Numerical method for multi-scale PDE: applications to weakly compressible fluids

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Outline

Physical and mathematical context

Gas dynamic: Euler equations

TONUS Team work's

Multi-scale in time/space models for plasmas: MHD, Vlasov equations.
Plasma: gas dynamic + electromagnetic.

- We propose to understand the problem on a "simpler" problem: gas dynamic.
- **Euler equation**:

$$\left\{\begin{array}{l}\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p I_d) = 0 \\ \partial_t E + \nabla \cdot (E \boldsymbol{u} + p \boldsymbol{u}) = 0\end{array}\right.$$

- with $\rho(t, \mathbf{x}) > 0$ the density, $\boldsymbol{u}(t, \mathbf{x})$ the velocity and $E(t, \mathbf{x}) > 0$ the total energy.
- The pressure $p(t, \mathbf{x})$ is defined by $p = \rho T$ (perfect gas law) with $T(t, \mathbf{x})$ the temperature.

Model

The system is an hyperbolic system which model nonlinear transport/waves. Physically it correspond to conservation laws.

Properties

No dissipation (no smoothing) processus. These systems can generated discontinuities.

Wave propagation and scales

- The model can be write on a general form

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}_x(\boldsymbol{U}) + \partial_y \boldsymbol{F}_y(\boldsymbol{U}) = \boldsymbol{G}(\boldsymbol{U})$$

$$\longrightarrow \partial_t \boldsymbol{U} + A_x(\boldsymbol{U})\partial_x \boldsymbol{U} + A_y(\boldsymbol{U})\partial_y \boldsymbol{U} = \boldsymbol{G}(\boldsymbol{U})$$

  for smooth solutions.
- These models propagate some complex waves with the velocities given by the eigenvalues of

$$A(\boldsymbol{U}) = A_x(\boldsymbol{U})n_x + A_y(\boldsymbol{U})n_y$$

  with $\mathbf{n}$ a normal vector.
- At the end three eigenvalues: $(\boldsymbol{u}, \mathbf{n})$ and $(\boldsymbol{u}, \mathbf{n}) \pm c$ with the sound speed $c^2 = \gamma \frac{p}{\rho}$.

Physic interpretation:

- the acoustic waves due to the pressure and normal velocity perturbations,
- the density, momentum and energy transport at the velocity $\boldsymbol{u}$.

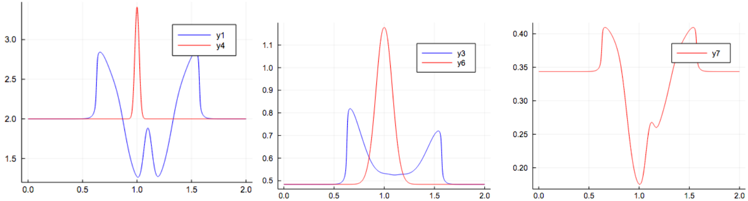
- **Two important scales**: $\boldsymbol{u}$ and c

Physical problem I: large acoustic waves.

- We introduce the Mach number $M = \frac{|u|}{c}$ the ratio between the two velocities/scales.
- We consider the initial state:

$$\rho = 2.0 + 0.05G(x), \quad u = 0.2, \quad p = 0.5 + \underbrace{\delta p}_{|\delta p| \approx 0.5}$$

- The large perturbation of p generate a large acoustic wave with $\delta u \approx 0.2$.
- This velocity gradient created by the waves generates an important compression density.



- Left: $\rho(t, \mathbf{x})$, Middle: $p(t, \mathbf{x})$, Right: Mach number

Conclusion:

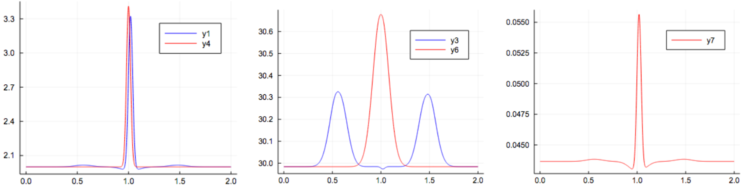
- We must correctly capture this large wave to capture the good behavior of the density.
- This phenomena is a one-scale phenomena since $M \approx 0.3$.

Physical problem II: small acoustic waves.

- We introduce the Mach number $M = \frac{|u|}{c}$ the ratio between the two velocities/scales.
- We consider the initial state:

$$\rho = 2.0 + 0.05G(x), \quad u = 0.2, \quad p = 30 + \underbrace{\delta p}_{|\delta p| \approx 0.5}$$

- The small perturbation of p generates a small gravity wave with $\delta u \approx 0.03$.
- This velocity gradient created by the waves is very small and and there is no compression of density (just advection).



- Left: $\rho(t, \mathbf{x})$, Middle: $p(t, \mathbf{x})$, Right: Mach number

Conclusion:

- Since $\partial_x u \approx \partial_x p \approx 0$, the main dynamic is given by: $\partial_t \rho + u \partial_x \rho = 0$
- No necessary to capture these small waves to capture the behavior of the density.
- This phenomena is a two-scale phenomena since $M \approx 0.05$.

Low mach limit

- We propose an asymptotic interpretation of the small acoustic waves case.
- We want obtain dimensionless equation. We rewrite the equation

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0 \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

- Normalization:
  - we introduce characteristic time t_0, velocity V, length L.
  - the characteristic velocity u_0 and pressure γp_0. The sound velocity is $c^2 = \frac{\gamma p_0}{\rho_0}$.

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$$\begin{cases} \partial_t \rho + \left[\dfrac{t_0 u_0}{L}\right] \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \left[\dfrac{t_0 u_0}{L}\right] \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \left[\dfrac{t_0 p_0}{\rho_0 u_0 L}\right] \nabla p = 0 \\ \partial_t p + \left[\dfrac{t_0 u_0}{L}\right] \boldsymbol{u} \cdot \nabla p + \left[\dfrac{\gamma t_0 u_0}{L}\right] p \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

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$$\left\{\begin{array}{l}\partial_t\rho+\left[\frac{u_0}{V}\right]\nabla\cdot(\rho\boldsymbol{u})=0\\ \rho\partial_t\boldsymbol{u}+\left[\frac{u_0}{V}\right]\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\left[\frac{c_0^2}{u_0V}\right]\nabla p=0\\ \partial_t p+\left[\frac{u_0}{V}\right]\boldsymbol{u}\cdot\nabla p+\left[\frac{\gamma u_0}{V}\right]p\nabla\cdot\boldsymbol{u}=0\end{array}\right.$$

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- We define the mach number: $M = \frac{u_0}{c_0}$. Using this we obtain

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \left[\frac{1}{M^2} \right] \nabla p = 0 \\ \partial_t p + \boldsymbol{u} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{u} = 0 \end{cases} \longrightarrow \begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{M^2} \nabla p = 0 \\ \partial_t E + \nabla \cdot (E \boldsymbol{u} + p \boldsymbol{u}) = 0 \end{cases}$$

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Low Mach limit

When M tends to zero, we obtain incompressible Euler equation:

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla \Pi = 0 \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

In 1D we have just advection of ρ.

Numerical problem I: time discretization.

- When we discretize in space a PDE we obtain an ODE:

$$\partial_t \boldsymbol{U} = A(\boldsymbol{U})$$

- Classical time discretization : explicit time scheme.

$$\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\Delta t} = A(\boldsymbol{U}^n)$$

- **Default of Explicit scheme**: the CFL condition $\Delta t < \frac{\Delta x}{\lambda}$ with lambda the maximal speed of the system.
- For low mach flow (small acoustic waves):
  - The fast phenomena: acoustic wave at velocity c
  - The important phenomena: transport at velocity u
  - Expected CFL: $\Delta t < \frac{\Delta x}{|u|}$, CFL in practice $\Delta t < \frac{\Delta x}{|c|}$
  - At the end we use a Δt divised by M compare to the expected Δt
- Solution: Implicit time scheme. No CFL condition

$$\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\Delta t} = A(\boldsymbol{U}^{n+1})$$

Idea

Taking a larger time step, the implicit scheme allows to filter the fast acoustic waves which are not useful in the low-Mach regime.

Implicit scheme and conditioning I

- Implicit time scheme:

$$M_i \boldsymbol{U}^{n+1} = (I_d + \Delta t A(I_d))\boldsymbol{U}^{n+1} = \boldsymbol{U}^n$$

- We must solve a nonlinear system and after linearization redsolve some linear systems.
- How solve a linear system:
  - Exact solver. Too costly for large problem (system, 3D, high order discretization).
  - iterative solver. Used in practice. Default: slow convergence for ill-conditioning matrix.
- Conditioning of a matrix M:

$$k(M) = \frac{\mid M^{-1} \mid}{\mid M \mid} \approx \frac{\lambda_{max}}{\lambda_{min}}$$

- Approximative conditioning

$$k(M_i) \approx 1 + O\left(\frac{\Delta t}{\Delta x^p M}\right)$$

Remark

- We recover the two scales in the conditioning number. The full implicit schemes are difficult to use for this reason.

Semi implicit scheme

First idea

- We explicit the slow scale (transport) and implicit the fast scales (acoustic)

- Euler equation in 1D:

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x(\rho u) = 0\\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0\\ \partial_t E + \partial_x(Eu) + \partial_x(pu) = 0\end{array}\right.$$

- We use an explicit scheme for convection (or we split the convection). Implicit acoustic step:

$$\left\{\begin{array}{l}\rho^{n+1} = \rho^n\\ (\rho u)^{n+1} = \rho^n u^n - \Delta t \partial_x p^{n+1} + Rhs_u\\ E^{n+1} = E^n - \Delta t \partial_x (p^{n+1} u^{n+1}) = Rhs_E\end{array}\right.$$

Plugging this in the second equation, we obtain

$$E^{n+1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho)$$

- Matrix-vector product to compute u^{n+1}.

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- Matrix-vector product to compute u^{n+1}.

Conclusion

- **Semi implicit**: only one scale in the implicit symmetric positive operator.
- Strong gradient of ρ generates ill-conditioning. Assembly at each time (costly).
- Nonlinear solver which bad convergence for if $\Delta t >> 1$ and $\partial_x p$ not so small.

Spatial discretization in space

Spatial discretization

- Finite Volume method (I don't explain how its work). Based on conservative form.
- First order method: error in space homogeneous to $O(\Delta x)$.
- **Two scale problem**: the naive VF method admit an error homogeneous to the fast scales for the two scales.

- Example: isolated contact:
  - varying density (gaussian)
  - constant pressure ($p = 1$) and velocity ($u << 1$)
- **Exact. solution**:

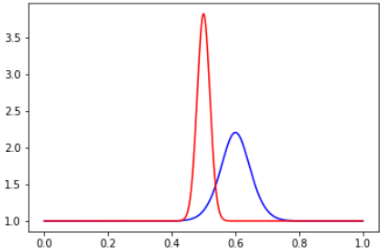
$$\partial_t \rho + u_0 \partial_x \rho = 0$$

which is equivalent to a translation of $u_0 t$.

- Ratio between transport and acoustic:

$$\frac{1}{M} \approx 20$$

- Naive scheme $T_f = 2$ $u_0 = 0.05$ and 1000 cells



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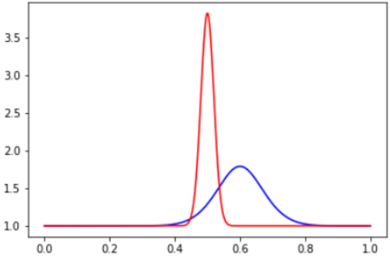

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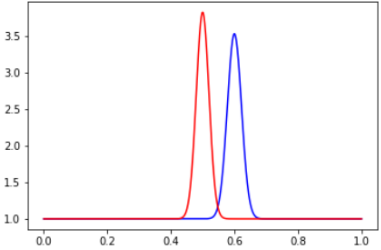
- Good scheme $T_f = 2$ $u_0 = 0.05$ and 1000 cells



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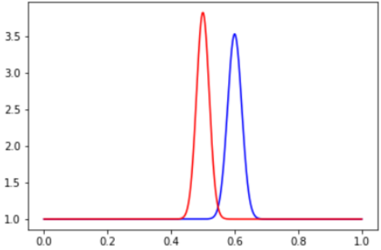


Relaxation method

Relaxation method

- **Problem**: the nonlinearity of the implicit acoustic step generate difficulties.
- Non conservative form and acoustic term:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x p = 0 \\ \partial_t p + u\partial_x p + \rho c^2 \partial_x u = 0 \end{cases}$$

- **Idea**: Relax only the acoustic part to linearized the implicit part.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{cases}$$

- **Limit**:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = \varepsilon\partial_x \left[A\partial_x p\right] \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon\partial_x \left[(A u \partial_x p) + B\partial_x u\right] \\ \partial_t E + \partial_x(E u + p u) = \varepsilon\partial_x \left[A E\partial_x p + A\partial_x \frac{p^2}{2} + B\partial_x \frac{u^2}{2}\right] \end{cases}$$

- with $A = \frac{1}{\rho}\left(\frac{\rho}{\phi} - 1\right)$ and $B = \left(\rho\phi\lambda^2 - \rho^2 c^2\right)$.
- **Stability**: $\phi\lambda > \rho c^2$ and $\rho > \phi$.

Avdantage

- We keep the conservative form for the original variables and obtain a fully linear acoustic.

Splitting

- If you want solve $\partial_t \boldsymbol{U} = A\boldsymbol{U}$ the solution is given by

$$\boldsymbol{U}(t) = e^{-At}\boldsymbol{U}(t=0) = e^{-(A_1+A_2)t}\boldsymbol{U}(t=0) \approx e^{-A_1 t}e^{-A_2 t}\boldsymbol{U}(t=0)$$

- A splitting scheme consists to solve two/or more parts of the system separately.

Aim

- For large acoustic waves (Mach number not small) we want capture all the phenomena. Consequently use an explicit scheme.
- For small/fast acoustic waves (low Mach number) we want filter acoustic. Consequently use an implicit scheme for acoustic.

Splitting: Explicit convective part/Implicit acoustic part.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \mathcal{M}^2(t)\Pi) = 0 \\ \partial_t E + \partial_x(E v + \mathcal{M}^2(t)\Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda_c^2 \partial_x v = 0 \\ \partial_t v + v\partial_x v + \frac{\mathcal{M}^2(t)}{\phi}\partial_x \Pi = 0 \end{cases}, \quad \begin{cases} \partial_t \rho = 0 \\ \partial_t(\rho u) + (1-\mathcal{M}^2(t))\partial_x \Pi = 0 \\ \partial_t E + (1-\mathcal{M}^2(t))\partial_x(\Pi v) = 0 \\ \partial_t \Pi + \phi(1-\mathcal{M}^2(t))\lambda_a^2 \partial_x v = 0 \\ \partial_t v + (1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x \Pi = 0 \end{cases}$$

with $\mathcal{M}(t) \approx max_x \frac{|u|}{c}$

- After each time step: we project $\Pi = p$ and $v = u$ (can be view as a discretization of the stiff source term)

Implicit time scheme

- We introduce the implicit scheme for the "acoustic part":

$$\begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} + \Delta t(1 - \mathcal{M}^2(t_n))\partial_x \Pi^{n+1} = (\rho u)^n \\ E^{n+1} + \Delta t(1 - \mathcal{M}^2(t_n))\partial_x (\Pi v)^{n+1} = E^n \\ \Pi^{n+1} + \Delta t\phi(1 - \mathcal{M}^2(t_n))\lambda_a^2 \partial_x v^{n+1} = \Pi^n \\ v^{n+1} + \Delta t(1 - \mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x \Pi^{n+1} = v^n \end{cases}$$

- We plug the equation on v in the equation on Π. We obtain the following algorithm:
  - Step 1: we solve

$$(I_d - (1 - \mathcal{M}^2(t_n))^2 \Delta t^2 \lambda_c^2 \partial_{xx})\Pi^{n+1} = \Pi^n - \Delta t(1 - \mathcal{M}^2(t_n))\phi\lambda_c^2 \partial_x v^n$$

  - Step 2: we compute

$$v^{n+1} = v^n - \Delta t(1 - \mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x \Pi^{n+1}$$

  - Step 3: we compute

$$(\rho u)^{n+1} = (\rho u)^n - \Delta t(1 - \mathcal{M}^2(t_n))\partial_x \Pi^{n+1}$$

  - Step 4: we compute

$$E^{n+1} = E^n - \Delta t(1 - \mathcal{M}^2(t_n))\partial_x (\Pi^{n+1} v^{n+1})$$

Advantage

- We solve only a constant Laplacian. We can assembly matrix one time.
- No problem of conditioning, which comes from to the strong gradient of ρ

Results I

- Smooth contact :

$$\begin{cases} \rho(t,x) = \chi_{x<x_0} + 0.1\chi_{x>x_0} \\ u(t,x) = 0.01 \\ p(t,x) = 1 \end{cases}$$

- Error

cells	Ex Rusanov	Ex LR	SI Rusanov	New SI Rus	New SI LR
250	0.042	$3.6E^{-4}$	$1.4E^{-3}$	$7.8E^{-4}$	$4.1E^{-4}$
500	0.024	$1.8E^{-4}$	$6.9E^{-4}$	$3.9E^{-4}$	$2.0E^{-4}$
1000	0.013	$9.0E^{-5}$	$3.4E^{-4}$	$2.0E^{-4}$	$1.0E^{-5}$
2000	0.007	$4.5E^{-5}$	$1.7E^{-4}$	$9.8E^{-5}$	$4.9E^{-5}$

- Suliciu: relaxation scheme different. The implicit Laplacian is not constant and depend of ρ^n.
- Comparison time scheme:

Scheme	λ	Δt
Explicit	$\max(\mid u-c \mid, \mid u+c \mid)$	$2.2E^{-4}$
SI Suliciu	$\max(\mid u - \mathcal{M}(t_n))\frac{\lambda}{\rho} \mid, \mid u + \mathcal{M}(t_n))\frac{\lambda}{\rho} \mid)$	0.0075
SI new relaxation	$\max(\mid v - \mathcal{M}(t_n))\lambda \mid, \mid v + \mathcal{M}(t_n))\lambda \mid)$	0.04

- Conditioning:

Schemes	Δt	conditioning
Si suliciu	0.00757	3000
Si new relax	0.041	9800
Si new relax	0.0208	2400
si new relax	0.0075	320

First 2D result I

- We take 100*100 cells $T_f = 1$ and

$$\begin{cases} \rho(t,\mathbf{x}) = G(\mathbf{x} - \boldsymbol{u}_0 t) \\ \boldsymbol{u}(t,\mathbf{x}) = \boldsymbol{u}_0, \quad \text{such that } \nabla \cdot \boldsymbol{u}_0 = 0 \text{ and } \mid \boldsymbol{u}_0 \mid \approx 10^{-3} \\ p(t,\mathbf{x}) = 1 \end{cases}$$

- Results:

Vars	Ex Rusanov	Ex LR	SI Rusanov	New SI LR
ρ	0.39	$1.9E^{-4}$	$8.4E^{-4}$	$7.5E^{-5}$
u	0.87	0.51	$5.3E^{-3}$	$2.7E^{-3}$
p	$9.6E^{-8}$	$5.5E^{-7}$	$1.8E^{-6}$	$7.2E^{-7}$
Δt	$4.2E^{-4}$	$4.4E^{-4}$	0.8	1(max 9)

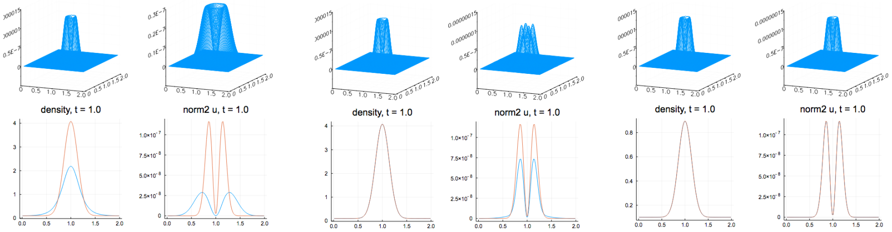
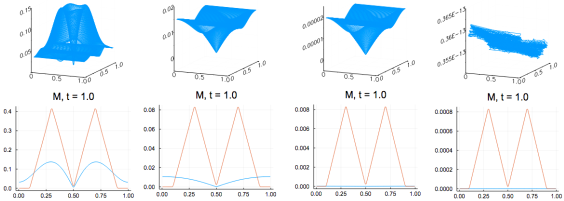


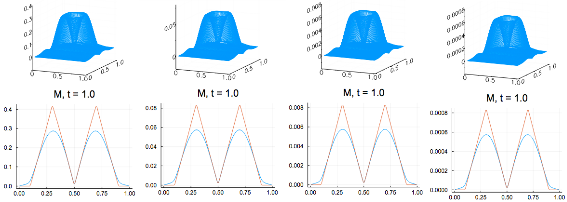
Figure: Explicit Rusanov scheme, Ex LR-Like, Semi Implicit relax

First 2D results II

- Gresho vortex: stationary vortex with varying Mach number and $\nabla \cdot \boldsymbol{u} = 0$.
- We plot the norm of $\boldsymbol{u}$

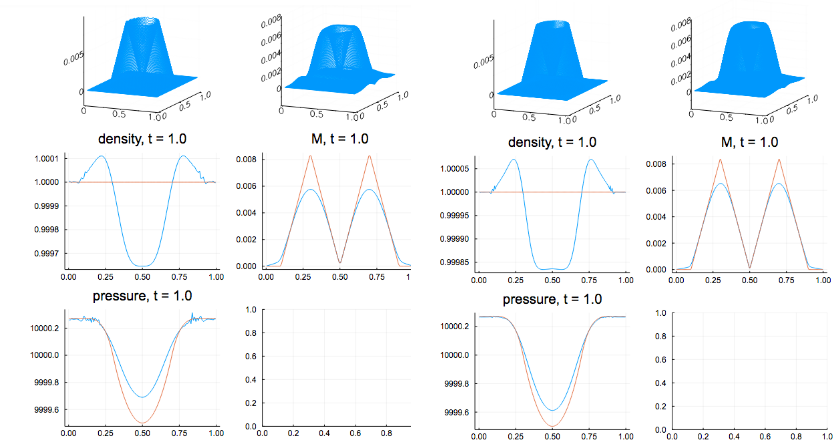


- Ex scheme: $M = 0.5$ ($\Delta t = 1.4E^{-3}$), $M = 0.1$ ($\Delta t = 3.5E^{-4}$), $M = 0.01$ ($\Delta t = 3.5E^{-5}$), $M = 0.001$ ($\Delta t = 3.5E^{-6}$)



- New scheme: $M = 0.5$ ($\Delta t = 2.5E^{-3}$), $M = 0.1$ ($\Delta t = 2.5E^{-3}$), $M = 0.01$ ($\Delta t = 2.5E^{-3}$), $M = 0.001$ ($\Delta t = 2.5E^{-3}$)

- Gresho vortex: stationary vortex with varying Mach number and $\nabla \cdot \boldsymbol{u} = 0$.
- Convergence for $\boldsymbol{u}$ and p

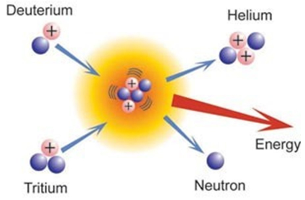


- Results with New-relax. Left: 120*120 cells, Right: 240*240 cells

Other multi-scale problems for plasma physics

Tokamak simulation and magnetized plasma

- **Fusion DT**: At sufficiently high energies deuterium and tritium (plasmas) can fuse to Helium. Free energy is released.
- **Plasma**: For very high temperature, the gas is ionized and give a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroïdal chamber where the plasma (10^8 Kelvin), is confined using magnetic fields. **Larger Tokamak**: Iter



Specificity for the Tokamak

- To stabilize the plasma we need very large magnetic field $\boldsymbol{B}$.
- This very large magnetic field generates time/space two scale problem between parallel and perpendicular (to $\boldsymbol{B}$) dynamic.

- Plasma description:
  - **Microscopic**: Newton laws for each particle. Coupled by external forces.
  - **Mesoscopic**: description by probability density. Probability to have a particle at the time t the position $\mathbf{x}$ and the velocity $\mathbf{v}$.
  - **Macroscopic**: description by macro quantities: density, velocity, pressure etc. Euler, Navier-Stokes, MHD equations.
- Dimensionless Vlasov- Maxwell equation:

$$\begin{cases} \partial_t f_i + \boldsymbol{v} \cdot \nabla_{\mathbf{x}} f_i + e\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_{\boldsymbol{v}} f_i = \dfrac{1}{\tau} Q(f_i, f_i) \\ \delta\left(\partial_t f_e + \boldsymbol{v} \cdot \nabla_{\mathbf{x}} f_e\right) - e\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_{\boldsymbol{v}} f_e = \dfrac{1}{\tau} Q(f_e, f_e) \\ \varepsilon^2 \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\mu_0 \boldsymbol{J} \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \\ \varepsilon^2 \nabla \cdot \boldsymbol{E} = n_i - n_e \end{cases}$$

with $\varepsilon \approx \frac{V_0}{c}$, $\tau = \frac{\lambda}{L}$ with λ the mean free path, $\delta = \frac{m_e}{m_i}$ the mass ratio.

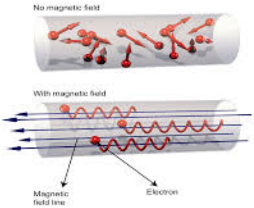
- **Limit** : $\tau \rightarrow 0$ ==> Euler-Maxwell bi-fluid.
- **Limit** : $\tau \rightarrow 0$, $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$ ==> Extended MHD.

Aim

- Aim $\tau \rightarrow 0$: filter collision and capture the equilibrium.
- Aim $\varepsilon \rightarrow 0$: filter fast electromagnetic waves (weak coupling with the rest).
- Aim $\delta \rightarrow 0$: filter inertial effect of electron. Main dynamic given by ions.

Gyro-kinetic limit and Anisotropic diffusion

- **Gyrokinetic model**: We consider the Vlasov Maxwell equations.
- For large magnetic field: two space/time scales:
  - fast rotation of ion around the magnetic field lines (radius, velocity depends of $\boldsymbol{B}$)
  - average transport of ion in the parallel direction.
- Gyrokinetic model filter the fast rotation. We can design aslo numerical scheme for Vlasov to filter this rotation if necessary.

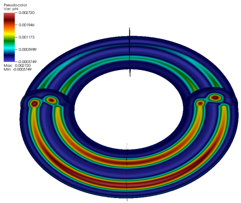


- **Thermal anisotropic diffusion**:

$$\partial_t T - \nabla \cdot (\kappa_{||} \boldsymbol{B} \otimes \boldsymbol{B} \nabla T) - \kappa_{iso} \Delta T = 0$$

- To avoid a strong CFL condition: implicit scheme.
- Conditioning of the matrix:

$$C \approx \frac{\kappa_{||}}{\kappa_{iso} \Delta^2} \approx 10^2 - 10^{10}$$



Conclusion

Problem

- We consider problem with two space/time scales.
- Sometimes we want solve the two scales. Sometimes we want filter (neglect) the fast one and capture the slow one.
- **Naive method**: we must capture the fast one to capture the slow one. Very important cost.

Euler equation

- Introducing Dynamic splitting scheme we separate the scales.
- Introducing implicit scheme for the acoustic wave we can filter these waves.
- Introducing relaxation we simplify at the maximum the implicit scheme.
- An adapted spatial scheme is also very important.

Announcement

- With some colleges we organize the summer school "Cemracs 2020".
- **Theme**: "Models and simulation of many passive/active particles". Physics particles, cells, population dynamic, crowd movement, smart city.