Semi implicit scheme for Low-Mach regime

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Outline

Physical and mathematical context

Gas dynamic: Euler equations

- **Multiscale fluid problem**: Low-Mach regime in for compressible equations: Euler/Navier-Stokes, MHD ideal/resistive, multi-fluid models etc, quasi-neutral limit for plasma, collisional limit for kinetic equations etc.
- **Euler equation**:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p I_d) = 0 \\ \partial_t E + \nabla \cdot (E \boldsymbol{u} + p \boldsymbol{u}) = 0 \end{cases}$$

- with $\rho(t, \mathbf{x}) > 0$ the density, $\boldsymbol{u}(t, \mathbf{x})$ the velocity and $E(t, \mathbf{x}) > 0$ the total energy.
- The pressure $p(t, \mathbf{x})$ is defined by $p = \rho T$ (perfect gas law) with $T(t, \mathbf{x})$ the temperature.

- **Hyperbolic system** which propagate some nonlinear waves.
- **Waves speed**: three eigenvalues: $(\boldsymbol{u}, \mathbf{n})$ and $(\boldsymbol{u}, \mathbf{n}) \pm c$ with the sound speed $c^2 = \gamma \frac{p}{\rho}$.

Physic interpretation:

- the isotropic acoustic waves due to the pressure and normal velocity perturbations,
- the fluid motion at the velocity $\boldsymbol{u}$.

- **Two important velocity scales**: $\boldsymbol{u}$ and c

Low mach limit

- We propose to obtain dimensionless equations. We rewrite the equation on the form:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0 \\ \partial_t p + \nabla \cdot (p \boldsymbol{u}) + (\gamma - 1) p \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

- Normalization:
  - we introduce characteristic time t_0, velocity V, length L.
  - the characteristic velocity u_0 and pressure γp_0. The sound velocity is $c^2 = \frac{\gamma p_0}{\rho_0}$.

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$$\begin{cases} \partial_t \rho + \left[\dfrac{t_0 u_0}{L}\right] \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \left[\dfrac{t_0 u_0}{L}\right] \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \left[\dfrac{t_0 p_0}{\rho_0 u_0 L}\right] \nabla p = 0 \\ \partial_t p + \left[\dfrac{t_0 u_0}{L}\right] \boldsymbol{u} \cdot \nabla p + \left[\dfrac{\gamma t_0 u_0}{L}\right] p \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

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$$\begin{cases} \partial_t \rho + \left[\dfrac{u_0}{V}\right] \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \rho \partial_t \boldsymbol{u} + \left[\dfrac{u_0}{V}\right] \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \left[\dfrac{c_0^2}{u_0 V}\right] \nabla p = 0 \\ \partial_t p + \left[\dfrac{u_0}{V}\right] \boldsymbol{u} \cdot \nabla p + \left[\dfrac{\gamma u_0}{V}\right] p \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

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- We want to focus on the fluid motion consequently we choose $V=u_0$.
- We define the mach number: $M=\frac{u_0}{c_0}$. Using this we obtain

$$\left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\left[\dfrac{1}{M^2}\right]\nabla p=0\\ \partial_t p+\boldsymbol{u}\cdot\nabla p+\gamma p\nabla\cdot\boldsymbol{u}=0\end{array}\right. \longrightarrow \left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})+\dfrac{1}{M^2}\nabla p=0\\ \partial_t E+\nabla\cdot(E\boldsymbol{u}+p\boldsymbol{u})=0\end{array}\right.$$

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Low Mach limit

When M tends to zero, we obtain incompressible Euler equation:

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0 \\ \rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p_2 = 0 \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

In 1D we have just advection of ρ.
Classical incompressible Euler equation if $p = p(\rho)$ or $\rho(t = 0) = cts$.

Numerical difficulties in space: Finite volume I

Finite Volumes

The Finite Volumes is the natural method to solve hyperbolic systems. The constant by cell approximation is very useful to preserve the maximum principle to capture shock waves.

- Default of FV scheme. Consistency :

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta x(\partial_x D(\boldsymbol{U})\partial_x \boldsymbol{U}) + O(\Delta x^2)$$

- We consider $\boldsymbol{U}_M$ the solution at the low mach limit.
- The scheme can be considered as not adapted/adapted for this regime if

$$lim_{M\rightarrow 0} \mid D(\boldsymbol{U}_M) \mid \approx M^{-p}, \quad lim_{M\rightarrow 0} \mid D(\boldsymbol{U}_M) \mid < C$$

- The simpler schemes are not adapted in general (following slide) [GV97].
- **Other problem**. At the limit we have $\nabla \cdot \boldsymbol{u}_M = 0$ and $\nabla p_{2,M}$ the Lagrange multiply associated. At the continuous level

$$Ker(\nabla \cdot) = Span\left\{\boldsymbol{u} = \nabla^{\perp}\phi\right\}, \quad Ker(\nabla) = Span\left\{p = cts\right\}$$

- It is not always true at the discrete level [DJOR16]-[BDJP19]. Example:

$$\partial_x p \approx \frac{p_{j+1} - p_{j-1}}{\Delta x} \text{ admit in the kernel } (1,-1,1,-1,....,1,-1)$$

Difficulties

Design a scheme with a good viscosity, stable and avoiding spurious mods.

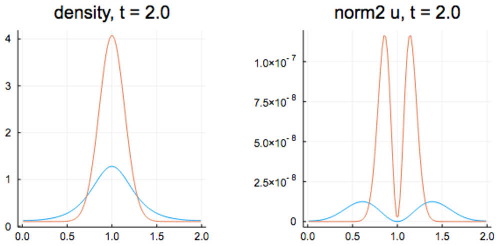
Numerical difficulties in space: Finite volume II

- VF method + Rusanov flux. Equivalent equation:

$$\begin{cases} \partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) = \dfrac{S\Delta x}{2}\Delta \rho \\ \rho\partial_t \boldsymbol{u} + \rho \boldsymbol{u}\cdot\nabla \boldsymbol{u} + \dfrac{1}{M^2}\nabla p = \dfrac{S\Delta x}{2}\Delta \boldsymbol{u} \\ \partial_t p + \boldsymbol{u}\cdot\nabla p + \gamma p \nabla\cdot \boldsymbol{u} = \dfrac{S\Delta x}{2}\Delta p \end{cases}$$

- **Problem**: S must be larger that $\frac{1}{M}$ for stability. Huge diffusion.

- Example: isolated contact $p = 1$, $\nabla\cdot\boldsymbol{u}_0 = 0$ and $\boldsymbol{u}_0$ constant in time.
- Rusanov scheme $T_f = 2 \mid \boldsymbol{u}_0 \mid \approx 0.001$ and 100*100 cells.



- Red: exact solution, Blue: numerical solution.

Numerical problem I: time discretization.

- **Explicit scheme**: the CFL condition $\Delta t < \frac{\Delta x}{\lambda}$ with λ the maximal speed of the system. For low mach flow:
  - The fast phenomena: acoustic waves at velocity c
  - The important phenomena: transport at velocity u
  - Expected CFL: $\Delta t < \frac{\Delta x}{|u|}$, CFL in practice $\Delta t < \frac{\Delta x}{|c|}$
  - At the end, we use a Δt divised by M compare to the expected Δt

First solution

Implicit time scheme. No CFL condition. Taking a larger time step, it allows to "filter" the fast acoustic waves which are not useful in the low-Mach regime.

- Implicit time scheme:

$$M_i \boldsymbol{U}^{n+1} = (I_d + \Delta t A(I_d))\boldsymbol{U}^{n+1} = \boldsymbol{U}^n$$

- We must solve a nonlinear system and after linearization solve some linear systems.

Problem

- Direct solver too costly. Approximative conditioning for iterative solver:

$$k(M_i) \approx 1 + O\left(\frac{\Delta t}{\Delta x^p M}\right)$$

- We recover the two scales in the conditioning number. The full implicit schemes are difficult to use for this reason.

Numerical problem II: time discretization.

First idea: Semi implicit scheme

- We explicit the slow scale (transport) and implicit the fast scale (acoustic) [CDK12]-[DLVD19]

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ \partial_t E + \partial_x(Eu) + \partial_x(pu) = 0 \end{cases}$$

Implicit acoustic step:

$$\begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} = \rho^n u^n - \Delta t \partial_x p^{n+1} + Rhs_u \\ E^{n+1} = E^n - \Delta t \partial_x (p^{n+1} u^{n+1}) = Rhs_E \end{cases}$$

Plugging this in the second equation, we obtain

$$E^{n+1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho)$$

- Matrix-vector product to compute u^{n+1}.

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- Matrix-vector product to compute u^{n+1}.

Conclusion

- **Semi implicit**: only one scale in the implicit symmetric positive operator.
- Strong gradient of ρ generates ill-conditioning. Assembly at each time (costly).
- Nonlinear solver can have bad convergence for if $\Delta t >> 1$ and $\partial_x p$ not so small.

Relaxation method

Relaxation method I

- **Relaxation** [XJ95]-[CGS12]-[BCG18]: a way to linearize and decouple the equations. Used to design new schemes.
- **Idea**: Approximate the model

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = 0, \text{ by } \quad \partial_t \mathbf{f} + \mathbf{A}(\mathbf{f}) = \frac{1}{\varepsilon}(Q(\mathbf{f}) - \mathbf{f})$$

- At the limit and taking $Pf = \boldsymbol{U}$ we obtain

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = \varepsilon \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + O(\varepsilon^2)$$

- **Time scheme**:
  - we solve

$$\frac{\mathbf{f}^* - \mathbf{f}^n}{\Delta t} + \mathbf{A}(\mathbf{f}^{*,n}) = 0$$

  - and after we approximate the stiff source term by

$$f^{n+1} = \mathbf{f}^* + \omega(Q(\mathbf{f}^*) - \mathbf{f}^*)$$

  with $\omega \in]0, 2]$.

Why ?

- In general, we construct **A** with a simpler structure than **F** to design numerical flux in FV.
- Here, we construct **A** with a simpler structure to design simple implicit scheme.

Relaxation method II

- **Problem**: the nonlinearity of the implicit acoustic step generates difficulties.
- Non conservative form and acoustic term:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t p+u\partial_x p+\rho c^2\partial_x u=0\partial_t u+u\partial_x u+\frac{1}{\rho}\partial_x p=0\end{array}\right.$$

- **Idea**: Relax only the acoustic part ([BCG18]) to linearize the implicit part.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho u v+\Pi)=0\\ \partial_t E+\partial_x(E v+\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda^2\partial_x v=\frac{1}{\varepsilon}(p-\Pi)\\ \partial_t v+v\partial_x v+\frac{1}{\phi}\partial_x\Pi=\frac{1}{\varepsilon}(u-v)\end{array}\right.$$

- **Limit**:

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho u)=\varepsilon\partial_x\left[A\partial_x p\right]\\ \partial_t(\rho u)+\partial_x(\rho u^2+p)=\varepsilon\partial_x\left[(Au\partial_x p)+B\partial_x u\right]\\ \partial_t E+\partial_x(Eu+pu)=\varepsilon\partial_x\left[AE\partial_x p+A\partial_x\frac{p^2}{2}+B\partial_x\frac{u^2}{2}\right]\end{array}\right.$$

- with $A=\frac{1}{\rho}\left(\frac{\rho}{\phi}-1\right)$ and $B=\left(\rho\phi\lambda^2-\rho^2c^2\right)$.
- **Stability**: $\phi\lambda>\rho c^2$ and $\rho>\phi$.

Avdantage

- We keep the conservative form for the original variables and obtain a fully linear acoustic.

Dynamical splitting

- **Splitting**: we solve sub-part of the system one by one. **Dynamic case**: Splitting time depending for low-mach [IDGH2018]
- For large acoustic waves (Mach number not small) we want capture all the phenomena. Consequently use an explicit scheme.
- For small/fast acoustic waves (low Mach number) we want filter acoustic. Consequently use an implicit scheme for acoustic.

Splitting: Explicit convective part/Implicit acoustic part.

$$\left\{\begin{array}{l} \partial_t\rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \mathcal{M}^2(t)\Pi) = 0 \\ \partial_t E + \partial_x(E v + \mathcal{M}^2(t)\Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda_c^2 \partial_x v = 0 \\ \partial_t v + v\partial_x v + \frac{\mathcal{M}^2(t)}{\phi}\partial_x \Pi = 0 \end{array}\right. , \quad \left\{\begin{array}{l} \partial_t\rho = 0 \\ \partial_t(\rho u) + (1-\mathcal{M}^2(t))\partial_x \Pi = 0 \\ \partial_t E + (1-\mathcal{M}^2(t))\partial_x(\Pi v) = 0 \\ \partial_t \Pi + \phi(1-\mathcal{M}^2(t))\lambda_a^2 \partial_x v = 0 \\ \partial_t v + (1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x \Pi = 0 \end{array}\right.$$

with $\mathcal{M}(t) \approx max\left(\mathcal{M}_{min}, min\left(max_x \frac{|u|}{c}, 1\right)\right)$

- Eigenvalues of Explicit part: v, $v \pm \mathcal{M}(t)\underbrace{\lambda_c}_{\approx c}$. Implicit part $v0$, $\pm(1-\mathcal{M}^2(t))\underbrace{\lambda_a}_{\approx c}$
- **At the end**: we make the projection $\Pi = p$ and $v = u$ (can be viewed as a discretization of the stiff source term).

Implicit time scheme

- We introduce the implicit scheme for the "acoustic part":

$$\begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\partial_x \Pi^{n+1} = (\rho u)^n \\ E^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\partial_x (\Pi v)^{n+1} = E^n \\ \Pi^{n+1} + \Delta t\phi(1-\mathcal{M}^2(t_n))\lambda_a^2 \partial_x v^{n+1} = \Pi^n \\ v^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x \Pi^{n+1} = v^n \end{cases}$$

- We plug the equation on v in the equation on Π. We obtain the following algorithm:
  - Step 1: we solve

$$(I_d - (1-\mathcal{M}^2(t_n))^2 \Delta t^2 \lambda_c^2 \partial_{xx})\Pi^{n+1} = \Pi^n - \Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_c^2 \partial_x v^n$$

  - Step 2: we compute

$$v^{n+1} = v^n - \Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x \Pi^{n+1}$$

  - Step 3: we compute

$$(\rho u)^{n+1} = (\rho u)^n - \Delta t(1-\mathcal{M}^2(t_n))\partial_x \Pi^{n+1}$$

  - Step 4: we compute

$$E^{n+1} = E^n - \Delta t(1-\mathcal{M}^2(t_n))\partial_x (\Pi^{n+1} v^{n+1})$$

Advantage

- We solve only a constant Laplacian. We can assembly matrix one time.
- No problem of conditioning, which comes from to the strong gradient of ρ

Spatial scheme in 1D

- **Idea**: FV upwind fluxes for the explicit part + Central fluxes for the implicit part.
- Main problem of the explicit part: design numerical flux.
- **First possibility**: since the maximal eigenvalue is $O(Mach)$ a Rusanov scheme.
- Other solution more efficient "pressure-convection" splittign scheme. The flux is given by

$$\begin{pmatrix} \rho v \\ \rho u v + \mathcal{M}^2(t)\Pi \\ E v + \mathcal{M}^2(t)(v\Pi) \\ \phi v \\ \frac{\mathcal{M}^2(t)}{\phi}\Pi \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho u v + \mathcal{M}^2(t)\Pi \\ E v + \mathcal{M}^2(t)(v\Pi) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \phi v \\ \frac{\mathcal{M}^2(t)}{\phi}\Pi \end{pmatrix}$$

- The new variables v and Π can be solved independently. We propose a numerical flux $G = \mathbf{G}_{\rho,\rho u,E} + \mathbf{G}_{\Pi,v}$.
- For the equations on Π, v we have a linear flux, so we apply an upwind fluxes on the eigenvectors to obtain:

$$\begin{cases} v_{j+\frac{1}{2}} = \frac{1}{2}\left(v_{j+1} + v_j\right) - \frac{\mathcal{M}(t)}{2\lambda_c \phi}\left(\Pi_{j+1} - \Pi_j\right) \\ \Pi_{j+\frac{1}{2}} = \frac{1}{2}\left(\Pi_{j+1} + \Pi_j\right) - \frac{\lambda_c \phi}{2\mathcal{M}(t)}\left(v_{j+1} - v_j\right) \end{cases}$$

- We use this flux for $\mathbf{G}_{\Pi,v}$ part but also for the Π, v in the other part. For the transport of the conservative variables at velocity v we use:

$$\rho_{j+\frac{1}{2}} = \frac{v_{j+\frac{1}{2}}}{2}\left(\rho_j + \rho_{j+1}\right) - \frac{\mid v_{j+\frac{1}{2}} \mid}{2}\left(\rho_{j+1} - \rho_j\right)$$

Viscosity and spurious mods

- Here, we propose to study the viscosity to understand the Low-Mach Behavior.
- Relaxation (first order in time) + Euler implicit with central FV + Euler explicit with new flux (1D fluxes in the normal direction).
- Viscosity 2D for two first equations (we neglect viscosity due to Euler time scheme):

$$\begin{cases} \partial_t \rho + .. \approx \nabla \cdot [(\Delta t A + \Delta x \rho)\nabla p + \Delta x \mid \mathbf{v} \mid \nabla \rho] \\ \partial_t(\rho \boldsymbol{u}) + .. \approx \nabla \cdot [(\Delta t A \boldsymbol{u} + \Delta x \rho \boldsymbol{u}) \otimes \nabla p + \Delta x \mid \mathbf{v} \mid \nabla(\rho \boldsymbol{u})] + \nabla(\Delta t B)\nabla \cdot \boldsymbol{u} + \frac{\Delta x}{2}\mathcal{M}(t)\lambda_c \phi \Delta \boldsymbol{u} \end{cases}$$

with $\mathcal{M}(t)\lambda_c \approx \max_x \mid \mathbf{v} \mid = \approx \max_x \mid \mathbf{u} \mid$.

Conclusion

- At low Mach we have:
  - slow transport of ρ,
  - $\nabla \cdot \boldsymbol{u} = 0$ and $p = p_0 + M^2 p_2$ with p_0 =cst and M the Mach number.
- The viscosities which destabilize these solutions are small, homegeneous to $O(Mach)$.

Mods

- The implicit central fluxes for acoustic does not add viscosity but add spurious mods.
- The 5 points implicit Laplacian allows to stabilization this. Perhaps it is not sufficient.
- We can use any good scheme for linear acoustic. Possible example: work's of W. Barsukow, C Klingenberg or J. Jung, V. Perrier.

Results 1D I: contact

- Smooth contact :

$$\begin{cases} \rho(t,x) = \chi_{x<x_0} + 0.1\chi_{x>x_0} \\ u(t,x) = 0.01 \\ p(t,x) = 1 \end{cases}$$

- Error

cells	Ex Rusanov	Ex LR	SI Rusanov	New SI Rus	New SI LR
250	0.042	$3.6E^{-4}$	$1.4E^{-3}$	$7.8E^{-4}$	$4.1E^{-4}$
500	0.024	$1.8E^{-4}$	$6.9E^{-4}$	$3.9E^{-4}$	$2.0E^{-4}$
1000	0.013	$9.0E^{-5}$	$3.4E^{-4}$	$2.0E^{-4}$	$1.0E^{-5}$
2000	0.007	$4.5E^{-5}$	$1.7E^{-4}$	$9.8E^{-5}$	$4.9E^{-5}$

- Suliciu: relaxation scheme different. The implicit Laplacian is not constant and depend of ρ^n.
- Comparison time scheme:

Scheme	λ	Δt
Explicit	$\max(\mid u-c\mid, \mid u+c \mid)$	$2.2E^{-4}$
SI Suliciu	$\max(\mid u-\mathcal{M}(t_n))\frac{\lambda}{\rho}\mid, \mid u+\mathcal{M}(t_n))\frac{\lambda}{\rho}\mid)$	0.0075
SI new relaxation	$\max(\mid v-\mathcal{M}(t_n))\lambda\mid, \mid v+\mathcal{M}(t_n))\lambda\mid)$	0.04

- Conditioning:

Schemes	Δt	conditioning
Si suliciu	0.00757	3000
Si new relax	0.041	9800
Si new relax	0.0208	2400
si new relax	0.0075	320

Results in 2D I: contact

- We take 100*100 cells $T_f = 1$ and

$$\left\{\begin{array}{l} \rho(t,\mathbf{x}) = G(\mathbf{x} - \boldsymbol{u}_0 t) \\ \boldsymbol{u}(t,\mathbf{x}) = \boldsymbol{u}_0, \quad \text{such that } \nabla\cdot\boldsymbol{u}_0 = 0 \text{ and } \mid \boldsymbol{u}_0 \mid \approx 10^{-3} \\ p(t,\mathbf{x}) = 1 \end{array}\right.$$

- Results:

Vars	Ex Rusanov	Ex LR	SI Rusanov	New SI LR
ρ	0.39	$1.9E^{-4}$	$8.4E^{-4}$	$7.5E^{-5}$
u	0.87	0.51	$5.3E^{-3}$	$2.7E^{-3}$
p	$9.6E^{-8}$	$5.5E^{-7}$	$1.8E^{-6}$	$7.2E^{-7}$
Δt	$4.2E^{-4}$	$4.4E^{-4}$	0.8	1(max 9)

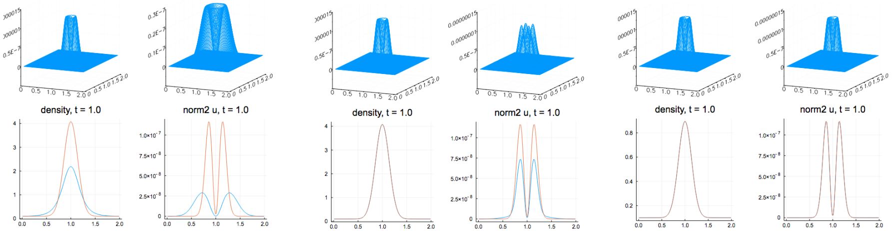
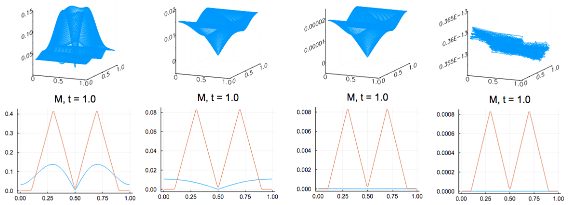


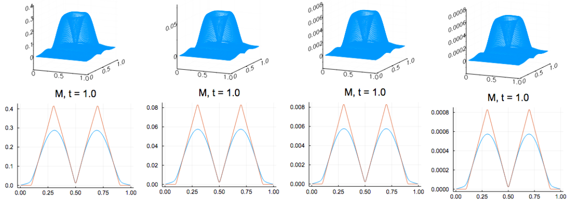
Figure: Explicit Rusanov scheme, Ex LR-Like, Semi Implicit relax

Results in 2D II: Gresho vortex

- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$
- We plot the norm of $\boldsymbol{u}$



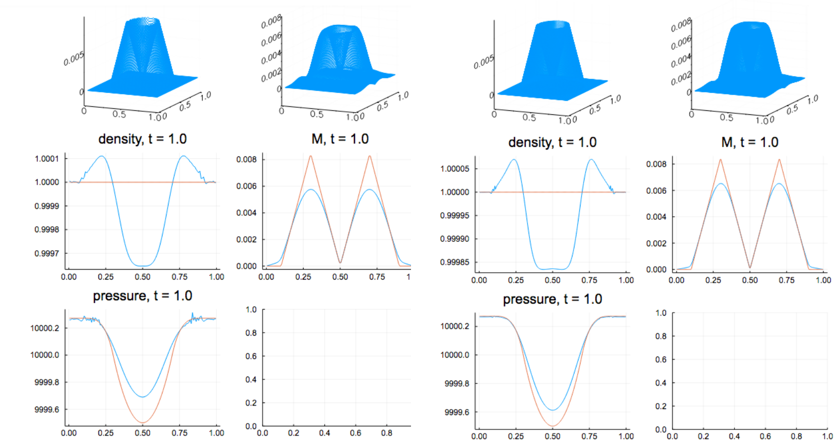
- Ex scheme: $M = 0.5$ ($\Delta t = 1.4E^{-3}$), $M = 0.1$ ($\Delta t = 3.5E^{-4}$), $M = 0.01$ ($\Delta t = 3.5E^{-5}$), $M = 0.001$ ($\Delta t = 3.5E^{-6}$)



- New scheme: $M = 0.5$ ($\Delta t = 2.5E^{-3}$), $M = 0.1$ ($\Delta t = 2.5E^{-3}$), $M = 0.01$ ($\Delta t = 2.5E^{-3}$), $M = 0.001$ ($\Delta t = 2.5E^{-3}$)

Results in 2D II: Gresho vortex

- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$
- Convergence for $\boldsymbol{u}$ and p



- Results with New-relax. Left: 120*120 cells, Right: 240*240 cells

Conclusion

Resume

- Introducing Dynamic splitting scheme we separate the scales.
- Introducing implicit scheme for the acoustic wave we can filter these waves.
- Introducing relaxation we simplify at the maximum the implicit scheme.
- A well-adapted spatial scheme is also very important.

Perspectives:

- **Understand stability** of the splitting + relaxation scheme. Not fully clear. Many option, what is the best ?
- Extension to High Order, MUSCL firstly and after DG and HDG schemes.
- Extension to Shallow-Water/Ripa models and MHD (main goal).
- Extension to second and third order terms. Example: visco-resistive extended MHD or compressible NS.

Announcement

- With some colleges we organize the summer school "Cemracs 2020".
- **Theme**: "Models and simulation of many passive/active particles". Physics particles, cells, population dynamic, crowd movement, smart city.

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