Towards a two-fluid model without gyroviscous cancellation

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Jorek Meeting 2019, Munich, Germany

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Outline

Kinetic models

- We begin by introduce the mesoscopic model to describe two species (ion, electron) plasma.
- $f_s(t, \boldsymbol{x}, \boldsymbol{v})$ is the density of particles (ion, electon) at time t, space $\boldsymbol{x}$ and velocity $\boldsymbol{v}$.

Vlasov-Maxwell 2 species

$$\left\{\begin{array}{l} \partial_t f_s + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(f_s) + \dfrac{q_s}{m_s}\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_{\boldsymbol{v}} f_s = \dfrac{1}{\varepsilon} C_s + C_{s,s'} \\ \dfrac{1}{c^2}\partial_t \boldsymbol{E} - \nabla\times\boldsymbol{B} = -\mu_0 \boldsymbol{J} \\ \partial_t \boldsymbol{B} = -\nabla\times\boldsymbol{E} \\ \nabla\cdot\boldsymbol{B} = 0 \\ \nabla\cdot\boldsymbol{E} = \dfrac{\sigma}{\varepsilon_0} \end{array}\right.$$

with $\sigma = q_i \int f_i + q_e \int f_e$ and $\boldsymbol{J} = q_i \int \boldsymbol{v} f_i + q_e \int \boldsymbol{v} f_e$

- ε is homogeneous to the collisional frequency.
- **Collisional limit**: we take $f_s = f_s^0 + \varepsilon f_s^1 + O(\varepsilon^2)$.
- Keeping only zero order terms we obtain two fluid Euler-Maxwell equations.
- Keeping only zero and first order terms we obtain two fluid Viscous Euler-Maxwell equations.

Two fluid Euler-Maxwell equations

Euler-Maxwell 2 species

- Final equations:

$$\begin{cases} \partial_t \rho_i + \nabla \cdot (\rho_i \boldsymbol{u}_i) = 0 \\ \partial_t \rho_e + \nabla \cdot (\rho_e \boldsymbol{u}_e) = 0 \\ \partial_t (\rho_i \boldsymbol{u}_i) + \nabla \cdot (\rho_i \boldsymbol{u}_i \otimes \boldsymbol{u}_i) + \nabla p_i = \sigma_i \mathbf{E} + \boldsymbol{J}_i \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i + \mathbf{R}_i \\ \partial_t (\rho_e \boldsymbol{u}_e) + \nabla \cdot (\rho_e \boldsymbol{u}_e \otimes \boldsymbol{u}_e) + \nabla p_e = \sigma_e \mathbf{E} + \boldsymbol{J}_e \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_e + \mathbf{R}_e \\ \partial_t \rho_i \epsilon_i + \nabla \cdot (\rho_i \epsilon_i \boldsymbol{u}_i + p_i \boldsymbol{u}_i) + \nabla \cdot \left(\mathbf{q}_i + \overline{\overline{\mathbf{\Pi}}}_i \cdot \boldsymbol{u}_i \right) = \sigma_i \boldsymbol{u}_i \cdot \boldsymbol{E} + \mathbf{R}_i \cdot \boldsymbol{u}_i + Q_{\Delta_i} \\ \partial_t \rho_e \epsilon_e + \nabla \cdot (\rho_e \epsilon_e \boldsymbol{u}_e + p_e \boldsymbol{u}_e) + \nabla \cdot \left(\mathbf{q}_e + \overline{\overline{\mathbf{\Pi}}}_e \cdot \boldsymbol{u}_e \right) = \sigma_e \boldsymbol{u}_e \cdot \boldsymbol{E} + \mathbf{R}_e \cdot \boldsymbol{u}_e + Q_{\Delta_e} \\ \varepsilon^2 \partial_t \boldsymbol{E} - \nabla \times \boldsymbol{B} = -\boldsymbol{J} \\ \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0 \\ \nabla \cdot \boldsymbol{B} = 0 \\ \varepsilon^2 \nabla \cdot \boldsymbol{E} = n_i - n_e \end{cases}$$

- with $\varepsilon \approx \frac{V_0}{c} << 1$.

- **Quasi neutral limit**: $\varepsilon \rightarrow 0$
- The generalized Ohm law is obtain using electron momentum equation:

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{J} - \frac{m_i}{\rho e} \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_e + \frac{m_i}{\rho e} \boldsymbol{J} \times \boldsymbol{B} - \frac{m_i}{\rho e} \nabla p_e + O\left(\frac{m_e}{m_i} \right).$$

- Taking $\frac{m_e}{m_i} \rightarrow 0$ and quai neutral limit we will obtain Extended MHD.

Extended MHD equations

- We take the two limits introduced previously.
- We define global quantities:

$$\rho = m_i n_i + m_e n_e, \quad \boldsymbol{u} = \frac{m_i n_i \boldsymbol{u}_i + m_e n_e \boldsymbol{u}_e}{m_i n_i + m_e n_e}, \quad p = p_i + p_e$$

Extended MHD

$$\left\{\begin{array}{l}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\
\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} + -\nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} \\
\\
\partial_t p_i + \boldsymbol{u} \cdot \nabla p_i + \gamma p_i \nabla \cdot \boldsymbol{u} + \kappa_i \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} \\
= 3(\gamma - 1) \dfrac{\rho_e}{\tau_e m_i} (T_i - T_e) \\
\\
\partial_t p_e + \boldsymbol{u} \cdot \nabla p_e + \gamma p_e \nabla \cdot \boldsymbol{u} + \kappa_e \nabla \cdot \mathbf{q}_e \\
= \dfrac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \dfrac{\nabla \rho}{\rho} \right) - 3(\gamma - 1) \dfrac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2 \\
\\
\partial_t \boldsymbol{B} = -\nabla \times \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \dfrac{m_i}{\rho e} \nabla p_e + \dfrac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\
\mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{B} = 0
\end{array}\right.$$

- Additionally we can assume that $T_i < T_e$ such that $p_i \approx p_e$.

Velocity approximation (D. Schnack)

- We introduce the Spatial ratio: $\delta = \frac{\rho_i^*}{L}$ with ρ_i^* the ion Larmor radius.
- Time ratio assumption: $\varepsilon = \frac{w_0}{w_{ci}} \approx \delta^2$ (idem for collisional frequency).
- Velocity small parameter: $\xi = \frac{V_0}{V_{T_i}} \approx \delta$

Firstly we take the ion velocity equation to obtain:

$$m_i n_i \partial_t \boldsymbol{u}_i + m_i n_i \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i + \nabla p_i = e n_i (\mathbf{E} + \boldsymbol{u}_i \times \boldsymbol{B}) - \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i + \boldsymbol{R}_i$$

After small computations we obtain which gives

$$\boldsymbol{u}_i = (\boldsymbol{u}_i \cdot \boldsymbol{B}) \frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{m_i}{e|\boldsymbol{B}|^2} \boldsymbol{B} \times (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i) + \frac{\boldsymbol{B}}{n_i e |\boldsymbol{B}|^2} \times (\nabla p_i + \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i - \boldsymbol{R}_i)$$

After the Ordering we obtain

$$\delta \boldsymbol{u} = \delta \frac{(\boldsymbol{u}_i \cdot \boldsymbol{B}) \boldsymbol{B}}{|\boldsymbol{B}|^2} + \delta \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \delta^3 (\partial_t \boldsymbol{u}_i + \boldsymbol{u}_i \cdot \nabla \boldsymbol{u}_i) + \frac{\boldsymbol{B}}{n |\boldsymbol{B}|^2} \times \left(\delta \nabla p_i + \underbrace{\delta^2 \nabla \cdot \overline{\overline{\mathbf{\Pi}}}_i}_{?} - \underbrace{\frac{\delta}{R_m} \boldsymbol{R}_i}_{\approx \delta^2} \right)$$

Final velocity

$$\boldsymbol{u} \approx (\boldsymbol{u}_i \cdot \boldsymbol{B}) \frac{\boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{B}}{n |\boldsymbol{B}|^2} \times \nabla p_i$$

Viscous tensor approximation I

Final velocity

$$\boldsymbol{u} \approx \frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid^2} v_{\parallel} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{B}}{n \mid \boldsymbol{B} \mid^2} \times \nabla p_i$$

- Classical decomposition of viscous tensor:

$$\overline{\overline{\Pi}} = \overline{\overline{\Pi}}_{\parallel} + \overline{\overline{\Pi}}_{c} + \overline{\overline{\Pi}}_{\perp} \approx \overline{\overline{\Pi}}_{\parallel} + \overline{\overline{\Pi}}_{gyro}$$

- Proposition of simplification by A. Zeiler (IPP report):
  - Viscosity:

$$\nabla \cdot \overline{\overline{\Pi}}_{\parallel} = G\boldsymbol{b} \cdot \nabla \boldsymbol{b} - \frac{1}{3} \nabla G + \nabla_{\parallel} G$$

  - Viscous heating:

$$\overline{\overline{\Pi}}_{\parallel} : \nabla \boldsymbol{u} = -\frac{1}{3\eta_0} G^2$$

with $G = -\eta_0(2\boldsymbol{b} \cdot \nabla v_{\parallel} - ((\boldsymbol{b} \cdot \nabla \boldsymbol{b}) \cdot \boldsymbol{u}_{\perp})$ and $\boldsymbol{b} = \frac{\boldsymbol{B}}{|\boldsymbol{B}|}$.

- We can neglect the viscous heating and in this we obtain a dissipation linked to G^2.

Viscous tensor approximation II

Final velocity

$$\boldsymbol{u} \approx \frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid^2} v_{\parallel} + \frac{\boldsymbol{E} \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \frac{\boldsymbol{B}}{n \mid \boldsymbol{B} \mid^2} \times \nabla p_i$$

- Proposition of simplification by A. Zeiler (IPP report):
  - Viscosity

$$\nabla \cdot \overline{\overline{\Pi}}_{gv} = -\rho \boldsymbol{u}_i^* \cdot \nabla \boldsymbol{u} + p_i \left(\nabla \times \frac{m_i \boldsymbol{b}}{e \parallel \boldsymbol{B} \parallel} \right) \cdot \nabla \boldsymbol{u}$$

$$+ \nabla_{\perp} \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla \cdot \boldsymbol{b} \times \boldsymbol{u} \right) + \boldsymbol{b} \times \nabla \left(\frac{m_i p_i}{2e \parallel \boldsymbol{B} \parallel} \nabla_{\perp} \cdot \boldsymbol{u} \right)$$

  - Principle of the Gyro-viscous cancelation: neglect the three last terms and kill the first one with a part of the advection part.
  - No Gyro viscous heating.
  - Proposition simple: no simplification or energy conserving simplification.
- We can prove that

$$\int \nabla \cdot \overline{\overline{\Pi}}_{gv} \boldsymbol{u} = 0$$

- It is also true for the two first terms. We can keep only the two first terms.

Final model

$$
\left\{
\begin{array}{l}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\
\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{J} \times \boldsymbol{B} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} - \nabla \cdot \overline{\overline{\boldsymbol{\Pi}}}_{gv} \\
\partial_t \dfrac{1}{\gamma - 1} p_i + \dfrac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_i + \dfrac{\gamma}{\gamma - 1} p_i \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_i + \overline{\overline{\boldsymbol{\Pi}}}_{\parallel} : \nabla \boldsymbol{u} + \overline{\overline{\boldsymbol{\Pi}}}_{gv} : \nabla \boldsymbol{u} \\
= 3 \dfrac{\rho_e}{\tau_e m_i} (T_i - T_e) \\
\\
\partial_t \dfrac{1}{\gamma - 1} p_e + \dfrac{1}{\gamma - 1} \boldsymbol{u} \cdot \nabla p_e + \dfrac{\gamma}{\gamma - 1} p_e \nabla \cdot \boldsymbol{u} + \nabla \cdot \mathbf{q}_e \\
= \frac{1}{\gamma - 1} \frac{m_i}{\rho e} \boldsymbol{J} \cdot \left(\nabla p_e - \gamma p_e \frac{\nabla \rho}{\rho} \right) - 3 \frac{\rho_e}{\tau_e m_i} (T_i - T_e) + \eta |\boldsymbol{J}|^2 \\
\\
\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \\
\boldsymbol{E} = \left(-\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{J} - \dfrac{m_i}{\rho e} \nabla p_e + \dfrac{m_i}{\rho e} (\boldsymbol{J} \times \boldsymbol{B}) \right) \\
\mu_0 \nabla \times \boldsymbol{B} = \boldsymbol{J}, \quad \nabla \cdot \boldsymbol{B} = 0
\end{array}
\right.
$$

with

$$
\left\{ \boldsymbol{u} = \boldsymbol{u}_{\perp} + \boldsymbol{u}_{\parallel}, \quad \boldsymbol{u}_{\perp} = \boldsymbol{u}_E + \boldsymbol{u}_i^*, \quad \boldsymbol{u}_i^* = \frac{m_i}{e \rho} \frac{\boldsymbol{B} \times \nabla p_i}{\mid \boldsymbol{B} \mid^2}, \quad \boldsymbol{u}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mid \boldsymbol{B} \mid^2}, \quad \boldsymbol{u}_{\parallel} = v_{\parallel} \frac{\boldsymbol{B}}{\mid \boldsymbol{B} \mid} \right.
$$

Reduction assumption

- To obtain a reduced model we write the equation as a potential decomposition and we write the equation on the potential.

Jorek Reduction

- For $\boldsymbol{B}$

$$\boldsymbol{B} = \frac{F_0}{R^2}\boldsymbol{e}_\phi + \frac{1}{R}\nabla\psi \times \boldsymbol{e}_\phi$$

- For $\boldsymbol{u}$

$$\boldsymbol{u} = -R\nabla u \times \boldsymbol{e}_\phi + v_{||}\boldsymbol{B} + \frac{m_i R}{\rho e F_0}\boldsymbol{e}_\phi \times \nabla p$$

M3DC1 Reduction

- For $\boldsymbol{B}$

$$\boldsymbol{B} = \frac{F}{R}\boldsymbol{e}_\phi + \nabla\psi \times \boldsymbol{e}_\phi$$

- For $\boldsymbol{u}$

$$\boldsymbol{u} = R^2\nabla u \times \boldsymbol{e}_\phi + R\omega\boldsymbol{e}_\phi + \frac{1}{R^2}\nabla_\perp\chi$$

- The first term to $\boldsymbol{u}$ seems equivalent to say that $\boldsymbol{E} = F_0\nabla u$.
- The diamagnetic term are not explicitly put in the M3DC1 velocity. The velocity in M3DC1 is linear compare to the scalar variables not the case in JOREK.

Projection assumption

- As say before we need projection to conclude the reduction

Jorek Projection

- for ψ: we take the equation on $\mathbf{A} = \phi \boldsymbol{e}_\phi$ and multiply by $\boldsymbol{e}_\phi$?
- for $\boldsymbol{u}$ to obtain poloidal velocity we apply

$$\boldsymbol{e}_\phi \nabla \times (R^2..)$$

and to obtain parallel velocity we apply $\boldsymbol{B} \cdot ()$

Compute the reduce model

Principle

- We put the reduction $\boldsymbol{u}$ and $\boldsymbol{B}$ in the full equation, put the projections and neglect some small terms.
- **Problem**: make that keeping the momentum and the energy conservation.
- Simplify some terms and keep the energy is difficult (for me).

Possible ways (for me)

- Write all the terms in pressure/velocity/$\boldsymbol{B}/\rho$ equations and hope that the projection does not broke the energy conservation.
- Take the conservation energy momentum and energy and after apply the reduction. After that compute the equations on pressure and potential.
- New model proposed by Nikulsin talk's correspond of these possibilities ?
- Work in the full variables and project with the weak form (B. Nkonga proposition) + no simplification.

Third way I

- B. Nkonga way to derivate the equation.
- We consider

$$\rho \partial_t \boldsymbol{u} + \nabla p = 0$$

- on the weak form

$$\int \rho \partial_t (\boldsymbol{u}, \mathbf{v}) dW + \int (\boldsymbol{u}, \nabla p) dW = 0$$

  with $\mathbf{v}$ a test function and $dW = RdV$
- We choose the test function as

$$\mathbf{v} = -R \nabla v_i \times \boldsymbol{e}_\phi$$

  with v_i a scalar basis function. We obtain

$$\int \rho \partial_t (\boldsymbol{u}, \mathbf{v}) dW = \int R^2 (\rho \nabla u, \nabla v_i) dW$$

  and

$$\int (\boldsymbol{u}, \nabla p) = \int (-R \nabla v_i \times \boldsymbol{e}_\phi, \nabla p) RdV = \int \frac{1}{R} \boldsymbol{e}_\phi \cdot \nabla \times (R^2 p) v_i RdV = \int \frac{1}{R} [R^2, p] v dW$$

- We obtain the weak form of JOREK.
- The choice of $\mathbf{v}$ can be view as the choice of projection.
- This way allow to derive the the model with the same way (only choice of velocity and projection can change)
- Without simplification we should be obtain a energy conserving weak for the previous full model.

Remarks for all models

- To finish: property that i understand for the other models.
- Model 303 (no diamagnetic terms)
  - No conservation in energy for the model: missign some small cross between the velocities.
  - Conservation in energy for time scheme if it is the case for the model.
  - No conservation in the linearization: we need to converge Newton/picard process for that.
- Model 199
  - Conservation in energy for the model.
  - Conservation in energy for time scheme.
  - No conservation in the linearization: we need to converge Newton/picard process for that.

Conclusion

- The full model introduced before seems a good candidate to begin all the reduction
- Construction of reduced using Boniface method seems good to obtain the different reduced models with energy conservation (it is depend of the simplification)
- With simplification we will obtain model in JOREK. Without we will obtain additional terms.
- Other possible advantage: To derivate all the models we begin with the full MHD. A interesting point will be to write the time scheme for full MHD (Crank-Nicolson, Semi implicit closed M3DC1, splitting scheme etc) which is more simple and reduced after.
- I can help Boniface, Guido, Javier, Matthias etc for the derivation of these models