Implicit kinetic relaxation schemes. Application to the plasma physic

D. Coulette[5], E. Franck[1,2], P. Helluy[1,2],
C. Courtes[2], L. Navoret[2], L. Mendoza[2], F. Drui[2]

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[1]Inria Nancy Grand Est, France
[2]IRMA, Strasbourg university, France
[3]Wurzburg university, Germany
[5]ENS Lyon, France

Outline

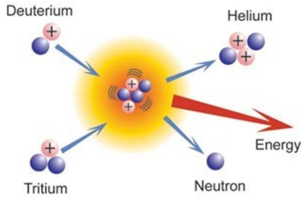
Physical and mathematical context

Iter Project and nuclear fusion

Applications

- Modeling and numerical simulation for the nuclear fusion.

- **Fusion DT**: At sufficiently high energies deuterium and tritium (plasmas) can fuse to Helium. Free energy is released.
- **Plasma**: For very high temperature, the gas is ionized and give a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroïdal chamber where the plasma (10^8 Kelvin), is confined using magnetic fields. **Larger Tokamak**: Iter



Difficulties:

- Plasma turbulence (center of the Tokamak).
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- Necessary to simulate these phenomena and test some controls in realistic geometries of Tokamak.

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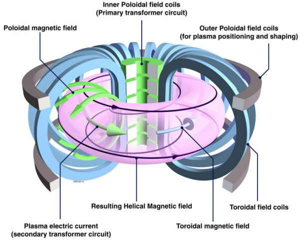

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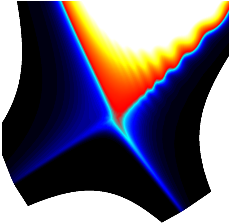

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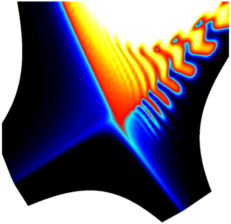

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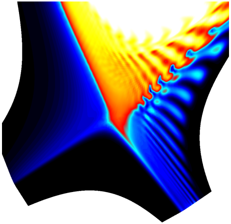

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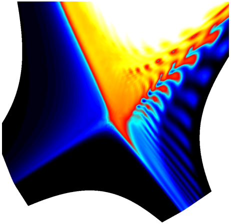

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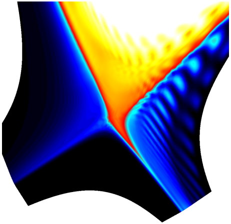

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Models

Model for the center: Gyro-kinetic model

$$\begin{cases} \partial_t(B_{\|} f) + \nabla \cdot \left(\frac{d\mathbf{x}_g}{dt} f\right) + \partial_{v_{\|}} \left(B_{\|} \frac{dv_{\|}}{dt} f\right) = 0 \\ -\nabla \cdot_{\perp} (\rho_e(\mathbf{x}) \nabla_{\perp} \phi) = \rho(\mathbf{x}) - 1 + S(\phi) \end{cases}$$

- The guiding center motion $\frac{d\mathbf{x}_g}{dt}$ and $\frac{dv_{\|}}{dt}$ depend of $\boldsymbol{B}_{\|}$ and $\nabla\phi$) and ρ is the density of the gyro-distribution f.
- **Other models**: Vlasov-Maxwell or Poisson.
- **Kinetic models** coupled with **elliptic model**.

Model for the edge: Resistive MHD

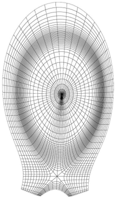
$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla \cdot \boldsymbol{\Pi} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = \nabla \cdot ((\kappa \boldsymbol{B} \otimes \boldsymbol{B} + \varepsilon I_d) \nabla T) + \eta \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \eta \nabla \times (\nabla \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- Scaling: $\nu, \eta << 1$ and $\kappa \mid \boldsymbol{B} \mid^2 >> 1 >> \varepsilon$.
- **Other models**: Reduced incompressible MHD model or Extended MHD.
- **Hyperbolic model** coupled with **parabolic model**.

Geometries and times schemes

Geometry

- **3D geometry**: Torus with a non circular section.
- **Poloidal geometry**: aligned with the magnetic surfaces of the equilibrium.
- Non structured grids and singularities.



Time schemes for kinetic model

- **Vlasov**: large kinetic velocities.
- **Vlasov**: large poloïdal velocities due to the electric field variation.
- Characteristic time larger that time associated to fast velocities. We need CFL-free schemes.
- **Turbulence**: We need high-order scheme and fine grids.

Time schemes for MHD model

- **Anisotropic diffusion**: We need CFL-free schemes.
- **Perp magneto-acoustic waves**: larger than characteristic velocity. Needs CFL-free schemes.
- **Usual schemes**: Implicit high-order schemes. Very hard to invert the nonlinear problem.

Kinetic model and SL schemes

Semi Lagrangian scheme

- One of the main scheme to treat transport and kinetic equations.
- **Idea**: use the characteristic method.

- Example: Backward SL

$$\partial_t f + a\partial_x f = 0$$

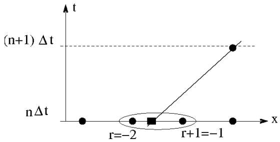
  - **Aim**: compute at the mesh point x_j:

$$f(t+\Delta t, x_j)$$

  - Solution:

$$f(t+\Delta t, x_j) = f(t, x_j - a\Delta t)$$

  - $x_n = x_j - a\Delta t$ is not a mesh point.
  - Using $f(t, x_i)$ we interpolate the function at x_n.



- **BSL/FSL**: follow the backward characteristic and interpolate/follow the forward characteristic and distribute on the mesh.
- **Different type of SL**: Classical SL (punctual values), Conservative SL (Average cell values), DG/CG SL (weak form of SL scheme).

Advantages/drawbacks

- **Advantages**: infinite/high order in time/space. **CFL-less and no matrix inversion**.
- **Drawbacks**: BC and Gibbs oscillations due to high-order methods.
- **Interesting works**: Positive SL (B. Després), Artificial diffusion for SL, limiting.

Aim

Aim:

Construct High-Order Solver like SL-Solver (no matrix inversion, no CFL) for the different type of PDE.

Approximate BGK model for hyperbolic systems

Relaxation scheme

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \lambda^2 \partial_x u = \frac{1}{\varepsilon}(F(u) - v) \end{cases}$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_x ((\lambda^2 - | \partial F(u) |^2) \partial_x u) + O(\varepsilon^2)$$

- **Stability**: the limit system is dissipative if $(\lambda^2 - | \partial F(u) |^2) > 0$.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\begin{cases} \partial_t f_- - \lambda \partial_x f_- = \frac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \partial_t f_+ + \lambda \partial_x f_+ = \frac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{cases}$$

- with $u = f_- + f_+$ and $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\lambda}$.

First Generalization

- **Main property**: the transport is diagonal (D1Q2 model) which can be easily solved.
- **Generalization**: one Xin-Jin or D1Q2 model by macroscopic variable.

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $U \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- We define the macroscopic variable by $P\boldsymbol{f} = \boldsymbol{U}$.

- Consistence condition (Natalini - Aregba [96-98-20], Bouchut [99-03]) :

$$\mathcal{C} \begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) & = \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) & = \boldsymbol{F}(\boldsymbol{U}) \end{cases}$$

- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial \boldsymbol{f} \, \boldsymbol{eq} - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

- Natural extension in 2D/3D.
- **General scheme**: $[D1Q2]^n$, one D1Q2 by macroscopic equation.

Time discretization

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Many schemes**: Jin-Filbet [10], Dimarco-Pareschi [11-14-17], Lafitte-Samaey [17] etc.
- **Main idea**: splitting scheme between transport and the relaxation (Dellar [13]).
- **Key point**: the macroscopic variables are conserved during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ explicit.
- **Scheme**: Theta-scheme for the relaxation and SL (or implicit DG) scheme for the transport.

First order scheme (first order transport)

- We define the two operators for each step :

$$T_{\Delta t} : (I_d + \Delta t \Lambda \partial_x I_d)\boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{\Delta t}{2}\partial_x(P\Lambda^2 \partial_x \boldsymbol{f}) + \left(\frac{(2-\omega)\Delta t}{2\omega}\right)\partial_x\left(D(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

- with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$ and $D(\boldsymbol{U}) = (P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq} - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2)$.

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First order scheme (exact transport)

- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n)$$

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High-Order time schemes

Second-order scheme

- **Order of convergence**: one for the kinetic variables. one or two ($\omega = 2$ and exact transport) for the macroscopic variables.
- **Second order scheme**: Strang Splitting + SL scheme

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t, \omega = 2) \circ T\left(\frac{\Delta t}{2}\right).$$

High order scheme: composition method

$$M_p(\Delta t) = \Psi(\gamma_1 \Delta t) \circ \Psi(\gamma_2 \Delta t) \circ ..... \circ \Psi(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

CV and new scheme

- All the schemes convergence only with the second order for the kinetic variables.
- Loose of order also for macroscopic variables (see numerical results).
- The 2th order scheme satisfies $\Psi(\Delta t) = \Psi^{-1}(-\Delta t)$ but not $\Psi(\Delta t = 0) \neq I_d$. Correction:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R(\Delta t, \omega = 2) \circ T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t, \omega = 2) \circ T\left(\frac{\Delta t}{4}\right)$$

Burgers: convergence results

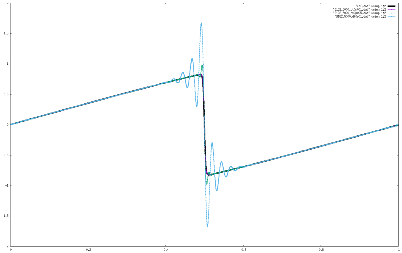
- **Model**: Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 2000 cells, degree 11.
- **Test**: $\rho(t=0,x) = sin(2\pi x)$. $T_f = 0.14$ (before the shock) and no viscosity.
- Scheme: splitting schemes and Suzuki composition + splitting.

	SPL 1, $\theta = 1$		SPL 1, $\theta = 0.5$		SPL 2, $\theta = 0.5$		Suzuki	
Δt	Error	order	Error	order	Error	order	Error	order
0.005	$2.6E^{-2}$	-	$1.3E^{-3}$	-	$7.6E^{-4}$	-	$4.0E^{-4}$	-
0.0025	$1.4E^{-2}$	0.91	$3.4E^{-4}$	1.90	$1.9E^{-4}$	2.0	$3.3E^{-5}$	3.61
0.00125	$7.1E^{-3}$	0.93	$8.7E^{-5}$	1.96	$4.7E^{-5}$	2.0	$2.4E^{-6}$	3.77
0.000625	$3.7E^{-3}$	0.95	$2.2E^{-5}$	1.99	$1.2E^{-5}$	2.0	$1.6E^{-7}$	3.89

- Scheme: second order splitting scheme.
- Same test after the shock:



Remark on the relaxation scheme

- **Classical result**: Strang Splitting + second order/exact scheme for relaxation converge at first order for $\varepsilon \approx 0$. SL solver + Strang splitting.

	CN		Exact		SSP RK	
	Error	Order	Error	Order	Error	Order
$\Delta t = 4.10^{-3}$	$4.8E^{-4}$	-	$2.0E^{-2}$	-	$2.0E^{-2}$	-
$\Delta t = 2.10^{-3}$	$1.2E^{-4}$	2.0	$1.1E^{-2}$	0.86	$1.1E^{-2}$	0.86
$\Delta t = 1.10^{-3}$	$2.9E^{-5}$	2.05	$5.7E^{-3}$	0.95	$5.5E^{-3}$	1.0
$\Delta t = 5.10^{-4}$	$7.4E^{-6}$	1.95	$2.9E^{-3}$	0.97	$2.8E^{-3}$	0.98

- **Conclusion**: we lose one order of cv with exact and SPP-RK solver.

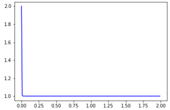
- **Scheme** for $\varepsilon \approx 0$:
- For Euler implicit, exact and SSP-RK2 schemes.

$$\boldsymbol{f}^{n+1} \approx \boldsymbol{f}^{eq}(\boldsymbol{U}^n) + O(\varepsilon)$$

- For Crank-Nicolson.

$$\boldsymbol{f}^{n+1} \approx 2\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n + O(\varepsilon)$$

- We solve the EDO $\partial_t u = \frac{1}{\varepsilon}(u_{eq} - u)$.



- Implicit Euler scheme. $\Delta t = 100\varepsilon$

Conclusion:

- If you begin far to $\boldsymbol{f}^{eq}$ the exact/SPP-RK solvers seems better.
- However, for high-order splitting scheme the over-relaxation (CN) seems important .
- At the limit $\varepsilon = 0$ this scheme is revertible contrary the other.

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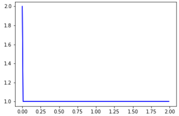

- Exact time scheme. $\Delta t = 100\varepsilon$

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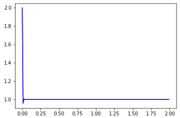

- SSP RK2 scheme. $\Delta t = 100\varepsilon$

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$\Delta t = 2.10^{-3}$	$1.2E^{-4}$	2.0	$1.1E^{-2}$	0.86	$1.1E^{-2}$	0.86
$\Delta t = 1.10^{-3}$	$2.9E^{-5}$	2.05	$5.7E^{-3}$	0.95	$5.5E^{-3}$	1.0
$\Delta t = 5.10^{-4}$	$7.4E^{-6}$	1.95	$2.9E^{-3}$	0.97	$2.8E^{-3}$	0.98

- **Conclusion**: we lose one order of cv with exact and SPP-RK solver.

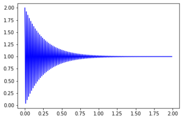
- **Scheme** for $\varepsilon \approx 0$:
- For Euler implicit, exact and SSP-RK2 schemes.

$$\boldsymbol{f}^{n+1} \approx \boldsymbol{f}^{eq}(\boldsymbol{U}^n) + O(\varepsilon)$$

- For Crank-Nicolson.

$$\boldsymbol{f}^{n+1} \approx 2\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n + O(\varepsilon)$$

- We solve the EDO $\partial_t u = \frac{1}{\varepsilon}(u_{eq} - u)$.



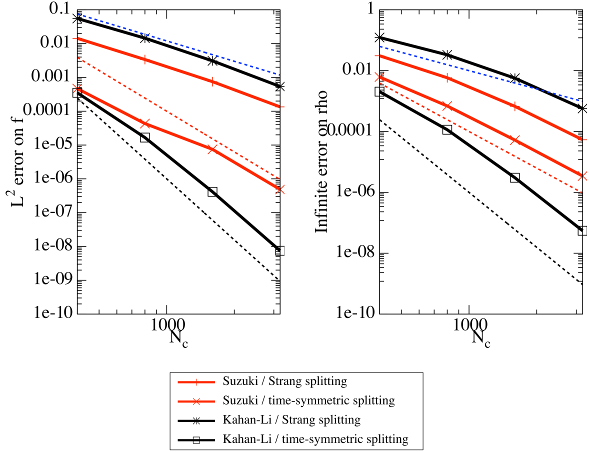
- Crank-Nicolson scheme. $\Delta t = 100\varepsilon$

Conclusion:

- If you begin far to $\boldsymbol{f}^{eq}$ the exact/SPP-RK solvers seems better.
- However, for high-order splitting scheme the over-relaxation (CN) seems important .
- At the limit $\varepsilon = 0$ this scheme is revertible contrary the other.

Convergence

- **Equation**: Euler isothermal
- Model $[D1Q2]^2$ High-order space scheme. Comparison of the time scheme.
- **Test case**: smooth solution. $\Delta t = \frac{\beta \Delta x}{\lambda}$ with $\beta = 50$



- With Strang splitting: only order 2 for f.
- Loss of convergence for macroscopic variables for Kahan-li + Strang splitting.

Numerical results: 2D MHD drifting vortex

- **Model** : compressible ideal MHD.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
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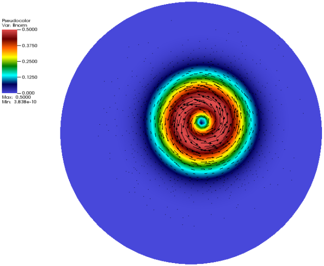
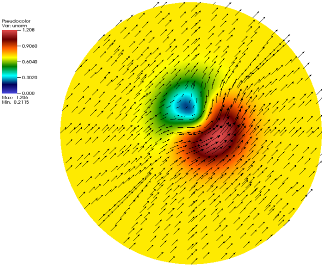
Magnetic field



Velocity



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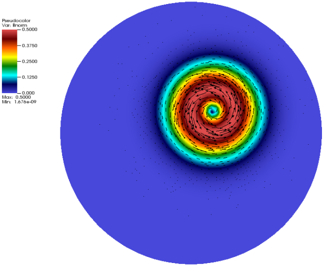

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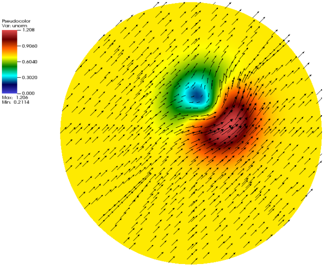

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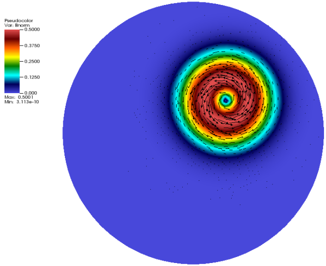

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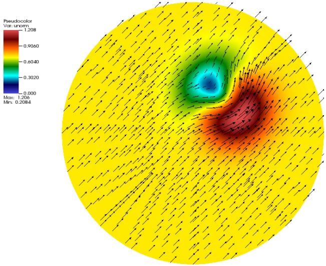

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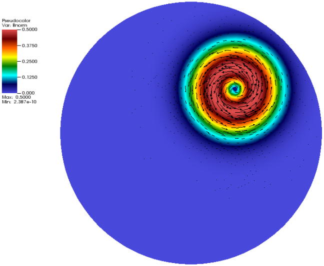

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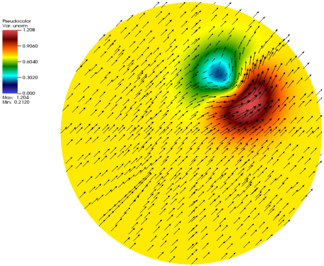

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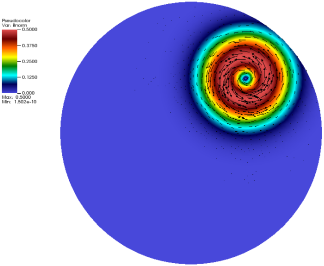
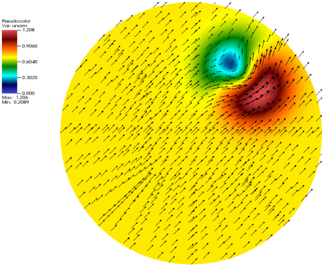


Velocity



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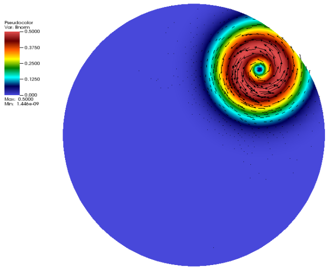

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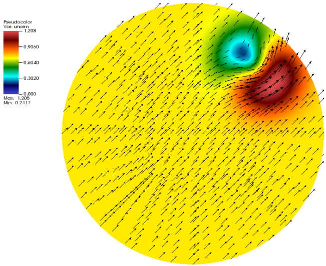

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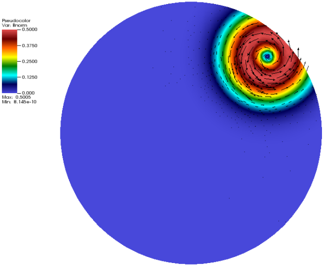

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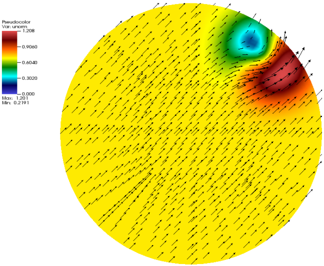

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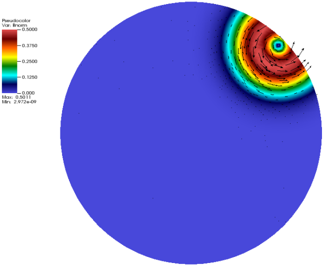

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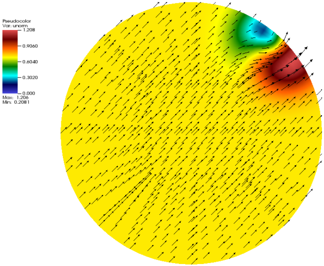

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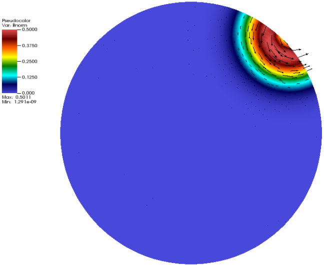
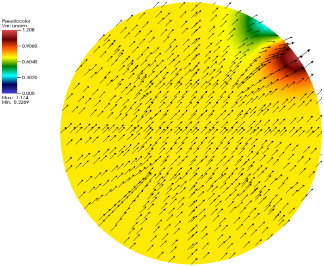


Velocity



Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2-Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

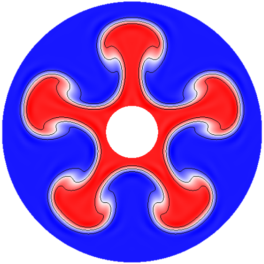
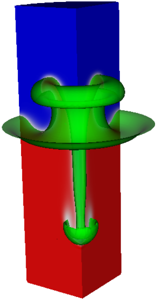


Figure: Plot of the mass fraction of gas

3D case in cylinder



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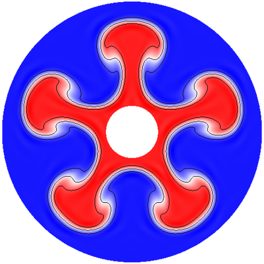


Figure: Plot of the mass fraction of gas

2D cut of the 3D case

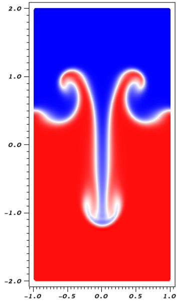


Figure: Plot of the mass fraction of gas

Limit of the method

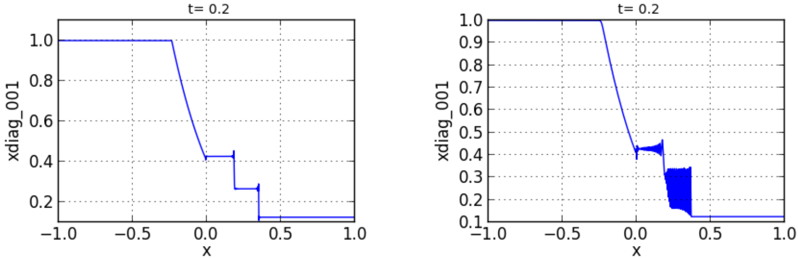
Limitation

- High-order extension allows to correct the main default of relaxation: large error.

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \sigma \Delta t \partial_x((\lambda^2 I_d - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + O(\Delta t^2 \lambda^3)$$

- In two situations the High-order extension is not sufficient:
  - For discontinuous solutions like shocks.
  - For strongly multi-scale problem like low-Mach problem.

- **Euler equation**: Sod problem.
- Second order time scheme + SL scheme:



- Left: density $\Delta t = 1.0^{-4}$. Right: density $\Delta t = 4.0^{-4}$
- **Conclusion**: shock and high order time scheme needs limiting methods.

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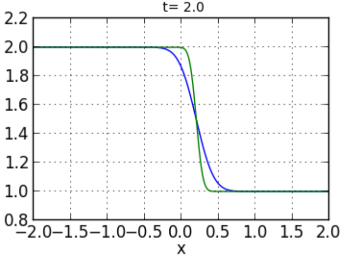

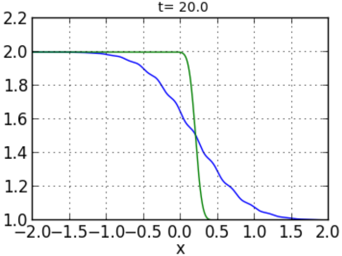

- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: First order method too much dissipative for low Mach flow (dissipation with acoustic coefficient).

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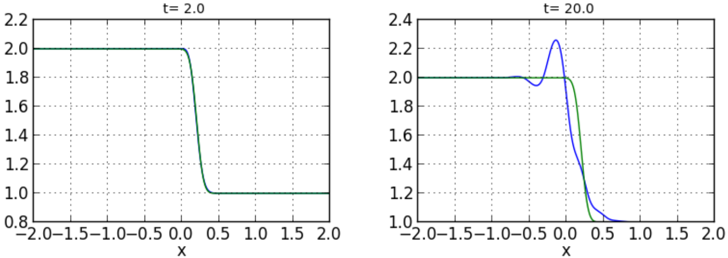

- Order 1 Left: $M = 0.1$. Right: $M = 0.01$
- **Conclusion**: Second order method too much dispersive for low Mach flow (dispersion with acoustic coefficient).

D1Q3 models and low Mach limit I

- Default of $[D1Q2]^n$ model: diffusion/dispersion homogenous to larger speed.
- **Aim**: reduce this error for the slow part of the waves. **Idea**: introduce slow or null velocity.

$[D1Q3]^N$ model

- Velocity set $V = [\lambda_-, 0, \lambda_+]$ and

$$\begin{cases} \boldsymbol{f}_-^{eq}(\boldsymbol{U}) = \dfrac{1}{\lambda_-}\boldsymbol{F}^-(\boldsymbol{U}) \\ \boldsymbol{f}_0^{eq}(\boldsymbol{U}) = \left(\boldsymbol{U} - \left(\dfrac{\boldsymbol{F}^+(\boldsymbol{U})}{\lambda_+} + \dfrac{\boldsymbol{F}^-(\boldsymbol{U})}{\lambda_-}\right)\right) \\ \boldsymbol{f}_+^{eq}(\boldsymbol{U}) = \dfrac{1}{\lambda_+}\boldsymbol{F}^+(\boldsymbol{U}) \end{cases}$$

- $\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{F}^+(\boldsymbol{U}) + \boldsymbol{F}^-(\boldsymbol{U})$ the "positive" and "negative" flux.

- **Example**: Burgers

$$F^-(\rho) = \frac{1}{2}\left(F(\rho) + \frac{\alpha}{\lambda}\int^{\rho}(\partial F(u))^2\right) \quad F^+(\rho) = \frac{1}{2}\left(F(\rho) + \frac{\alpha}{\lambda}\int^{\rho}(\partial F(u))^2\right)$$

- f $\alpha = \frac{\lambda}{|\rho|}$ we obtain

$$\partial_t\rho + \partial_x\left(\frac{\rho^2}{2}\right) = \sigma\Delta t\partial_x\left(\left(|\ \rho\ |\ \lambda - \rho^2\right)\partial_x\rho\right)$$

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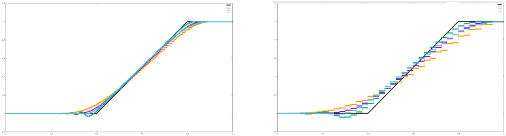
- If $\alpha = 1$ we obtain

$$\partial_t \rho + \partial_x\left(\frac{\rho^2}{2}\right) = O(\Delta t^2)$$

D1Q3 models and low Mach limit II

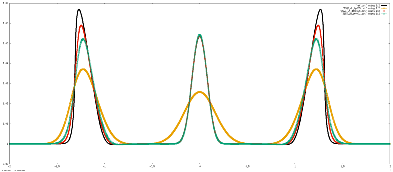
- Rarefaction wave. First order scheme in time.



- Left/right $\Delta t = 0.002/0.01$. D1Q2 (yellow), $\alpha = \frac{\lambda}{|\rho|}, 1, 2$ (blue, green, violet).
- Euler equation: Flux splitting for low-mach flow.
- **Idea**: Splitting of the flux (Zha-Bilgen, Toro-Vasquez):

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} (\rho)u \\ (\rho u)u + p \\ (E)u + pu \end{pmatrix}$$

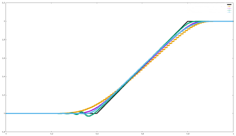
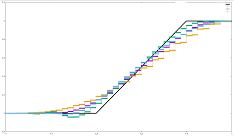
- **Test case**: Acoustic wave. SL order 11, 4000 cells.



- D1Q2 $\Delta t = 0.005$ (yellow), D1Q3 $\Delta t = 0.005/0.01$ (red, green). Contact captured.

D1Q3 models and low Mach limit II

- Rarefaction wave. First order scheme in time.

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- Euler equation: Flux splitting for low-mach flow.
- Idea: Lax-Wendroff Flux splitting for convection and AUSM-type for the pressure term.

$$\boldsymbol{F}^{\pm}(\boldsymbol{U}) = \frac{1}{2}\begin{pmatrix} (\rho u \pm \alpha \frac{u^2}{\lambda}\rho) + p \\ (\rho u^2 \pm \alpha \frac{u^2}{\lambda} q) + p(1 \pm \gamma \frac{u}{\lambda}) \\ (E u \pm \alpha \frac{u^2}{\lambda} E) + (pu \pm \frac{1}{\lambda}\gamma(u^2 + \lambda^2)p) \end{pmatrix}$$

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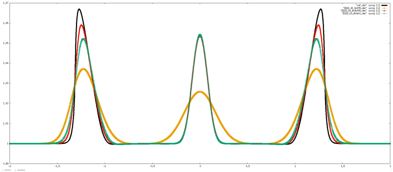


- D1Q2 $\Delta t = 0.005$ (yellow), D1Q3 $\Delta t = 0.005/0.01$ (red, green). Contact captured.

BC : preliminary results

- **Question**: What BC for the kinetic variables. How keep the order ?

First result

- The second order symmetric scheme (Ψ_{ap}) for the equation

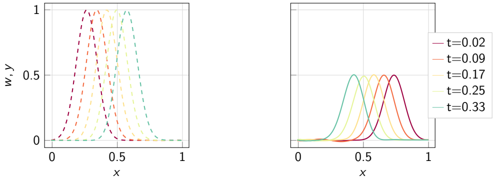
$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

is consistant with

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\Delta t^2) \\ \partial_t \boldsymbol{W} - \partial \boldsymbol{F}(\boldsymbol{U}) \partial_x \boldsymbol{W} = O(\Delta t^2) \end{cases}$$

with $\boldsymbol{W} = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}$.

- **Natural BC**: in condition for $\boldsymbol{U}$ and $\boldsymbol{W} = 0$ or $\partial_x \boldsymbol{W} = 0$.
- Example: $F(u) = cu$ (transport):



- Transport of the w (dashed lines) and $y = z - f(w)$ (plain lines) quantities.

BC : preliminary results

- **Question**: What BC for the kinetic variables. How keep the order ?

First result

- The second order symmetric scheme (Ψ_{ap}) for the equation

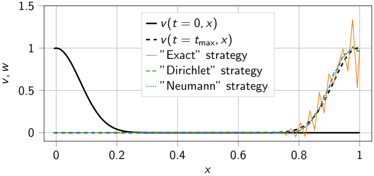
$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

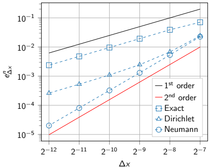
is consistant with

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\Delta t^2) \\ \partial_t \boldsymbol{W} - \partial \boldsymbol{F}(\boldsymbol{U}) \partial_x \boldsymbol{W} = O(\Delta t^2) \end{cases}$$

with $\boldsymbol{W} = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}$.

- **Natural BC**: in condition for $\boldsymbol{U}$ and $\boldsymbol{W} = 0$ or $\partial_x \boldsymbol{W} = 0$.
- Example: $F(u) = cu$ (transport):





- Initial state and comparison of the final states. Gaussian initial profile, $\Delta x = 2^{-7}$.

Approximate BGK model for parabolic systems

Relaxation scheme for diffusion

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u - \nu \partial_{xx} u = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \dfrac{\lambda^2}{\varepsilon^2} \partial_x u = -\dfrac{1}{\varepsilon^2} v \end{cases}$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u - \partial_x (\lambda^2 \partial_x u) = \varepsilon^2 \partial_{xxxx} u + O(\varepsilon^4)$$

- **Consistency**: Choosing $\lambda^2 = \nu$ we obtain the initial solution.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\begin{cases} \partial_t f_- - \dfrac{\lambda}{\varepsilon} \partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^- - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon} \partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+ - f_+) \end{cases}$$

- with $u = f_- + f_+$ and $f_{eq}^{\pm} = \frac{u}{2}$.

First Generalization

- **Main property**: the transport is diagonal (D1Q2 model) which can be easily solved.

Generalisation

- We consider the equation

$$\partial_t u - \partial_x (D(x,u)\partial_x u) = 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities and $u = \sum_i^{n_v} w_i f_i$
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \frac{\Lambda}{\varepsilon}\partial_x \boldsymbol{f} = \frac{R(u,x)}{\varepsilon^2}(\boldsymbol{f}^{eq}(u) - \boldsymbol{f})$$

Limit system

- We assume $\sum_i v_i f_i^{eq} = 0$, $\sum_i v_i^2 f_i^{eq} = \alpha u$
- $\sum_i (R(\boldsymbol{f}^{eq} - \boldsymbol{f}))_i = 0$,
- and $\sum_i v_i (R(\boldsymbol{f}^{eq} - \boldsymbol{f}))_i = -\alpha D(x,u)^{-1} v$. In this case we have an equivalence with

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \frac{\alpha}{\varepsilon^2}\partial_x u = -\frac{\alpha}{D(x,\rho)\varepsilon^2} v \end{cases}$$

- which gives at the limit

$$\partial_t u - \partial_x (D(x,u)\partial_x u) = O(\varepsilon^2)$$

Discretization

Consistency analysis

- We consider $\partial_t \rho - D\partial_{xx}\rho = 0$.
- We define the two operators for each step:

$$T_{\Delta t} : e^{\Delta t \frac{\Lambda}{\varepsilon}\partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \rho = \Delta t \partial_x \left(\left(\frac{1-\omega}{\omega} + \frac{1}{2} \right) \frac{\lambda^2}{\varepsilon^2} \partial_x \rho \right) + O(\Delta t^2)$$

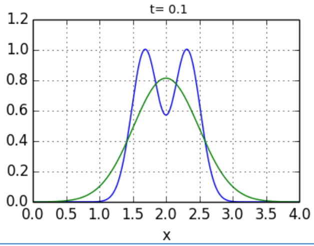
- We don't have convergence for all ε. The splitting scheme is not AP
- Taking $D = \lambda^2$, $\theta = 0.5$ and $\varepsilon = \sqrt{\Delta t}$ we obtain the diffusion equation.
- **Question**: When you choose like this. Consistence or not ?
- **First results** (for these choices of parameters):
  - Second order at the numerical level.
  - At the minimum the first order theoretically.

Numerical results for diffusion equation

- Heat equation. Scheme with $\varepsilon = \Delta t^{\gamma}$ and very high order SL + fine grid.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
	Error	order	Error	order	Error	order
$\Delta t = 0.04$	$1.87E^{-2}$	-	1.43	-	1.43	-
$\Delta t = 0.02$	$6.57E^{-3}$	1.50	0.2	0	0.23	0
$\Delta t = 0.01$	$1.85E^{-3}$	1.82	0.2	0	0.23	0
$\Delta t = 0.005$	$3.6E^{-4}$	2.36	0.2	0	0.23	0
$\Delta t = 0.0025$	$7.3E^{-5}$	2.30	0.2	0	0.23	0

- We want solve the equation: $\partial_t \rho - \partial_{xx} \rho = 0$
- $\Delta t = 0.1$. The scheme oscillate. We cannot take very large time step.



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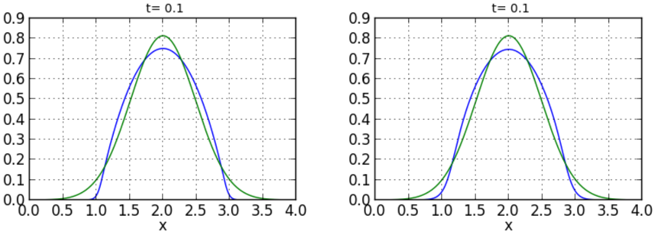
- We want solve the equation: $\partial_t \rho - \partial_{xx} D(\rho) = 0$
- $p = 1$ (green) $p = 2$ (blue). Left $\Delta t = 0.001$. Right $\Delta t = 0.005$.



Numerical results for diffusion equation

- Heat equation. Scheme with $\varepsilon = \Delta t^{\gamma}$ and very high order SL + fine grid.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
	Error	order	Error	order	Error	order
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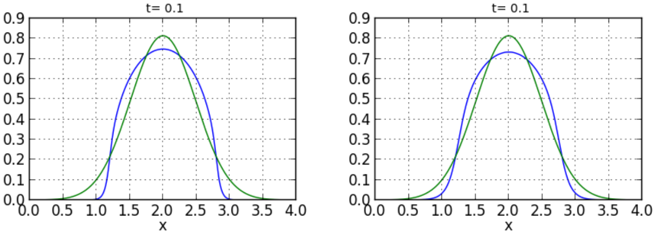
- We want solve the equation: $\partial_t \rho - \partial_{xx} D(\rho) = 0$
- $p = 1$ (green) $p = 3$ (blue). Left $\Delta t = 0.001$. Right $\Delta t = 0.005$.



Numerical results for diffusion equation

- Heat equation. Scheme with $\varepsilon = \Delta t^{\gamma}$ and very high order SL + fine grid.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
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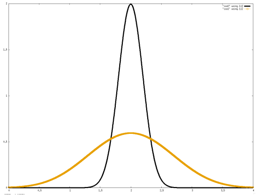
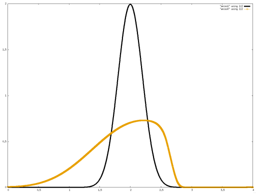
- We want solve the equation: $\partial_t \rho = \partial_x(A(x)\partial_x \rho)$.





- Left: $A(x) = 1$. Right: $\frac{1}{2}(1 - erf(5(x - x_0))$. Black : initial data. Yellow: final data.

Conclusion

Time scheme for BGK

- **High order Method**: Composition + Strang Splitting (or modified version) + Crank-Nicolson scheme for relaxation.
- **Default**: scheme not accurate (compare to Jin-Filbet/Pareschi-Gimarco schemes) far to the equilibrium.
- **Advantage**: independent transport equation. Useful with implicit transport solver.

Implicit Kinetic relaxation schemes

- We can approximate hyperbolic/parabolic PdE by small BGK models (Elliptic also).
- Using this, we propose high-order scheme with large time step algorithm (SL method).
- This algorithm is very competitive against implicit scheme (no matrices, no solvers).

Future works

- Validate nonlinear/anisotropic at the order 2 and try to reduce the constant error.
- Study the scheme for elliptic problems.
- 1D scheme for low-mach. Extension in 2D/3D and improve stability.
- Application to MHD and anisotropic diffusion for plasma.
- Continue the study for the BC.
- Propose artificial viscosity method for the total scheme (relaxation and SL steps) to avoid oscillations.