Relaxation Schemes for low-Mach Problems

F. Bouchut[3], D. Coulette[4], E. Franck[12], P.Helluy[12], L. Navoret[12]

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[1]Inria Nancy Grand Est, France
[2]IRMA, Strasbourg university, France
[3]Marne la Vallée, university, France
[4]ENS Lyon, France

Outline

Physical and mathematical context

Gas dynamic: Euler equations

- **Context**: Plasma simulation with Euler/MHD equations.
- **Euler equation**:

$$\left\{\begin{array}{l}\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u})=0\\ \partial_t(\rho\boldsymbol{u})+\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}+pI_d)=0\\ \partial_t E+\nabla\cdot(E\boldsymbol{u}+p\boldsymbol{u})=0\end{array}\right.$$

- with $\rho(t,\mathbf{x})>0$ the density, $\boldsymbol{u}(t,\mathbf{x})$ the velocity and $E(t,\mathbf{x})>0$ the total energy.
- The pressure p is defined by $p=\rho T$ (perfect gas law) with T the temperature.
- **Hyperbolic system** with nonlinear waves. **Waves speed**: three eigenvalues: $(\boldsymbol{u},\mathbf{n})$ and $(\boldsymbol{u},\mathbf{n})\pm c$ with the sound speed $c^2=\gamma\frac{p}{\rho}$.

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Physic interpretation:

- **Two important velocity scales**: $\boldsymbol{u}$ and c and the ratio (Mach number) $M=\frac{|\boldsymbol{u}|}{c}$.
- When M tends to zero, we obtain incompressible Euler equation:

$$\left\{\begin{array}{l}\partial_t\rho+\boldsymbol{u}\cdot\nabla\rho=0\\ \rho\partial_t\boldsymbol{u}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p_2=0\\ \nabla\cdot\boldsymbol{u}=0\end{array}\right.$$

  In 1D we have just advection of ρ.
- **Aim**: construct an scheme (AP) valid at the limit with a uniform cost.

Numerical difficulties in space: VF in 1D

- Second method: Finite volume and DG method
  - VF method + Rusanov flux. Equivalent equation:

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x(\rho u) = \dfrac{S\Delta x}{2}\partial_{xx}\rho \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \dfrac{1}{M^2}\partial_x p = \dfrac{S\Delta x}{2}\partial_{xx}(\rho u) \\ \partial_t E + \partial_x(E u) + \partial_x(p u) = \dfrac{S\Delta x}{2}\partial_{xx} E\end{array}\right.$$

  - **Problem**: S must be larger than $\frac{1}{M}$ for stability. Huge diffusion.

- Example: isolated contact $p = 1$ and $u = 0.1$.
- **Exact. solution**:

$$\partial_t \rho + u_0 \partial_x \rho = 0$$

- **Rusanov scheme**:

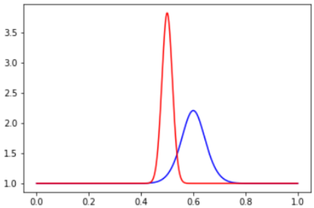
$$\partial_t \rho + u_0 \partial_x \rho = \frac{S\Delta x}{2}\partial_{xx}\rho$$

with $S > u_0 + c \approx 1.5$

- **Upwind scheme for limit**:

$$\partial_t \rho + u_0 \partial_x \rho = \frac{u_0 \Delta x}{2}\partial_{xx}\rho$$

- Rusanov scheme $T_f = 2$ $u_0 = 0.05$ and 1000 cells



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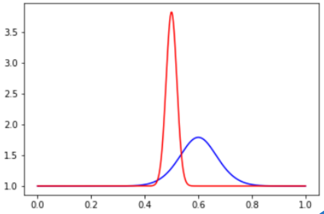

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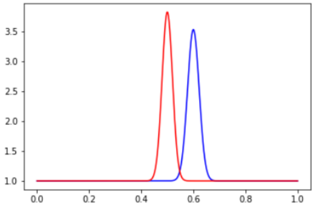

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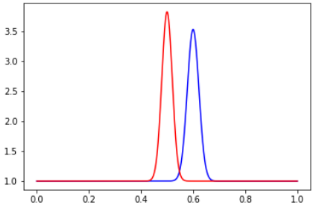

Numerical difficulties in space: VF in 2D

- Same analysis in 2D.
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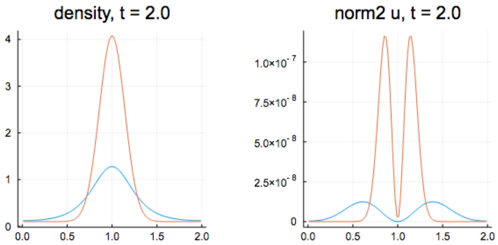
  - **Problem**: S must be larger that $\frac{1}{M}$ for stability. Huge diffusion.
- Example: isolated contact $p = 1$, $\nabla \cdot \boldsymbol{u}_0 = 0$ and $\boldsymbol{u}_0$ constant in time.
- Rusanov scheme $T_f = 2 \mid \boldsymbol{u}_0 \mid \approx 0.001$ and 100*100 cells.



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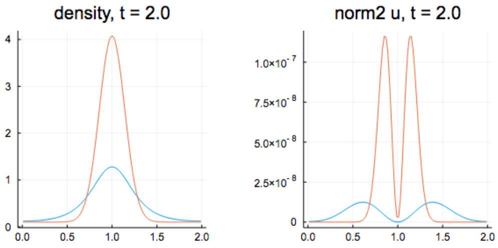

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Numerical problem I: time discretization.

- **Explicit scheme**: the CFL condition for low mach flow:
  - The fast phenomena: acoustic waves at velocity c
  - The important phenomena: transport at velocity u
  - Expected CFL: $\Delta t < \frac{\Delta x}{|u|}$, CFL in practice $\Delta t < \frac{\Delta x}{|c|}$
  - At the end, we use a Δt divided by M compared to the expected Δt

First solution

Implicit time scheme. No CFL condition. Taking a larger time step, it allows to "filter" the fast acoustic waves which are not useful in the low-Mach regime.

- Implicit time scheme:

$$M_i \boldsymbol{U}^{n+1} = (I_d + \Delta t A(I_d))\boldsymbol{U}^{n+1} = \boldsymbol{U}^n$$

- We must solve a nonlinear system and after linearization solve some linear systems.

Problem

- Direct solver too costly. Approximative conditioning for the iterative solvers:

$$k(M_i) \approx 1 + O\left(\frac{\Delta t}{\Delta x^p M}\right)$$

- We recover the two scales in the conditioning number. The full implicit schemes are difficult to use for this reason.

Numerical problem II: time discretization.

First idea: Semi implicit scheme

- We explicit the slow scale (transport) and implicit the fast scale (acoustic) [CDK12]-[DLVD19]

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0 \\ \partial_t E + \partial_x(Eu) + \partial_x(pu) = 0 \end{cases}$$

Implicit acoustic step:

$$\begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} = \rho^n u^n - \Delta t \partial_x p^{n+1} + Rhs_u \\ E^{n+1} = E^n - \Delta t \partial_x (p^{n+1} u^{n+1}) = Rhs_E \end{cases}$$

Plugging this in the second equation, we obtain

$$E^{n+1} - \Delta t^2 \partial_x \left(\frac{p^{n+1}}{\rho^n} \partial_x p^{n+1} \right) = Rhs(E^n, u^n, \rho)$$

- Matrix-vector product to compute u^{n+1}.

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Conclusion

- **Semi implicit**: only one scale in the implicit symmetric positive operator.
- Strong gradient of ρ generates ill-conditioning. Assembly at each time (costly).
- Nonlinear solver can have bad convergence for if $\Delta t >> 1$ and $\partial_x p$ not so small.

Relaxation method

- **Relaxation** [XJ95]-[CGS12]-[BCG18]: a way to linearize and decouple the equations. Used to design new schemes.
- **Idea**: Approximate the model

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = 0, \text{ by } \quad \partial_t \mathbf{f} + \mathbf{A}(\mathbf{f}) = \frac{1}{\varepsilon}(Q(\mathbf{f}) - \mathbf{f})$$

- At the limit and taking $P\mathbf{f} = \boldsymbol{U}$ we obtain

$$\partial_t \boldsymbol{U} + \partial_x \mathbf{F}(\boldsymbol{U}) = \varepsilon \partial_x (D(\boldsymbol{U}) \partial_x \boldsymbol{U}) + O(\varepsilon^2)$$

- **Time scheme**:
  - we solve

$$\frac{\mathbf{f}^* - \mathbf{f}^n}{\Delta t} + \mathbf{A}(\mathbf{f}^{*,n}) = 0$$

  - and after we approximate the stiff source term by

$$\mathbf{f}^{n+1} = \mathbf{f}^* + \omega(Q(\mathbf{f}^*) - \mathbf{f}^*)$$

  with $\omega \in]0, 2]$.

Why ?

- In general, we construct **A** with a simpler structure than **F** to design numerical flux in FV.
- Here, we construct **A** with a simpler structure to design simple implicit scheme.

Full-Implicit relaxation method

Xin-Jin relaxation method

- We consider the following nonlinear hyperbolic system

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = 0$$

- with $\boldsymbol{U}$ a vector of N functions.
- **Aim**: Find a way to approximate this system with a sequence of simple systems.
- **Idea**: Xin-Jin relaxation method (very popular in the hyperbolic and Finite Volume community) [JX95]-[Nat96]-[ADN00].

$$\begin{cases} \partial_t \boldsymbol{U} + \partial_x \boldsymbol{V} = 0 \\ \partial_t \boldsymbol{V} + \lambda^2 \partial_x \boldsymbol{U} = \frac{1}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}) - \boldsymbol{V}) \end{cases}$$

Limit of the hyperbolic relaxation scheme

- The limit scheme of the relaxation system is

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \varepsilon \partial_x((\lambda^2 - \mid A(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + o(\varepsilon^2)$$

- with $A(\boldsymbol{U})$ the Jacobian of $\boldsymbol{F}(\boldsymbol{U})$.

- **Conclusion**: the relaxation system is an approximation of the original hyperbolic system (error in ε).

Xin-Jin implicit scheme

Main property

- **Relaxation system**: "the nonlinearity is local and the non-locality is linear".
- **Main idea**: **splitting scheme** between implicit transport and **implicit** relaxation.
- **Key point**: the $\partial_t \boldsymbol{U} = 0$ during the relaxation step. Therefore $\boldsymbol{F}(\boldsymbol{U})$ **is explicit**.

- Relaxation step:

$$\begin{cases} \boldsymbol{U}^{n+1} = \boldsymbol{U}^n \\ \boldsymbol{V}^{n+1} = \theta \frac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^{n+1}) - \boldsymbol{V}^{N+1}) + (1-\theta)\frac{\Delta t}{\varepsilon}(\boldsymbol{F}(\boldsymbol{U}^n) - \boldsymbol{V}^n) \end{cases}$$

- Transport step (order 1) :

$$I_d + \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \boldsymbol{U}^{n+1} \\ \boldsymbol{V}^{n+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{U}^n \\ \boldsymbol{V}^n \end{pmatrix}$$

  - We plug the equation on $\boldsymbol{V}$ in the equation on $\boldsymbol{U}$.
  - We obtain the implicit part:

$$(I_d - \Delta t^2 \lambda^2 \partial_{xx})\boldsymbol{U}^{n+1} = \boldsymbol{U}^n - \Delta t \partial_x \boldsymbol{V}^n$$

  - We apply a matrix-vector product

$$\boldsymbol{V}^{n+1} = -\Delta t \lambda^2 \partial_x \boldsymbol{U}^{n+1}$$

- Natural extension at the second order in time. **In space**: FV (used here) or DG/FE.

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Advantages and defauts

Advantages

- If we have N equations, we obtain N independent wave systems.
- Each substep can be solved implicitly with one inversion of constant elliptic problem and one matrix-vector product.
- Uniform cost in Mach number with a good-preconditioning (multigrids).

Numerical error

- Error for the first order splitting scheme:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \left(\frac{2-\omega}{\omega}\right) \partial_x((\lambda^2 I_d - \mid A(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + O(\Delta t^2)$$

- In Low Mach regime $\partial_x u \approx M$, $\partial_x p \approx M$ and $c \approx \frac{1}{M}$ consequently

$$\partial_t \rho + \partial_x(\rho u) \approx \Delta t \left(\frac{2-\omega}{\omega}\right) (\partial_x(c^2 - u^2)\partial_x \rho) + O(\Delta t^2)$$

- **Conclusion**: Huge diffusion for the contact wave.
- In a 2D case:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \approx \left(\frac{2-\omega}{\omega}\right) \frac{\Delta t}{2M^2} \mid \mathbf{u} \mid^2 \Delta \mathbf{u} + O(\Delta t^2)$$

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- Error for the first order splitting scheme:

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \Delta t \left(\frac{2-\omega}{\omega}\right) \partial_x((\lambda^2 I_d - \mid A(\boldsymbol{U}) \mid^2)\partial_x \boldsymbol{U}) + O(\Delta t^2)$$

- In Low Mach regime $\partial_x u \approx M$, $\partial_x p \approx M$ and $c \approx \frac{1}{M}$ consequently

$$\partial_t \rho + \partial_x(\rho u) \approx \Delta t \left(\frac{2-\omega}{\omega}\right) u^2 \left(\partial_x(\frac{1}{M^2} - 1)\partial_x \rho\right) + o(\Delta t^2)$$

- **Conclusion**: Huge diffusion for the contact wave.
- In a 2D case:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \approx \left(\frac{2-\omega}{\omega}\right) \frac{\Delta t}{2M^2} \mid \mathbf{u} \mid^2 \Delta \mathbf{u} + O(\Delta t^2)$$

- **Gresho vortex**: The initial data are given by $\rho(t=0,\mathbf{x}) = 1 + M^2\rho_2(\mathbf{x})$,

$$\mathbf{u}(t=0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \text{with } \nabla\cdot\mathbf{u}_0 = 0,$$

$\parallel \mathbf{u}_0 \parallel \approx 1$ and $\rho(t,\mathbf{x}) = \rho_0 + M^2\rho_2(\mathbf{x})$ and $p(t,\mathbf{x}) = \frac{1}{\gamma M^2}$.

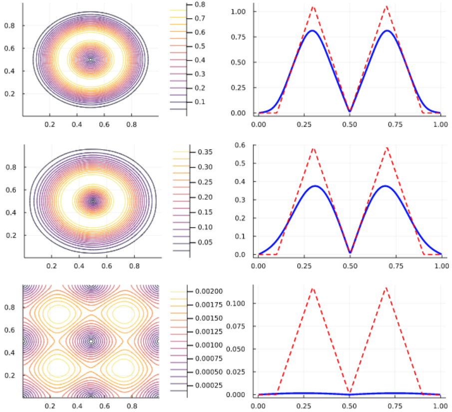


Figure: Norm of the spatial Mach number for the first order implicit Xin-Jin relaxation scheme. Top: $M=0.9$, middle: $M=0.5$, bottom: $M=0.1$.

Results: AP correction for isothermal case

- **Error**:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \approx \Delta t \left(\frac{2-\omega}{\omega} \right) \frac{\Delta t}{2M^2} \Delta \mathbf{u} + O(\Delta t^2)$$

- **Idea**: take $\omega = 2 - M^2$

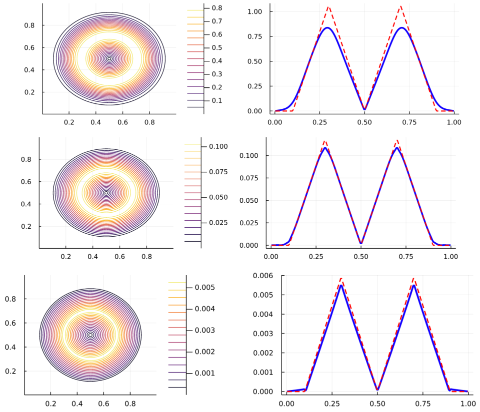


Figure: Norm of the spatial Mach number for the first order adaptive implicit Xin-Jin relaxation scheme. Top: $M = 0.9$, middle top: $M = 0.1$, middle bottom: $M = 0.03$ bottom: $M = 0.005$.

Results: AP correction for the full case

- This correction is sufficient ?
- Contact wave in 1D for $\omega = 2$:

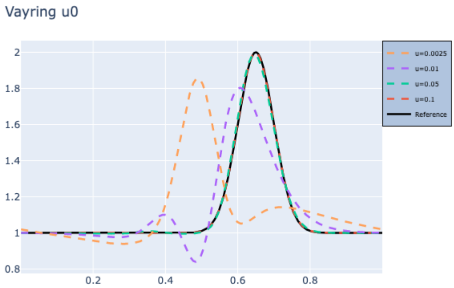


Figure: Density given by second order implicit scheme varying u_0 in the relaxation.

- Results for $u_0 = 0.1$ ($M \approx \frac{1}{10}$) and $u_0 = 0.05$ ($M \approx \frac{1}{20}$) are quite convincing.
- Not for smaller Mach number. Too much dispersive effects.
- **Conclusion**: The correction modify the diffusion to avoid the Mach number dependency but it is not the case in the dispersion (of the splitting and/or time scheme).

Semi-Implicit relaxation method

Suliciu-type Relaxation method

- **Problem**: the nonlinearity of the implicit acoustic step generates difficulties.
- Non-conservative form and acoustic term:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t p + u\partial_x p + \rho c^2 \partial_x u = 0 \\ \partial_t u + u\partial_x u + \frac{1}{\rho}\partial_x p = 0 \end{array}\right.$$

- **Idea**: Relax only the acoustic part ([BCG18]) to linearize the implicit part.

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u v + \Pi) = 0 \\ \partial_t E + \partial_x(E v + \Pi v) = 0 \\ \partial_t \Pi + v\partial_x \Pi + \phi\lambda^2 \partial_x v = \frac{1}{\varepsilon}(p - \Pi) \\ \partial_t v + v\partial_x v + \frac{1}{\phi}\partial_x \Pi = \frac{1}{\varepsilon}(u - v) \end{array}\right.$$

- **Limit**:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = \varepsilon\partial_x \left[A\partial_x p\right] \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon\partial_x \left[(A u \partial_x p) + B\partial_x u\right] \\ \partial_t E + \partial_x(E u + p u) = \varepsilon\partial_x \left[A E\partial_x p + A\partial_x \frac{p^2}{2} + B\partial_x \frac{u^2}{2}\right] \end{array}\right.$$

- with $A = \frac{1}{\rho}\left(\frac{\rho}{\phi} - 1\right)$ and $B = \left(\rho\phi\lambda^2 - \rho^2 c^2\right)$.
- **Stability**: $\phi\lambda > \rho c^2$ and $\rho > \phi$.

Avdantage

- We keep the conservative form for the original variables and obtain a fully linear acoustic.

Dynamical splitting

- **Splitting**: we solve sub-part of the system one by one. **Dynamic case**: Splitting time depending for low-Mach [IDGH2018]
- For large acoustic waves (Mach number not small) we want capture to all the phenomena. Consequently use an explicit scheme.
- For small/fast acoustic waves (low Mach number) we want filter acoustic. Consequently use an implicit scheme for acoustic.

Splitting: Explicit convective part/Implicit acoustic part.

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x(\rho v)=0\\ \partial_t(\rho u)+\partial_x(\rho u v+\mathcal{M}^2(t)\Pi)=0\\ \partial_t E+\partial_x(Ev+\mathcal{M}^2(t)\Pi v)=0\\ \partial_t\Pi+v\partial_x\Pi+\phi\lambda_c^2\partial_x v=0\\ \partial_t v+v\partial_x v+\frac{\mathcal{M}^2(t)}{\phi}\partial_x\Pi=0\end{array}\right. , \quad \left\{\begin{array}{l}\partial_t\rho=0\\ \partial_t(\rho u)+(1-\mathcal{M}^2(t))\partial_x\Pi=0\\ \partial_t E+(1-\mathcal{M}^2(t))\partial_x(\Pi v)=0\\ \partial_t\Pi+\phi(1-\mathcal{M}^2(t))\lambda_a^2\partial_x v=0\\ \partial_t v+(1-\mathcal{M}^2(t))\frac{1}{\phi}\partial_x\Pi=0\end{array}\right.$$

with $\mathcal{M}(t)\approx max\left(\mathcal{M}_{min}, min\left(max_x\frac{|u|}{c},1\right)\right)$

- Eigenvalues of Explicit part: v, $v\pm\mathcal{M}(t)\underbrace{\lambda_c}_{\approx c}$. Implicit part 0, $\pm(1-\mathcal{M}^2(t))\underbrace{\lambda_a}_{\approx c}$
- **At the end**: we make the projection $\Pi=p$ and $v=u$ (can be viewed as a discretization of the stiff source term).

Implicit time scheme

- We introduce the implicit scheme for the "acoustic part":

$$\begin{cases} \rho^{n+1} = \rho^n \\ (\rho u)^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\partial_x \Pi^{n+1} = (\rho u)^n \\ E^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\partial_x (\Pi v)^{n+1} = E^n \\ \Pi^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2 \partial_x v^{n+1} = \Pi^n \\ v^{n+1} + \Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x \Pi^{n+1} = v^n \end{cases}$$

- We plug the equation on v in the equation on Π. We obtain the following algorithm:
  - Step 1: we solve

$$(I_d - (1-\mathcal{M}^2(t_n))^2 \Delta t^2 \lambda_a^2 \partial_{xx})\Pi^{n+1} = \Pi^n - \Delta t(1-\mathcal{M}^2(t_n))\phi\lambda_a^2 \partial_x v^n$$

  - Step 2: we compute

$$v^{n+1} = v^n - \Delta t(1-\mathcal{M}^2(t_n))\frac{1}{\phi}\partial_x \Pi^{n+1}$$

  - Step 3: we compute

$$(\rho u)^{n+1} = (\rho u)^n - \Delta t(1-\mathcal{M}^2(t_n))\partial_x \Pi^{n+1}$$

  - Step 4: we compute

$$E^{n+1} = E^n - \Delta t(1-\mathcal{M}^2(t_n))\partial_x (\Pi^{n+1} v^{n+1})$$

Advantage

- We solve only a constant Laplacian. We can assembly matrix once.
- No problem of conditioning, which comes from to the strong gradient of ρ

Spatial scheme in 1D

- **Idea**: FV Godunov fluxes for the explicit part + Central fluxes for the implicit part.
- Main problem of the explicit part: design numerical flux.
- **First possibility**: since the maximal eigenvalue is $O(Mach)$ a Rusanov scheme.
- Other solution: construct a Godunov scheme for the relaxation system. Principle:
  - eigenvalues: $v-\mathcal{E}(t)\lambda_c, v(x3), v+\mathcal{E}(t)\lambda_c$
  - Strong invariants of external waves:

$$\partial_t(v\pm\phi\lambda_c\pi)+(v\pm\mathcal{E}(t)\lambda_c)\partial_x(v\pm\phi\lambda_c\pi)=0$$

  - Strong invariants of central waves:

$$\partial_t\left(\frac{1}{\rho}+\frac{\pi}{\rho\phi\lambda^2}\right)+v\partial_x\left(\frac{1}{\rho}+\frac{\pi}{\rho\phi\lambda^2}\right)=0$$

$$\partial_t\left(u-\frac{\phi}{\rho}v\right)+v\partial_x\left(u-\frac{\phi}{\rho}v\right)=0$$

$$\partial_t\left(\rho e+\frac{\pi^2}{2\rho\phi\lambda_c^2}+\frac{(v-u)^2}{2(\frac{\rho}{\phi}-1)}\right)+v\partial_x\left(\rho e+\frac{\pi^2}{2\rho\phi\lambda_c^2}+\frac{(v-u)^2}{2(\frac{\rho}{\phi}-1)}\right)=0$$

  - **Important**: strong invariant are weak invariant (conserved) on the other waves. **Exemple**: (π, v) preserved on central wave.
  - We obtain all the intermediary states using these previous results.

Results 1D I: contact

- Smooth contact :

$$\begin{cases} \rho(t,x) = \chi_{x<x_0} + 0.1\chi_{x>x_0} \\ u(t,x) = 0.01 \\ p(t,x) = 1 \end{cases}$$

- Error

cells	Ex Rusanov	Ex LR	Old relax Rusanov	Relax Rus	Relax PC-FVS
250	0.042	$3.6E^{-4}$	$1.4E^{-3}$	$7.8E^{-4}$	$4.1E^{-4}$
500	0.024	$1.8E^{-4}$	$6.9E^{-4}$	$3.9E^{-4}$	$2.0E^{-4}$
1000	0.013	$9.0E^{-5}$	$3.4E^{-4}$	$2.0E^{-4}$	$1.0E^{-5}$
2000	0.007	$4.5E^{-5}$	$1.7E^{-4}$	$9.8E^{-5}$	$4.9E^{-5}$

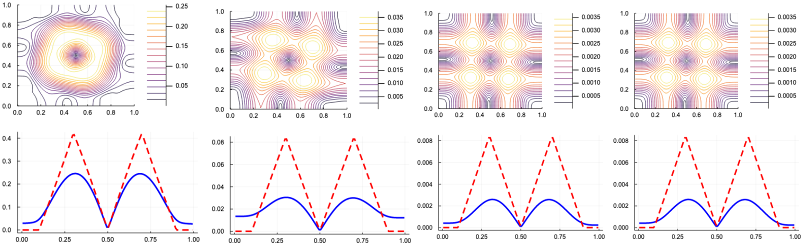
- **Old relax**: other relaxation scheme where the implicit Laplacian is not constant and depend of ρ^n.
- Comparison time scheme:

Scheme	λ	Δt
Explicit	$\max(\mid u-c\mid,\mid u+c\mid)$	$2.2E^{-4}$
SI Old relax	$\max(\mid u-\mathcal{M}(t_n))\frac{\lambda}{\rho}\mid,\mid u+\mathcal{M}(t_n))\frac{\lambda}{\rho}\mid)$	0.0075
SI new relaxation	$\max(\mid v-\mathcal{M}(t_n))\lambda\mid,\mid v+\mathcal{M}(t_n))\lambda\mid)$	0.04

- Conditioning:

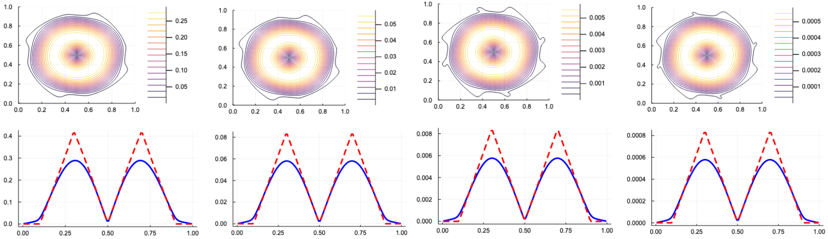
Schemes	Δt	conditioning
Si old relax	0.00757	3000
Si new relax	0.041	9800
Si new relax	0.0208	2400
si new relax	0.0075	320

- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$



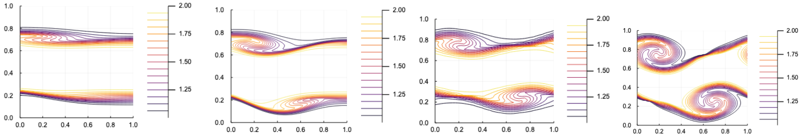
- Explicit Lagrange+remap scheme Norm of the velocity (2D plot). 1D initial (red) and final (blue) time .From left to right: $M_0 = 0.5$ ($\Delta t = 1.4E^{-3}$), $M_0 = 0.1$ ($\Delta t = 3.5E^{-4}$), $M_0 = 0.01$ ($\Delta t = 3.5E^{-5}$), $M_0 = 0.001$ ($\Delta t = 3.5E^{-6}$).

- Gresho vortex: $\nabla \cdot \boldsymbol{u} = 0$ and $p = \frac{1}{M^2} + p_2(\mathbf{x})$



- Relaxation scheme. Norm of the velocity (2D plot). 1D initial (red) and final (blue) times. From left to right: $M = 0.5$, $\Delta t = 2.5E^{-3}$, $M = 0.1$, $\Delta t = 2.5E^{-3}$, $M = 0.01$, $\Delta t = 2.5E^{-3}$, $M = 0.001$, $\Delta t = 2.5E^{-3}$.

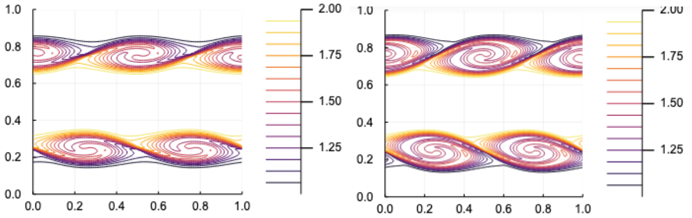
- Kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 1$, $M_0 = 0.1$. Explicit Lagrange-Remap scheme with 120×120 (left) and 360×360 cells (middle left), SI two-speed relaxation scheme ($\lambda_c = 18$, $\lambda_a = 15$, $\phi = 0.98$) with 42×42 (middle right) and 120×120 cells (right).

Results in 2D: Kelvin helmholtz

- Kelvin-Helmholtz instability. Density:



- Density at time $T_f = 3$, $k = 2$, $M_0 = 0.01$ with SI two-speed relaxation scheme ($\lambda_c = 180$, $\lambda_a = 150$, $\phi = 0.98$). Left: 120×120 cells. Right: 240×240 cells.

Well-balanced extension for Ripa model

Ripal model and steady states

- To finish we propose to see if the method can be combined with WB property to solve flow around equilibrium.
- Ripa equation:

$$\left\{\begin{array}{l}\partial_t h + \partial_x(hu) = 0,\\ \partial_t(hu) + \partial_x(hu^2 + \frac{p(h,\Theta))}{\mathcal{F}_r^2} = -\frac{gh}{\mathcal{F}_r^2}\Theta\partial_x z,\\ \partial_t(h\Theta) + \partial_x(h\Theta u) = 0,\end{array}\right. \quad (1)$$

- where $h(x,t)$ is the water height, $u(x,t)$ the velocity, $\Theta(x,t)$ the temperature and $z(x)$ the topography, the pressure law is given by: $p(h,\Theta) = g\Theta\frac{1}{2}h^2$ and the Froud number $\mathcal{F}_r = u/\sqrt{gh}$.
- Steady state:

$$\left\{\begin{array}{l}u = 0,\\ \Theta = Cst,\\ h + z = Cst,\end{array}\right. \quad \left\{\begin{array}{l}u = 0,\\ z = Cst,\\ \Theta\frac{h^2}{2} = Cst,\end{array}\right. \quad \left\{\begin{array}{l}u = 0,\\ h = Cst,\\ z + \frac{h}{2}\ln(\Theta) = Cst.\end{array}\right. \quad (2)$$

- Aim: solve flows like

$$u = O(\mathcal{F}_r), \quad \Theta = Cst + O(\mathcal{F}_r), \quad h + z = Cst + O(\mathcal{F}_r), \quad (3)$$

with $\mathcal{F}_r \ll 1$. In that case, the perturbation has a small amplitude but moves with a large propagation speed of order $O(1/\mathcal{F}_r)$.

Splitting scheme

- **Idea**: use the same scheme as for Euler equation coupling with WB approach. Splitting:

$$(C)\quad \left\{\begin{array}{l} \partial_t h + \partial_x(hv) = 0, \\ \partial_t(hu) + \partial_x(huv + \mathcal{F}^2\Pi) = -\mathcal{F}^2\, gh\Theta\partial_x z, \\ \partial_t(h\Theta) + \partial_x(h\Theta v) = 0, \\ \partial_t\Pi + v\partial_x\Pi + h_m\lambda^2\partial_x v = 0 \\ \partial_t v + v\partial_x v + \frac{\mathcal{F}^2}{h_m}\partial_x\Pi = -\mathcal{F}^2\frac{h}{h_m} g\Theta\partial_x z \end{array}\right.$$

$$(W)\quad \left\{\begin{array}{l} \partial_t h = 0, \\ \partial_t(hu) + (1-\mathcal{F}^2)\left(\partial_x\Pi + hg\partial_x z\right) = 0, \\ \partial_t h\Theta = 0 \\ \partial_t\Pi + (1-\mathcal{F}^2)h_m\lambda^2\partial_x v = 0 \\ \partial_t v + \frac{(1-\mathcal{F}^2)}{h_m}\left(\partial_x\Pi + hg\partial_x z\right) = 0 \end{array}\right.$$

$$(R)\quad \left\{\partial_t\Pi = \frac{1}{\varepsilon}\left(p(h,\Theta) - \Pi\right), \quad \partial_t v = \frac{1}{\varepsilon}\left(u - v\right),\right.$$

where $\mathcal{F} = \max\left(\mathcal{F}_{\min}, \min\left(\frac{u}{\sqrt{h\Theta g}}, 1\right)\right)$ and

$$\left(\frac{h}{h_m} - 1\right) > 0, \quad \gamma = \left(h_m\lambda^2 - hc^2\right) > 0.$$

Well-balanced property

- **Explicit part**: we plug the source term into the flux (Jin Levermore technic).
- Specific discretization of the steady states at the interface: centered gradient for $\partial_x z$, average mean for h, entropic mean for Θ.
- **Implicit part**: The final algorithm writes:
  - Step 1: solve

$$\left(\Pi_j^{n+1} - (1-\mathcal{F}^2)^2 \Delta t^2 \lambda^2 \frac{\Pi_{j+1}^{n+1} - 2\Pi_j^{n+1} + \Pi_{j-1}^{n+1}}{\Delta x^2}\right) =$$

$$\Pi_j^n - \Delta t(1-\mathcal{F}^2)\lambda^2 \frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} + (1-\mathcal{F}^2)^2 \Delta t^2 \lambda^2 \frac{1}{\Delta x}\left(S_{j+\frac{1}{2}}^n - S_{j-\frac{1}{2}}^n\right),$$

  with

$$S_{j+\frac{1}{2}}^n = h_{j+\frac{1}{2}}^n \Theta_{j+\frac{1}{2}}^n \frac{z_{j+1} - z_j}{\Delta x},$$

  computed as for the explicit.
  - Step 2: compute

$$v_j^{n+1} = v_j^n - (1-\mathcal{F}^2)\frac{\Delta t}{h_m}\frac{\Pi_{j+1}^{n+1} - \Pi_{j-1}^{n+1}}{2\Delta x} - (1-\mathcal{F}^2)\frac{\Delta t}{h_m}\frac{g}{2}\left(S_{j+\frac{1}{2}}^n - S_{j-\frac{1}{2}}^n\right),$$

$$(hu)_j^{n+1} = (hu)_j^n - \Delta t(1-\mathcal{F}^2)\frac{\Pi_{j+1}^{n+1} - \Pi_{j-1}^{n+1}}{2\Delta x} - \frac{g\Delta t}{2}(1-\mathcal{F}^2)\left(S_{j+\frac{1}{2}}^n - S_{j-\frac{1}{2}}^n\right).$$

  - If the steady state is preserved at time n it still be preserved after an implicit step

WB property

$$\begin{array}{llll}(ST1) & z(x)=0.1+G_{x_0,\sigma}(x), & h_0(x)=8.0-z(x), & \Theta_0(x)=1,\\ (ST2) & z(x)=1, & h_0(x)=1.0+0.2G_{x_0,\sigma}(x), & \Theta_0(x)=\frac{1}{gh_0(x)^2},\\ (ST3) & z(x)=x(1-x), & h_0(x)=1, & \Theta_0(x)=2e^{-x(1-x)}.\end{array}$$

Δt/Error	Tests	Rusanov	SI WB Ex	SI two-speed WB Imp
ST1	Error h	$1.5E^{-2}$	$1.5E^{-17}$	$3.6E^{-13}$
	Error u	$5.9E^{-3}$	$1.5E^{-15}$	$6.7E^{-13}$
	Error Θ	0.0	0.0	0.0
	Δt	$8.1E^{-4}$	$7.1E^{-4}$	$1.42E^{-1}$
ST2	Error h	$9.3E^{-2}$	0.0	$8.4E^{-12}$
	Error u	$7.3E^{-9}$	0.0	$1.3E^{-13}$
	Error Θ	0.13	$1.8E^{-17}$	$6.0E^{-12}$
	Δt	$2.5E^{-3}$	$2.3E^{-3}$	$4.7E^{-1}$
ST3	Error h	0.59	0.0	$1.38E^{-12}$
	Error u	0.65	$1.6E^{-15}$	$4.4E^{-14}$
	Error Θ	0.19	0.0	$1.4E^{-12}$
	Δt	$2.4E^{-3}$	$1.8E^{-3}$	0.49

Numerical results

Wave perturbation:

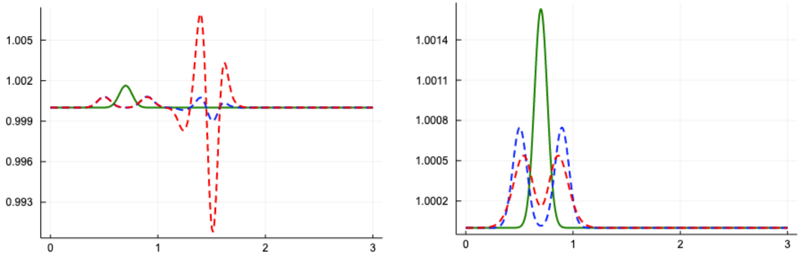


Figure: Left: explicit Rusanov scheme; In green the initial data. In red the solution on a semi-coarse grid (1200 cells), in blue the solution on a fine grid (12000 cells). Right: SI two-speed WB; in green the initial data. In red the solution on a coarse grid (600 cells), in blue the solution on a semi-coarse grid (4800 cells).

Conclusion

Resume

- Introducing Dynamic splitting scheme we separate the scales.
- Introducing implicit scheme for the acoustic wave we can filter these waves.
- Introducing relaxation we simplify at the maximum the implicit scheme.
- A well-adapted spatial scheme is also very important.
- **At the end**: we capture the incompressible limit.

Perspectives:

- **To avoid some spurious mods**: Use compatible discretization for the linear wave part (mimetic/staggered DF, compatible finite element).
- Extension to High Order, MUSCL firstly and after DG and HDG schemes.
- Extension to MHD (main goal). For MHD the relaxation it is ok but the splitting is less clear.