Implicit/semi-implicit schemes based on relaxation methods for compressible flows

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Outline

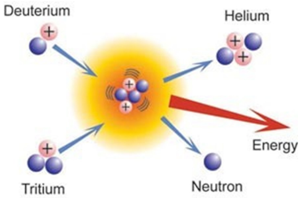
Physical and mathematical context

Iter Project and nuclear fusion

Applications

- Modeling and numerical simulation for the nuclear fusion.

- **Fusion DT**: At sufficiently high energies deuterium and tritium (plasmas) can fuse to Helium. Free energy is released.
- **Plasma**: For very high temperature, the gas is ionized and give a plasma which can be controlled by magnetic and electric fields.
- **Tokamak**: toroïdal chamber where the plasma (10^8 Kelvin), is confined using magnetic fields. **Larger Tokamak**: Iter



Difficulties:

- Plasma turbulence (Tokamak center) ==> Kinetic models.
- Plasma instabilities (Tokamak edge) ==> Fluid models.
- Necessary to simulate these phenomena and test some controls in realistic geometries of Tokamak.

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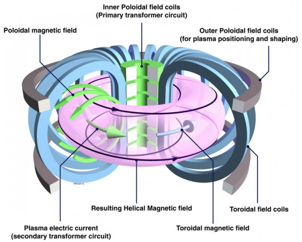

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MHD equations

- MHD equation (non conservative form):

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \\ \partial_t \boldsymbol{B} + \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- with ρ the density, $\boldsymbol{u}$ the velocity, p the pressure and $\boldsymbol{B}$ the magnetic field.
- We can write the model on conservative form. It is a hyperbolic system admitting a entropy dissipation equation.

$$\partial_t \rho S + \nabla \cdot (\rho S \boldsymbol{u}) \leq \nabla \cdot \boldsymbol{B}$$

- Eigen-structure:
  - Material waves at the velocity $(\boldsymbol{u}, \boldsymbol{n})$
  - Alfven waves at the velocity $v_A = \sqrt{\frac{|\boldsymbol{B}|^2}{\rho}}$
  - Slow and Fast Magneto-acoustic waves: depending of v_A and $c = \sqrt{\frac{\gamma p}{\rho}}$ the sound speed.
- The ratio between the wave speeds can be huge. The MHD is a strongly multi-scale problem in time.
- For tokamak simulation the phenomena are strongly anisotropic with $\boldsymbol{B}$ as dominant direction.

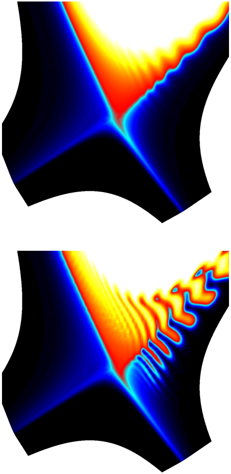
MHD equilibrium and instabilities

- **In tokamak** we want maintains the plasma around an equilibrium

$$\underbrace{(\nabla \times \boldsymbol{B})}_{\boldsymbol{J}} \times \boldsymbol{B} = \nabla p,$$

with $\boldsymbol{u} = 0$.

- Some instabilities can appear and damages the device.
- It important to simulate these instabilities and the possible methods to control them.
- Physical regime:
  - **Low** β: $c << V_a$
  - compressible in parallel direction: $\boldsymbol{u}_{||} \approx c$
  - incompressible in perpendicular direction: $\boldsymbol{u}_{\perp} << c$
- To treat this regime and the strong diffusion in parallel direction we need implicit/semi implicit scheme.



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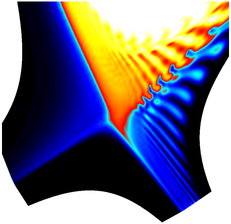

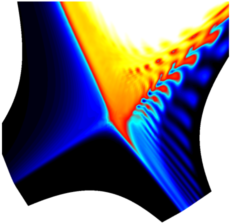

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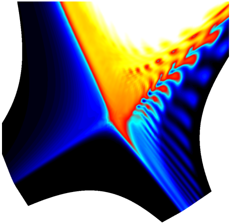

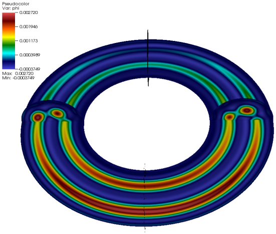

Euler equation and Low-Mach regime

- To treat the MHD problem we need a scheme efficient for compressible flow (parallel part) and nearly incompressible flow (perpendicular part).
- **Simplify problem**: Construct schemes for compressible Euler equations able to treat the two regimes.
- Equations:

$$\left\{\begin{array}{l}\partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) = 0\\ \rho\partial_t \boldsymbol{u} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = 0\\ \partial_t p + \nabla\cdot(p\boldsymbol{u}) + (\gamma-1)p\nabla\cdot\boldsymbol{u} = 0\end{array}\right.$$

- Normalization:
  - we introduce characteristic time t_0, velocity V, length L.
  - the characteristic velocity u_0 and pressure γp_0. The sound velocity is $c^2 = \frac{\gamma p_0}{\rho_0}$.

Application

- Astrophysics with the Euler equations (additional gravity term in general).
- Nuclear fission with multi-phase models.
- Nuclear fusion in Tokamak with the MHD model.

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$$\left\{\begin{array}{l}\partial_t\rho+\left[\dfrac{t_0u_0}{L}\right]\nabla\cdot(\rho\boldsymbol{u})=0\\ \rho\partial_t\boldsymbol{u}+\left[\dfrac{t_0u_0}{L}\right]\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\left[\dfrac{t_0p_0}{\rho_0u_0L}\right]\nabla p=0\\ \partial_t p+\left[\dfrac{t_0u_0}{L}\right]\boldsymbol{u}\cdot\nabla p+\left[\dfrac{\gamma t_0u_0}{L}\right]p\nabla\cdot\boldsymbol{u}=0\end{array}\right.$$

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$$\begin{cases} \partial_t \rho + \left[\frac{u_0}{V}\right] \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \left[\frac{u_0}{V}\right] \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \left[\frac{c_0^2}{u_0 V}\right] \nabla p = 0 \\ \partial_t p + \left[\frac{u_0}{V}\right] \boldsymbol{u} \cdot \nabla p + \left[\frac{\gamma u_0}{V}\right] p \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

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- Normalization:
  - we introduce characteristic time t_0, velocity V, length L.
  - the characteristic velocity u_0 and pressure γp_0. The sound velocity is $c^2=\frac{\gamma p_0}{\rho_0}$.
- We want to focus on the fluid motion consequently we choose $V=u_0$.
- We define the mach number: $M=\frac{u_0}{c_0}$. Using this we obtain

$$\left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\left[\frac{1}{M^2}\right]\nabla p=0\\ \partial_t p+\boldsymbol{u}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{u}=0\end{array}\right. \longrightarrow \left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\frac{1}{M^2}\nabla p=0\\ \partial_t E+\nabla\cdot(E\boldsymbol{u})+\nabla\cdot(p\boldsymbol{u})=0\end{array}\right.$$

- with $E=\frac{p}{\gamma-1}+M^2\frac{\rho|\boldsymbol{u}|^2}{2}$.

Low-Mach limit

Limit in 2D

- We consider $\boldsymbol{u} = \boldsymbol{u}_0 + M^2 \boldsymbol{u}_1$, We consider $p = p_0 + M^2 p_1$. The limit is:

$$\begin{cases} \partial_t \rho_0 + \boldsymbol{u}_0 \cdot \nabla \rho_0 = 0 \\ \partial_t \boldsymbol{u}_0 + \boldsymbol{u}_0 \cdot \nabla \boldsymbol{u}_0 + \dfrac{1}{\rho_0} \nabla p_1 = 0 \\ \nabla \cdot \boldsymbol{u}_0 = 0 \end{cases}$$

- If $\rho_0 = cts$ we obtain the classical incompressible Euler equation.
- **Interpretation**: Fluid motion around the acoustic equilibrium : $\nabla \cdot \boldsymbol{u}_0 = 0$, $\nabla p_0 = 0$.

Limit in 1D

- We consider $u = u_0 + M^2 u_1$, We consider $p = p_0 + M^2 p_1$. The limit is:

$$\begin{cases} \partial_t \rho_0 + u_0 \partial_x \rho_0 = 0 \\ \partial_t u_0 + \partial_x p_1 = 0 \\ \partial_x u_0 = 0 \end{cases}$$

- **Interpretation**: Fluid motion (isolated contact) around the acoustic equilibrium : $\partial_x u_0 = 0$ and $\partial_x p_0 = 0$.

Aim

- A scheme which has a good behavior in the limit regime (around the acoustic equilibrium).

Numerical difficulties in space: VF and DG

- **Methods used**: VF/DG (FE also but not here). Principle of VF method:

$$\partial_t \boldsymbol{U}(t,x) + \partial_x \boldsymbol{F}(\boldsymbol{U}(t,x)) = 0$$

$$\int_{\Omega_j} \partial_t \boldsymbol{U}(t,x) + \int_{\Omega_j} \partial_x \boldsymbol{F}(\boldsymbol{U}(t,x)) = 0$$

- with Ω_j a cell. Easily we obtain:

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- We consider $\boldsymbol{U}(t,x) = \sum_j \boldsymbol{U}_j \chi_{\Omega_j}$ with $\boldsymbol{U}_j(t) = \frac{1}{|\Omega_j|} \int_{\Omega} \boldsymbol{U}(t,x)$

$$\mid \Omega_j \mid \partial_t \boldsymbol{U}_j(t) + \boldsymbol{F}(\boldsymbol{U}(t,x_{j+\frac{1}{2}})) - F(\boldsymbol{U}(t,x_{j-\frac{1}{2}})) = 0$$

- The quantities $\boldsymbol{F}(\boldsymbol{U}(t,x_{j\pm\frac{1}{2}}))$ are unknown. **VF** idea: $\boldsymbol{F}(\boldsymbol{U}(t,x_{j\pm\frac{1}{2}})) \approx \boldsymbol{G}(\boldsymbol{U}_j, \boldsymbol{U}_{j+1})$. We speak about of numerical fluxes.
- Classical fluxes : centered (unstable with explicit scheme):

$$\boldsymbol{G}(\boldsymbol{U}_j, \boldsymbol{U}_{j+1}) = \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{U}_j) + \boldsymbol{F}(\boldsymbol{U}_{j+1})\right)$$

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- Classical fluxes : upwind, Riemann Solver etc:

$$\boldsymbol{G}(\boldsymbol{U}_j, \boldsymbol{U}_{j+1}) = \frac{1}{2}\left(\boldsymbol{F}(\boldsymbol{U}_j) + \boldsymbol{F}(\boldsymbol{U}_{j+1})\right) - A(\boldsymbol{U}_j, \boldsymbol{U}_{j+1})(\boldsymbol{U}_{j+1} - \boldsymbol{U}_j)$$

- Discrete scheme:

$$\partial_t \boldsymbol{U}(t) + D_{2h}(\boldsymbol{U}(t)) - \Delta x D_h(A(\boldsymbol{U}) D_h \boldsymbol{U}) = 0$$

- with D_{kh} the k order discrete derivative;

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- Equivalent equation:

$$\partial_t \boldsymbol{U}(t,x) + \partial_x \boldsymbol{F}(\boldsymbol{U}(t,x)) - \Delta x \partial_x (A(\boldsymbol{U}) \partial_x \boldsymbol{U}) = O(\Delta x^2)$$

- We speak about numerical diffusion.

Numerical difficulties in space: VF and DG II

- **Properties of hyperbolic systems**: these models can generate discontinuities. No unicity of the weak solution.
- To obtain uniqueness and stability we introduce additional entropy equation:

$$\partial_t \eta(\boldsymbol{U}) + \partial_x \mathbf{Q}(\boldsymbol{U}) \leq 0 \rightarrow \partial_t \int \eta(\boldsymbol{U}) \leq 0$$

- with $\eta(\boldsymbol{U})$ a convex function, $\zeta(\boldsymbol{U})$ the entropic flux such that $\eta^{'}(\boldsymbol{U})\boldsymbol{F}^{'}(\boldsymbol{U}) = Q^{'}(\boldsymbol{U})$. The left part is exactly zero for smooth solution.
- Stability of the scheme:

$$\partial_t \eta(\boldsymbol{U}) + D_h \zeta(\boldsymbol{U}) \leq 0 \rightarrow \partial_t \int \eta(\boldsymbol{U}) \leq 0$$

- Approximated model:

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Conclusion

- The structure of the numerical diffusion play an important role in the stability.
- Aim of scheme: find a scheme with a viscosity matrix which minimize the error for the solutions or some particular solutions (low mach flow, steady state etc) and keeping the stability properties.

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- if $\eta^{''}(\boldsymbol{U}) A(\boldsymbol{U}) \geq 0$.

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Numerical difficulties in space: VF in 1D

- Second method: Finite volume and DG method
  - VF method + Rusanov flux. Equivalent equation:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = \dfrac{S\Delta x}{2}\partial_{xx}\rho \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \dfrac{1}{M^2}\partial_x p = \dfrac{S\Delta x}{2}\partial_{xx}(\rho u) \\ \partial_t E + \partial_x(Eu) + \partial_x(pu) = \dfrac{S\Delta x}{2}\partial_{xx}E \end{cases}$$

  - **Problem**: S must be larger that $\frac{1}{M}$ for stability. Huge diffusion.

- Example: isolated contact $p = 1$ and $u = 0.1$.
- **Exact. solution**:

$$\partial_t \rho + u_0 \partial_x \rho = 0$$

- **Rusanov scheme**:

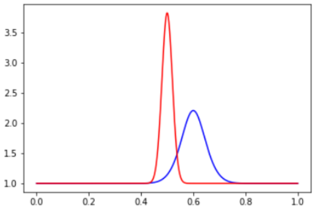
$$\partial_t \rho + u_0 \partial_x \rho = \frac{S\Delta x}{2}\partial_{xx}\rho$$

with $S > u_0 + c \approx 1.5$

- **Upwind scheme for limit**:

$$\partial_t \rho + u_0 \partial_x \rho = \frac{u_0 \Delta x}{2}\partial_{xx}\rho$$

- Rusanov scheme $T_f = 2$ $u_0 = 0.05$ and 1000 cells



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$$\partial_t \rho + u_0 \partial_x \rho = 0$$

- **Rusanov scheme**:

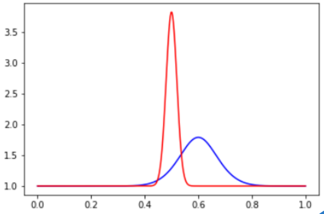
$$\partial_t \rho + u_0 \partial_x \rho = \frac{S\Delta x}{2}\partial_{xx}\rho$$

with $S > u_0 + c \approx 1.5$

- **Upwind scheme for limit**:

$$\partial_t \rho + u_0 \partial_x \rho = \frac{u_0 \Delta x}{2}\partial_{xx}\rho$$

- Rusanov scheme $T_f = 5$ $u_0 = 0.02$ and 1000 cells



Numerical difficulties in space: VF in 1D

- Second method: Finite volume and DG method
  - VF method + Rusanov flux. Equivalent equation:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = \frac{S\Delta x}{2}\partial_{xx}\rho \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \frac{1}{M^2}\partial_x p = \frac{S\Delta x}{2}\partial_{xx}(\rho u) \\ \partial_t E + \partial_x(Eu) + \partial_x(pu) = \frac{S\Delta x}{2}\partial_{xx}E \end{cases}$$

  - **Problem**: S must be larger that $\frac{1}{M}$ for stability. Huge diffusion.

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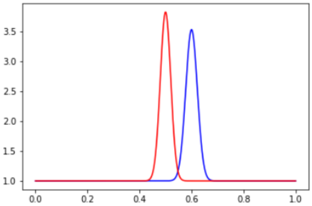
- Lagrange+remap scheme $T_f = 2$ $u_0 = 0.05$ and 1000 cells



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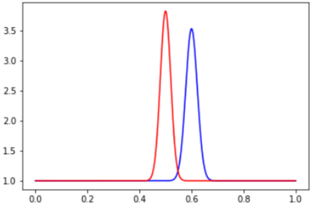


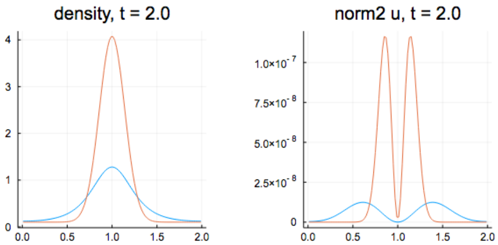
Numerical difficulties in space: VF in 2D

- Same analysis in 2D.
  - VF method + Rusanov flux. Equivalent equation:

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) = \dfrac{S\Delta x}{2}\Delta\rho \\ \partial_t(\rho\boldsymbol{u}) + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \dfrac{1}{M^2}\nabla p = \dfrac{S\Delta x}{2}\Delta(\rho\boldsymbol{u}) \\ \partial_t E + \nabla\cdot(E\boldsymbol{u}) + \nabla\cdot(p\boldsymbol{u}) = \dfrac{S\Delta x}{2}\Delta E \end{cases}$$

  - **Problem**: S must be larger that $\frac{1}{M}$ for stability. Huge diffusion.

- Example: isolated contact $p = 1$, $\nabla\cdot\boldsymbol{u}_0 = 0$ and $\boldsymbol{u}_0$ constant in time.
- Rusanov scheme $T_f = 2 \mid \boldsymbol{u}_0 \mid \approx 0.001$ and 100*100 cells.



- Red: exact solution, Blue: numerical solution.

Numerical difficulties in space: VF in 2D

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$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = \frac{S\Delta x}{2}\Delta \rho \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u}\cdot \nabla \boldsymbol{u} + \frac{1}{M^2}\nabla p = \frac{S\Delta x}{2}\Delta \boldsymbol{u} \\ \partial_t p + \boldsymbol{u}\cdot\nabla p + \gamma p \nabla\cdot \boldsymbol{u} = \frac{S\Delta x}{2}\nabla p \end{cases}$$

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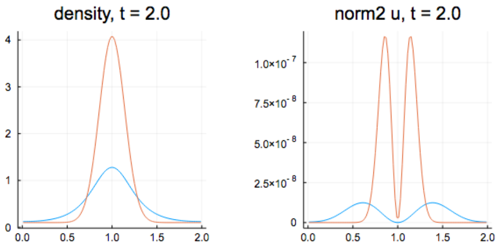

- Red: exact solution, Blue: numerical solution.

Numerical difficulties in time

Explicit time scheme

- **Low-Mach regime**: fast and small acoustic waves. Weak/no coupling with the fluid motion.
- **Explicit scheme**: CFL condition

$$\max_x \left(u + \frac{c}{M}\right) \Delta t \leq h$$

- Δt is very small and allows to capture the fast waves. We want/can filter the fast waves.
- **Solution**: full implicit/semi implicit time schemes.

Implicit time scheme

- Nonlinear problem to invert: Newton/picard + linear solver.
- **Drawbacks**: matrix to assembly, to store and to invert.
- Operator to invert:

$$(I_d h - \Delta t A) \approx A, \quad \text{for } h << 1 \text{ and } \Delta t >> 1$$

with A the discrete spatial scheme of the Jacobian.

  - **Full implicit**: Eigenvalues of A: $(u - \frac{c}{M}, u, u + \frac{c}{M})$. So ill-conditioning.
  - In 2D additional zero eigenvalue (shear wave) which generate ill-conditioning.
  - Strong gradient of p and ρ generate also ill-conditioning.

Classical implicit scheme

- We use an explicit scheme for convection (or we split the convection).
- Implicit acoustic step:

$$\left\{\begin{array}{l}\rho^{n+1}=\rho^{n}\\(\rho u)^{n+1}=\rho^{n}u^{n}-\Delta t\partial_{x}p^{n+1}+Rhs_{u}\\E^{n+1}=E^{n}-\Delta t\partial_{x}(p^{n+1}u^{n+1})=Rhs_{E}\end{array}\right.$$

Plugging this in the second equation, we obtain

$$E^{n+1}-\Delta t^{2}\partial_{x}\left(\frac{p^{n+1}}{\rho^{n}}\partial_{x}p^{n+1}\right)=Rhs(E^{n},u^{n},\rho)$$

- Matrix-vector product to compute u^{n+1}.
- Works with similar idea: [DegondTang09]-[DLV17]-[DDLV18].

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- Implicit acoustic step:

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Plugging this in the second equation, we obtain

$$\frac{p^{n+1}}{\gamma - 1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho)$$

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Conclusion

- **Semi implicit**: We have only one scale in the implicit operator. The operator is symmetric and positive.
- Strong gradient of p and ρ generate also ill-conditioning. The matrix must be assembled at each time (costly).
- Nonlinear solver which bad convergence for if $\Delta t >> 1$ and the gradient of p not so small.

Numerical difficulties in time

Aim

- Design implicit/semi implicit VF/DG scheme without problem of conditioning/inverting etc.
- **Solution proposed**: construct new model larger, but simpler (relaxation model) with approximate the original model and write the scheme for the new model to obtain the scheme for the original one.

Linear and full implicit relaxation scheme

General principle

- We consider the following nonlinear hyperbolic system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- with $\boldsymbol{U}$ a vector of N functions.
- **Aim**: Find a way to approximate this system with a sequence of simple systems.
- **Idea**: Xin-Jin relaxation method (very popular in the hyperbolic and Finite Volume community) [JX95]-[Nat96]-[ADN00].

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

Limit of the hyperbolic relaxation scheme

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x((\lambda^2 - \mid A(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + o(\varepsilon^2)$$

- with $A(\boldsymbol{U})$ the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$.

- **Conclusion**: the relaxation system is an approximation of the hyperbolic original system (error in ε).

Specific kinetic model: stability

- **First order stability**: we consider the first order approximation

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x \left(\left(\lambda^2 I_n - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) + O(\varepsilon^2)$$

$$\partial_t \eta(\boldsymbol{U}) + \partial_x \mathbf{Q}(\boldsymbol{U}) - \varepsilon \partial_x \left(\eta^{'}(\boldsymbol{U}) \left(\lambda^2 I_n - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right) \partial_x \boldsymbol{U} \right) \leq 0 + O(\varepsilon^2)$$

- The second equation is true if $\eta^{''}(\boldsymbol{U}) A(\boldsymbol{U}) \geq 0$.
- Finally, we have the entropy property at the first order if

$$\lambda > vp_{max} \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid, \quad \text{with } A(\boldsymbol{U}) = \left(\lambda^2 I_n - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2 \right).$$

- **Entropy stability**: For the model [Jin95]

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + \lambda^2 \partial_x u = \dfrac{1}{\varepsilon}(F(u) - v) \end{cases}$$

we obtain

$$\partial_t \Phi(u,v) + \partial_x \Psi(u,v) \leq -\frac{1}{\varepsilon} \partial_v \Phi(u,v) \cdot (v - f(u)) \leq 0$$

with $\Phi(u,v) = h_+(v + \lambda u) + h_-(v - \lambda u)$, $\Psi(u,v) = \lambda(h_+(v + \lambda u) - h_-(v - \lambda u))$ and

$$h_\pm(F(u) \pm \lambda u) = \frac{1}{2} \left(\eta(u) \pm \frac{Q(u)}{\lambda} \right)$$

- The inequality is true if $\Phi(u,v)$ convex compare to v and $\partial_v \Phi(u, v = f(u)) = 0$.
- It is true if $\mid F^{'}(u) \mid < \lambda$. The situation seems the same for systems.

XIn-Jin implicit scheme

Main property

- **Relaxation system**: "the nonlinearity is local and the non locality is linear".
- **Main idea**: **splitting scheme** between implicit transport and **implicit** relaxation [Paru15].
- **Key point**: the $\partial_t \boldsymbol{U} = 0$ during the relaxation step. Therefore $\boldsymbol{f}^{eq}(\boldsymbol{U})$ **is explicit**.

- Relaxation step:

$$\begin{cases} \boldsymbol{U}^{n+1} = \boldsymbol{U}^n \\ \boldsymbol{V}^{n+1} = \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^{n+1}) - \boldsymbol{V}^{N+1}) + (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^n) - \boldsymbol{V}^n) \end{cases}$$

- Transport step (order 1) :

$$I_d + \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \boldsymbol{U}^{n+1} \\ \boldsymbol{V}^{n+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{U}^n \\ \boldsymbol{V}^n \end{pmatrix}$$

  - We plug the equation on $\boldsymbol{V}$ in the equation on $\boldsymbol{U}$.
  - We obtain the implicit part:

$$(I_d - \Delta t^2 \lambda^2 \partial_{xx})\boldsymbol{U}^{n+1} = \boldsymbol{U}^n - \Delta t \partial_x \boldsymbol{V}^n$$

  - We apply a matrix-vector product

$$\boldsymbol{V}^{n+1} = -\Delta t \lambda^2 \partial_x \boldsymbol{U}^{n+1}$$

- **Advantages**: N independent elliptic equations with constant coefficient.
- Natural extension at the second order in time. **In space**: FV (used here) or DG/FE.

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Time discretization

- Consistency analysis of the scheme : splitting + CN for relaxation + Euler implicit for transport.

First order scheme (first order transport)

- We define the two operators for each step :

$$T_{\Delta t} : (I_d + \Delta t A \partial_x I_d)\boldsymbol{f}^{n+1} = \boldsymbol{f}^n$$

$$R_{\Delta t} : \boldsymbol{f}^{n+1} + \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^{n+1}) = \boldsymbol{f}^n - (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{f}^{eq}(\boldsymbol{U}) - \boldsymbol{f}^n)$$

- **Final scheme**: $T_{\Delta t} \circ R_{\Delta t}$ is consistent with

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \frac{\Delta t}{2}\lambda^2 \partial_{xx}\boldsymbol{U} + \left(\frac{(2-\omega)\Delta t}{2\omega}\right)\partial_x\left(D(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\Delta t^2)$$

- with $\omega = \frac{\Delta t}{\varepsilon + \theta \Delta t}$ and $D(\boldsymbol{U}) = (\lambda^2 I_n - \mid \partial \boldsymbol{F}(\boldsymbol{U}) \mid^2)$.

- **Order 2**: If we choose $\varepsilon = 0$ + $\theta = 0.5$ for the relaxation (so we have $\omega = 2$) + Crank-Nicolson for transport part + Strang splitting. No numerical diffusion but numerical dispersion.

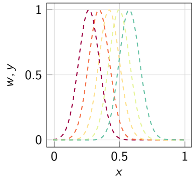
BC : results

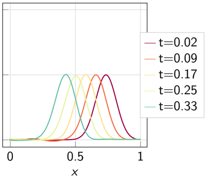
- **Question**: What BC for the kinetic variables. How keep the order ?

First result

- The second order symmetric (modified version tot he previous scheme) scheme for the Xin-Jin relaxation:
$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$
is consistent with
$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = O(\Delta t^2) \\ \partial_t \boldsymbol{W} - \partial \boldsymbol{F}(\boldsymbol{U}) \partial_x \boldsymbol{W} = O(\Delta t^2) \end{cases}$$
with $\boldsymbol{W} = \boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}$.

- **Natural BC**: entering condition for $\boldsymbol{U}$ and $\boldsymbol{W} = 0$ or $\partial_x \boldsymbol{W} = 0$.
- Example: $F(u) = cu$ (transport):





- Transport of the u (dashed lines) and $w = v - f(u)$ (plain lines) quantities.
- Same results for the Euler equations.

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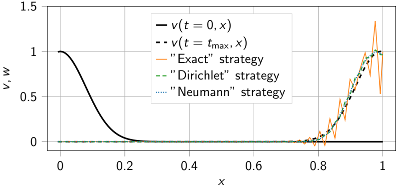
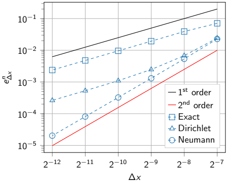




- Same results for the Euler equations.

Xin-Jin relaxation: limit of the method

Numerical error

- Error for the first order splitting scheme:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \partial_x ((\lambda^2 I_d - \mid A(\boldsymbol{U}) \mid^2) \partial_x \boldsymbol{U}) + o(\Delta t^2)$$

- **Low-Mach Euler equation**: we take $\lambda > c$. For the density equation, we obtain

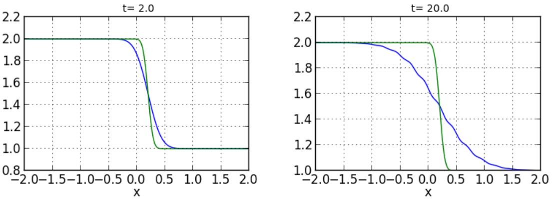
$$\partial_t \rho + \partial_x(\rho u) = \Delta t \partial_x ((\lambda^2 - u^2)\partial_x \rho - \rho \partial_x u^2 - \partial_x p) + o(\Delta t^2)$$

- In Low mach regime $\partial_x u \approx M$, $\partial_x p \approx M$ and $u \approx M$ consequently

$$\partial_t \rho + \partial_x(\rho u) \approx \Delta t \partial_x (c^2 \partial_x \rho) - O(M)\partial_{xx}\rho + o(\Delta t^2)$$

- **Conclusion**: Huge diffusion for the contact wave.

Test: smooth contact. First order time scheme. $T_f = \frac{2}{M}$. $\Delta t = T_f/100$.



Order 1. Left: $M = 0.1$. Right: $M = 0.01$

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- Error for the first order splitting scheme:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \partial_x((\lambda^2 I_d - \mid A(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + o(\Delta t^2)$$

- **Low-Mach Euler equation**: we take $\lambda > c$. For the density equation, we obtain

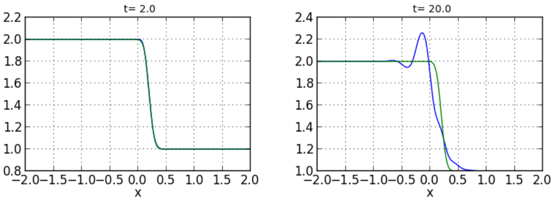
$$\partial_t \rho + \partial_x(\rho u) = \Delta t \partial_x((\lambda^2 - u^2)\partial_x \rho - \rho \partial_x u^2 - \partial_x p) + o(\Delta t^2)$$

- In Low mach regime $\partial_x u \approx M$, $\partial_x p \approx M$ and $u \approx M$ consequently

$$\partial_t \rho + \partial_x(\rho u) \approx \Delta t \partial_x(c^2 \partial_x \rho) - O(M)\partial_{xx}\rho + o(\Delta t^2)$$

- **Conclusion**: Huge diffusion for the contact wave.

item **Test**: smooth contact. Second order time scheme. $T_f = \frac{2}{M}$. $\Delta t = T_f/100$.



Order 1 Left: $M = 0.1$. Right: $M = 0.01$

Possible solution: Relaxation with central wave

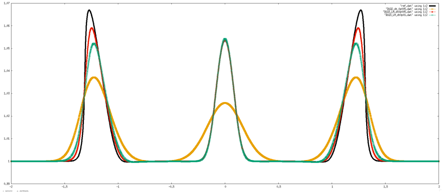
- Relaxation methods with a central wave [Bou09]-[Nat96]-[ADN00].

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \partial_x \boldsymbol{W} = \dfrac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \\ \partial_t \boldsymbol{W} + \lambda^2 \partial_x \boldsymbol{V} = \dfrac{1}{\varepsilon}(\lambda(\boldsymbol{F}^+(\boldsymbol{U}) - \boldsymbol{F}^-(\boldsymbol{U})) - \boldsymbol{W}) \end{cases}$$

  with $\boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{F}^+(\boldsymbol{U}) + \boldsymbol{F}^-(\boldsymbol{U})$. Additional zero wave.
- **Limit**:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \partial_x(\lambda(A^+(\boldsymbol{U}) - A^-(\boldsymbol{U})) - \mid \partial F(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + o(\Delta t^2)$$

- **Question**: choice of the flux splitting.
- **Test case**: Acoustic wave. Very high-order, 4000 cells.



- Xin-Jin $\Delta t = 0.005$ (yellow), Splitting-Relaxation $\Delta t = 0.005/0.01$ (red, green). Contact captured.
- **Conclusion**: Relaxation with central Can preserve contact wave and the low mach limit. BUT Stability not clear.

Semi implicit relaxation scheme

First Semi implicit scheme I

- **Previous approach** difficult to relax the two scales correctly and keep stability.
- **Idea**: Relax only the acoustic part to linearized the implicit part [CGS11]-[CC12]
- **Suliciu approach**: relax the pressure which is a strongly nonlinear function of macroscopic variables.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t(\rho u)+\partial_x(\rho u^2+\Pi)=0\\ \partial_t E+\partial_x(Eu+\Pi u)=0\\ \partial_t(\rho\Pi)+\partial_x(\rho\Pi u)+\lambda^2\partial_x u=\frac{\rho}{\varepsilon}(p-\Pi)\end{array}\right.$$

- **Limit**:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t(\rho u)+\partial_x(\rho u^2+p)=\varepsilon\partial_x\left(\left(\lambda^2-\rho^2c^2\right)\partial_x u\right)\\ \partial_t E+\partial_x(Eu+pu)=\varepsilon\partial_x\left(\left(\lambda^2-\rho^2c^2\right)\partial_x\frac{u^2}{2}\right)\end{array}\right.$$

- **Stability**: $\lambda>\rho c$.
- Contact waves:

$$\left\{\begin{array}{l}\partial_t\rho+u\partial_x\rho=0\\ \partial_x u=0\\ \partial_x p=0\end{array}\right.$$

- redare preserved by the relaxation approximation.
- Another way to say that : the contact waves are also solution of the relaxation model if $\pi(t=0)=p(t=0)$.
- For the low-mach flow (around the contact waves) the relaxation model is a very accurate approximation.

First Semi implicit scheme II

- **Idea**: splitting + implicit scheme for acoustic part [IDG18];
- Splitting scheme: convective part

$$(C) = \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \mathcal{E}^2(t)\Pi) = 0 \\ \partial_t E + \partial_x(Eu + \mathcal{E}^2(t)\Pi u) = 0 \\ \partial_t(\rho\Pi) + \partial_x(\rho\Pi u) + \lambda_c^2 \partial_x u = \frac{\rho}{\varepsilon}(p - \Pi) \end{cases}$$

- The eigenvalues: $(u - \mathcal{E}(t)\frac{\lambda}{\rho}, u, u + \mathcal{E}(t)\frac{\lambda}{\rho})$.
- Splitting scheme: acoustic part

$$(A) = \begin{cases} \partial_t \rho = 0 \\ \partial_t(\rho u) + (1 - \mathcal{E}^2(t))\partial_x \Pi = 0 \\ \partial_t E + (1 - \mathcal{E}^2(t))\partial_x(\Pi u) = 0 \\ \partial_t(\rho\Pi) + (1 - \mathcal{E}^2(t))\lambda_a^2 \partial_x u = \frac{\rho}{\varepsilon}(p - \Pi) \end{cases}$$

- The eigenvalues: $(-(1 - \mathcal{E}^2(t))\frac{\lambda}{\rho}, 0, (1 - \mathcal{E}^2(t))\frac{\lambda}{\rho})$
- with $\lambda^2 = \lambda_c^2 + (1 - \mathcal{E}^2(t))\lambda_a^2$.
- **Important point**:

$$\mathcal{E}^2(t) \approx \min\left(\mathcal{E}_{min}, \max\left(\frac{u}{c}, 1\right)\right)^2.$$

First Semi implicit scheme III

- Spatial scheme for convective part: Rusanov scheme:
  - Principle of Rusanov scheme. Diffusion matrix:

$$A(\boldsymbol{U}) = \frac{S}{2} Id \boldsymbol{U}$$

    with S larger that the maximal wave speed.
  - For the full explicit scheme $S >\mid u \mid + c \approx c$ in low mach regime.
  - For the splitting implicit scheme $S >\mid u \mid + \mathcal{E}(t) \approx 2u$ in low mach regime.
  - **Conclusion**: the density is slowly damped as a classical scheme for advection. Good behavior of scheme for low mach flow.
  - Since is never zero. The scheme doesn't preserve steady contact wave (u=0).
  - For high-mach flow the full model is explicit and we obtain classical scheme.
- Spatial scheme for the acoustic part: centered scheme. The stability is preserved since this part will be implicit.

First Semi implicit scheme IV

- Time scheme:

$$\begin{cases} (\rho u)^{n+1} = \rho^n u^n - \Delta t(1-\mathcal{E}^2(t))\partial_x \Pi^{n+1} = 0 \\ E^{n+1} = E^n - \Delta t(1-\mathcal{E}^2(t))\partial_x(\Pi^{n+1}u^{n+1}) = 0 \\ \rho^n \Pi^{n+1} = \rho^n \Pi^n - (1-\mathcal{E}^2(t))\Delta t \lambda_a^2 \partial_x u^{n+1} = 0 \end{cases}$$

The last equation can be rewritten as

$$u^{n+1} = u^n - \Delta t(1-\mathcal{E}^2(t))\frac{1}{\rho^n}\partial_x \Pi^{n+1} = 0$$

Plugging this in the second equation, we obtain

$$\Pi^{n+1} - \Delta t^2(1-\mathcal{E}^2(t))^2 \frac{1}{\rho^n}\partial_x\left(\frac{1}{\rho^n}\lambda_a^2 \partial_x \Pi^{n+1}\right) = b(\Pi^n, u^n)$$

- Matrix-vector product to compute u and E.
- **Advantages Implicit part**: just one linear elliptic problem to invert.
- **Defaults**: conditioning depending of the density and need to be assembly at each time.
- **Problem**: velocity is a nonlinear function of ρ and ρu.

Second Semi implicit scheme I

- **Idea**: Relax only the acoustic part to linearized the implicit part.
- **New approach**: relax the pressure and velocity (acoustic variables).

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{cases}$$

- **Limit**:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = \varepsilon\partial_x\left[\frac{1}{\rho}\left(\frac{\rho}{\phi} - 1\right)\partial_x p\right] \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon\partial_x\left[\frac{1}{\rho}\left[u\left(\frac{\rho}{\phi} - 1\right)\partial_x p + \left(\rho\phi\lambda^2 - \rho^2 c^2\right)\partial_x u\right]\right] \\ \partial_t E + \partial_x(E u + p u) = \varepsilon\partial_x\left[\frac{1}{\rho}\left[E\left(\frac{\rho}{\phi} - 1\right)\partial_x p + \left(\frac{\rho}{\phi} - 1\right)\partial_x \frac{p^2}{2} + \left(\rho\phi\lambda^2 - \rho^2 c^2\right)\partial_x \frac{u^2}{2}\right]\right] \end{cases}$$

- **Stability**: $\phi\lambda > \rho c^2$ and $\rho > \phi$.
- Contact waves:

$$\begin{cases} \partial_t \rho + u\partial_x \rho = 0 \\ \partial_x u = 0 \\ \partial_x p = 0 \end{cases}$$

- redare preserved by the relaxation approximation.
- The contact waves are also solutions if $\pi(t=0) = p(t=0)$ and $v(t=0) = u(t=0)$.
- For the low-mach flow (around the contact waves) the relaxation model is a very accurate approximation.

Second Semi implicit scheme II

- **First order stability**: we consider the first order approximation

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon\partial_x \left(A(\boldsymbol{U})\partial_x \boldsymbol{U}\right) + O(\varepsilon^2)$$

$$\partial_t \eta(\boldsymbol{U}) + \partial_x \mathbf{Q}(\boldsymbol{U}) - \varepsilon\partial_x \left(\eta^{'}(\boldsymbol{U})A(\boldsymbol{U})\partial_x \boldsymbol{U}\right) \leq 0 + O(\varepsilon^2)$$

- The second equation is true if $\eta^{''}(\boldsymbol{U})A(\boldsymbol{U}) \geq 0$. It is true for the matrix associated with relaxation scheme if

$$\phi\lambda^2 > \rho c^2, \quad \rho > \phi.$$

- **Entropy stability**: We rewrite the model as

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t(\rho\Pi) + \partial_x(\rho v \Pi) + ab\partial_x v = \frac{\rho}{\varepsilon}(p - \Pi) \\ \partial_t(\rho v) + \partial_x(\rho v^2) + \frac{a}{b}\partial_x \Pi = \frac{\rho}{\varepsilon}(u - v) \\ \partial_t a + \partial_x(a v) = 0 \\ \partial_t b + \partial_x(b v) = 0 \end{cases}$$

- with $a(t=0) = \rho\lambda$ and $b(=0) = \phi\lambda$.
- **Idea** : comparison principle. We consider S the entropy and $\hat{S}$ the function such that

$$\partial_t \hat{S} + v\partial_x \hat{S} = 0, \quad \text{with } \hat{S}(t=0) = S(t=0)$$

- We prove using the equations that $S(\rho, s) \leq \hat{e}$ and using specific invariants that $\hat{e} > e(\rho, \hat{s})$. We deduce that

$$S(\rho, e) > \hat{S}, \rightarrow \int S(t) \geq \int \hat{S}(t) = \int S(t=0)$$

Second Semi implicit scheme III

- **Idea**: splitting + implicit scheme for acoustic part.
- Splitting scheme: convective part

$$
(C) = \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \mathcal{E}^2(t)\Pi) = 0 \\ \partial_t E + \partial_x(E v + \mathcal{E}^2(t)\Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda_c^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{\mathcal{E}^2(t)}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{cases}
$$

- The eigenvalues: $(v - \mathcal{E}(t)\lambda, v, v + \mathcal{E}(t)\lambda)$.
- Splitting scheme: acoustic part

$$
(A) = \begin{cases} \partial_t \rho = 0 \\ \partial_t(\rho u) + (1 - \mathcal{E}^2(t))\partial_x \Pi = 0 \\ \partial_t E + (1 - \mathcal{E}^2(t))\partial_x(\Pi v) = 0 \\ \partial_t \Pi + (1 - \mathcal{E}^2(t))\phi\lambda_a^2 \partial_x v = \frac{}{\varepsilon}(p - \Pi) \\ \partial_t v + (1 - \mathcal{E}^2(t))\frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{cases}
$$

- The eigenvalues: $(-(1 - \mathcal{E}^2(t))\lambda, 0, (1 - \mathcal{E}^2(t))\lambda$
- with $\lambda^2 = \lambda_c^2 + (1 - \mathcal{E}^2(t))\lambda_a^2$.
- **Important point**:

$$
\mathcal{E}^2(t) \approx \min\left(\mathcal{E}_{min}, \max\left(\frac{u}{c}, 1\right)\right)^2.
$$

Second Semi implicit scheme IV

- Spatial scheme for convective part: Rusanov scheme:
  - Diffusion matrix for this scheme:

$$\partial_x(A(\boldsymbol{U})\partial_x \boldsymbol{U}) = \frac{S}{2}\partial_{xx}\boldsymbol{U}$$

    with S larger that the maximal wave speed.
  - For the full explicit scheme $S >| u | + c \approx c$ in low mach regime.
  - For the splitting implicit scheme $S >| u | + \mathcal{E}(t) \approx 2u$ in low mach regime.
  - **Conclusion**: the density is slowly damped as a classical scheme for advection. Good behavior of scheme for low mach flow.
- Spatial scheme for convective part: LR-like scheme:
  - Diffusion matrix for this scheme:

$$\partial_x(A(\boldsymbol{U})\partial_x \boldsymbol{U}) = \begin{pmatrix} \partial_x(| u | \partial_x \rho) + \partial_x(\rho \partial_x p) \\ \partial_x(| u | \partial_x(\rho u)) + \partial_x(\rho u \partial_x p) + \mathcal{E}(t)\frac{\phi\lambda}{2}\partial_{xx}u \\ \partial_x(| u | \partial_x E) + \partial_x(E\partial_x p) + \frac{\phi\lambda}{2}(\mathcal{E}(t)\partial_x(u\partial_x p) + \mathcal{E}(t)^3\partial_x(p\partial_x u)) \end{pmatrix}$$

  - **Conclusion**: the density is slowly damped as a classical scheme for advection. Good behavior of scheme for low mach flow.
  - This scheme is less dissipative for the density and preserve exactly stationary contact.
- Spatial scheme for the acoustic part: centered scheme. The stability is preserved since this part will be implicit.

Second Semi implicit scheme V

- Time scheme:

$$(A1) = \begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} = (\rho u)^n - \Delta t(1-\mathcal{E}^2(t))\partial_x \Pi^{n+1} \\ E^{n+1} = E^n - \Delta t(1-\mathcal{E}^2(t))\partial_x(\Pi^{n+1} v^{n+1}) \\ \\ \Pi^{n+1} + (1-\mathcal{E}(t))\Delta t \phi \lambda_a^2 \partial_x v^{n+1} = \Pi^n \\ v^{n+1} + (1-\mathcal{E}(t))\Delta t \frac{1}{\phi}\partial_x \Pi^{n+1} = v^n \end{cases}$$

We consider the equation on the new velocity

$$v^{n+1} = -\Delta t(1-\mathcal{E}^2(t))\frac{1}{\phi}\partial_x \Pi^{n+1} + v^n$$

We plug into the equation on Π and we obtain

$$\left(I_d - \theta^2(1-\mathcal{E}^2(t))^2 \Delta t^2 \lambda_a^2 \partial_{xx}\right)\Pi^{n+1} = R(\Pi^n, v^n)$$

- Matrix-vector product to compute v, E and ρu.
- **Advantages Implicit part**: just one linear and constant elliptic problem to invert.
- The matrix can be constructed once and the conditioning does not depend of ρ.

Results I

- Smooth contact

$$\begin{cases} \rho(t,x) = \chi_{x<x_0} + 0.1\chi_{x>x_0} \\ u(t,x) = 0.01 \\ p(t,x) = 1 \end{cases}$$

- Error

cells	Ex Rusanov	Ex LR	I Xin-jin	SI Rusanov	New SI Rus	New SI LR
250	0.042	$3.6E^{-4}$	0.32	$1.4E^{-3}$	$7.8E^{-4}$	$4.1E^{-4}$
500	0.024	$1.8E^{-4}$	0.24	$6.9E^{-4}$	$3.9E^{-4}$	$2.0E^{-4}$
1000	0.013	$9.0E^{-5}$	0.17	$3.4E^{-4}$	$2.0E^{-4}$	$1.0E^{-5}$
2000	0.007	$4.5E^{-5}$	0.12	$1.7E^{-4}$	$9.8E^{-5}$	$4.9E^{-5}$

- Comparison time scheme:

Scheme	λ	Δt
Explicit	$\max(\mid u-c \mid, \mid u+c \mid)$	$2.2E^{-4}$
Xin-Jin	-	0.0052
SI Suliciu	$\max(\mid u-\mathcal{E}(t)\frac{\lambda}{\rho} \mid, \mid u+\mathcal{E}(t)\frac{\lambda}{\rho} \mid)$	0.0075
SI new relaxation	$\max(\mid v-\mathcal{E}(t)\lambda \mid, \mid v+\mathcal{E}(t)\lambda \mid)$	0.04

- Conditioning:

Schemes	Δt	conditioning
Si suliciu	0.00757	3000
Si new relax	0.041	9800
Si new relax	0.0208	2400
si new relax	0.0075	320

2D extension

- 2D extension:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{v}) + \nabla \Pi = 0 \\ \partial_t E + \nabla \cdot (E \boldsymbol{v} + \Pi \boldsymbol{v}) = 0 \\ \partial_t \Pi + \boldsymbol{v} \cdot \nabla \Pi + \phi \lambda^2 \nabla \cdot \boldsymbol{v} = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{1}{\phi} \nabla \Pi = \frac{1}{\varepsilon}(\boldsymbol{u} - \boldsymbol{v}) \end{cases}$$

- **Limit**:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = \varepsilon \nabla \cdot \left[\frac{1}{\rho} \left(\frac{\rho}{\phi} - 1 \right) \nabla \boldsymbol{p} \right] \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{v}) + \nabla \boldsymbol{p} = \varepsilon \nabla \cdot \left[\frac{1}{\rho} u \left(\frac{\rho}{\phi} - 1 \right) \nabla \boldsymbol{p} \right] + \varepsilon \nabla \left[\frac{1}{\rho} \left(\rho \phi \lambda^2 - \rho^2 c^2 \right) \nabla \cdot \boldsymbol{u} \right] \\ \partial_t E + \nabla \cdot ((E + p) \boldsymbol{u}) = \varepsilon \nabla \cdot \left[\frac{1}{\rho} \left[E \left(\frac{\rho}{\phi} - 1 \right) \nabla \boldsymbol{p} + \left(\frac{\rho}{\phi} - 1 \right) \nabla \frac{p^2}{2} \right] \right] \\ + \varepsilon \nabla \cdot \left[\frac{1}{\rho} \left(\rho \phi \lambda^2 - \rho^2 c^2 \right) \boldsymbol{u} \nabla \cdot \boldsymbol{u} \right] \end{cases}$$

- **Remark**: This diffusion approximate of the relaxation model preserve the acoustic steady states and consequently the low mach limit.

Scheme

- Splitting "convection" (Euler explicit) + "acoustic" (theta scheme).
- **Convective part**: Lagrange+remap-like scheme on Cartesian meshes.
- **Acoustic part**: centered scheme based also on nodal method.

First 2D result I

- We take 100*100 cells $T_f = 1$ and

$$\begin{cases} \rho(t,\mathbf{x}) = G(\mathbf{x} - \boldsymbol{u}_0 t) \\ \boldsymbol{u}(t,\mathbf{x}) = \boldsymbol{u}_0, \quad \text{such that } \nabla \cdot \boldsymbol{u}_0 = 0 \text{ and } \mid \boldsymbol{u}_0 \mid \approx 10^{-3} \\ p(t,\mathbf{x}) = 1 \end{cases}$$

- Results:

Vars	Ex Rusanov	Ex LR	SI Rusanov	New SI Rus...	New SI LR
ρ	0.39	$1.9E^{-4}$	$8.4E^{-4}$	$7.3E^{-4}$	$7.5E^{-5}$
u	0.87	0.51	$5.3E^{-3}$	$4.8E^{-3}$	$2.7E^{-3}$
p	$9.6E^{-8}$	$5.5E^{-7}$	$1.8E^{-6}$	$7.2E^{-7}$	$7.2E^{-7}$
Δt	$4.2E^{-4}$	$4.4E^{-4}$	0.8	1(max 9)	1(max 9)

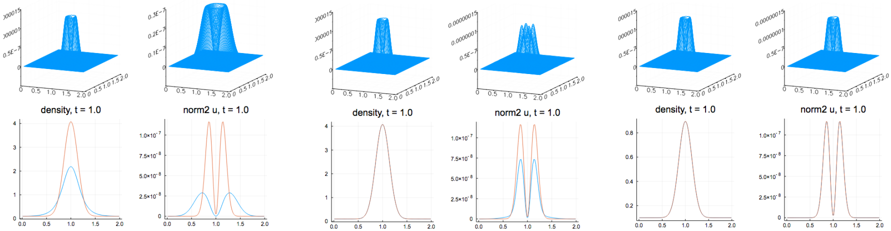


Figure: Explicit Rusanov scheme, ex Lr-Like, Semi Implicit relax

First 2D results II

- Gresho vortex: stationary vortex with varying Mach number.
- Classical test case for Low-Mach flow for Euler equation.

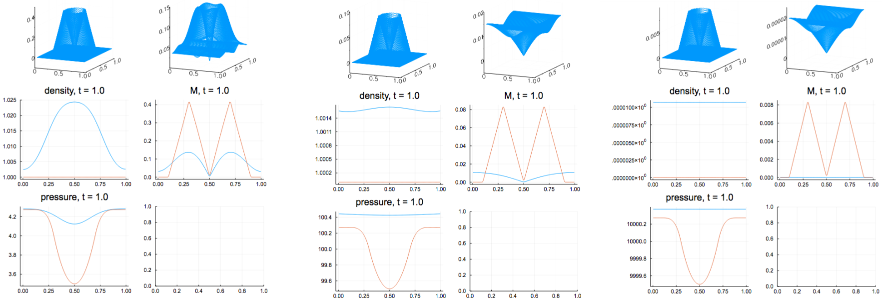


Figure: Results with Rusanov: $M = 0.5$ $(\Delta t = 1.4E^{-3})$, $M = 0.1$ $(\Delta t = 3.5E^{-4})$, $M = 0.01$ $(\Delta t = 3.5E^{-4})$

First 2D results II

- Gresho vortex: stationary vortex with varying Mach number.
- Classical test case for Low-Mach flow for Euler equation.

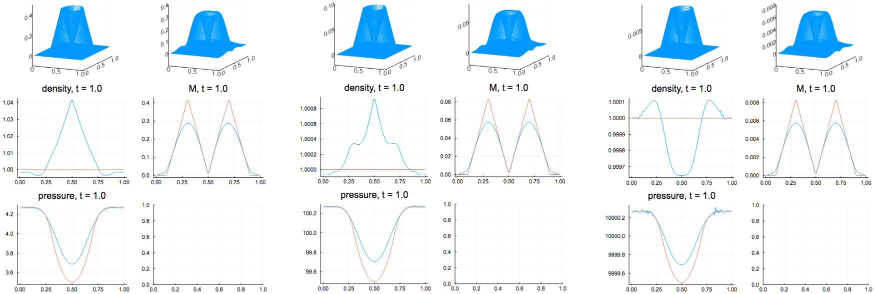


Figure: Results with New-relax: $M = 0.5$ ($\Delta t = 2.5E^{-3}$), $M = 0.1$ ($\Delta t = 2.5E^{-3}$), $M = 0.01$ ($\Delta t = 2.5E^{-3}$)

Conclusion

Full implicit schemes

- The Xin-Jin model + high order scheme gives good results.
- **Drawback**: Not sufficiently accurate in the Low -mach regime.
- First relaxation method with central wave as solution.
- **Future works**: understand the stability of these relaxation methods for low-mach flow and extend in 2D.
- All these relaxation models can be rewritten/generalized on a diagonal form (approximated BGK methods) with very high-order schemes and Semi-Lagrangian schemes.

Semi implicit schemes

- Relaxation + Splitting + VF allows to preserve contact wave and low Mach regime with a simple implicit step.
- **Stability**: Possible modification of the scheme to obtain discrete entropy inequality.
- **Future works**:
  - High accuracy for acoustic wave with a theta scheme for relaxation and implicit.
  - Modification splitting: Problem of time step if $\partial_t \mathcal{E}(t) >> 1$.
  - DG Extension in 1D/2D. Which limiting ? MOOD ? Subcell etc ?
  - MHD, Exner, Euler with gravity extension in 1D.
  - MHD in 2D. Large difficulty to be accurate around the magneto-acoustic steady state.

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