Implicit kinetic relaxation schemes. Application to the plasma physics

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Outline

Physical and mathematical context

Plasma and models

Plasma

- **Plasma**: ionized gas (high energies).
- Strong coupling between hydrodynamics and electromagnetic (nuclear fusion, astrophysics) etc.

Kinetic modeling

$$\begin{cases} \partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot(\nabla\phi f) = 0 \\ \\ -\Delta\phi = \rho(\mathbf{x}) \end{cases}$$

- **Vlasov-Poisson**. Other models: Vlasov-Maxwell or gyrokinetic.
- **Kinetic models** coupled with **elliptic model**.

Fluid modeling: MHD

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) = 0, \\ \rho\partial_t\boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = (\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nu\Delta\boldsymbol{u} \\ \partial_t p + \nabla\cdot(p\boldsymbol{u}) + (\gamma-1)p\nabla\cdot\boldsymbol{u} = \nabla\cdot((\kappa\boldsymbol{B}\otimes\boldsymbol{B} + \varepsilon I_d)\nabla T) \\ \partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \eta\nabla\times(\nabla\times\boldsymbol{B}) \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$

- **Hyperbolic model** coupled with **parabolic model**.

Times scales and time schemes

Problem

- In general the plasma dynamic is a strongly multiscale problem.
- Kinetic model:
  - Strong oscillations of the electric potential generate very large velocities compare to the average velocity.
- Fluid model:
  - Fast magnetosonic waves (pressure and magnetic field pertubation) generate very large velocities compare to the fluid velocity.
- Anisotropic diffusion:
  - Very large diffusion in one direction compare to the other.
- **Conclusion**: for the models we need CFL-free method.
- **Implicit schemes**: these problem are ill-conditioned.

Aim:

Construct High-Order Solver CFL-free and without matrix invertion for the different type of PDE.

Kinetic model and SL schemes

Semi Lagrangian scheme

- One of the main scheme to treat transport and kinetic equations.
- **Idea**: use the characteristic method.

- Example: Backward SL

$$\partial_t f + a\partial_x f = 0$$

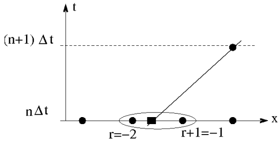
  - **Aim**: compute at the mesh point x_j:

$$f(t+\Delta t, x_j)$$

  - Solution:

$$f(t+\Delta t, x_j) = f(t, x_j - a\Delta t)$$

  - $x_n = x_j - a\Delta t$ is not a mesh point.
  - Using $f(t, x_i)$ we interpolate the function at x_n.



- **BSL/FSL**: follow the backward characteristic and interpolate/follow the forward characteristic and distribute on the mesh.
- **Different type of SL**: Classical SL (punctual values), Conservative SL (Average cell values), DG/CG SL (weak form of SL scheme).

Advantages/drawbacks

- **Advantages**: infinite/high order in time/space. CFL-less and no matrix inversion.
- **Drawbacks**: BC and Gibbs oscillations due to high-order methods.

Case I: hyperbolic systems

Relaxation scheme

- We consider the classical Xin-Jin [95] relaxation for a scalar system $\partial_t u + \partial_x F(u) = 0$:

$$\left\{ \begin{array}{l} \partial_t u + \partial_x v = 0 \\ \partial_t v + \lambda^2 \partial_x u = \dfrac{1}{\varepsilon}(F(u) - v) \end{array} \right.$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u + \partial_x F(u) = \varepsilon \partial_x ((\lambda^2 - \mid \partial F(u) \mid^2) \partial_x u) + O(\varepsilon^2)$$

- **Stability**: the limit system is dissipative if $(\lambda^2 - \mid \partial F(u) \mid^2) > 0$.
- **Sub-characteristic**: $\lambda > \mid \partial F(u) \mid$ with λ the velocity of the new model.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\left\{ \begin{array}{l} \partial_t f_- - \lambda \partial_x f_- = \dfrac{1}{\varepsilon}(f_{eq}^- - f_-) \\ \partial_t f_+ + \lambda \partial_x f_+ = \dfrac{1}{\varepsilon}(f_{eq}^+ - f_+) \end{array} \right.$$

- with $u = f_- + f_+$ and $f_{eq}^{\pm} = \frac{u}{2} \pm \frac{F(u)}{2\lambda}$.
- **New system**: D1Q2 kinetic model (diagonal transport).

Generic kinetic relaxation scheme

Kinetic relaxation system

- **Considered model**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- **Lattice**: $W = \{\lambda_1, ..., \lambda_{n_v}\}$ a set of velocities.
- **Mapping matrix**: P a matrix $n_c \times n_v$ $(n_c < n_v)$ such that $\boldsymbol{U} = P\boldsymbol{f}$, with $\boldsymbol{U} \in \mathbb{R}^{n_c}$.
- **Kinetic relaxation system**:

$$\partial_t \boldsymbol{f} + \Lambda \partial_x \boldsymbol{f} = \frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})$$

- Consistence condition (Natalini - Aregba [96-98-20], Bouchut [99-03]) :

$$\mathcal{C} \begin{cases} P\boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{U} \\ P\Lambda \boldsymbol{f}^{eq}(\boldsymbol{U}) = \boldsymbol{F}(\boldsymbol{U}) \end{cases}$$

- **In 1D** : same property of stability that the classical relaxation method.
- **Limit of the system**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

- Natural extension in 2D/3D.
- **General scheme**: $[D1Q2]^n$, one D1Q2 by macroscopic equation.

Time discretization

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Main idea**: **splitting scheme** between transport and the relaxation (Dellar [13]).
- **Key point**: the macroscopic variables are conserved during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ **is explicit**:

$$P\left(\partial_t \boldsymbol{f}\right) = P\left(\frac{1}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f})\right), \longrightarrow \partial_t \boldsymbol{U} = \frac{1}{\varepsilon}(P\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{U}) = 0,$$

- **Scheme**: Theta-scheme for the relaxation and SL scheme for the transport.

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First order scheme (exact transport)

- We define the two operators for each step :

$$T_{\Delta t} : e^{\Delta t \Lambda \partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \left(\frac{(2-\omega)\Delta t}{2\omega}\right) \partial_x \left(D(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

- with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$ and $D(\boldsymbol{U}) = \left(P\Lambda^2 \partial_{\boldsymbol{U}} \boldsymbol{f}^{eq}(\boldsymbol{U}) - \partial \boldsymbol{F}(\boldsymbol{U})^2\right)$.

High-Order time schemes

Second-order scheme

- **Order of convergence**: one for the kinetic variables. one or two ($\omega = 2$ and exact transport) for the macroscopic variables.
- **Second order scheme**: Strang Splitting + SL scheme

$$\Psi(\Delta t) = T\left(\frac{\Delta t}{2}\right) \circ R(\Delta t, \omega = 2) \circ T\left(\frac{\Delta t}{2}\right).$$

High order scheme: composition method

$$M_p(\Delta t) = \Psi(\gamma_1 \Delta t) \circ \Psi(\gamma_2 \Delta t) \circ ..... \circ \Psi(\gamma_s \Delta t)$$

- with $\gamma_i \in [-1, 1]$, we obtain a p-order schemes.
- Susuki scheme : $s = 5$, $p = 4$. Kahan-Li scheme: $s = 9$, $p = 6$.

CV and new scheme

- All the schemes convergence only with the second order for the kinetic variables.
- Loss of order also for macroscopic variables (see numerical results).
- The 2th order scheme satisfies $\Psi(\Delta t) = \Psi^{-1}(-\Delta t)$ but not $\Psi(\Delta t = 0) \neq I_d$. Correction:

$$\Psi_{ap}(\Delta t) = T\left(\frac{\Delta t}{4}\right) \circ R\left(\frac{\Delta t}{2}, \omega = 2\right) \circ T\left(\frac{\Delta t}{2}\right) \circ R\left(\frac{\Delta t}{2}, \omega = 2\right) \circ T\left(\frac{\Delta t}{4}\right)$$

Burgers: convergence results

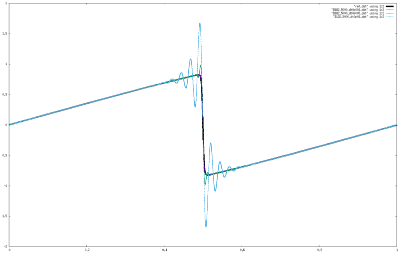
- **Model**: Burgers equation

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- Spatial discretization: SL-scheme, 2000 cells, degree 11.
- **Test**: $\rho(t=0,x) = sin(2\pi x)$. $T_f = 0.14$ (before the shock) and no viscosity.
- Scheme: splitting schemes and Suzuki composition + splitting.

	SPL 1, $\theta = 1$		SPL 1, $\theta = 0.5$		SPL 2, $\theta = 0.5$		Suzuki	
Δt	Error	order	Error	order	Error	order	Error	order
0.005	$2.6E^{-2}$	-	$1.3E^{-3}$	-	$7.6E^{-4}$	-	$4.0E^{-4}$	-
0.0025	$1.4E^{-2}$	0.91	$3.4E^{-4}$	1.90	$1.9E^{-4}$	2.0	$3.3E^{-5}$	3.61
0.00125	$7.1E^{-3}$	0.93	$8.7E^{-5}$	1.96	$4.7E^{-5}$	2.0	$2.4E^{-6}$	3.77
0.000625	$3.7E^{-3}$	0.95	$2.2E^{-5}$	1.99	$1.2E^{-5}$	2.0	$1.6E^{-7}$	3.89

- Scheme: second order splitting scheme.
- Same test after the shock:



Remark on the relaxation scheme

- **Classical result**: Strang Splitting + second order/exact scheme for relaxation converge at first order for $\varepsilon \approx 0$. SL solver + Strang splitting.

	CN		Exact		SSP RK2	
	Error	Order	Error	Order	Error	Order
$\Delta t = 4.10^{-3}$	$4.8E^{-4}$	-	$2.0E^{-2}$	-	$2.0E^{-2}$	-
$\Delta t = 2.10^{-3}$	$1.2E^{-4}$	2.0	$1.1E^{-2}$	0.86	$1.1E^{-2}$	0.86
$\Delta t = 1.10^{-3}$	$2.9E^{-5}$	2.05	$5.7E^{-3}$	0.95	$5.5E^{-3}$	1.0
$\Delta t = 5.10^{-4}$	$7.4E^{-6}$	1.95	$2.9E^{-3}$	0.97	$2.8E^{-3}$	0.98

- **Conclusion**: we lose one order of cv with exact and SPP-RK solver.

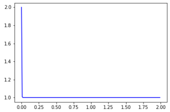
- **Schemes** for $\varepsilon \approx 0$:
- For Euler implicit, exact and SSP-RK2 schemes.

$$\boldsymbol{f}^{n+1} \approx \boldsymbol{f}^{eq}(\boldsymbol{U}^n) + O(\varepsilon)$$

- For Crank-Nicolson.

$$\boldsymbol{f}^{n+1} \approx 2\boldsymbol{f}^{eq}(\boldsymbol{U}^n) - \boldsymbol{f}^n + O(\varepsilon)$$

- We solve the EDO $\partial_t u = \frac{1}{\varepsilon}(u_{eq} - u)$.



- Implicit Euler scheme. $\Delta t = 100\varepsilon$

Conclusion:

- If you start far from $\boldsymbol{f}^{eq}$ the exact/SPP-RK solvers seems better.
- However, for high-order splitting schemes the over-relaxation (CN) seems important.

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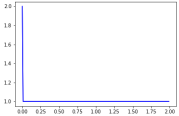

- Exact time scheme. $\Delta t = 100\varepsilon$

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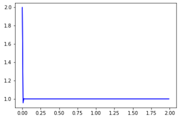

- SSP RK2 scheme. $\Delta t = 100\varepsilon$

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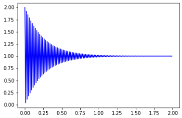

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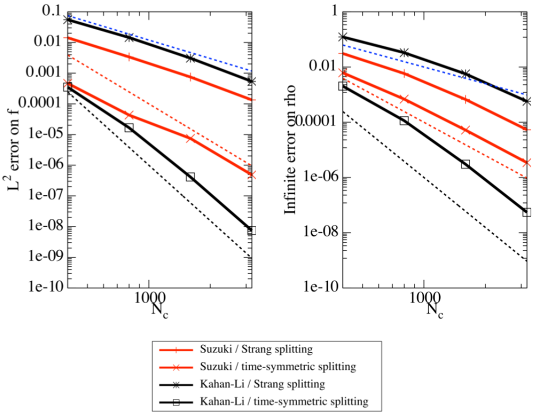
Convergence

- **Equation**: Euler isothermal
- Model $[D1Q2]^2$ High-order space scheme. Comparison of the time scheme.
- **Test case**: smooth solution. $\Delta t = \frac{\beta \Delta x}{\lambda}$ with $\beta = 50$



- With Strang splitting: only order 2 for f.
- Loss of convergence for macroscopic variables for Kahan-li + Strang splitting.

Numerical results: 2D-3D fluid models

- **Model** : liquid-gas Euler model with gravity.
- **Kinetic model** : $(D2 - Q4)^n$. Symmetric Lattice.
- **Transport scheme** : 2 order Implicit DG scheme. 3th order in space. CFL around 6.
- **Test case** : Rayleigh-Taylor instability.

2D case in annulus

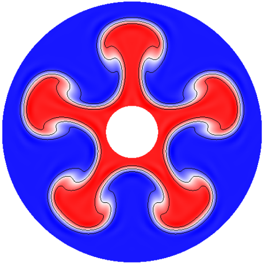
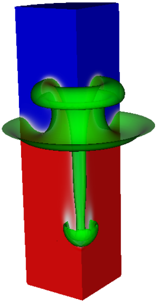


Figure: Plot of the mass fraction of gas

3D case in cylinder



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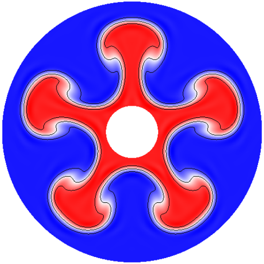


Figure: Plot of the mass fraction of gas

2D cut of the 3D case

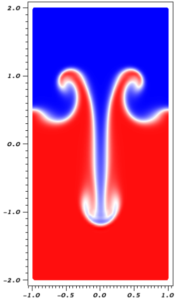


Figure: Plot of the mass fraction of gas

BC : preliminary results

- **Question**: What BC for the kinetic variables. How keep the order ?

First result

- The second order symmetric scheme (Ψ_{ap}) for the following equation (equivalent to $[D1Q2]^n$ kinetic model):

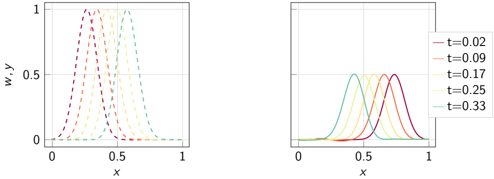
$$\left\{\begin{array}{l} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \frac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{array}\right.$$

is consistant with

$$\left\{\begin{array}{l} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\Delta t^2) \\ \partial_t \boldsymbol{W} - \partial \boldsymbol{F}(\boldsymbol{U}) \partial_x \boldsymbol{W} = O(\Delta t^2) \end{array}\right.$$

with $\boldsymbol{W} = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}$.

- **Natural BC**: entering condition for $\boldsymbol{U}$ and $\boldsymbol{W} = 0$ or $\partial_x \boldsymbol{W} = 0$.
- Example: $F(u) = cu$ (transport):



- Transport of the u (dashed lines) and $w = v - f(u)$ (plain lines) quantities.

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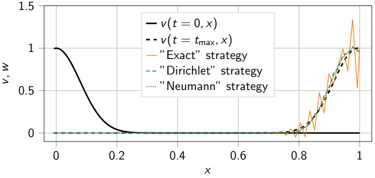

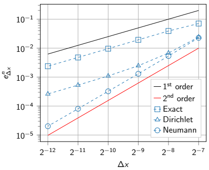

- Initial state and comparison of the final states. Gaussian initial profile, $\Delta x = 2^{-7}$.

Case II: parabolic systems

Relaxation scheme for diffusion

- We consider the classical Xin-Jin relaxation for a scalar system $\partial_t u - \nu \partial_{xx} u = 0$:

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \dfrac{\lambda^2}{\varepsilon^2}\partial_x u = -\dfrac{1}{\varepsilon^2} v \end{cases}$$

Limit

- The limit scheme of the relaxation system is

$$\partial_t u - \partial_x(\lambda^2 \partial_x u) = \varepsilon^2 \partial_{xxxx} u + O(\varepsilon^4)$$

- **Consistency**: Choosing $\lambda^2 = \nu$ we obtain the initial solution.

- We diagonalize the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}$ to obtain

$$\begin{cases} \partial_t f_- - \dfrac{\lambda}{\varepsilon}\partial_x f_- = \dfrac{1}{\varepsilon^2}(f_{eq}^-(u) - f_-) \\ \partial_t f_+ + \dfrac{\lambda}{\varepsilon}\partial_x f_+ = \dfrac{1}{\varepsilon^2}(f_{eq}^+(u) - f_+) \end{cases}$$

- with $u = f_- + f_+$ and $f_{eq}(u)^{\pm} = \frac{u}{2}$.
- **Many schemes for this limit**. Hyperbolic case: Jin-Levermore [96], Gosse-Toscani [00] etc. Kinetic case: Lemou-Cresetto and al [09-14-17], Pareschi-Dimarco [07-10-14] etc.

Consistency analysis

- We consider $\partial_t \rho - \nu \partial_{xx}\rho = 0$. We define the two operators for each step:

$$T_{\Delta t} : e^{\Delta t \frac{\Lambda}{\varepsilon}\partial_x} \boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon^2}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \rho = \Delta t \partial_x \left(\left(\frac{1-\omega}{\omega} + \frac{1}{2} \right) \frac{\lambda^2}{\varepsilon^2} \partial_x \rho \right) + O(\Delta t^2)$$

- We don't have convergence for all ε. The splitting scheme is not AP
- Taking $\nu = \lambda^2$, $\theta = 0.5$ and $\varepsilon = \sqrt{\Delta t}$ we obtain the diffusion equation.
- **Cv**: First order at the theoretical level and second order at the numerical one.

- Heat equation. Scheme with $\varepsilon = \Delta t^\gamma$ and very high order SL + fine grid.

	$\gamma = \frac{1}{2}$		$\gamma = 1$		$\gamma = 2$	
	Error	order	Error	order	Error	order
$\Delta t = 0.04$	$1.87E^{-2}$	-	1.43	-	1.43	-
$\Delta t = 0.02$	$6.57E^{-3}$	1.50	0.2	0	0.23	0
$\Delta t = 0.01$	$1.85E^{-3}$	1.82	0.2	0	0.23	0
$\Delta t = 0.005$	$3.6E^{-4}$	2.36	0.2	0	0.23	0
$\Delta t = 0.0025$	$7.3E^{-5}$	2.30	0.2	0	0.23	0

Conclusion

Time scheme for Kinetic BGK model

- **High order time scheme**: Composition + Strang Splitting (or modified version) + Crank-Nicolson scheme for relaxation.
- **Default**: scheme not accurate (compare to Jin-Filbet/Pareschi-Gimarco schemes) far from the equilibrium.
- **Advantage**: independent transport equation so high parallelism.

Implicit Kinetic relaxation schemes

- We can approximate hyperbolic/parabolic PDE by small kinetic models.
- Using SL scheme in space + previous time scheme we obtain high-order space/time method CFL-free without matrix invertion.
- This algorithm is very competitive against classical implicit schemes (no matrices, no solvers).

Future works

- Apply method tononlinear/anisotropic diffusion equation and increase order of convergence.
- 1D scheme for low-mach Euler equation. Extension in 2D/3D and improve stability.
- Application to MHD and anisotropic diffusion for plasma.
- Continue the study for the BC.