast scale}}$$

- The convective part associated at the slow scale. The acoustic part associated at the fast scale.
- **Smooth contact**: We take 200 time step and $\Delta t = \frac{0.001}{u}$:

Error	$u = 10^{-1}$	$u = 10^{-2}$	$u = 10^{-3}$	$u = 10^{-4}$
$\alpha = 1$	$2.5E^{-3}$	$2.5E^{-3}$	$2.5E^{-3}$	$2.5E-3$
λ_s	2	0.2	0.02	0.002
λ_f	2	20	200	2000

Conclusion

- **Conclusion**: the error would be homogeneous to

$$\mid \rho^n(x) - \rho(t,x) \mid \approx \left[O((\alpha - 1)\Delta t u^2) + O(\Delta t^2 u \lambda_s^q) \right] T_f .$$

- with λ_s which can be taken as $O(u)$.
- **Drawback**: For the stability it seems necessary to have

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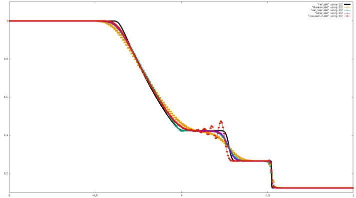
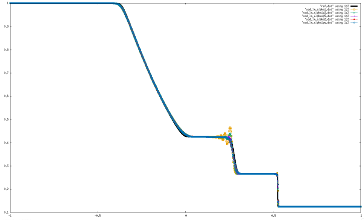
1D Euler equations IV

- Test case: Sod problem. 4000 cells, First order is space and time.
- Comparison of schemes:



- Reference (black), Rusanov (orange), Van-Leer (green), Osher (violet), Low-Mach with $\alpha = 1$ (red).
- Comparison of low-mach scheme for different values of α:



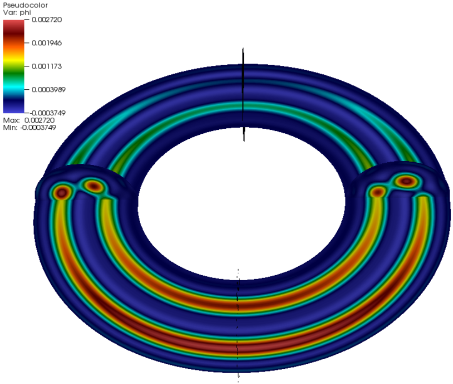
- Results for $\omega = 1.0$, $\Delta t = 0.001$. Reference (black), $\alpha = 1$ (orange), $\alpha = 1.2$ (green), $\alpha = 1.5$ (violet), $\alpha = 2$ (red), $\alpha = 1 + u$ (blue).

Kinetic relaxation method for Diffusion problem

Main parabolic problem

- Coupling anisotropic diffusion + resistivity.

$$\begin{cases} \partial_t T - \nabla \cdot ((\boldsymbol{B} \otimes \boldsymbol{B}) \nabla T + \varepsilon \nabla T) = 0 \\ \partial_t \boldsymbol{B} - \eta \nabla \times (T^{-\frac{5}{2}} \nabla \times \boldsymbol{B}) = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$



- The temperature T for the case $\eta = 0$ and $\boldsymbol{B}$ given by magnetic equilibrium.

Kinetic model and scheme for diffusion I

- We solve the equation: $\partial_t \rho + \partial_x(u\rho) = D\partial_{xx}\rho$
- D1Q2 Kinetic system proposed (S. Jin, F. Bouchut):

$$\begin{cases} \partial_t f_- - \frac{\lambda}{\varepsilon}\partial_x f_- = \frac{1}{\varepsilon^2}(f_{eq}^- - f_-) \\ \partial_t f_+ + \frac{\lambda}{\varepsilon}\partial_x f_+ = \frac{1}{\varepsilon^2}(f_{eq}^+ - f_+) \end{cases}$$

- with $f_{eq}^{\pm} = \frac{\rho}{2} \pm \frac{\varepsilon(u\rho)}{2\lambda}$. The limit is given by:

$$\partial_t \rho + \partial_x(u\rho) = \partial_x((\lambda^2 - \varepsilon^2 \mid u \mid^2)\partial_x \rho) + \lambda^2\varepsilon^2\partial_x(\partial_{xx}(u\rho) + u\partial_{xx}\rho) - \lambda^2\varepsilon^2\partial_{xxxx}\rho$$

- We introduce $\alpha >\mid u \mid$. Choosing $D = \lambda^2 - \varepsilon^2\alpha^2$ we obtain

$$\partial_t \rho + \partial_x(u\rho) = \partial_x(D\partial_x \rho) + O(\varepsilon^2)$$

- We can choose $\varepsilon = \Delta t^{\gamma}$ and $\omega = 2$.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
	Error	order	Error	order	Error	order
$\Delta t = 0.04$	$1.87E^{-2}$	-	1.43	-	1.43	-
$\Delta t = 0.02$	$6.57E^{-3}$	1.50	0.2	0	0.23	0
$\Delta t = 0.01$	$1.85E^{-3}$	1.82	0.2	0	0.23	0
$\Delta t = 0.005$	$3.6E^{-4}$	2.36	0.2	0	0.23	0
$\Delta t = 0.0025$	$7.3E^{-5}$	2.30	0.2	0	0.23	0

- The splitting scheme is not AP.

Kinetic model and scheme for diffusion II

Consistency analysis

- We consider $\partial_t \rho - D\partial_{xx}\rho = 0$.
- We define the two operators for each step:

$$T_{\Delta t} : e^{\Delta t \frac{\Lambda}{\varepsilon}\partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \rho = \Delta t \partial_x \left(\left(\frac{1-\omega}{\omega} + \frac{1}{2} \right) \frac{\lambda^2}{\varepsilon^2} \partial_x \rho \right) + O(\Delta t^2)$$

- Taking $D = \lambda^2$, $\theta = 0.5$ and $\varepsilon = \sqrt{\Delta t}$ we obtain the diffusion equation.
- **Question**: what is the error term is this case ?
- **First results** (for these choices of parameters):
  - Second order at the numerical level.
  - At the minimum the first order theoretically.
- **Problem**: For a large time step, the scheme oscillate. **How reduce this** ?

Kinetic scheme for anisotropic/nonlinear diffusion

- We consider the diffusion equation with $\partial_t \rho - \partial_x (A(\rho, x)\partial_x \rho) = 0$ with $D(\rho, x) > 0$.
- We consider a kinetic system

$$\partial_t \boldsymbol{f} + \frac{\Lambda}{\varepsilon}\partial_x \boldsymbol{f} = \frac{R(x,\rho)}{\varepsilon^2}(\boldsymbol{f}^{eq} - \boldsymbol{f})$$

- We define $P\boldsymbol{f} = \sum_i^N f_i = \rho$ and $Q\boldsymbol{f} = \frac{1}{\varepsilon}\sum_i^N v_i f_i = u$.
- If

$$P\boldsymbol{f}_{eq} = \rho, \quad Q\boldsymbol{f}^{eq} = 0, \quad \sum_i^N v_i^2 f_i^{eq} = \alpha\rho$$

and

$$P\left[R(x,\rho)(\boldsymbol{f}^{eq} - \boldsymbol{f})\right] = 0, \quad Q\left[R(x,\rho)(\boldsymbol{f}^{eq} - \boldsymbol{f})\right] = -\alpha D^{-1} Q\boldsymbol{f}$$

- We obtain the equivalence with the following model (which gives at the limit the diffusion model)

$$\begin{cases} \partial_t \rho + \partial_x v = 0 \\ \partial_t v + \dfrac{\alpha}{\varepsilon^2}\partial_x \rho = -\dfrac{\alpha}{D(x,\rho)\varepsilon^2} v \end{cases}$$

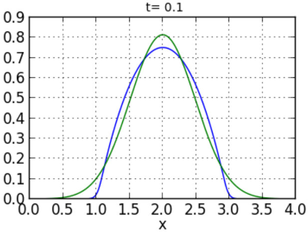
- **Example**: D1Q2

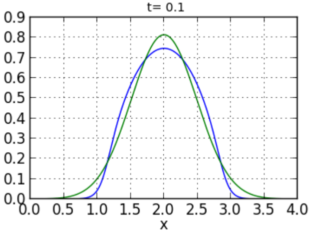
$$\begin{cases} \partial_t f_+ + \dfrac{1}{\varepsilon}\partial_x f_+ = \dfrac{1}{D(x,\rho)\varepsilon^2}(f_+^{eq} - f_+) \\ \partial_t f_- - \dfrac{1}{\varepsilon}\partial_x f_- = \dfrac{1}{D(x,\rho)\varepsilon^2}(f_-^{eq} - f_-) \end{cases}$$

- with $f_\pm^{eq} = \frac{1}{2}\rho$.

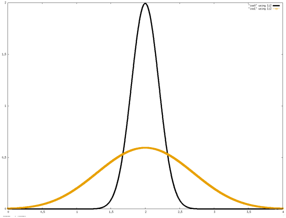
Results for anisotropic/nonlinear diffusion

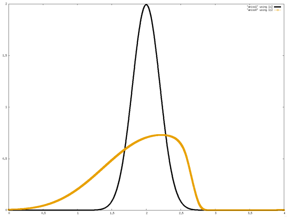
- We want solve the equation: $\partial_t \rho - \partial_{xx} D(\rho) = 0$
- $p = 1$ (green) $p = 2$ (blue). Left $\Delta t = 0.001$. Right $\Delta t = 0.005$.





- The second kinetic scheme allows to treat also nonlinear diffusion.
- We want solve the equation: $\partial_t \rho = \partial_x (A(x) \partial_x \rho)$.





- Left: $A(x) = 1$. Right: $\frac{1}{2}(1 - erf(5(x - x_0)))$. Black : initial data. Yellow: final data.

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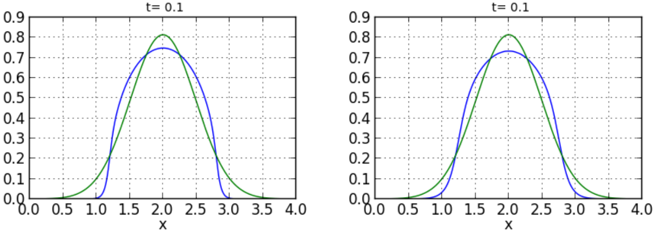

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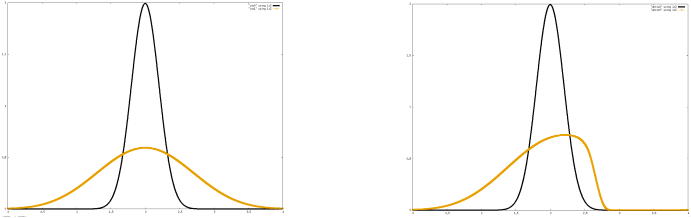

- Left: $A(x) = 1$. Right: $\frac{1}{2}(1 - erf(5(x - x_0)))$. Black : initial data. Yellow: final data.

Conclusion

LBM as relaxation scheme

- **LBM** method can be rewritten as a specific scheme for BGK model.
- Using this, we propose high-order scheme with large time step algorithm (SL method).
- This algorithm is very competitive against implicit scheme (no matrices, no solvers).

D1Q3/5 schemes

- The $[D1Q3]^n$ schemes allows to reduce the error compared to $[D1Q2]^n$.
- Using the flux-vector splitting FV method we obtain new $[D1Q3]^n$.
- The $[D1Q3]^n$ Osher scheme is generic for hyperbolic systems.
- We propose a new $[D1Q3]^n$ scheme for low-Mach. **Problem**: stability for $\omega \approx 1$. Modification ?

Kinetic scheme and LBM for diffusion

- These methods allows to treat also the diffusion equations using the splitting error.
- Poor/correct accuracy for anisotropic diffusion/heat equation. Need to be increased.

Future works

- 2D/3D diffusion and low-Mach, MHD, BC, Dispersive waves, Limiting methods, Machine Learning.