Splitting scheme for full and reduced MHD systems. Application to JOREK

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Outline

Splitting and nonlinear solver: Full MHD

Fusion and Tokamak Instabilities

Time scheme in JOREK

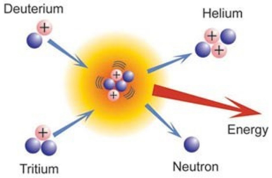
- Crank-Nicolson or BD2 scheme time scheme.
- **Solver**: GMRES + Specific Block-Jacobi preconditioning.

Advantages/ drawbacks

- **Advantages**: very efficient in the linear phase. Accurate time scheme.
- **Drawbacks**: less accurate in the nonlinear phase. Huge memory consumption for Jacobian and PC.

Physic-Based preconditioning

- **Idea**: splitting + parabolization + Jacobian-free method.
- **Drawbacks**: Not easy to understand, to modify.
- **Aim**: time splitting scheme for MHD.



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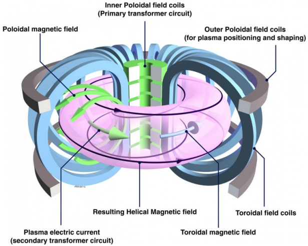

Figure: Tokamak

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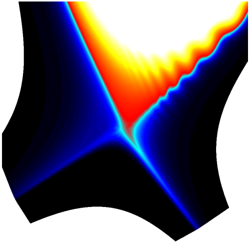

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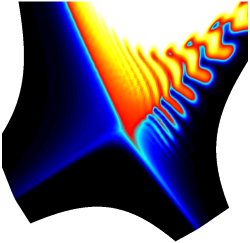

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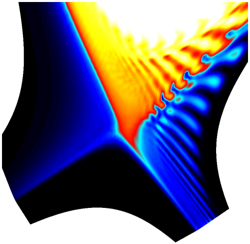

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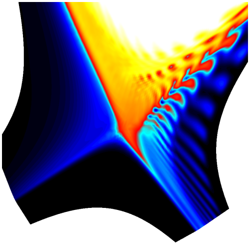

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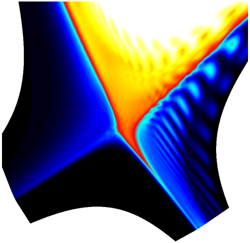

Model

- Resistive MHD model for Tokamak:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla \cdot \boldsymbol{\Pi} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + \gamma p \nabla \cdot \boldsymbol{u} = \nabla \cdot ((k_\parallel (\boldsymbol{B} \otimes \boldsymbol{B}) + k_\perp I_d) \nabla T) + \eta(T) \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \eta(T) \nabla \times (\nabla \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, $\boldsymbol{u}$ the velocity , p and T the pressure and temperature, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{\Pi} = \boldsymbol{\Pi}(\nabla \boldsymbol{u}, \boldsymbol{B})$ the stress tensor.
- with ν the viscosity, $k_\parallel$, $k_\perp$ the thermal conductivities and η the resistivity.

Important Properties

- Conservation in time: $\nabla \cdot \boldsymbol{B} = 0$ and

$$\frac{d}{dt} \int \left(\rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{\mid \boldsymbol{B} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

Possible simplification

- $\nabla \cdot \boldsymbol{\Pi} \approx \Delta \boldsymbol{u}$.
- Ohmic ($\eta \mid \nabla \times \boldsymbol{B} \mid^2$) and viscous heating $\nu \boldsymbol{\Pi} : \nabla \boldsymbol{u}$ neglected.

Two stage Energy conserving Splitting

Idea

Separate the convection diffusion and magneto-acoustic part. Parabolize the magneto-acoustic part.

- Convection - diffusion step:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u} \\ \partial_t p = \nabla \cdot \boldsymbol{q} + \eta(T) \mid \nabla \times \boldsymbol{B} \mid^2 + \nu \boldsymbol{\Pi} : \nabla \boldsymbol{u} \\ \partial_t \boldsymbol{B} = \eta(T) \nabla \times (\nabla \times \boldsymbol{B}) \end{cases}$$

- Energy balance

$$\partial_t \int \left(\frac{\mid \boldsymbol{B} \mid^2}{2} + \rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{p}{\gamma - 1} \right) = 0$$

- Magneto-Acoustic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

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- **Splitting and Equilibrium**: the balance between pressure gradient and Lorentz force is preserved.

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$$\partial_t \int \left(\frac{\mid \boldsymbol{B} \mid^2}{2} + \rho \frac{\mid \boldsymbol{u} \mid^2}{2} + \frac{p}{\gamma - 1} \right) \leq 0$$

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Three stage Energy conserving Splitting

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- Acoustic step:

$$\begin{cases} \partial_t\rho = 0, \\ \rho\partial_t\boldsymbol{u} + \nabla p = 0 \\ \partial_t p + \nabla\cdot(p\boldsymbol{u}) + (\gamma-1)p\nabla\cdot\boldsymbol{u} = 0 \\ \partial_t\boldsymbol{B} = 0 \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

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- Magnetic step:

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Nonlinear solver for convection-diffusion step

- **First possibility**: classical Newton method.
- **Other possibility**: less accurate method but simpler to solve.

Idea

Use a specific Picard nonlinear solver which allows to decouple the equation.

- We consider a Crank- Nicolson scheme to obtain:

$$\left\{\begin{array}{l}
\rho^{n+1}+\Delta t\theta\nabla\cdot(\rho^{n+1}\boldsymbol{u}^{n+1})=R_{\rho},\\
\rho^{n}\boldsymbol{u}^{n+1}+\Delta t\theta\rho^{n+1}\boldsymbol{u}^{n+1}\cdot\nabla\boldsymbol{u}^{n+1}-\Delta t\theta\nu\Delta\boldsymbol{u}^{n+1}=R_{\boldsymbol{u}}\\
p^{n+1}-\Delta t\theta\nabla\cdot\boldsymbol{q}^{n+1}-\Delta t\theta\eta(T^{n+1})\mid\nabla\times\boldsymbol{B}^{n+1}\mid^{2}-\Delta t\theta\nu\nabla\boldsymbol{u}^{n+1}:\nabla\boldsymbol{u}^{n+1}=R_{p}\\
\boldsymbol{B}^{n+1}-\Delta t\theta\eta(T^{n+1})\nabla\times(\nabla\times\boldsymbol{B}^{n+1})=R_{\boldsymbol{B}}
\end{array}\right.$$

- with $\boldsymbol{q}^{n+1}=(k_{\|}(\boldsymbol{B}^{n+1}\otimes\boldsymbol{B}^{n+1})+k_{\perp}I_{d})\nabla T^{n+1}$

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Use a specific Picard nonlinear solver which allows to decouple the equation.

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- with $\boldsymbol{q}^{k+1}=(k_{\parallel}(\boldsymbol{B}^{k+1}\otimes\boldsymbol{B}^{k+1})+k_{\perp}I_{d})\nabla T^{k+1}$
- **Algorithm** (at each linear step):
  - we solve two independent equations on the magnetic and velocity fields.
  - Using $\boldsymbol{B}^{k+1}$ and $\boldsymbol{u}^{k+1}$ we compute the two last equations on density and after pressure.

Nonlinear solver for magneto acoustic step

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- **Algorithm** (at each linear step):
  - We plug the pressure and magnetic field in the velocity equation. We obtain

$$\rho^n \boldsymbol{u}^{k+1} - (\Delta t^2\theta^2)\, \nabla \left(\nabla \cdot (p^k \boldsymbol{u}^{k+1}) + (\gamma - 1)p^k \nabla \cdot \boldsymbol{u}^{k+1}\right) + \left(\nabla \times \nabla \times \left(\boldsymbol{u}^{k+1} \times \boldsymbol{B}^k\right)\right) \times \boldsymbol{B}^k]$$

  - We solve this equation and after we compute p^{k+1} and $\boldsymbol{B}^{k+1}$ using $\boldsymbol{u}^{k+1}$ (matrix vector product).

Compatible isogeometric analysis

Compatible space I

- **Isogeometric analysis**: use the same basis functions to represent the geometry and physical unknowns.
- **B-Splines**: functions of arbitrary degree p and regularity between C^0 and C^{p-1}.
- **B-Splines**: by 1D tensor product. Complex geometries obtained by global mapping.
- **Compatible space**: DeRham sequence

3D Vector fields

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\boldsymbol{grad}} & H(\mathbf{curl},\Omega) & \xrightarrow{\boldsymbol{curl}} & H(div,\Omega) & \xrightarrow{div} & L^2(\Omega) \\
 & & & & & & \\
H^1(\mathcal{P}) & \xleftarrow{\widetilde{\boldsymbol{grad}}^*} & H(\mathbf{curl},\mathcal{P}) & \xleftarrow{\widetilde{\boldsymbol{curl}}^*} & H(div,\mathcal{P}) & \xleftarrow{\widetilde{div}^*} & L^2(\mathcal{P})
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3D Vector fields approximations

$$
\begin{array}{ccccccc}
H^1(\mathcal{P}) & \overset{\textbf{grad}}{\longrightarrow} & H(\textbf{curl},\mathcal{P}) & \overset{\textbf{curl}}{\longrightarrow} & H(div,\mathcal{P}) & \overset{div}{\longrightarrow} & L^2(\mathcal{P}) \\
\widetilde{\Pi}^h_{h1}\downarrow & & \widetilde{\Pi}^h_{curl}\downarrow & & \widetilde{\Pi}^h_{div}\downarrow & & \widetilde{\Pi}_{L2}\downarrow \\
V^h & \overset{\textbf{grad}^h}{\longrightarrow} & V^h_{curl} & \overset{\textbf{curl}^h}{\longrightarrow} & V^h_{div} & \overset{div^h}{\longrightarrow} & X^h \\
\mathcal{S}^{p,p,p} & & \begin{pmatrix} \mathcal{S}^{p-1,p,p} \\ \mathcal{S}^{p,p-1,p} \\ \mathcal{S}^{p,p,p-1} \end{pmatrix} & & \begin{pmatrix} \mathcal{S}^{p,p-1,p-1} \\ \mathcal{S}^{p-1,p,p-1} \\ \mathcal{S}^{p-1,p-1,p} \end{pmatrix} & & \mathcal{S}^{p-1,p-1,p-1}
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2D Vector fields 1

$$
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H^1(\Omega) & \xrightarrow{\boldsymbol{grad}} & H(\mathbf{curl},\Omega) & \xrightarrow{\boldsymbol{rot}} & L^2(\Omega) \\
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\end{array}
$$

2D Vector fields 2

$$
\begin{array}{ccccc}
H^1(\Omega) & \xrightarrow{\boldsymbol{curl}} & H(div,\Omega) & \xrightarrow{\mathbf{div}} & L^2(\Omega) \\
H^1(\mathcal{P}) & \xleftarrow{\widetilde{\boldsymbol{curl}}^*} & H(div,\mathcal{P}) & \xleftarrow{\widetilde{\mathbf{div}}^*} & L^2(\mathcal{P})
\end{array}
$$

- We can, as in 3D, construct a Discrete DeRham sequence.

Compatible space II

- Advantage of Compatible B-Splines space:
  - High degree, high regularity.
  - Preservation of the properties (3D case here)

$$div_h(\boldsymbol{Curl}_h) = 0, \quad \boldsymbol{Curl_h}(\boldsymbol{grad}_h) = 0$$

  and

$$\boldsymbol{Curl}_h^* = \boldsymbol{Curl}_h, \quad \boldsymbol{grad}_h^* = div_h$$

  - Dual properties useful for energy conservation, kernel properties for constraints and avoid spurious modes.
- **Other point**: strong form (equation verified at the coefficient level). Example: Explicit Maxwell.

$$\begin{cases} \boldsymbol{E}^{n+1} = \boldsymbol{E}^n + \Delta t \nabla \times \boldsymbol{B}^n = 0 \\ \boldsymbol{B}^{n+1} = \boldsymbol{B}^n - \Delta t \nabla \times \boldsymbol{E}^n = 0 \\ \nabla \cdot \boldsymbol{B}^{n+1} = 0, \nabla \cdot \boldsymbol{E}^{n+1} = \rho \end{cases}$$

- We take the $\boldsymbol{B}$ equation, choose $\boldsymbol{E} \in H(curl)$ and consequently $\boldsymbol{B} \in H(div)$, multiply by test function and integrate to obtain

$$M\boldsymbol{B}_h^{n+1} = M\boldsymbol{B}_h^n + \Delta t C \boldsymbol{E}_h^n$$

- with M the mass matrix and C the weak curl matrix.

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- Property of the space: $C = M\boldsymbol{Curl}_h$ therefore we have the following strong form

$$\boldsymbol{B}_h^{n+1} = \boldsymbol{B}_h^n + \Delta t \boldsymbol{Curl}_h \boldsymbol{E}_h^n$$

- Applying div_h we obtain $div_h \boldsymbol{B}_h^{n+1} = 0$.

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$$\begin{cases} \boldsymbol{E}^{n+1} = \boldsymbol{E}^n + \Delta t \nabla \times \boldsymbol{B}^n = 0 \\ \boldsymbol{B}^{n+1} = \boldsymbol{B}^n - \Delta t \nabla \times \boldsymbol{E}^n = 0 \\ \nabla \cdot \boldsymbol{B}^{n+1} = 0, \nabla \cdot \boldsymbol{E}^{n+1} = \rho \end{cases}$$

- Taking $\boldsymbol{B} \in H(div)$ we don't have compatibility with the first equation since we have $\nabla \times \boldsymbol{B}$. Idea: integrate by part the first equation (weak form)

$$\int (\boldsymbol{E}^{n+1}, \boldsymbol{C}) = \int (\boldsymbol{E}^n, \boldsymbol{C}) + \Delta t \int (\boldsymbol{B}^n, \nabla \times \boldsymbol{C})$$

- Taking $\boldsymbol{C} \in H(curl)$ we obtain a consistent equation.

Compatible space II

- Advantage of Compatible B-Splines space:
  - High degree, high regularity.
  - Preservation of the properties (3D case here)

$$div_h(\boldsymbol{Curl}_h) = 0, \quad \boldsymbol{Curl_h}(\boldsymbol{grad}_h) = 0$$

  and

$$\boldsymbol{Curl}_h^* = \boldsymbol{Curl}_h, \quad \boldsymbol{grad}_h^* = div_h$$

  - Dual properties useful for energy conservation, kernel properties for constraints and avoid spurious modes.
- **Other point**: strong form (equation verified at the coefficient level). Example: Explicit Maxwell.

$$\begin{cases} \boldsymbol{E}^{n+1} = \boldsymbol{E}^n + \Delta t \nabla \times \boldsymbol{B}^n = 0 \\ \boldsymbol{B}^{n+1} = \boldsymbol{B}^n - \Delta t \nabla \times \boldsymbol{E}^n = 0 \\ \nabla \cdot \boldsymbol{B}^{n+1} = 0, \nabla \cdot \boldsymbol{E}^{n+1} = \rho \end{cases}$$

- At the matrix level, we obtain

$$M_{curl}\boldsymbol{E}^{n+1} = M_{curl}\boldsymbol{E}^n + \Delta t Curl_h^T M_{div}\boldsymbol{B}^n$$

- Taking $\boldsymbol{C} \in H(curl)$ we obtain a consistent equation.

Compatible space III

- Additionally we need the commutative projection.
- The 3D projectors are defined by:

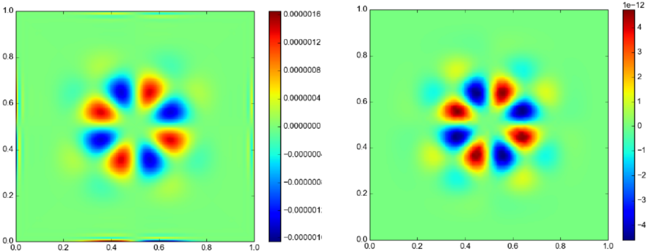
$$\widetilde{\Pi}^h_{h1} := \begin{cases} \widetilde{\Pi}^h_{h1} f = f^0_p \in V^h \\ f^0_p(\mathbf{x}_k) = \mathbf{x}_k, \quad \forall \mathbf{x}_k \in N_h \end{cases} \qquad \widetilde{\Pi}^h_{L2} := \begin{cases} \widetilde{\Pi}^h_{L2} \mathbf{f} = \mathbf{f}^3_p \in X^h \\ \displaystyle\int_{v_k} \mathbf{f}^3_p = \int_{s_k} \mathbf{f}, \quad \forall v_k \in \Omega_h \end{cases}$$

- with N_h the nodes of the mesh. Ω_h the cells of the mesh.

$$\widetilde{\Pi}^h_{curl} := \begin{cases} \widetilde{\Pi}^h_{curl} \mathbf{f} = \mathbf{f}^1_p \in V^h_{curl} \\ \displaystyle\int_{e_k} \mathbf{f}^1_p \cdot \mathbf{t} = \int_{e_k} \mathbf{f} \cdot \mathbf{t}, \quad \forall e_k \in E_h \end{cases} \qquad \widetilde{\Pi}^h_{div} := \begin{cases} \widetilde{\Pi}^h_{div} \mathbf{f} = \mathbf{f}^2_p \in V^h_{div} \\ \displaystyle\int_{f_k} \mathbf{f}^2_p \cdot \mathbf{n} = \int_{f_k} \mathbf{f} \cdot \mathbf{n}, \quad \forall f_k \in F_h \end{cases}$$

- with E_h the edges of the mesh. Ω_h the faces of the mesh.

- Exemple: $\rho_2 = \nabla \times (2x(1-x)y(1-y))$. Comparison between L^2 and commutative projection in $H(div)$:



Compatible space IV

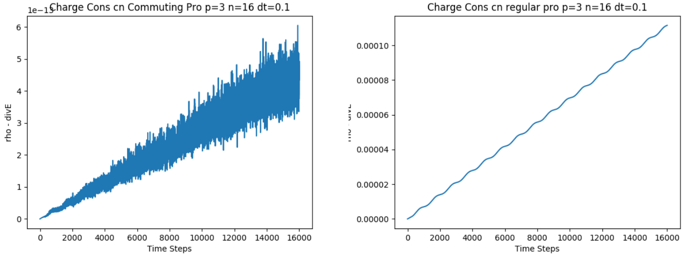
- Numerical example: 2D Maxwell model:

$$\begin{cases} \boldsymbol{E}^{n+1} = \boldsymbol{E}^n + \Delta t CurlB^n - \mu_0 \boldsymbol{J} \\ B^{n+1} = B^n - \Delta trot(\boldsymbol{E}^n) \\ \nabla \cdot \boldsymbol{B}^{n+1} = 0, \nabla \cdot \boldsymbol{E}^{n+1} = \rho \end{cases}$$

- with $CurlB = \begin{pmatrix} \partial_y B \\ -\partial_x B \end{pmatrix}$ and $Rot(\boldsymbol{E}) = \partial_x E_y - \partial_y E_x$.
- Property to preserve

$$\nabla \cdot \partial_t \boldsymbol{E} = \partial_t \rho, \quad \text{since} \quad \partial_t \rho + \nabla \cdot \boldsymbol{J} = 0.$$

- Charge conservation for Implicit scheme with 16*16 cells. Order 3



- Left: Compatible space with commutative projection. Right: Compatible space without commutative projection.

Three stage Energy conserving Splitting

- Magnetic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \partial_t p = 0 \\ \partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- Strong conservation of $\nabla \cdot \boldsymbol{B} = 0 ==> \boldsymbol{B}_h \in H(div)$
- Multiply by $\boldsymbol{v}$ and integrate by part

$$\partial_t \int (\boldsymbol{u}_h, \boldsymbol{v}) + \int (\boldsymbol{B}_h, \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_h)) = 0$$

- So $(\boldsymbol{u}_h \times \boldsymbol{B}_h), (ref\, \boldsymbol{v}) \in H(Curl) ===> \boldsymbol{u}_h \in H(Curl)$
- Commutative projection needed.

- Acoustic step:

$$\begin{cases} \partial_t \rho = 0, \\ \rho \partial_t \boldsymbol{u} + \nabla p = 0 \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \\ \partial_t \boldsymbol{B} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- $\boldsymbol{u}_h \in H(Curl) ==> p \in H_1$
- Multiply by q and integrate by part

$$\partial_t \int p_h q - \int (p \boldsymbol{u}, \nabla q) - (\gamma - 1)(\boldsymbol{u}, \nabla(p_h q)) = 0$$

- Commutative projection needed.

- Convection - diffusion step:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = 0 \\ \partial_t p = 0 \\ \partial_t \boldsymbol{B} = 0 \end{cases}$$

- $\rho \boldsymbol{u} \in H(Div) ===> \rho_h \in L^2$.
- We that to $\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla \times \boldsymbol{u} \times \boldsymbol{u} + \nabla(\boldsymbol{u}, \boldsymbol{u})$ to obtain

$$\int \rho(\partial_t \boldsymbol{u}, \boldsymbol{v}) + \int (\nabla \times \boldsymbol{u} \times \boldsymbol{u}, \boldsymbol{v}) - \frac{1}{2} \mid \boldsymbol{u} \mid^2 \nabla \cdot (\rho \boldsymbol{v})$$

- Energy: strong form of ρ equation multiply $\frac{1}{2} \mid \boldsymbol{u}_h \mid^2$ and integrate.

Splitting and nonlinear solver: reduced MHD 199

Model

- Resistive reduced MHD 1999 model for Tokamak:

$$\begin{cases} \partial_t \rho + \boldsymbol{u}_\perp \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u}_\perp = \nabla \cdot (D(\psi) \nabla T) \\ \partial_t T + \boldsymbol{u}_\perp \cdot \nabla T + (\gamma - 1) T \nabla \cdot \boldsymbol{u}_\perp = \nabla \cdot (K(\psi) \nabla T) \\ \\ \nabla \times (\hat{\rho} \partial_t \boldsymbol{u}_\perp) \cdot e_{\boldsymbol{\phi}} + \nabla \times (\hat{\rho} \boldsymbol{u}_\perp \cdot \nabla \boldsymbol{u}_\perp) \cdot e_{\boldsymbol{\phi}} + (\nabla R^2 \times \nabla p) \cdot e_{\boldsymbol{\phi}} + \boldsymbol{B} \cdot \nabla j = \nabla \cdot (\nu \nabla w) \\ \\ \dfrac{1}{R^2} \partial_t \psi + \boldsymbol{B} \cdot \nabla u = \dfrac{\eta(T)}{R^2} j \\ j = \Delta^* \psi \\ w = \Delta_{pol} u \end{cases}$$

- with u the electric potential, ψ the poloidal magnetic flux, w the vorticity and j the toroidal current. $\hat{\rho} = R^2 \rho$.

- Reduced operators:
  - $\boldsymbol{B} = \frac{F_0}{R} e_{\boldsymbol{\phi}} + \frac{1}{R} \nabla \psi \times e_{\boldsymbol{\phi}}$
  - $\boldsymbol{u} = \boldsymbol{u}_\perp = -R \nabla \nabla u \times e_{\boldsymbol{\phi}}$

- Reduced operators:
  - $\boldsymbol{B} \cdot \nabla I_d = -\frac{1}{R} [\psi, I_d] + \frac{F_0}{R^2} \partial_\phi I_d$
  - $\boldsymbol{u}_\perp \cdot \nabla = -R [I_d, u]$
  - $\nabla \cdot \boldsymbol{u}_\perp = -\frac{1}{R} [R^2, u] = -2 \partial_Z u$

Energy balance

$$\frac{d}{dt} E(t) = \frac{d}{dt} \int \left(\hat{\rho} \frac{\mid \nabla u \mid^2}{2} + \frac{\mid \nabla \psi \mid^2}{2R^2} + \frac{p}{\gamma - 1} \right) \leq \int \eta(T) j^2 + \nu \int w^2$$

Two stage Energy conserving Splitting

Idea

Separate the convection diffusion and magneto-acoustic part. Parabolize the magneto-acoustic part.

- Convection - diffusion step:

$$\begin{cases} \partial_t \rho = \nabla\cdot(D(\psi)\nabla T) \\ \partial_t T = \nabla\cdot(K(\psi)\nabla T) \\ \nabla\times(\hat\rho\partial_t \boldsymbol{u}_\perp)\cdot e_\phi + \nabla\times(\hat\rho \boldsymbol{u}_\perp\cdot\nabla\boldsymbol{u}_\perp)\cdot e_\phi = \nabla\cdot(\nu\nabla w) \\ \dfrac{1}{R^2}\partial_t\psi = \dfrac{\eta(T)}{R^2} j \\ j = \Delta^*\psi \\ w = \Delta_{pol} u \end{cases}$$

  - Energy balance

$$\frac{d}{dt}E(t) \leq 0$$

- Magneto-Acoustic step:

$$\begin{cases} \partial_t\rho + \boldsymbol{u}_\perp\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u}_\perp = 0 \\ \partial_t T + \boldsymbol{u}_\perp\cdot\nabla T + (\gamma-1)T\nabla\cdot\boldsymbol{u}_\perp = 0 \\ \nabla\times(\hat\rho\partial_t\boldsymbol{u}_\perp)\cdot e_\phi + (\nabla R^2\times\nabla p)\cdot e_\phi + \boldsymbol{B}\cdot\nabla j = 0 \\ \dfrac{1}{R^2}\partial_t\psi + \boldsymbol{B}\cdot\nabla u = 0 \\ j = \Delta^*\psi \\ w = \Delta_{pol} u \end{cases}$$

  - Energy balance

$$\frac{d}{dt}E(t) = 0$$

- **Splitting**: does not preserved the energy balance. Possible to modify the splitting to assure the balance.

Nonlinear solver for convection-diffusion step

- **First possibility**: classical Newton method.
- **Other possibility**: less accurate method but simpler to solve.

Idea

Use a specific Picard nonlinear solver which allows to decouple the equation.

- We consider a Crank- Nicolson scheme to obtain:

$$\begin{cases} \rho^{n+1} - \theta\Delta t\nabla\cdot(D(\psi^{n+1})\nabla\rho^{n+1}) = R_\rho \\ T^{n+1} - \theta\Delta t\nabla\cdot(K(\psi^{n+1})\nabla T^{n+1}) = R_T \\ \nabla\times(\hat{\rho}^n\boldsymbol{u}_\perp^{n+1})\cdot\boldsymbol{e_\phi} + \theta\Delta t\nabla\times(\hat{\rho}^{n+1}\boldsymbol{u}_\perp^{n+1}\cdot\nabla\boldsymbol{u}_\perp^{n+1})\cdot\boldsymbol{e_\phi} - \theta\Delta t\nabla\cdot(\nu\nabla w^{n+1}) = R_u \\ \\ \dfrac{1}{R^2}\psi^{n+1} - \dfrac{\eta(T^{n+1})}{R^2}\Delta^*\psi^{n+1} = R_\psi \\ \\ w^{n+1} = \Delta_{pol}u^{n+1} \end{cases}$$

- **First possibility**: classical Newton method.
- **Other possibility**: less accurate method but simpler to solve.

Idea

Use a specific Picard nonlinear solver which allows to decouple the equation.

- We write the following picard algorithm:

$$\left\{\begin{array}{l} \rho^{k+1} - \theta\Delta t\nabla\cdot(D(\psi^{k+1})\nabla T^{k+1}) = R_\rho \\ T^{k+1} - \theta\Delta t\nabla\cdot(K(\psi^{k+1})\nabla T^{k+1}) = R_T \\ \nabla\times(\hat{\rho}^n \boldsymbol{u}_\perp^{k+1})\cdot \boldsymbol{e_\phi} + \theta\Delta t\nabla\times(\hat{\rho}^k \boldsymbol{u}_\perp^k\cdot\nabla \boldsymbol{u}_\perp^{k+1})\cdot \boldsymbol{e_\phi} - \theta\Delta t\nabla\cdot(\nu\nabla\Delta_{pol} u^{k+1}) = R_u \\ \dfrac{1}{R^2}\psi^{k+1} - \theta\Delta t\dfrac{\eta(T^k)}{R^2}\Delta^*\psi^{k+1} = R_\psi \end{array}\right.$$

- **Algorithm** (at each linear step):
  - we solve two independent equations on the poloidal magnetic flux and potential.
  - Using ψ^{k+1} and u^{k+1} we compute the two last equations on density and temperature.

Nonlinear solver for magneto acoustic step

- **First possibility**: classical Newton method. **Problem**: complicate to couple with parabolization.

Idea

Use a specific Picard nonlinear solver which allows to decouple the equation.

- We consider a Crank- Nicolson scheme to obtain:

$$
\begin{cases}
\rho^{n+1} + \theta\Delta t \boldsymbol{u}_{\perp}^{n+1}\cdot\nabla\rho^{n+1} + \theta\Delta t\rho^{n+1}\nabla\cdot\boldsymbol{u}_{\perp}{}_{n+1} = R_{\rho} \\
T^{n+1} + \theta\Delta t \boldsymbol{u}_{\perp}^{n+1}\cdot\nabla T^{n+1} + \theta\Delta t(\gamma-1)T^{n+1}\nabla\cdot\boldsymbol{u}_{\perp}^{n+1} = R_T \\
\nabla\times(\hat{\rho}^n\boldsymbol{u}_{\perp}^{n+1})\cdot \boldsymbol{e}_{\phi} + \theta\Delta t(\nabla R^2\times\nabla p^{n+1})\cdot\boldsymbol{e}_{\phi} + \boldsymbol{B}^{n+1}\cdot\nabla j^{n+1} = R_u \\
\dfrac{1}{R^2}\psi^{n+1} + \theta\Delta t\boldsymbol{B}^{n+1}\cdot\nabla u^{n+1} = R_{\psi} \\
j = \Delta^*\psi
\end{cases}
$$

Nonlinear solver for magneto acoustic step

- **First possibility**: classical Newton method. **Problem**: complicate to couple with parabolization.

Idea

Use a specific Picard nonlinear solver which allows to decouple the equation.

- We write the following picard algorithm:

$$\begin{cases} \rho^{k+1} + \theta\Delta t \boldsymbol{u}_{\perp}^{k+1}\cdot\nabla\rho^{k} + \theta\Delta t\rho^{k}\nabla\cdot\boldsymbol{u}_{\perp}^{k+1} = R_{\rho} \\ T^{k+1} + \theta\Delta t \boldsymbol{u}_{\perp}^{k+1}\cdot\nabla\rho^{k} + \theta\Delta t(\gamma-1)T^{k}\nabla\cdot\boldsymbol{u}_{\perp}^{k+1} = R_{T} \\ \nabla\times(\hat{\rho}^{n}\boldsymbol{u}_{\perp}^{k+1})\cdot\boldsymbol{e}_{\phi} + \theta\Delta t(\nabla R^{2}\times\nabla p^{k+1})\cdot\boldsymbol{e}_{\phi} + \boldsymbol{B}^{k}\cdot\nabla\Delta^{*}\psi^{k+1} = R_{u} \\ \dfrac{1}{R^{2}}\psi^{k+1} + \theta\Delta t\boldsymbol{B}^{k}\cdot\nabla u^{k+1} = R_{\psi} \\ j = \Delta^{*}\psi \end{cases}$$

- **Algorithm** (at each linear step):
  - We plug the pressure and poloidal flux equations in the potential equation. We obtain

$$\nabla\times(\hat{\rho}^{n}\boldsymbol{u}_{\perp}^{k+1})\cdot\boldsymbol{e}_{\phi} - (\Delta t^{2}\theta^{2})\left[T^{k}R^{2}\times\nabla\left(\rho^{k}\nabla\cdot\boldsymbol{u}_{\perp}^{k+1}\right) + \rho^{k}R^{2}\times\nabla\left(T^{k}\nabla\cdot\boldsymbol{u}_{\perp}^{k+1}\right)\right.$$
$$\left. + \left(\boldsymbol{B}^{k}\cdot\nabla\left(\Delta^{*}\boldsymbol{B}^{k}\cdot\nabla u^{k+1}\right)\right)\right]$$

  - We solve this equation and after we compute p^{k+1} and ψ^{k+1} using u^{k+1} (matrix vector product).

Numerical results in JOREK I

- **Aim**: compare growth rates of the splitting/ full scheme for different Δt.
- **Tearing mode**. Circular domain. $n_{Tor} = 3$
- Convergence Δt / growth rates

Δt	Full scheme
$\Delta t = 3000$	$3.77E^{-4}$
$\Delta t = 1000$	$3.45E^{-4}$
$\Delta t = 500$	$3.43E^{-4}$
$\Delta t = 250$	$3.41E^{-4}$
$\Delta t = 125$	$3.41E^{-4}$

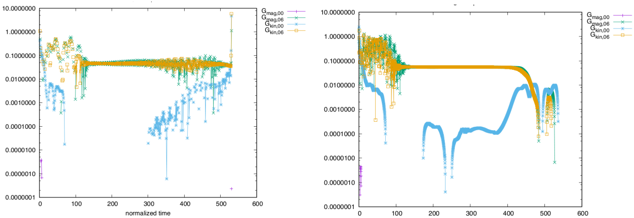
Δt	Splitting scheme
$\Delta t = 1500$	$1.23E^{-4}$
$\Delta t = 1000$	$1.76E^{-4}$
$\Delta t = 500$	$2.64E^{-4}$
$\Delta t = 200$	$3.24E^{-4}$
$\Delta t = 100$	$3.37E^{-4}$

- **Conclusion**: We obtain similar result with a time step 5 times smaller. The splitting under estimate the GR and the full over estimate a little bit.
- **Comment**: test made without picard scheme on the two steps. Picard scheme validate for the convection step not for the magneto-acoustic step.

- **Aim**: compare growth rates of the splitting/ full scheme for different Δt.
- **Inxflow test case**. D-Shape geometry. $n_{Tor} = 3$
- Convergence Δt / growth rates

Δt	Full	Spl scheme
1	$6.1E^{-2}$	$4.2 - 5.8E^{-2}$, $4.5 - 4.8E^{-2}$
0.5	$6.1E^{-2}$	$4.5 - 5.5E^{-2}$, $5.1 - 5.2E^{-2}$
0.25	$6.1E^{-2}$	$5.5 - 5.7E^{-2}$, $5.6 - 5.7E^{-2}$
0.125	$6.1E^{-2}$	$6.0 - 6.1E^{-2}$, $6.05 - 6.1E^{-2}$

- **Conclusion**: Convergence of the two methods. We obtain similar result with a time step 5 times smaller. For large time step oscillation of the growth rate.
- **Comment**: test made without picard scheme on the two steps.



- Left: Growth rates for the splitting scheme with $\Delta t = 1$. Right: Growth rates for the splitting scheme with $\Delta t = 0.25$

Default of the splitting: resistivity

- **Resistivity**: play an important role in the instability growth-rate.
- **Resistivity dependency**: the resistivity is given by:

$$\eta(T) = \eta_0 T^{-\frac{5}{2}}$$

- Strong nonlinear dependency between resistivity and temperature.

main Remark

- **Main evolution for T**: magneto-acoustic step. The evolution of the temperature is important in this step.
- **Main problem**: the resistivity term is in the other step. This splitting can explain the error on the growth-rate.
- Better result if resistivity is in the magneto -acoustic step but no compatible with parabolization.

Possible solution

- **Keep a part (constant ?) of the resistivity effect in the magneto acoustic part**. Predictor-corrector or other method.
- **Iterative splitting**. Each step is solver more than one during one time step but simpler step.

Conclusion

Full mhd

- Energy preserving Splitting + compatible space allows:
  - Preserve energy at the discrete level in ideal case. More stability ?
  - Preserve strongly $\nabla \cdot \boldsymbol{B} = 0$.
  - In each step we solve simple problems (elliptic or advection diffusion) + matrix vector product.
  - High-order and High-regularity.
  - **Needs**: stabilization for advection and preconditioning for elliptic solvers.

Splitting for Reduced mhd

- Splitting less efficient as full solver but useful.
- Time step divise by around 5 for a similar accuracy.
- Simpler problems to solve at each step.

Following work for reduced MHD

- Picard solver for Magnetic-acoustic step (June).
- Validation of the scheme with Picard solver for each step (July).
- Parabolization for magneto-acoustic step and september (July - September).
- Extension to the model 303 and PeTsc for each sub step.