HOCHSCHILD–KOSTANT–ROSENBERG ISOMORPHISM FOR DERIVED DELIGNE–MUMFORD STACKS

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Abstract. We prove a Hochschild–Konstant–Rosenberg (HKR) theorem for arbitrary derived Deligne–Mumford (DM) stacks, extending the results of Arinkin–Căldăraru–Hablicsek in the smooth, global quotient case, although with different methods. To formulate our result, we introduce the notion of *orbifold inertia* stack of a derived DM stack; this supplies a finely tuned derived enhancement of the classical inertia stack, which does not always coincide with the classical truncation of the free loop space. We show that, in characteristic 0, given a derived DM stack, the shifted tangent bundle of its orbifold inertia stack is equivalent to its free loop space. This yields a canonical HKR isomorphism of algebras between the Hochschild homology of a derived DM stack and the cohomology of differential forms on its orbifold inertia stack. Moreover, this isomorphism intertwines the natural circle action and the de Rham differential. Similarly, HKR theorems for derived DM stacks are established for Hochschild cohomology, cyclic homology, negative cyclic homology and periodic cyclic homology. As applications, we provide a rich supply of computations of Hochschild homology and Hochschild cohomology of interesting derived DM stacks, for instance, weighted projective lines, root stacks, quotients by algebraic groups, mapping stacks, etc.

Contents

1. Introduction 1
2. Some results on mapping stacks 7
3. Orbifold inertia for derived Deligne-Mumford stacks 12
4. HKR isomorphism for derived Deligne–Mumford stacks 18
5. Circle action and de Rham differential 25
6. Examples and applications 31
Appendix A. Recollection on derived fixed points 43
References 46

1. Introduction

In this article, we establish the Hochschild–Konstant–Rosenberg (HKR) equivalence for derived Deligne–Mumford (DM) stacks over a field of characteristic 0, previously known only in the smooth global quotient case thanks to the work of Arinkin–Căldăraru–Hablicsek [7]. Our result generalizes theirs in three directions: (1) we allow arbitrarily singular and even derived Deligne–Mumford stacks; (2) we go beyond the global quotient case; (3) we establish the compatibility between the natural circle action and the de Rham differential operator. Along the way, we introduce a derived enhancement of the classically defined inertia stack, named *orbifold inertia* in this paper, and show that it coincides with the classical inertia for smooth DM stacks and can have a non-trivial derived structure already for underived singular DM stacks.

1.1. **Historical context.** The Hochschild homology can be defined for any associative algebra A, as the derived tensor product of A with itself as an (A, A)-bimodule:

$$\mathsf{HH}_*(A) \simeq A \otimes^{\mathrm{L}}_{A\otimes A^{\mathrm{op}}} A \ .$$

In their seminal 1962 paper [26], Hochschild, Konstant and Rosenberg proved that, the Hochschild homology of a *smooth commutative* algebra A over a field of characteristic 0 is equivalent to its graded algebra of differential forms:

$$\mathsf{HH}_*(A) \simeq \bigoplus_i \Omega^i_A \ .$$

The HKR theorem shows in particular that the algebra of differential forms, a subtle geometric invariant, can be computed in terms of a general construction in homological algebra which makes sense even outside the commutative world. The HKR theorem is, from this perspective, one of the cornerstones of non-commutative geometry. In the mid-1980's, Connes [18] and Tsygan [59] discovered that the de Rham differential operator can also be recovered purely in terms of homological algebra, as a manifestation of a natural S^1-action on $\mathsf{HH}_*(A)$. This finding opened the way to many subsequent developments, showing that essentially the whole calculus of differential forms and polyvector fields can be transported to the non-commutative setting.

The HKR theorem can be extended to the global geometric case. In [60], the definition of the Hochschild homology is extended to all schemes. The classical formulation of the HKR equivalence in this geometric setting, due to Swan [54, Corollary 2.6], states that if X is a smooth quasi-projective scheme over a field of characteristic zero, its Hochschild homology is equivalent to the hyper-cohomology of differential forms

$$\mathsf{HH}_{-i}(X) \simeq \bigoplus_{q-p=i} H^q(X, \Omega^p_X) \ . \tag{1.1.1}$$

The smoothness assumption can be dropped, at the price of working with the derived exterior powers of the cotangent complex, as first shown in the work of André [3] and Quillen [44]. Within derived algebraic geometry, it is possible to reinterpret Hochschild homology in terms of a basic geometric construction, the *free loop space*. If X is a derived stack, its free loop space is the mapping stack

$$\mathscr{L}X \coloneqq \mathbf{Map}(S^1, X) \ ,$$

and the Hochschild homology of X is equivalent to the algebra of (derived) functions on $\mathscr{L}X$:

$$\mathsf{HH}_*(X) \simeq \mathcal{O}(\mathscr{L}X) \ .$$

As first observed by Ben–Zvi and Nadler in [11], the equivalence (1.1.1) can be understood as the linearization of a more fundamental equivalence of stacks. In characteristic 0, if X is a scheme there is a canonical equivalence

$$\exp\colon \mathsf{T}[-1]X \to \mathscr{L}X \ , \tag{1.1.2}$$

where $\mathsf{T}[-1]X \coloneqq \mathrm{Spec}_X\left(\mathrm{Sym}_{\mathcal{O}_X} \mathbb{L}_X[1]\right)$ is (the total space of) the shifted tangent complex of X. It is equally possible to understand the de Rham differential and compare it with the S^1-action on $\mathscr{L}X$ in this setting, see [58]. The cohomology of the structure sheaf of $\mathsf{T}[-1]X$ computes the right-hand side of (1.1.1); thus the HKR isomorphism (1.1.1) can be obtained from (1.1.2) by taking global functions. Furthermore, in [11] it was also proven that (1.1.2), despite being well-defined for derived Artin stacks, it is typically *not* an equivalence; rather it is an equivalence only on formal completions (along the zero section on the source and at constant loops on the target).

In between schemes and Artin stacks, there are Deligne–Mumford stacks. Once again, the map (1.1.2) cannot be an equivalence in general: already at the truncated level, the right hand side

decomposes into connected components, known as *twisted sectors*, whereas the left hand side only knows information on the principal sector. Intuitively, this phenomenon arises because specifying a map $S^1 \to X$ amounts to choose a point $x \in X$ together with monodromy data at x, i.e. a conjugacy class in the isotropy group of x; when X is a Deligne–Mumford stack, isotropy groups are finite, and monodromy becomes a discrete parameter.

Nevertheless, it was shown by Arinkin–Căldăraru–Hablicsek in [7] that the situation is not as hopeless as it might seem. Restricting their attention to Deligne–Mumford stacks of the form

$$X = [Y/G] \ ,$$

where G is a finite group acting on a smooth (hence underived) scheme Y, they showed that $\mathsf{HH}_*(X)$ can be computed in terms of differential forms on the *inertia stack* of X. Recall that the inertia stack $\mathsf{I}X$ is the self-intersection of the diagonal in the category of classical (i.e. underived) stacks,

$$\mathsf{I}X = X \times^{\mathrm{cl}}_{X \times X} X \ .$$

Note that, since X is underived, the inertia is equivalent to the classical truncation of the free loop space

$$\mathsf{I}X \simeq \mathrm{t}_0(\mathscr{L}X) \ . \tag{1.1.3}$$

Additionally, since X is a global quotient, the inertia can be easily computed via the following formula

$$\mathsf{I}X \simeq \coprod_{g \in G/G} [Y^g/Z(g)] \ . \tag{1.1.4}$$

Here G/G is the set of conjugacy classes of G, Y^g denotes the classical locus of points in Y fixed under the action of g, and $Z(g)$ is the centralizer of g. The main result of [7] is the construction of a canonical equivalence, for X of the special form $[Y/G]$ as above,

$$\mathsf{exp} : \mathsf{T}[-1]\mathsf{I}X \to \mathscr{L}X \ .$$

After linearizing by taking the global sections of the structure sheaf, one obtains the equivalence

$$\mathsf{HH}_{-i}([Y/G]) \simeq \bigoplus_{g \in G/G} \bigoplus_{q-p=i} H^q(Y^g, \Omega^p_{Y^g})^{Z(g)} \ , \tag{1.1.5}$$

which is a generalization of the global HKR theorem for schemes (1.1.1).

1.2. **Main results.** Our main goal is to establish the HKR equivalence for *all* derived DM stacks in characteristic 0. We do not require the stacks to be smooth, or to admit a presentation as global quotients under the action of a finite or even algebraic group. In this respect, our result is new also in the setting of classical DM stacks. Our methods are quite different from the ones in [7], which relied on the construction of a splitting of the derived self-intersection of the diagonal. The extra generality in which we place ourselves in makes that approach insufficient.

In fact, the very definition of the inertia stack of a *derived* DM stack is a non-trivial construction that we propose in this paper. Notice that for general derived DM stacks equivalences (1.1.3) and (1.1.4) are no longer available, and it is therefore a priori unclear how to define $\mathsf{I}X$ at this level of generality. To formulate our definition, let C_r be the cyclic group of order r.

Definition 1.1. Let X be a derived DM stack. The *orbifold inertia* stack of X is defined as

$$\mathsf{I}^{\mathsf{DM}}X \coloneqq \operatorname*{colim}_r \mathbf{Map}(\mathsf{B}\mathsf{C}_r, X) \ ,$$

where **Map** stands for the mapping stack of derived stacks, the colimit is taken in the category of derived stacks and over positive integers with respect to the divisibility relation.

This definition is clearly inspired by the work of Abramovich, Graber and Vistoli [2], and it has been kindly suggested to us by Charanya Ravi. After the first version of this paper became available, the above notion has been considered by Kinjo [29] for Artin stacks as well (under the name of *torsion loop stack*), proving a multiplicative version of the dimensional reduction theorem.

We list below the main features of this construction:

(1) There are isomorphisms of classical truncations:

$$\mathrm{t}_0(\mathsf{I}^{\mathsf{DM}}X) \simeq \mathrm{t}_0(\mathscr{L}X) \simeq \mathsf{I}(\mathrm{t}_0(X)) \ .$$

See Theorem 3.3.

(2) If X is a classical smooth (hence underived) DM stack, then $\mathsf{I}^{\mathsf{DM}}X$ coincides with the classical inertia stack $\mathsf{I}X$, hence it is in particular underived. See Theorem 3.4.

(3) If $X \simeq [Y/G]$ is a derived global quotient DM stack, where G is a finite group acting on a derived scheme Y, there are equivalences

$$\mathsf{I}^{\mathsf{DM}}X \simeq \coprod_{g\in G/G} [Y^g/Z(g)] \simeq \left[\left(\coprod_{g\in G} Y^g\right)/G\right] \ , \tag{1.2.1}$$

where Y^g denotes the *derived* fixed locus (Theorem 6.1) of the automorphism g on Y, and $Z(g)$ is the centralizer of g. Note that the equivalence (1.2.1) generalizes (1.1.4) to global quotient derived DM stacks. See Theorem 6.5.
Moreover, (1.2.1) can be further generalized to derived DM stacks of the form $[Y/G]$ for Y a derived algebraic space and G a linear algebraic group acting on Y. See Theorem 6.18.

(4) We give in Theorem 6.7 an underived but singular DM stack X such that $\mathsf{I}^{\mathsf{DM}}X$ is a nontrivial derived enhancement of $\mathsf{I}X$, showing that the our definition reveals new features even in the classical setting: the deviation of the canonical map $\mathsf{I}X \to \mathsf{I}^{\mathsf{DM}}X$ from being an equivalence contains information on the singularities of classical DM stacks.

The following is our main result:

Theorem 1.2 (HKR for derived DM stacks, Theorem 4.9)**.** *For every derived Deligne–Mumford stack X locally almost of finite presentation over a commutative $\mathbb{Q}$-algebra, there is a canonical equivalence of derived DM stacks over X:*

$$\begin{array}{ccc} \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X & \xrightarrow[\simeq]{\widehat{\mathsf{aff}}^*} & \mathscr{L}X \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

This equivalence is furthermore functorial in X.

Theorem 1.2 recovers the main result of Arinkin–Căldăraru–Hablicsek [7, Theorem 1.15] when X is a classical, smooth, global quotient DM stack, since in that setting, our map $\widehat{\mathsf{aff}}^*$ coincides with the comparison map exp constructed in *loc. cit.* In full generality, the map $\widehat{\mathsf{aff}}^*$ is induced by a natural formality equivalence $\mathsf{C}^*_{\mathsf{sing}}(S^1;k) \simeq H^*(S^1;k)$ as cocommutative Hopf algebras.

Taking global sections of the structure sheaves on both sides of the equivalence in Theorem 1.2, we obtain the following HKR theorem for Hochschild homology, which is a direct generalization of (1.1.1) to derived DM stacks.

Corollary 1.3 (HKR for Hochschild homology, Theorem 4.12 and Theorem 5.8)**.** *Let X be a derived Deligne–Mumford stack locally almost of finite presentation over a commutative $\mathbb{Q}$-algebra. Then we have an equivalence of* algebra objects *in the ∞-category Mod_k:*

$$\mathsf{HH}_*(X) \simeq \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1])) \ . \tag{1.2.2}$$

where the Hochschild homology $\mathsf{HH}_*(X)$ is equipped with its natural algebra structure, and $\Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]))$ is equipped with the natural algebra structure induced from the algebra structure on the symmetric algebra.
Moreover, the equivalence (1.2.2) intertwines the natural S^1-algebra structure on

$$\mathcal{O}(\mathscr{L}X) \simeq \mathsf{HH}_*(X)$$

and the natural mixed structure on

$$\mathcal{O}(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X) \simeq H^*(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]))$$

induced by the de Rham differential.

From Theorem 1.2, we also deduce an HKR theorem for Hochschild cohomology for derived DM stacks:

Corollary 1.4 (HKR for Hochschild cohomology, Theorem 4.15)**.** *Let X be a derived DM stack locally almost of finite presentation over a commutative $\mathbb{Q}$-algebra. There is a canonical equivalence of objects in the ∞-category Mod_k:*

$$\mathsf{HH}^*(X) \simeq \Gamma(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X, q^!(\mathcal{O}_X)) \ ,$$

where $q\colon \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X \to X$ is the natural morphism.

If X is moreover of finite presentation and lci, then

$$\mathsf{HH}^*(X) \simeq \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]) \otimes i^*\omega_X^\vee[-\dim(X)]) \ . \tag{1.2.3}$$

where ω_X is the dualizing sheaf of X, and $i\colon \mathsf{I}^{\mathsf{DM}}X \to X$ is the canonical morphism.

The compatibility of the circle action and the de Rham differential in Theorem 1.3 yields the following HKR theorems for cyclic homology and its variants:

Corollary 1.5 (HKR for cyclic, negative cyclic and periodic cyclic homology, Theorem 5.10)**.** *Let X be a derived DM stack locally almost of finite presentation over a commutative $\mathbb{Q}$-algebra. Let DR be the derived de Rham complex of its orbifold inertia $\mathsf{I}^{\mathsf{DM}}X$. For any integer i, let $\mathsf{DR}^{\geq i} \coloneqq \mathrm{Fil}^i_{\mathsf{Hdg}}\mathsf{DR}$ and $\mathsf{DR}^{\leq i} \coloneqq \mathrm{cofib}(\mathrm{Fil}^i_{\mathsf{Hdg}}\mathsf{DR} \to \mathsf{DR})$, where $\mathrm{Fil}^\bullet_{\mathsf{Hdg}}$ is the Hodge filtration. There are canonical equivalences of objects in the ∞-category Mod_k:*

$$\mathsf{HC}(X) \simeq \bigoplus_{i\geq 0} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}^{\leq i}[2i]\right) \ ;$$

$$\mathsf{HN}(X) \simeq \prod_{i\in\mathbb{Z}} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}^{\geq i}[2i]\right) \simeq \prod_{i\leq 0} H^*_{\mathrm{dR}}(\mathsf{I}^{\mathsf{DM}}X)[2i] \times \prod_{i=1}^{\infty} \left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}^{\geq i}[2i]\right) \ ;$$

$$\mathsf{HP}(X) \simeq \prod_{i\in\mathbb{Z}} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}[2i]\right) \simeq \prod_{i\in\mathbb{Z}} H^*_{\mathrm{dR}}(\mathsf{I}^{\mathsf{DM}}X)[2i] \ .$$

Remark 1.6. In mixed and positive characteristics, the HKR isomorphism fails in general already for schemes; see Remark 4.11 for a fuller discussion of these aspects. So in particular our Theorem 1.2 cannot be expected to hold in that setting. However, some of our results can be formulated in a way that also remains valid in mixed and positive characteristics. A detailed treatment of our constructions in these more general settings will appear in a forthcoming work, where we will extend the construction of the *HKR filtration* on Hochschild homology, which was studied in [35] for affine schemes, to the broader context of derived DM stacks.

1.3. **Applications.** Our results enable computations of the free loop space, hence the Hochschild (co)homology, of DM stacks in a much broader range of examples that were previously inaccessible to existing methods. In Section 6, we work out explicitly some consequences in interesting geometric situations.

First of all, for derived DM stacks which admit a presentation $X \simeq [Y/G]$ as a global quotient of a derived scheme Y by a finite group G, we give an explicit formula in Theorem 6.5 for its orbifold inertia in terms of derived fixed loci; we have already presented this as equation (1.2.1) above. This allows us to prove in Theorem 6.6 the explicit HKR decomposition of the Hochschild (co)homology of $[Y/G]$, generalizing (1.1.5).

In Section 6.3, we turn our attention to DM stacks that cannot be presented as global quotients by finite group actions. In Theorem 6.11, we give a detailed treatment of two simple but instructive examples: Thurston's football and teardrop, two weighted projective lines which cannot be obtained as global quotients by finite groups. More generally, we compute the Hochschild homology and Hochschild cohomology of arbitrary weighted projective lines (Theorem 6.12) and root stacks $\sqrt[n]{(X, D)}$; see Theorem 6.15 and Theorem 6.17. These examples are particularly relevant for us as they fall beyond the scope of the HKR theorems which are currently available in the literature. As such, they show the reach of our new techniques and findings.

Although most DM stacks are not global quotients by finite group actions, many of them are global quotients by linear algebraic groups ([20, Theorem 2.18]), most notably, for moduli stacks constructed via Geometric Invariant Theory. In Section 6.4, we extend the formula (1.1.5) to derived DM stacks of the form $[Y/G]$ with G a linear algebraic group acting on a derived algebraic space Y; see Theorem 6.19.

Finally, Hochschild cohomology of a stack controls its (possibly non-commutative, stacky, and derived) deformations. Therefore our main result on Hochschild cohomology Theorem 1.4 opens the way of studying the non-commutative/stacky/derived deformation theory of some interesting Deligne–Mumford stacks that are not global quotients. To exemplify the great potential in this direction, let us just mention one instance, namely, the moduli stacks of Deligne–Mumford stable curves. Corollary 1.4 implies in particular a formula for Hochschild cohomology conjectured by Okawa and Sano in [37, Question 2.23 and (2.27)]. Thanks to our work, their result [37, Corollary 3.21] becomes unconditional: for any pairs of integers $g \geq 1$ and $n \geq 0$ such that $g + n \geq 5$, the moduli stack of Deligne–Mumford stable curves of genus g with n marked points $\overline{\mathscr{M}}_{g,n}$ satisfies

$$\mathsf{HH}^2(\overline{\mathscr{M}}_{g,n}) = 0.$$

In particular, $\overline{\mathscr{M}}_{g,n}$ does not admit non-commutative deformations, generalizing a theorem of Hacking [23] that $\overline{\mathscr{M}}_{g,n}$ is rigid, i.e., it does not admit (commutative) deformations.

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Conventions. Throughout the paper, k is a commutative $\mathbb{Q}$-algebra. All the derived Deligne–Mumford stacks considered in this paper are assumed to be 1-stacks. Even when we do not state

it explicitly, all derived schemes, algebraic spaces and Deligne–Mumford stacks are assumed to be locally almost of finite presentation.

2. Some results on mapping stacks

In this section we collect some foundational results that we could not locate in the exact form that we needed in the literature. There is an evident overlap of intention with [10, §3.2] and [24, Appendix A], but the results discussed below are slightly more general. We systematically use the theory of tensor products in $\mathbf{Pr}^{\mathrm{L}}_k$, which was not yet fully available at the time [10, 24] were firstly written. We refer the reader to [33, §4.8.1] and to [39, §I.4] for background on this topic.

2.1. **Categorical finiteness properties.** We introduce the following finiteness conditions on (the ∞-categories of quasi-coherent complexes of) derived stacks.

Definition 2.1. Let $F \in \mathsf{dSt}_k$ and write $p\colon F \to \mathrm{Spec}(k)$ for the structural morphism.

(1) The derived stack F is called $\otimes$-*universal* if for any derived stack $X \in \mathsf{dSt}_k$, the canonical morphism
$$\boxtimes\colon \mathsf{QCoh}(F) \otimes_k \mathsf{QCoh}(X) \longrightarrow \mathsf{QCoh}(F \times X)$$
is an equivalence.

(2) The derived stack F is called *categorically quasi-compact* if the functor
$$p_*\colon \mathsf{QCoh}(F) \longrightarrow \mathsf{Mod}_k$$
commutes with filtered colimits.

(3) The derived stack F is called *categorically perfect* if the functor
$$p^*\colon \mathsf{Mod}_k \longrightarrow \mathsf{QCoh}(F)$$
admits a left adjoint p_+.

Example 2.2. Let $F \in \mathsf{dSt}_k$.

(1) If $\mathsf{QCoh}(F)$ is dualizable in $\mathbf{Pr}^{\mathrm{L}}_k$, then F is $\otimes$-universal. Indeed, in this case $\mathsf{QCoh}(F)\otimes_k(-)$ commutes with limits, and therefore both source and target $\boxtimes$ satisfy descent in X and so it is enough to test that it is an equivalence when X is affine. In this case, the claim follows from [10, Proposition 4.13] (applied with $Y = \mathrm{Spec}(k)$).

(2) Let $K \in \mathbf{Spc}$ be an ∞-groupoid, seen as a constant derived stack. Then
$$\mathsf{QCoh}(K) \simeq \mathrm{Fun}(K, \mathsf{Mod}_k)$$
is compactly generated, and for every $A \in \mathsf{dCAlg}_k$ we have
$$\mathsf{QCoh}(K) \otimes_k \mathsf{Mod}_A \simeq \mathrm{Fun}(K, \mathsf{Mod}_A)\ ,$$
as it follows from [33, Proposition 4.8.1.17]. Thus, K is $\otimes$-universal. If K is in addition a compact object in $\mathbf{Spc}$, then the constant derived stack K is categorically quasi-compact.

Lemma 2.3. *Assume that F is categorically quasi-compact, that $\mathsf{QCoh}(F) \simeq \mathrm{Ind}(\mathrm{Perf}(F))$ and that $p_*\colon \mathsf{QCoh}(F) \to \mathsf{Mod}_k$ preserves perfect complexes. Then F is categorically perfect, and moreover the left adjoint p_+ is characterized by the fact that for any $M \in \mathrm{Perf}(F)$, we have*
$$p_+(M) \simeq (p_*(M^\vee))^\vee.$$

Proof. Under these assumptions, projection formula holds, that is, the natural comparison morphism

$$p_*(M) \otimes N \longrightarrow p_*(M \otimes p^*(N))$$

is an equivalence. The same computation of [41, Proposition 7.11] shows that defining p_+ by the given formula on perfect complexes and taking the left Kan extension along $\mathrm{Perf}(F) \to \mathrm{Ind}(\mathrm{Perf}(F))$ produces indeed a left adjoint for p^*. See also [34, Proposition 6.4.5.3]. □

Lemma 2.4. *Let $F \in \mathsf{dSt}_k$ be $\otimes$-universal and categorically quasi-compact. Then for every morphism $f\colon X \to Y$ in dSt_k, the square*

$$\begin{array}{ccc} \mathsf{QCoh}(Y) & \xrightarrow{f^*} & \mathsf{QCoh}(X) \\ \downarrow{\scriptstyle p_Y^*} & & \downarrow{\scriptstyle p_X^*} \\ \mathsf{QCoh}(F \times Y) & \xrightarrow{(\mathrm{id}_F \times f)^*} & \mathsf{QCoh}(F \times X) \end{array}$$

is vertically right adjointable, that is, the natural Beck-Chevalley transformation

$$f^* \circ p_{Y,*} \longrightarrow p_{X,*} \circ (\mathrm{id}_F \times f)^*$$

is an equivalence. If in addition F is categorically perfect, then the above square is also vertically left adjointable.

Proof. Since F is $\otimes$-universal, we find canonical identifications

$$(\mathrm{id}_F \times f)^* \simeq \mathrm{id}_{\mathsf{QCoh}(F)} \otimes f^* \ , \qquad p_X^* \simeq p^* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ , \qquad q_X^* \simeq q^* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ .$$

Since F is categorically quasi-compact, the adjunction $p^* \dashv p_*$ holds in $\mathbf{Pr}^{\mathrm{L}}_k$. The functoriality of the tensor product in $\mathbf{Pr}^{\mathrm{L}}_k$ guarantees therefore that the adjunctions

$$p^* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \dashv p_* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \qquad \text{and} \qquad p^* \otimes \mathrm{id}_{\mathsf{QCoh}(Y)} \dashv p_* \otimes \mathrm{id}_{\mathsf{QCoh}(Y)}$$

hold. The uniqueness of the adjoint therefore provides two more identifications

$$p_{X,*} \simeq p_* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ , \qquad q_{X,*} \simeq q_* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ .$$

At this point, the conclusion is obvious. The same argument shows that if there exists a left adjoint p_+ for p^*, then the adjunction $p_+ \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \dashv p_X^*$ holds as well, and therefore the base change holds as well. □

Notation 2.5. Let

$$\begin{array}{ccc} F & \xrightarrow{\quad f \quad} & G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathrm{Spec}(k) & \end{array}$$

be a diagram in dSt_k. The associated $*$-Beck-Chevalley transformation is the natural transformation

$$q_* \longrightarrow q_* f_* f^* \simeq p_* f^* \ .$$

If furthermore F and G are categorically perfect, the associated $+$-Beck-Chevalley transformation is the natural transformation

$$p_+ f^* \longrightarrow p_+ f^* q^* q_+ \simeq p_+ p^* q_+ \longrightarrow q_+ \ .$$

Lemma 2.6. *Let $f\colon F \to G$ be a morphism in dSt_k. For every $X \in \mathsf{dSt}_k$, let*

$$p_X\colon F \times X \longrightarrow X \ , \qquad q_X\colon G \times X \longrightarrow X$$

be the canonical projections. Assume that:

(1) both F and G are $\otimes$-universal and categorically quasi-compact.

(2) the $*$-Beck-Chevalley transformation associated to f

$$q_* \longrightarrow p_* f^*$$

is an equivalence.

Then for every $X \in \mathsf{dSt}_k$, the triangle

$$\begin{array}{ccc} \mathsf{QCoh}(G \times X) & \xrightarrow{(f \times \mathrm{id}_X)^*} & \mathsf{QCoh}(F \times X) \\ & {\scriptstyle q_X^*} \nwarrow \quad \nearrow {\scriptstyle p_X^*} & \\ & \mathsf{QCoh}(X) & \end{array}$$

is vertically right adjointable. If F and G are moreover categorically perfect, and $+$-Beck-Chevalley transformation associated to f is an equivalence, then the above triangle is vertically left adjointable as well.

Proof. Assumption (1) provides canonical identifications

$$(f \times \mathrm{id}_X)^* \simeq f^* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ , \qquad p_X^* \simeq p^* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ , \qquad q_X^* \simeq q^* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ .$$

As in the proof of Theorem 2.4, we furthermore obtain identifications

$$p_{X,*} \simeq p_* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ , \qquad q_{X,*} \simeq q_* \otimes \mathrm{id}_{\mathsf{QCoh}(X)} \ .$$

Thus, the conclusion follows from (3) and the functoriality of the tensor product in $\mathbf{Pr}_k^{\mathrm{L}}$. The proof of left adjointability follows along the same lines. □

The following proof is completely standard, and well-documented in the literature. See [47, Proposition B.3.5] or [36, Proposition 1.4]. The proof is elementary, so we include it for the convenience of the reader:

Proposition 2.7. *Let $F, X \in \mathsf{dSt}_k$ be derived stacks. Assume that F is $\otimes$-universal and categorically perfect. Assume that X admits a global cotangent complex and is infinitesimally cohesive. Then $\mathbf{Map}(F, X)$ admits a global cotangent complex, given by the formula*

$$\mathbb{L}_{\mathbf{Map}(F,X)} \simeq \pi_+ \mathrm{ev}^*(\mathbb{L}_X) \ ,$$

where $\mathrm{ev} \colon X \times \mathbf{Map}(F, X) \to X$ is the evaluation map and $\pi \colon X \times \mathbf{Map}(F, X) \to \mathbf{Map}(F, X)$ is the canonical projection.

Proof. Let $S \in \mathrm{dAff}_k$ be a test derived affine scheme and fix a morphism $S \to \mathbf{Map}(F, X)$ classifying a morphism $f \colon S \times F \to X$. Let $M \in \mathsf{QCoh}(S)_{\geqslant 0}$ and write $S[M]$ the split square-zero extension determined by M. By definition of mapping stack, the space of liftings

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{Map}(F, X) \\ \downarrow & \nearrow & \\ S[M] & & \end{array}$$

is equivalent to the space of liftings

$$\begin{array}{ccc} S \times F & \xrightarrow{f} & X \\ \downarrow & \nearrow & \\ S[M] \times F & & \end{array} \ .$$

However, $S[M] \times F \simeq (S \times F)[p_S^*(M)]$, where $p_S \colon S \times F \to S$ is the canonical projection. Therefore, the space of such liftings is equivalent to

$$\mathrm{Map}_{\mathsf{QCoh}(S\times F)}(f^*\mathbb{L}_X, p_S^* M) \ . \tag{2.1.1}$$

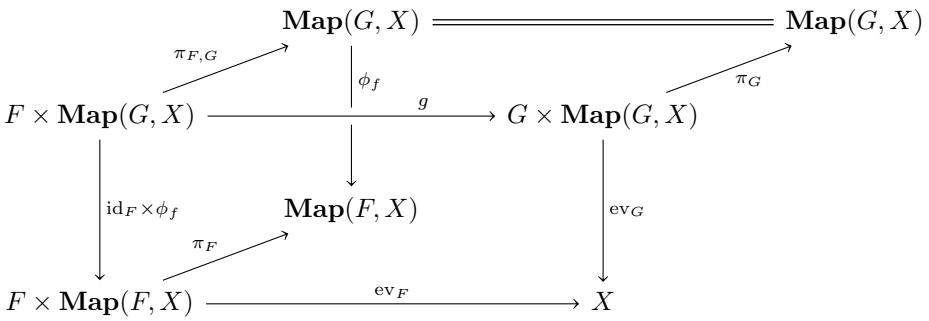
Since p^* admits a left adjoint and F is $\otimes$-universal, the same argument given in Theorem 2.4 implies that p_S^* also admits a left adjoint $p_{S,+}$. This shows that $\mathbf{Map}(F, X)$ admits a cotangent complex at the given point, which is furthermore given by the formula $p_{S,+}f^*(\mathbb{L}_X)$. The base change for the plus pushforward proven in Theorem 2.4 readily implies that $\mathbf{Map}(F, X)$ has a global cotangent complex, given by the claimed formula. □

Variant 2.8. Let $X \in \mathsf{dSt}_k$ be a derived stack and assume that it is infinitesimally cohesive and formally étale over k. Then for any derived stack F, $\mathbf{Map}(F, X)$ admits a global cotangent complex, given by 0. More generally, if $p \colon X \to Y$ is an infinitesimally cohesive and formally étale morphism in dSt_k, then for any derived stack F, the induced morphism $\mathbf{Map}(F, X) \to \mathbf{Map}(F, Y)$ is again formally étale. The same argument given in Theorem 2.7 supplies an identification of the space of derivations with (2.1.1). The formal étaleness assumption now supplies $f^*\mathbb{L}_p \simeq 0$, so the conclusion is automatic.

Construction 2.9. Let $f \colon F \to G$ be a morphism in dSt_k. Assume that F is $\otimes$-universal and that both F and G are categorically perfect. Fix as well $X \in \mathsf{dSt}_k$ admitting a cotangent complex and consider the induced morphism

$$\phi_f \colon \mathbf{Map}(G, X) \longrightarrow \mathbf{Map}(F, X) \ ,$$

which is part of the following commutative diagram:



where we set $g \coloneqq f \times \mathrm{id}_{\mathbf{Map}(G,X)}$. Since F is $\otimes$-universal and categorically perfect, Theorem 2.4 guarantees that the Beck-Chevalley transformation

$$\phi_f^* \circ \pi_{F,+} \longrightarrow \pi_{F,G,+} \circ (\mathrm{id}_F \times \phi_f)^*$$

is an equivalence. Besides, the top square induces a second Beck-Chevalley transformation

$$\mathsf{BC} \colon \pi_{F,G,+} \circ g^* \longrightarrow \pi_{G,+} \ , \tag{2.1.2}$$

which, paired with the above equivalence gives rise to a natural transformation

$$\phi_f^* \circ \pi_{F,+} \circ \mathrm{ev}_F^* \longrightarrow \pi_{G,+} \circ \mathrm{ev}_G^* \ .$$

Evaluating on $\mathbb{L}_X$ gives rise via to the natural comparison map

$$\phi_f^* \mathbb{L}_{\mathbf{Map}(F,X)} \longrightarrow \mathbb{L}_{\mathbf{Map}(G,X)} \ . \tag{2.1.3}$$

Proposition 2.10. *Let $F, G \in \mathsf{dSt}_k$ be $\otimes$-universal and categorically perfect derived stacks. Write $p \colon F \to \mathrm{Spec}(k)$ and $q \colon G \to \mathrm{Spec}(k)$ for the structural morphisms. Let $f \colon F \to G$ be a morphism and assume that the $+$-Beck-Chevalley transformation*

$$p_+ f^* \longrightarrow p_+ f^* q^* q_+ \simeq p_+ p^* q_+ \longrightarrow q_+$$

is an equivalence. Then for every derived stack $X \in \mathsf{dSt}_k$ admitting a cotangent complex, the induced transformation

$$\phi_f \colon \mathbf{Map}(G, X) \longrightarrow \mathbf{Map}(F, X)$$

is formally étale, that is, it admits a cotangent complex which is moreover zero.

Proof. Note that $\mathbb{L}_{\phi_f}$ is canonically identified with the cofiber of the map (2.1.3). It is therefore enough to prove that (2.1.3) is an equivalence. Tracing the definition given in Theorem 2.9, and applying Theorem 2.7, we see that if the Beck-Chevalley transformation (2.1.2) is an equivalence, then the same goes for (2.1.3). Since F and G are $\otimes$-universal and categorically perfect, the conclusion follows directly from Theorem 2.6. □

2.2. **Étale codescent.** The second goal of this section is to prove the following codescent property:

Theorem 2.11. *Let G be a connected and underived algebraic group. Let X be a derived Deligne–Mumford stack and let $X_{\text{ét}}$ be the small étale site of X. Then the functor*

$$\mathbf{Map}(\mathsf{B}G, -) \colon X_{\text{ét}} \longrightarrow \mathsf{dSt}_k$$

satisfies étale codescent.

Remark 2.12. The assumption that the source of the mapping stack is of the form $\mathsf{B}G$ with G connected cannot be easily weakened. For instance, $\mathbf{Map}(S^1, -)$ only satisfies codescent with respect to *representable* étale covers.

The proof of Theorem 2.11 heavily relies on the following technical result, that is also a key ingredient in the proof of Theorem 4.8 later in the paper:

Lemma 2.13. *Let G be a connected and underived algebraic group. For any underived affine scheme $S \in \mathrm{Aff}_k$ and any derived Deligne–Mumford stack X, the canonical morphism*

$$\mathrm{Map}(S \times \mathsf{B}G, X) \longrightarrow \mathrm{Map}(S, X) \tag{2.2.1}$$

induced by precomposition with $S \to S \times \mathsf{B}G$, is an equivalence.

Proof. Observe that, since S and $S \times \mathsf{B}G$ are underived, we can replace X by $\mathrm{t}_0(X)$, or, equivalently, assume that X is underived from the very beginning.

Since (2.2.1) is clearly surjective on π_0, it suffices to argue that it has contractible fibers. Fix therefore $x \colon S \to X$ a morphism. The fiber of (2.2.1) at x is described as the fiber of

$$\mathrm{Map}_{\mathsf{St}_{S/}}(S \times \mathsf{B}G, X) \longrightarrow \mathrm{Map}_{\mathsf{St}_{S/}}(S, X) \ ,$$

or equivalently as fiber of

$$\mathrm{Map}_{\mathsf{St}_{S/\!\!/S}}(S \times \mathsf{B}G, S \times X) \longrightarrow \mathrm{Map}_{\mathsf{St}_{S/\!\!/S}}(S, S \times X) \ . \tag{2.2.2}$$

Following [33, Example 5.2.6.13], we find an equivalence

$$S \times \mathsf{B}G \simeq \mathsf{B}_S(S \times G) \simeq \mathsf{Bar}^{(1)}(S \times G) \ ,$$

where the bar construction is performed in the ∞-topos $\mathsf{St}_{/S}$ of *underived* stacks over S. Similarly, S can be seen as the bar construction of the trivial group over S. Combining [33, Notation 5.2.6.11 & Remark 5.2.6.12], we see that

$$\mathsf{Bar}^{(1)} \colon \mathsf{Mon}_{\mathbb{E}_1}(\mathsf{St}_{S/\!\!/S}) \longrightarrow \mathsf{St}_{S/\!\!/S}$$

admits $\mathsf{Cobar}^{(1)}(-) \simeq \Omega^1(-)$ as a right adjoint. Here $\Omega^1(-)$ denotes the based loop space functor in $\mathsf{St}_{S/\!\!/S}$. We can therefore rewrite the map (2.2.2) as

$$\mathrm{Map}_{\mathsf{Mon}_{\mathbb{E}_1}(\mathsf{St}_{/S})}(S \times G, \Omega^1(S \times X)) \longrightarrow \mathrm{Map}_{\mathsf{Mon}_{\mathbb{E}_1}(\mathsf{St}_{/S})}(S, \Omega^1(S \times X)) \ ,$$

induced by the unit section $S \to S \times G$, seen as a morphism of groups. Notice that these mapping spaces are discrete. Thus, in order to conclude the argument it is therefore enough to argue that any morphism of S-groups

$$S \times G \longrightarrow \Omega^1(S \times X)$$

factors through the unit section of $\Omega^1(S \times X)$.

Observe now that

$$\Omega^1(S \times X) \coloneqq S \times_{S \times X} S \simeq S \times_{S \times X} (\mathsf{I}X \times S) \ ,$$

and that the unit section of this S-group is the pullback of the diagonal embedding

$$X \times S \longrightarrow \mathsf{I}X \times S \ .$$

Since X is a Deligne–Mumford stack, the diagonal $X \to X \times X$ is unramified. This implies that the map $X \to \mathsf{I}X$ is an open immersion, see [53, Tag 02GE]. It follows that the unit section $S \to \Omega^1(S \times X)$ is an open immersion. In particular, a morphism $S \times G \to \Omega^1(S \times X)$ factors through the unit section if and only if it factors topologically. In order to check this latter statement, it is enough to assume that S is the spectrum of a field. In this case, since G is connected by assumption, and the morphism is assumed to be a morphism of groups, it must factor through the connected component of the identity in $\Omega^1(S \times X)$, which is reduced to the unit element itself. □

Proof of Theorem 2.11. Since the functor $\mathbf{Map}(\mathsf{B}G, -)$ commutes with limits, it commutes also with the formation of Čech nerves. In particular, it suffices to show that if $f \colon U \to V$ is an étale epimorphism, the same goes for

$$\phi_f \colon \mathbf{Map}(\mathsf{B}G, U) \longrightarrow \mathbf{Map}(\mathsf{B}G, V) \ .$$

First, we observe that Theorem 2.8 implies that ϕ_f is formally étale. It immediately follows that ϕ_f is an effective epimorphism if and only if its truncation $\mathrm{t}_0(\phi_f)$ is. Besides, since G is underived, for any derived stack Y we have a canonical identification

$$\mathrm{t}_0\mathbf{Map}(\mathsf{B}G, Y) \simeq \mathrm{t}_0\mathbf{Map}(\mathsf{B}G, \mathrm{t}_0(Y)) \ .$$

We can therefore assume from the very beginning that X (and hence U and V) is underived.

Fix an affine test scheme $S \in \mathrm{Aff}_k$, and a morphism $S \to \mathbf{Map}(\mathsf{B}G, V)$, corresponding to a morphism $S \times \mathsf{B}G \to V$. We have to show that there exists an étale cover $S' \to S$ such that the restriction

$$S' \times \mathsf{B}G \longrightarrow S \times \mathsf{B}G \longrightarrow V$$

factors through $f \colon U \to V$. It follows from Theorem 4.8 that

$$\mathrm{Map}(S \times \mathsf{B}G, V) \longrightarrow \mathrm{Map}(S, V)$$

is an equivalence, and similarly for U in place of V. Thus, the conclusion follows from the fact that $f \colon U \to V$ was an effective epimorphism to begin with. □

3. Orbifold inertia for derived Deligne-Mumford stacks

Let X be a derived Deligne–Mumford stack (see for example [38] for the basic definition). We emphasize that in this work we only consider 1-stacks. Also, as we stated in the Convention section, all DM stacks are assumed to be locally almost of finite presentation. We typically think of X as a derived stack, but we will occasionally use its representation as a structured ∞-topos.

For an underived Deligne-Mumford stack X, it is customary to define the inertia stack of X as

$$\mathsf{I}X \coloneqq X \times_{X \times X} X \ ,$$

where the fiber product is taken in the category St_k of underived stacks. It can equivalently be realized as the (underived) mapping stack from the constant stack associated to $S^1 \in \mathbf{Spc}$:

$$\mathsf{I}X \simeq \mathbf{Map}_{\mathsf{St}_k}(S^1, X) \ . \tag{3.0.1}$$

The very same constructions performed in dSt_k yields the free loop space $\mathscr{L}X$:

$$\mathscr{L}X \simeq \mathbf{Map}(S^1, X) \simeq X \times^d_{X\times X} X \ , \tag{3.0.2}$$

where the fiber product is taken in the category of derived stacks. To generalize the HKR theorem, one needs to introduce a meaningful construction of the inertia stack.

3.1. **(Derived) orbifold inertia.** Our definition is motivated by the work of Abramovich–Graber–Vistoli [1] and has been suggested to us by Charanya Ravi, to whom we are very grateful. For every integer $r > 0$, let C_r be the cyclic group of order r, seen as a constant group scheme over $\mathrm{Spec}(k)$. When r divides r', there is a canonical group homomorphism $\mathsf{C}_{r'} \twoheadrightarrow \mathsf{C}_r$, and we set

$$\widehat{\mathbb{Z}} \coloneqq \text{“}\lim_r\text{”}\, \mathsf{C}_r \in \mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_1}\big(\mathrm{Pro}(\mathsf{dSt}_k)\big) \ .$$

Passing to the classifying stack, we also set

$$\widehat{S}^1 \coloneqq \text{“}\lim_r\text{”}\, \mathsf{BC}_r \in \mathrm{Pro}(\mathsf{dSt}_k) \ .$$

Notice that the canonical homomorphisms $\mathbb{Z} \twoheadrightarrow \mathsf{C}_r$ induce at the level of classifying stacks the map

$$\varpi_r \colon S^1 \longrightarrow \mathsf{BC}_r \ ,$$

and all these maps assemble into a morphism

$$\varpi \colon S^1 \longrightarrow \widehat{S}^1 \ . \tag{3.1.1}$$

Definition 3.1 (Orbifold inertia)**.** Let $X \in \mathsf{dSt}_k$ be a derived stack.

(1) The *r-th orbifold inertia of X* is the mapping stack

$$\mathsf{I}^{(r)}X \coloneqq \mathbf{Map}(\mathsf{BC}_r, X) \ .$$

(2) The *orbifold inertia of X* is the mapping stack

$$\mathsf{I}^{\mathsf{DM}}X \coloneqq \mathbf{Map}(\widehat{S}^1, X) \ .$$

Unraveling the definitions, we see that

$$\mathsf{I}^{\mathsf{DM}}X \simeq \operatorname*{colim}_r \mathsf{I}^{(r)}X \ ,$$

where the filtered colimit is computed in dSt_k.

Notation 3.2. The maps ϖ and ϖ_r induce well-defined morphisms

$$\iota_\varpi \colon \mathsf{I}^{\mathsf{DM}}X \longrightarrow \mathscr{L}X \qquad \text{and} \qquad \iota_r \colon \mathsf{I}^{(r)}X \longrightarrow \mathscr{L}X \ .$$

We also let

$$\pi \colon \mathscr{L}X \longrightarrow X \ , \qquad \pi^{\mathsf{DM}} \colon \mathsf{I}^{\mathsf{DM}}X \longrightarrow X \ , \qquad \pi_r \colon \mathsf{I}^{(r)}X \longrightarrow X$$

be the natural projections.

The main result of this section is the following properties of $\mathsf{I}^{\mathsf{DM}}X$:

Theorem 3.3. *Assume that X is a derived 1-Artin stack locally almost of finite presentation. Then the following holds.*

(1) For every positive integer r, the map

$$\iota_r \colon \mathsf{I}^{(r)}X \longrightarrow \mathscr{L}X$$

is a closed immersion. In particular, $\mathsf{I}^{(r)}X$ is a derived Artin stack.

(2) *If the cotangent complex of X is perfect (resp. has tor-amplitude in $[a,b]$), the same holds for the cotangent complex of $\mathsf{I}^{(r)}X$. In particular, if X is smooth (resp. lci [1]), the same holds for $\mathsf{I}^{(r)}X$.*

(3) *For positive integers $r \mid r'$, the transition map $\mathsf{I}^{(r)}X \to \mathsf{I}^{(r')}X$ is an open and closed immersion.*

Assume furthermore that X is a derived Deligne–Mumford stack locally almost of finite presentation. Then:

(4) *$\mathsf{I}^{\mathsf{DM}}X$ is a derived Deligne–Mumford stack locally almost of finite presentation and the map*
$$\iota_{\varpi} : \mathsf{I}^{\mathsf{DM}}X \longrightarrow \mathscr{L}X$$
induces an isomorphism on the truncation: $\mathrm{t}_0\mathsf{I}^{\mathsf{DM}}X \simeq \mathrm{t}_0\mathscr{L}X \simeq \mathsf{I}(\mathrm{t}_0(X))$.

Corollary 3.4. *Let X be a smooth Deligne–Mumford stack locally almost of finite presentation. Then $\mathsf{I}^{\mathsf{DM}}X$ is also smooth. In particular, it is underived and therefore it coincides with*
$$\mathsf{I}X = X \times_{X\times X} X \ ,$$
where the fiber product is computed in the category of underived *stacks St_k.*

Proof. The smoothness follows by combining items (2) and (3) of Theorem 3.3. The fact that it coincides with the usual inertia stack follows from item (4). □

Remark 3.5. Theorem 3.3 recovers the considerations of [2, §3.1] in the derived setting. The proof below might seem weirdly long compared to the arguments given in *loc. cit.* The ultimate reason is that whereas in the classical setting a point of $\mathsf{I}^{(r)}X$ is the datum of a pair (x,g) where x is a point of X and g is an automorphism of x whose order divides r, it is actually challenging to obtain a similar description of $\mathsf{I}^{(r)}X$ in the derived setting. This difficulty can be precisely quantified in terms of a basic computation in homotopy theory, see Theorem 3.12 below.

Remark 3.6. In general, when X is underived but singular, $\mathsf{I}^{\mathsf{DM}}X$ can be a non-trivial derived enhancement of $\mathsf{I}X$. See Theorem 6.7.

3.2. **Deformation theory of orbifold inertia.** We start by some general considerations on the cotangent complex of mapping stacks. The material covered here will be crucial in the proof of Theorem 3.3 (and especially item (4) of that theorem), but it will also be used later on in the proof of the HKR theorem.

For the finiteness conditions appearing in the following lemma, see Section 2.

Lemma 3.7. *For a positive integer r, the stack BC_r is $\otimes$-universal, categorically quasi-compact and categorically perfect.*

Proof. Following [12, Theorem 5.4] (see also [14, Theorem 5.25]), we obtain a canonical decomposition
$$\mathsf{QCoh}(\mathsf{BC}_r) \simeq \prod_{\zeta\in\mu_r} \mathsf{QCoh}_{\zeta}(\mathsf{BC}_r) \ , \tag{3.2.1}$$
and moreover we have canonical identifications $\mathsf{QCoh}_{\zeta}(\mathsf{BC}_r) \simeq \mathsf{Mod}_k$. It follows that $\mathsf{QCoh}(\mathsf{BC}_r)$ is compactly generated, and therefore BC_r is $\otimes$-universal. It is also categorically quasi-compact, as the pushforward is canonically identified with taking the component indexed by $\zeta = 1$ in

[1] In this paper, the terminology *lci* stands for *derived lci*, a.k.a. *quasi-smooth*, which means that the tor-amplitude of the cotangent complex belongs to $[-1,1]$. Notice that an underived stack X is derived lci if and only if X is lci in the classical sense, see for instance [42, Lemma 2.4] (whose proof works verbatim in the algebraic setting).

the decomposition (3.2.1). Finally, this operation preserves perfect complexes, and therefore we deduce that BC_r is categorically perfect as a consequence of Theorem 2.3. □

Lemma 3.8. *Let r and r' be positive integers such that r divides r', and let $f\colon \mathsf{BC}_{r'} \to \mathsf{BC}_r$ be the natural homomorphism. Let $p\colon \mathsf{BC}_{r'} \to \mathrm{Spec}(k)$ and $q\colon \mathsf{BC}_r \to \mathrm{Spec}(k)$ be the structural morphisms. Then the natural $+$-Beck-Chevalley transformation*

$$p_+ \circ f^* \longrightarrow q_+$$

is an equivalence.

Proof. Under the character decomposition (3.2.1) for $\mathsf{QCoh}(\mathsf{BC}_r)$, the functor f^* correspond to the map

$$\prod_{\mu_r} \mathsf{Mod}_k \longrightarrow \prod_{\mu_{r'}} \mathsf{Mod}_k$$

induced by the inclusion $\mu_r \subset \mu_{r'}$. Given $\zeta \in \mu_r$, write k_ζ for the element $k \in \mathsf{Mod}_k \simeq \mathsf{QCoh}_\zeta(\mathsf{Mod}_k)$. We therefore have $f^*(k_\zeta) \simeq k_\zeta$, and it suffices at this point to observe that

$$q_+(k_\zeta) \simeq q_*(k_{\zeta^{-1}})^\vee \simeq \begin{cases} k & \text{if } \zeta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Proposition 3.9. *Let X be a derived Artin stack. Then $\mathsf{I}^{(r)}X$ is infinitesimally cohesive, nilcomplete and admits a cotangent complex. Moreover:*

(1) if $\mathbb{L}_X$ is perfect of tor-amplitude $[a, b]$, the same holds for $\mathbb{L}_{\mathsf{I}^{(r)}X}$;

(2) if $r \mid r'$, the induced morphism $\mathsf{I}^{(r)}X \to \mathsf{I}^{(r')}X$ is formally étale.

Proof. Since X is Artin, it is itself infinitesimally cohesive and nilcomplete. Therefore, the same holds for $\mathbf{Map}(F, X)$ for any derived stack F, in particular for $\mathsf{I}^{(r)}X$. The existence of the cotangent complex follows by combining Theorem 3.7 with Theorem 2.7. Again by Theorem 2.7, the cotangent complex of $\mathsf{I}^{(r)}X$ is computed by the formula

$$\mathbb{L}_{\mathsf{I}^{(r)}X} \simeq \pi_+ \mathrm{ev}^*(\mathbb{L}_X).$$

Since π_+ consists of taking the dual of C_r-invariants (and since in characteristic zero this operation is t-exact), we see that point (1) holds. As for statement (2), it follows from the combination of Theorem 3.8 and Theorem 2.10. □

3.3. **A computation in homotopy theory.** We will reduce the proof of Theorem 3.3 to a computation in homotopy theory. We start by introducing some notations.

Notation 3.10. Let $r \in \mathbb{Z}$ be an integer. We denote by

$$\mathsf{cov}_r\colon S^1 \to S^1$$

the standard degree-r covering map, corresponding to the element $r \in \mathbb{Z} \simeq \pi_1(S^1)$. We write

$$\Gamma_r \coloneqq \mathrm{cofib}(\mathsf{cov}_r) \in \mathbf{Spc}\ .$$

Notice that the commutative square

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{r} & \mathbb{Z} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{C}_r \end{array}$$

of groups induces passing to classifying stacks a commutative square

$$\begin{array}{ccc} S^1 & \xrightarrow{\mathsf{cov}_r} & S^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathsf{B}\mathsf{C}_r \end{array} ,$$

and therefore a canonical comparison map

$$\gamma_r \colon \Gamma_r \longrightarrow \mathsf{B}\mathsf{C}_r \ .$$

Example 3.11. For $r = 2$, $\Gamma_2 \simeq \mathbb{RP}^2$, $\mathsf{B}\mathsf{C}_2 \simeq \mathbb{RP}^\infty$ and the map γ_2 is the natural inclusion.

Proposition 3.12.

(1) The space Γ_r admits the bouquet $(S^2)^{\vee r-1}$ as a universal cover.

(2) The map γ_r exhibits $\mathsf{B}\mathsf{C}_r$ as the 1-Postnikov truncation of Γ_r.

Proof. We represent S^1 as the complex circle, and we compute Γ_r as the strict pushout of the diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{z \mapsto z^r} & S^1 \\ \downarrow & & \downarrow \\ \mathbb{D} & \longrightarrow & \Gamma_r \end{array} ,$$

where $\mathbb{D} \coloneqq \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. We can alternatively describe Γ_r as follows. Let B_r be the strict pushout

$$\begin{array}{ccc} S^1 \times \mu_r & \longrightarrow & \mathbb{D} \times \mu_r \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & B_r \end{array}$$

There is a canonical action of μ_r on B_r given by

$$\zeta \cdot (z, \zeta') \coloneqq (\zeta \cdot z, \zeta\zeta') \ ,$$

and Γ_r coincides with the quotient of B_r by this action. Furthermore, this action is free and properly discontinuous and therefore the quotient map

$$B_r \longrightarrow \Gamma_r$$

is a covering map. Finally, it is easy to see that B_r is geometrically the gluing of r copies of $\mathbb{D}$ along their boundaries and hence homotopically equivalent to the bouquet of $r-1$ copies of S^2. This proves statement (1). It also implies that B_r is connected and

$$\pi_n(B_r) \simeq \begin{cases} \mathsf{C}_r & \text{if } n = 1 \\ \pi_n((S^2)^{\vee r-1}) & \text{if } n \geqslant 2 \ . \end{cases}$$

Statement (2) follows at once. □

Precomposition with cov_r induces for every $F \in \mathsf{dSt}_k$ a well-defined operation

$$(-)^r \colon \mathscr{L}F \longrightarrow \mathscr{L}F$$

over F. Write

$$\delta_F \colon F \longrightarrow \mathscr{L}F$$

for the relative diagonal.

Corollary 3.13. *Assume that F is a 1-Artin derived stack. Then the square*

$$\begin{array}{ccc} \mathsf{I}^{(r)}F & \longrightarrow & \mathscr{L}F \\ \downarrow & & \downarrow{\scriptstyle (-)^r} \\ F & \xrightarrow{\delta_F} & \mathscr{L}F \end{array}$$

is a pullback square on truncations.

Proof. Unraveling the definitions, we see that for every $F \in \mathsf{dSt}_k$, the square

$$\begin{array}{ccc} \mathbf{Map}(\Gamma_r, F) & \longrightarrow & \mathscr{L}F \\ \downarrow & & \downarrow{\scriptstyle (-)^r} \\ F & \xrightarrow{\delta_F} & \mathscr{L}F \end{array}$$

is a pullback square. Since $\mathrm{t}_0 \colon \mathsf{dSt}_k \to \mathsf{St}_k$ preserves limits, it is therefore enough to argue that the map

$$\mathsf{I}^{(r)}F = \mathbf{Map}(\mathsf{BC}_r, F) \longrightarrow \mathbf{Map}(\Gamma_r, F)$$

becomes an equivalence after applying t_0. Equivalently, we have to check that for every underived test affine scheme $S \in \mathrm{Aff}_k$, the map

$$\mathbf{Map}(\mathsf{BC}_r, F)(S) \longrightarrow \mathbf{Map}(\Gamma_r, F)(S)$$

is an equivalence. Since both BC_r and Γ_r are the constant stacks, we can identify this map with the map

$$\mathrm{Map}_{\mathbf{Spc}}(\mathsf{BC}_r, F(S)) \longrightarrow \mathrm{Map}_{\mathbf{Spc}}(\Gamma_r, F(S))$$

induced by $\gamma_r \colon \Gamma_r \to \mathsf{BC}_r$. Since F is a 1-Artin stack and S is underived, $F(S)$ is a 1-groupoid. Therefore, the conclusion follows directly from the fact that BC_r coincides with the 1-Postnikov truncation of Γ_r, as proven in Theorem 3.12-(2). □

3.4. **Proof of Theorem 3.3.** We are now ready for the proof of the main theorem of this section. We start by collecting the consequences of the results obtained so far.

Corollary 3.14. *Let X be a 1-Artin stack locally almost of finite presentation and with separated diagonal. Then:*

(1) for every r the map

$$\iota_r \colon \mathsf{I}^{(r)}X \longrightarrow \mathscr{L}X$$

is a closed immersion.

(2) the derived stack $\mathsf{I}^{(r)}X$ is 1-Artin and locally almost of finite presentation.

(3) For any positive integers $r \mid r'$, the induced morphism

$$\mathsf{I}^{(r)}X \longrightarrow \mathsf{I}^{(r')}X$$

is an open and closed immersion, and therefore $\mathsf{I}^{\mathsf{DM}}X$ is a derived 1-Artin stack locally almost of finite presentation.

Proof. For (1), it is enough to prove that ι_r is a closed immersion after passing to classical truncations. Thanks to Theorem 3.13, it is enough to argue that $\delta_X \colon X \to \mathscr{L}X$ is a closed immersion, which immediately follows from the assumption that the diagonal of X is separated. In particular, $\mathrm{t}_0(\mathsf{I}^{(r)}X)$ is a 1-Artin stack of finite presentation (in the underived sense). Point (2) follows combining Theorem 3.9 with Lurie's representability theorem. As for point (3), we already know as a consequence of (1) that the morphism in question is a closed immersion. Theorem 3.9 implies that it is formally étale, and therefore that it must be an open immersion as well. □

We complete the proof of Theorem 3.3 by showing the following:

Proposition 3.15. *Let X be a Deligne–Mumford stack locally almost of finite presentation. Then the map*

$$\iota_r \colon \mathsf{I}^{(r)} X \longrightarrow \mathscr{L} X$$

induces an open immersion on truncations. Besides, the morphism

$$\iota_{\varpi} \colon \mathsf{I}^{\mathsf{DM}} X \longrightarrow \mathscr{L} X$$

is an equivalence on truncations.

Proof. First observe that

$$\mathrm{t}_0(\mathscr{L} X) \simeq \mathrm{t}_0(\mathscr{L}(\mathrm{t}_0(X)) \simeq \mathsf{I}(\mathrm{t}_0(X)) \ .$$

To prove the first statement, we can therefore assume from the very beginning that X is underived. Thanks to Theorem 3.13 it is sufficient to argue that $\delta_X \colon X \to \mathrm{t}_0(\mathscr{L} X)$ is an open immersion. As already remarked in the proof of Theorem 2.13, since X is a Deligne–Mumford stack, the diagonal $X \to X \times X$ is unramified and therefore the map $X \to \mathsf{I}X$ is an open immersion, see [53, Tag 02GE].

As for the second statement, it suffices to argue that ι_{ϖ} is an effective epimorphism. This follows directly since any element of the stabilizers of X has finite order. □

4. HKR isomorphism for derived Deligne–Mumford stacks

In this section we prove the HKR theorem for derived Deligne–Mumford stacks. The strategy consists of two steps. First, we consider the natural map

$$S^1 \longrightarrow \mathsf{B}\mathbb{G}_a \times \widehat{S}^1,$$

and prove that when applying the mapping stacks against a derived Deligne–Mumford stack X locally almost of finite presentation, it gives rise to an equivalence

$$\mathbf{Map}(\mathsf{B}\mathbb{G}_a \times \widehat{S}^1, X) \xrightarrow{\simeq} \mathbf{Map}(S^1, X) \simeq \mathscr{L} X \ .$$

Then, we conclude by establishing a canonical identification

$$\mathbf{Map}(\mathsf{B}\mathbb{G}_a \times \widehat{S}^1, X) \simeq \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}} X \ ,$$

supplied by Theorem 4.8 below.

4.1. Unipotent loops.

Let k be a commutative $\mathbb{Q}$-algebra.

Proposition 4.1. *The derived stack $\mathsf{B}\mathbb{G}_a$ is perfect in the sense of* [10, Definition 3.2]. *In particular, the stable ∞-category $\mathsf{QCoh}(\mathsf{B}\mathbb{G}_a)$ satisfies*

$$\mathsf{QCoh}(\mathsf{B}\mathbb{G}_a) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathsf{B}\mathbb{G}_a)) \ .$$

In other words, perfect complexes on $\mathsf{B}\mathbb{G}_a$ are compact, and they generate the whole $\mathsf{QCoh}(\mathsf{B}\mathbb{G}_a)$.

Proof. This is a particular case of [10, Corollary 3.22]. □

Corollary 4.2. *The derived stack $\mathsf{B}\mathbb{G}_a$ is $\otimes$-universal and categorically quasi-compact.*

Proof. The $\otimes$-universality follows from Theorem 2.2-(1), while categorical quasi-compactness follows from the fact that $\mathcal{O}_{\mathsf{B}\mathbb{G}_a}$ is perfect, and therefore compact in $\mathsf{QCoh}(\mathsf{B}\mathbb{G}_a) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathsf{B}\mathbb{G}_a))$ thanks to Theorem 4.1. □

Recollection 4.3. There is a natural group morphism

$$\mathbb{Z} \longrightarrow \mathbb{G}_a \ ,$$

which induces, after applying the delooping functor B, a morphism

$$\mathsf{aff}\colon S^1 \longrightarrow \mathsf{B}\mathbb{G}_a \ . \tag{4.1.1}$$

Write $p\colon S^1 \to \mathrm{Spec}(k)$ and $q\colon \mathsf{B}\mathbb{G}_a \to \mathrm{Spec}(k)$ for the structural morphisms. Since we are in characteristic zero, [56, Lemma 2.2.5] implies that the natural comparison map

$$q_*(\mathcal{O}_{\mathsf{B}\mathbb{G}_a}) \longrightarrow p_*(\mathcal{O}_{S^1})$$

is an equivalence. In particular, aff exhibits $\mathsf{B}\mathbb{G}_a$ as the affinization of S^1.

Lemma 4.4. *With the notations of Theorem 4.3, the canonical comparison map*

$$q_*(\mathcal{F}) \longrightarrow p_*(\mathsf{aff}^*(\mathcal{F}))$$

is an equivalence for every eventually coconnective $\mathcal{F} \in \mathsf{QCoh}(\mathsf{B}\mathbb{G}_a)$. *In particular, it is an equivalence on* $\mathrm{Perf}(\mathsf{B}\mathbb{G}_a)$.

Proof. Write

$$A \coloneqq \Gamma(\mathsf{B}\mathbb{G}_a, \mathcal{O}_{\mathsf{B}\mathbb{G}_a}) \ .$$

It follows from [56, Lemme 2.2.5] that $\mathsf{B}\mathbb{G}_a \simeq \mathsf{cSpec}(A)$. As in [32, §4.5], we have a natural cocontinuous symmetric monoidal functor

$$\theta\colon \mathsf{Mod}_A \longrightarrow \mathsf{QCoh}(\mathsf{B}\mathbb{G}_a) \ ,$$

Since both $\mathsf{B}\mathbb{G}_a$ and S^1 are categorically quasi-compact (in virtue of Theorem 4.2 and Theorem 2.2-(2), respectively), it follows that both source and target of the Beck-Chevalley transformation $q_* \to p_* \circ \mathsf{aff}^*$ commute with arbitrary colimits, since in a stable category commuting with filtered colimits implies commuting with all colimits. Furthermore, since k has characteristic zero, we see that statement is true for $\mathcal{F} \coloneqq \mathcal{O}_{\mathsf{B}\mathbb{G}_a} \simeq \theta(A)$. Since A is a generator for Mod_A, it follows that the result holds for every $\mathcal{F}$ in the essential image of θ. Recall now from [32, Proposition 4.5.2] that θ is t-exact and that it restricts to an equivalence

$$(\mathsf{Mod}_A)_{\leqslant n} \simeq \mathsf{QCoh}(\mathsf{B}\mathbb{G}_a)_{\leqslant n}$$

where we are using homological indexing conventions. This immediately implies the first statement, and the second one follows from the fact that since $\mathsf{B}\mathbb{G}_a$ is smooth and quasi-compact, every perfect complex is eventually coconnective. □

Proposition 4.5. *The derived stack* $\mathsf{B}\mathbb{G}_a$ *is categorically perfect.*

Proof. We keep writing $q\colon \mathsf{B}\mathbb{G}_a \to \mathrm{Spec}(k)$ for the structural morphism. Since

$$\mathsf{QCoh}(\mathsf{B}\mathbb{G}_a) \simeq \mathrm{Ind}(\mathrm{Perf}(\mathsf{B}\mathbb{G}_a))$$

by Theorem 4.1 and since $\mathsf{B}\mathbb{G}_a$ is categorically quasi-compact by Theorem 4.2, Theorem 2.3 shows that it suffices to prove that

$$q_*\colon \mathsf{QCoh}(\mathsf{B}\mathbb{G}_a) \longrightarrow \mathsf{Mod}_k$$

preserves perfect complexes. Let therefore $\mathcal{F} \in \mathrm{Perf}(\mathsf{B}\mathbb{G}_a)$. By Theorem 4.4, we know that the natural comparison morphism

$$q_*(\mathcal{F}) \longrightarrow p_*\mathsf{aff}^*(\mathcal{F})$$

is an equivalence. It is then enough to observe that $\mathcal{G} \coloneqq \mathsf{aff}^*(\mathcal{F})$ can be identified with a functor $G\colon S^1 \to \mathrm{Perf}_k$, and $p_*(\mathcal{G})$ is identified with the limit of G. Since S^1 is a compact object in **Spc**, the limit remains perfect, whence the conclusion. □

Remark 4.6. Since $\mathsf{B}\mathbb{G}_a$ is $\otimes$-universal, the functoriality of the tensor product of ∞-categories shows that Theorem 4.5 recovers the integration map for $\mathsf{B}\mathbb{G}_a$ of [36, §5.3].

4.2. **Shifted tangents.** Let X be a derived Artin stack. For every integer $n \in \mathbb{Z}$ we set

$$\mathsf{T}[n]X \coloneqq \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[-n])) \ .$$

Since

$$\begin{array}{ccc} \mathbb{L}_X[-n] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}_X[-n+1] \end{array}$$

is a pushout in $\mathsf{QCoh}(X)$, we see that

$$\begin{array}{ccc} \mathsf{T}[n-1]X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathsf{T}[n]X \end{array}$$

is a pullback square in dSt_k, where both maps $X \to \mathsf{T}[n]X$ are the inclusion of the zero section.

Consider the ordinary split square-zero extension

$$k[\varepsilon] \coloneqq k \oplus k\varepsilon \ ,$$

with $\varepsilon^2 = 0$. We set

$$\mathbb{D}_0 \coloneqq \mathrm{Spec}(k[\varepsilon]) \ .$$

For every derived Artin stack X, we have a canonical equivalence

$$\mathbf{Map}(\mathbb{D}_0, X) \simeq \mathsf{T}X \ ,$$

see for instance [57, Proposition 1.4.1.6]. Define $\mathbb{D}_{-1}$ as the following pushout in dSt_k:

$$\begin{array}{ccc} \mathbb{D}_0 & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathbb{D}_{-1} \ . \end{array}$$

Remark 4.7. In fact, it follows from [36, Proposition 5.11] that there is a canonical identification $\mathbb{D}_{-1} \simeq \mathsf{B}\widehat{\mathbb{G}}_{\mathsf{a}}$.

It immediately follows from this discussion that

$$\mathbf{Map}(\mathbb{D}_{-1}, X) \simeq X \times_{\mathsf{T}X} X \simeq \mathsf{T}[-1]X \ .$$

The natural inclusion $\mathbb{D}_0 \to \mathbb{G}_a$ induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(k) & \longleftarrow & \mathbb{D}_0 & \longrightarrow & \mathrm{Spec}(k) \\ \| & & \downarrow & & \| \\ \mathrm{Spec}(k) & \longleftarrow & \mathbb{G}_a & \longrightarrow & \mathrm{Spec}(k) \ , \end{array}$$

and therefore a canonical comparison map

$$u \colon \mathbb{D}_{-1} \longrightarrow \mathsf{B}\mathbb{G}_a \ . \tag{4.2.1}$$

In particular, for every derived stack X, we obtain a natural comparison morphism

$$u^* \colon \mathbf{Map}(\mathsf{B}\mathbb{G}_a, X) \longrightarrow \mathbf{Map}(\mathbb{D}_{-1}, X) \simeq \mathsf{T}[-1]X \ .$$

Proposition 4.8. *For X a derived Deligne–Mumford stack locally almost of finite presentation, the morphism u^* is an equivalence.*

Proof. It is shown in [36, Proposition 5.22] that the map u^* is formally étale. It suffices therefore to argue that u^* is an isomorphism on truncations: this will imply that $\mathbf{Map}(\mathsf{B}\mathbb{G}_a, X)$ is geometric, and that u^* is an equivalence. For this, it is enough to argue that for any underived test scheme $S \in \mathrm{Aff}_k$, the morphism induced by precomposition with $S \to S \times \mathsf{B}\mathbb{G}_a$

$$\mathrm{Map}(S \times \mathsf{B}\mathbb{G}_a, X) \longrightarrow \mathrm{Map}(S, X) \tag{4.2.2}$$

is an equivalence. Since $\mathbb{G}_a$ is connected, this follows from Theorem 2.13. □

4.3. **The HKR equivalence for derived Deligne–Mumford stacks.** Consider the natural morphism

$$\widehat{\mathsf{aff}} \coloneqq (\mathsf{aff}, \varpi)\colon S^1 \longrightarrow \mathsf{B}\mathbb{G}_a \times \widehat{S}^1 \ .$$

See (3.1.1) and (4.1.1) for the definitions of ϖ and aff. For any derived Deligne–Mumford stack X locally almost of finite presentation, precomposition with $\widehat{\mathsf{aff}}$ induces a natural transformation

$$\mathbf{Map}(\mathsf{B}\mathbb{G}_a \times \widehat{S}^1, X) \longrightarrow \mathbf{Map}(S^1, X) = \mathscr{L}X \ .$$

We can furthermore rewrite

$$\mathbf{Map}(\mathsf{B}\mathbb{G}_a \times \widehat{S}^1, X) \simeq \mathbf{Map}(\mathsf{B}\mathbb{G}_a, \mathbf{Map}(\widehat{S}^1, X)) \simeq \mathbf{Map}(\mathsf{B}\mathbb{G}_a, \mathsf{I}^{\mathsf{DM}}X) \ .$$

Since $\mathsf{I}^{\mathsf{DM}}X$ is again a derived Deligne–Mumford stack by Theorem 3.3-(4), Theorem 4.8 provides an equivalence:

$$\mathbf{Map}(\mathsf{B}\mathbb{G}_a, \mathsf{I}^{\mathsf{DM}}X) \simeq \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X \ .$$

The goal of this section is to prove the following result.

Theorem 4.9 (HKR for derived DM stacks)**.** *Let X be a derived Deligne–Mumford stack locally almost of finite presentation, the comparison map*

$$\widehat{\mathsf{aff}}^* : \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X \longrightarrow \mathscr{L}X$$

is an equivalence of derived stacks over X.

We denote by aff_r the natural morphism $(\mathsf{aff}, \varpi_r)\colon S^1 \longrightarrow \mathsf{B}\mathbb{G}_a \times \mathsf{BC}_r$.

Proposition 4.10. *Fix a positive integer $r > 0$ and write $p\colon S^1 \to \mathrm{Spec}(k)$ and*

$$q_r\colon \mathsf{B}\mathbb{G}_a \times \mathsf{BC}_r \longrightarrow \mathrm{Spec}(k)$$

for the structural morphisms. The $+$-Beck-Chevalley transformation

$$p_+\mathsf{aff}_r^* \longrightarrow q_{r,+} \ .$$

induced by $\mathsf{aff}_r\colon S^1 \longrightarrow \mathsf{B}\mathbb{G}_a \times \mathsf{BC}_r$ is an equivalence.

Proof. Observe that both source and target of the $+$-Beck-Chevalley transformation commute with colimits. In addition

$$\mathsf{QCoh}(\mathsf{B}\mathbb{G}_a \times \mathsf{BC}_r) \simeq \mathsf{QCoh}(\mathsf{B}\mathbb{G}_a) \otimes \mathsf{QCoh}(\mathsf{BC}_r) \ ,$$

and since we are in characteristic zero, both $\mathsf{B}\mathbb{G}_a$ and BC_r are perfect stacks. In particular, it suffices to prove that for $\mathcal{F} \in \mathrm{Perf}(\mathsf{B}\mathbb{G}_a)$ and $\mathcal{G} \in \mathrm{Perf}(\mathsf{BC}_r)$, the $+$-Beck-Chevalley transformation is an equivalence on $\mathcal{F} \boxtimes \mathcal{G}$. Fix $\mathcal{G} \in \mathrm{Perf}(\mathsf{BC}_r)$. Reasoning as in the proof of Theorem 4.4, with the help of the functor θ, we see that it is enough to argue that

$$p_+\mathsf{aff}_r^*(\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes \mathcal{G}) \longrightarrow q_{r,+}(\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes \mathcal{G})$$

is an equivalence. Now, [12, Theorem 5.4] (see also [14, Theorem 5.25] for a proof in the ∞-categorical setting) supplies a canonical identification

$$\mathrm{Perf}(\mathsf{BC}_r) \simeq \prod_{\zeta \in \mu_r} \mathrm{Perf}_\zeta(\mathsf{BC}_r) \ ,$$

where $\mathrm{Perf}_\zeta(\mathsf{BC}_r)$ denotes the full subcategory of ζ-homogeneous objects. We have further identifications $\mathrm{Perf}_\zeta(\mathsf{BC}_r) \simeq \mathrm{Perf}_k$, and we write k_ζ for k seen as an element in $\mathrm{Perf}_\zeta(\mathsf{BC}_r)$. Notice that $\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes k_\zeta$ is sent by aff_r^* to the local system having k as stalk and monodromy given by multiplication by ζ. We still denote this local system by k_ζ. Hence

$$p_+\mathsf{aff}_r^*(\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes k_\zeta) \simeq p_+(k_\zeta) = p_*(k_{\zeta^{-1}})^\vee \ .$$

In particular, $p_+\mathsf{aff}_r^*(\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes k_\zeta)$ is the colimit

$$\begin{array}{ccc} k \oplus k & \xrightarrow{(1,-\zeta)} & k \\ \downarrow{\scriptstyle (1,-1)} & & \downarrow \\ k & \longrightarrow & p_+\mathsf{aff}_r^*(\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes k_\zeta) \ . \end{array}$$

From here, we immediately obtain the identification

$$p_+\mathsf{aff}_r^*(\mathcal{O}_{\mathsf{B}\mathbb{G}_a} \boxtimes k_\zeta) \simeq \mathrm{cofib}(1-\zeta \colon k \to k) \ ,$$

This implies that it is zero whenever $\zeta \neq 1$. We are therefore reduced to check the statement for $k_1 \simeq \mathcal{O}_{\mathsf{BC}_r}$, and in this case the statement is obvious. □

We are now ready for to prove our main result.

Proof of Theorem 4.9. Combining Theorem 4.10 and Theorem 2.10, we deduce that $\widehat{\mathsf{aff}}^*$ is formally étale. As a consequence of Theorem 3.3, both source and target are derived Deligne–Mumford stacks, and therefore it is sufficient to prove that $\widehat{\mathsf{aff}}^*$ induces an equivalence on truncations. This is guaranteed by Theorem 3.3-(4). □

Remark 4.11 (HKR in positive characteristics)**.** The HKR isomorphism does not hold in positive characteristics in general, even for schemes. In particular our Theorem 4.9 cannot hold in that setting, as stated. Let us briefly summarize what is known in that direction:

- The HKR isomorphism holds inconditionally for smooth affine schemes in any characteristics.
- The HKR isomorphism holds for smooth projective schemes of dimension less or equal to p over $\mathbb{F}_p$ [61], [5].
- There are examples of schemes of dimension $2p$ over $\mathbb{F}_p$ for which the HKR isomorphism fails [4].

Let us explain what the main issues are from the standpoint of our argument. Our formulation of the HKR equivalence, Theorem 4.9, states that there is a natural map

$$\widehat{\mathsf{aff}}^* : \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X \longrightarrow \mathscr{L}X$$

and that this map is an equivalence. The definition of $\widehat{\mathsf{aff}}^*$ rests on two ingredients. The first ingredient is the natural map

$$S^1 \to \mathsf{B}\mathbb{G}_a$$

which was constructed in Recollection 4.3. This map exists in any characteristics, however it induces an equivalence between the affinization of S^1 and $\mathsf{B}\mathbb{G}_a$ only in characteristic 0.[2] The second ingredient is Proposition 4.8. This also fails in positive characteristics, as $\mathsf{B}\mathbb{G}_a$ corepresents the shifted tangent $\mathsf{T}[-1]X$ only in characteristic 0. Thus, it is only in characteristic 0 that one can meaningfully define the map $\widehat{\mathsf{aff}}^*$; let alone prove that it is an equivalence. In the setting

[2]One way to see why this fails is that these two stacks have different coherent cohomology. The coherent cohomology of $\mathsf{B}\mathbb{G}_a$ is equivalent to the free commutative algebra in degree -1, which in characteristic p has non-vanishing cohomology in all degrees; whereas the coherent cohomology of S^1 coincides with singular cohomology, which is concentrated in degree 0 and 1.

of affine schemes, a replacement of the HKR isomorphism which holds in any characteristics is the *HKR filtration*, which was studied in [35]. In a precise sense, the existence of the HKR isomorphism is equivalent to the triviality of the HKR filtration (which holds for instance in characteristic 0). In forthcoming work, we shall explain how to extend the HKR filtration of [35] to the Hochschild homology of derived DM stacks. The key ingredient will be given by variants of the filtered circle studied in [35], which are adapted to the study of DM stacks.

4.4. **Multiplicative HKR isomorphism for Hochschild homology.** Recall that for a derived Deligne–Mumford stack X, its *Hochschild homology* is defined as the following complex of k-modules, viewed as an object in the stable ∞-category Mod_k:

$$\mathsf{HH}_*(X) \coloneqq \Gamma(X \times X, \Delta_* \mathcal{O}_X \otimes \Delta_* \mathcal{O}_X), \tag{4.4.1}$$

where $\Delta\colon X \to X \times X$ is the diagonal morphism. As it happens for schemes, derived base change for the derived pullback square

$$\begin{array}{ccc} \mathscr{L}X & \xrightarrow{p} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[\Delta]{} & X \times X \ . \end{array}$$

yields the usual interpretation of $\mathsf{HH}_*(X)$ as functions on $\mathscr{L}X$:

$$\mathsf{HH}_*(X) \simeq \Gamma(X, \Delta^* \Delta_* \mathcal{O}_X) \simeq \Gamma(\mathscr{L}X, \mathcal{O}_{\mathscr{L}X}).$$

In particular, it inherits an algebra structure coming from the right hand side. In more classical terms, for any integer i, the i-th *Hochschild homology group* of X is defined as

$$\mathsf{HH}_i(X) \coloneqq H^{-i}(\mathsf{HH}_*(X)) \simeq H^{-i}(X \times X, \Delta_* \mathcal{O}_X \otimes \Delta_* \mathcal{O}_X) \simeq H^{-i}(\mathscr{L}X, \mathcal{O}_{\mathscr{L}X}).$$

Consequently, $\bigoplus_i \mathsf{HH}_i(X)$ admits a natural graded algebra structure.

Theorem 4.12. *Let X be a derived Deligne–Mumford stack locally almost of finite presentation. Then we have an equivalence in the ∞-category $\mathsf{Alg}(\mathsf{Mod}_k)$:*

$$\mathsf{HH}_*(X) \simeq \Gamma(\mathsf{I}^{\mathsf{DM}} X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}} X}[1])). \tag{4.4.2}$$

where the Hochschild homology $\mathsf{HH}_*(X)$ is equipped with its natural algebra structure as derived functions on $\mathscr{L}X$ and $\Gamma(\mathsf{I}^{\mathsf{DM}} X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}} X}[1]))$ is equipped with the natural algebra structure induced from the algebra structure on the symmetric algebra.

Proof. Write $\pi\colon \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}} X \to \mathsf{I}^{\mathsf{DM}} X$ for the natural projection. Since the equivalence of Theorem 4.9 is over X, it induces a canonical equivalence $\pi_*(\mathcal{O}_{\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}} X}) \simeq \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}} X}[1])$ in $\mathsf{Alg}(\mathsf{QCoh}(X))$. Applying the lax-monoidal ∞-functor of global sections, we conclude. □

4.5. **HKR isomorphism for Hochschild cohomology.** Theorem 4.9 also allows to obtain a statement for Hochschild *co*homology. To formulate it let us denote by

$$p\colon \mathscr{L}X \longrightarrow X \ , \qquad q\colon \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}} X \longrightarrow X$$

the natural projections.

Notation 4.13. Let $f\colon Y \to X$ be a morphism. We denote by

$$\mathsf{Coh}^{\mathsf{b}}(Y/X)$$

the full subcategory of $\mathsf{QCoh}(Y)$ spanned by almost perfect complexes on Y having finite tor-amplitude relative to X.

Example 4.14. Assume that X is a smooth Deligne–Mumford stack and that $f\colon Y \to X$ is representable by quasi-compact algebraic spaces. Then unraveling the definitions we see that $\mathsf{Coh}^{\mathsf{b}}(Y/X)$ canonically coincide with $\mathsf{Coh}^{\mathsf{b}}(Y)$.

The morphisms p and q induce well-defined morphisms

$$p_*\colon \mathrm{Ind}(\mathsf{Coh}^{\mathsf{b}}(\mathscr{L}X/X)) \longrightarrow \mathsf{QCoh}(X)\ , \qquad q_*\colon \mathrm{Ind}(\mathsf{Coh}^{\mathsf{b}}(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X/X)) \longrightarrow \mathsf{QCoh}(X)\ .$$

By construction, both these functors commute with (filtered and hence all) colimits. Furthermore, since p and q are proper, the definition of $\mathsf{Coh}^{\mathsf{b}}(-/X)$ implies that p_* and q_* preserve compact objects. It follows that they both admit continuous right adjoints that we denote

$$p^!\colon \mathsf{QCoh}(X) \longrightarrow \mathrm{Ind}(\mathsf{Coh}^{\mathsf{b}}(\mathscr{L}X/X))\ , \qquad q^!\colon \mathsf{QCoh}(X) \longrightarrow \mathrm{Ind}(\mathsf{Coh}^{\mathsf{b}}(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X/X))\ .$$

In general, we set

$$\mathcal{HH}^*(X) \coloneqq p_*p^!(\mathcal{O}_X)\ ,$$

and we refer to it as the *Hochschild cohomology sheaf* of X, whose (derived) global sections recover the *Hochschild cohomology* of X:

$$\mathsf{HH}^*(X) \coloneqq \Gamma(X, \mathcal{HH}^*(X)) = \Gamma(\mathscr{L}X, p^!\mathcal{O}_X).$$

Corollary 4.15. *Let X be a derived DM stack locally almost of finite presentation. There is a canonical isomorphism in $\mathsf{QCoh}(X)$*

$$\mathcal{HH}^*(X) \simeq q_*q^!(\mathcal{O}_X)\ ,$$

and a canonical isomorphism in the stable ∞-category Mod_k

$$\mathsf{HH}^*(X) \simeq \Gamma(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X, q^!(\mathcal{O}_X))\ .$$

If X is moreover lci, then we have canonical isomorphisms

$$\begin{aligned}\mathsf{HH}^*(X) &\simeq \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]) \otimes i^*\omega_X^\vee[-\dim(X)]) \\ &\simeq \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{T}_{\mathsf{I}^{\mathsf{DM}}X}[-1]) \otimes \det \mathsf{N}[-c]),\end{aligned} \tag{4.5.1}$$

where ω_X is the dualizing sheaf of X, $i\colon \mathsf{I}^{\mathsf{DM}}X \to X$ is the canonical morphism,

$$\mathsf{N} \coloneqq \mathrm{cofib}(\mathbb{T}_{\mathsf{I}^{\mathsf{DM}}X} \to i^*\mathbb{T}_X)\ , \quad c \coloneqq \dim(X) - \dim(\mathsf{I}^{\mathsf{DM}}X)\ ,$$

and $\dim(-)$ is the dimension viewed as a locally constant $\mathbb{Z}$-valued function.

Proof. Since p_* and q_* are canonically identified by Theorem 4.9, we immediately obtain the first assertion. Applying Γ, it yields the second assertion. Assume now that X is a lci derived DM stack of finite presentation. By Theorem 3.3, $\mathsf{I}^{\mathsf{DM}}X = \mathsf{I}^{(r)}X$ for some r, which is also lci. We decompose the morphism q into the composition

$$q\colon \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X \xrightarrow{\pi} \mathsf{I}^{\mathsf{DM}}X \xrightarrow{i} X.$$

The standard fiber sequence of cotangent complexes for the map π shows that the dualizing sheaf $\omega_{\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X}$ is trivial. As a result, the relative dualizing sheaf ω_q is isomorphic to $q^*\omega_X^\vee$ and $\dim(q) = -\dim(X)$. Therefore,

$$q^!\mathcal{O}_X \simeq \omega_q[\dim(q)] \simeq q^*\omega_X^\vee[-\dim(X)]$$

We conclude by the following computation:

$$\begin{aligned}\mathsf{HH}^*(X) &\simeq \Gamma(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X, q^!(\mathcal{O}_X)) \\ &\simeq \Gamma(\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X, q^*\omega_X^\vee[-\dim(X)]) \\ &\simeq \Gamma(\mathsf{I}^{\mathsf{DM}}X, \pi_*\pi^*i^*\omega_X^\vee[-\dim(X)]) \\ &\simeq \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]) \otimes i^*\omega_X^\vee[-\dim(X)]),\end{aligned}$$

where the last step uses the projection formula. Finally, the last isomorphism in the statement □

In the classical case, Corollary 4.15 can be reformulated in a much more concrete way, generalizing [7, Corollary 1.17] to the case where X is not necessarily a global quotient by a finite group. The following consequence is one instance:

Corollary 4.16. *Assume that X is a smooth (hence underived) DM stack. Consider the connected-component decomposition of $\mathsf{I}^{\mathsf{DM}}X = \mathsf{I}X$:*

$$\mathsf{I}X = \bigsqcup_{i\in I} Z_i,$$

where I is the set of connected components of $\mathsf{I}X$. Let $c_i = \dim(X) - \dim(Z_i)$, and let $\omega_{Z_i/X}$ be the relative dualizing sheaf of the natural map $Z_i \to X$. Then we have an isomorphism of graded vector spaces:

$$\mathsf{HH}^*(X) \simeq \bigoplus_{i\in I} \bigoplus_{p+q=*} H^{p-c_i}\Big(Z_i, \bigwedge^q \mathsf{T}_{Z_i} \otimes \omega_{Z_i/X}\Big).$$

Proof. We apply (4.5.1) and use the relative canonical bundle formula $\omega_X^\vee|_{Z_i} \simeq \omega_{Z_i}^\vee \otimes \omega_{Z_i/X}$ and the canonical isomorphism $\Omega^p_{Z_i} \otimes \omega_{Z_i}^\vee \simeq \bigwedge^{\dim(Z_i)-p} \mathsf{T}_{Z_i}$. □

5. Circle action and de Rham differential

In the case of derived schemes over a field k of characteristic zero, the HKR equivalence $\mathsf{T}[-1]X \simeq \mathscr{L}X$ is more structured than a plain equivalence of derived schemes over X:

(1) there is a canonical action of S^1 on $\mathscr{L}X$ and of $\mathsf{B}\mathbb{G}_a$ on $\mathsf{T}[-1]X$;

(2) the morphism $\mathsf{aff}\colon S^1 \to \mathsf{B}\mathbb{G}_a$ is canonically a morphism of groups, and it allows to see $\mathsf{T}[-1]X$ as a S^1-derived scheme;

(3) the HKR equivalence can be promoted to an equivalence of S^1-derived schemes.

Furthermore, in concrete terms, the action of S^1 on $\mathscr{L}X$ encodes the Connes operator on Hochschild homology, whereas the action of $\mathsf{B}\mathbb{G}_a$ on $\mathsf{T}[-1]X$ encodes the de Rham differential. The goal of this section is to generalize these statements to the Deligne–Mumford setting.

5.1. S^1-objects and mixed objects.

Before discussing the details of the generalization, let us briefly recall how one can make the previous statements precise:

Recollection 5.1. In [58], Toën and Vezzosi introduced two ∞-categories:

(1) The ∞-category of S^1-*algebras*, that can be defined as

$$S^1\text{-CAlg}_k \coloneqq \mathsf{CoMod}_{\mathsf{C}^*_{\mathsf{sing}}(S^1;k)}(\mathrm{CAlg}_k)\ .$$

(2) The ∞-category of *mixed algebras*, that can be defined as

$$\varepsilon\text{-CAlg}_k \coloneqq \mathsf{CoMod}_{k\oplus k[-1]}(\mathrm{CAlg}_k)\ ,$$

where $k\oplus k[-1]$ is the split square-zero extension equipped with its natural Hopf structure.

Taking these as definitions, it is clear that any equivalence

$$\phi\colon \mathsf{C}^*_{\mathsf{sing}}(S^1;k) \simeq k\oplus k[-1]$$

as *Hopf algebras* induces an equivalence A_ϕ fitting in the following commutative diagram

$$\begin{array}{ccc} S^1\text{-}\mathsf{CAlg}_k & \xrightarrow[\sim]{A_\phi} & \varepsilon\text{-}\mathsf{CAlg}_k \\ & {\scriptstyle U_{S^1}} \searrow \quad \swarrow {\scriptstyle U_\varepsilon} & \\ & \mathsf{CAlg}_k\ , & \end{array} \tag{5.1.1}$$

where U_{S^1} and U_ε are the natural forgetful functors. From this point of view (which is slightly different from the one taken in [58]), the main result of [58] can be stated as follows: the comonadic functors U_{S^1} and U_ε are also monadic, and their left adjoints L_{S^1} and L_ε admit the following description. For any $A \in \mathsf{CAlg}_k$,

(a) $U_{S^1}L_{S^1}(A) \simeq S^1 \otimes A \simeq A \otimes_{A\otimes_k A} A$, so $L_{S^1}(A)$ is canonically identified with the Hochschild homology complex of A (considered only up to *quasi-isomorphism*) with the free S^1-action.

(b) $U_\varepsilon L_\varepsilon(A) \simeq \mathsf{DR}(A) \coloneqq \mathrm{Sym}_A(\mathbb{L}_{A/k}[1])$, so $L_\varepsilon(A)$ is identified with the derived de Rham algebra of A, with mixed structure given by the de Rham differential.

It follows that A_ϕ exchanges the S^1-action (a.k.a. $\mathsf{C}^*_{\mathsf{sing}}(S^1;k)$-coaction) on Hochschild homology of A with the mixed structure (a.k.a. $(k \oplus k[-1])$-coaction) on $\mathsf{DR}(A)$ given by the de Rham differential.

Remark 5.2 (Connes' operator)**.** Let $A \in S^1\text{-}\mathsf{CAlg}_k$. The S^1-action has an underlying coaction morphism

$$\gamma\colon A \longrightarrow A \otimes \mathsf{C}^*_{\mathsf{sing}}(S^1;k)\ .$$

Under the formality $\phi\colon \mathsf{C}^*_{\mathsf{sing}}(S^1;k) \simeq k \oplus k[-1]$, we can associate to γ a map $d_\gamma\colon A \to A \oplus A[-1]$, which, as a consequence of the counitality, is a shifted self-derivation of A. We refer to d_γ as the *Connes operator* associated to the S^1-structure on A. Forgetting all the higher coherences, the equivalence A_ϕ of Theorem 5.1 sends the pair (A,γ) to the pair (A, d_γ).

Remark 5.3 (Choice of formality)**.** Although the above statements work for any choice of formality ϕ, in order to globalize them it is convenient to fix one such choice of geometric origin, that we now describe. Note that Robalo proved in [45] that the space of choices of formality is essentially contractible. Fix a base field k (that we will soon assume of characteristic zero) and consider the morphism (4.1.1)

$$\mathsf{aff}\colon S^1 \longrightarrow \mathsf{B}\mathbb{G}_a\ .$$

Inspection reveals that it is induced via delooping by the morphism of commutative groups $\mathbb{Z} \to \mathbb{G}_a$. In particular, aff is canonically a morphism of derived group stacks, i.e. in $\mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_1}(\mathsf{dSt}_k)$. On the other hand, the morphism (4.2.1) $u\colon \mathbb{D}_{-1} \to \mathsf{B}\mathbb{G}_a$ induces a morphism

$$\bar{u}\colon \mathrm{Aff}(\mathbb{D}_{-1}) \longrightarrow \mathsf{B}\mathbb{G}_a\ ,$$

which, when the characteristic of k is *zero*, is an equivalence.

Lemma 5.4. *The space*

$$\{\mathsf{B}\mathbb{G}_a\} \times_{(\mathsf{dSt}_k)^{\simeq}_*} \mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_\infty}(\mathsf{dSt}_k)^{\simeq}$$

of $\mathbb{E}_\infty$-group structures on $\mathsf{B}\mathbb{G}_a$ is contractible.

Proof. The canonical atlas $\mathrm{Spec}(k) \to \mathsf{B}\mathbb{G}_a$ induces an isomorphism on π_0, and therefore

$$\mathsf{B}\mathbb{G}_a \in \mathsf{dSt}_k^{\geq 1}\ .$$

Notice furthermore that the inclusion $\mathsf{dSt}_k^{\geq 1} \hookrightarrow \mathsf{dSt}_k$ commutes with products. We can therefore consider the following ladder of pullbacks

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathsf{dSt}_k^{\geq 1}) & \longrightarrow & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathsf{dSt}_k) \\ \downarrow & & \downarrow & & \downarrow \\ \{*\} & \xrightarrow{\mathsf{B}\mathbb{G}_a} & (\mathsf{dSt}_k)^{\geq 1}_* & \longrightarrow & (\mathsf{dSt}_k)_* \end{array} ,$$

and conclude that

$$X^{\simeq} \simeq \{\mathsf{B}\mathbb{G}_a\} \times_{(\mathsf{dSt}_k)^{\simeq}_*} \mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_\infty}(\mathsf{dSt}_k)^{\simeq}$$

is the space of $\mathbb{E}_\infty$-group structures on $\mathsf{B}\mathbb{G}_a$. We can focus on the square on the left. Consider the commutative rectangle

$$\begin{array}{ccccc} X & \longrightarrow & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathsf{dSt}_k^{\geq 1}) & \xrightarrow{\Omega} & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_\infty}(\mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathsf{dSt}_k)) \\ \downarrow & & \downarrow & & \downarrow \\ \{*\} & \xrightarrow{\mathsf{B}\mathbb{G}_a} & (\mathsf{dSt}_k)^{\geq 1}_* & \xrightarrow{\Omega} & \mathrm{Mon}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathsf{dSt}_k) \end{array}$$

May's delooping theorem [33, Theorem 5.2.6.15] implies that the horizontal morphism in the square on the right are equivalences. In particular, the square in question is a pullback. As a consequence, it is enough to compute the outer pullback.

Observe now that $\Omega(\mathsf{B}\mathbb{G}_a) \simeq \mathbb{G}_a$ and that this is a discrete object in dSt_k. Furthermore, the induced $\mathbb{E}_1$-group structure on $\mathbb{G}_a$ coincides with the additive one. We now observe that, since $\mathbb{G}_a$ is discrete and since the $\mathbb{E}_1$-structure is fixed, being $\mathbb{E}_\infty$ is now a property rather than a structure. In other words, we see that X is either empty or contractible. As the additive group structure on $\mathbb{G}_a$ is commutative, we see that it is indeed the latter case. □

Corollary 5.5. *In characteristic zero, the zig-zag*

$$\mathbb{D}_{-1} \xrightarrow{u} \mathsf{B}\mathbb{G}_a \xleftarrow{\mathsf{aff}} S^1$$

induces a formality equivalence

$$\phi\colon \mathsf{C}^*_{\mathsf{sing}}(S^1; k) \simeq k \oplus k[-1]$$

as cocommutative Hopf algebras. Equivalently, upon passing to the affinization it induces an equivalence of derived $\mathbb{E}_\infty$-group stacks

$$\mathrm{Aff}(\mathbb{D}_{-1}) \xrightarrow{\overline{u}} \mathsf{B}\mathbb{G}_a \xleftarrow{\overline{\mathsf{aff}}} \mathrm{Aff}(S^1) .$$

In virtue of the above discussion, we can consider the zig-zag

$$\mathrm{Aff}(\mathbb{D}_{-1}) \xrightarrow{\overline{u}} \mathsf{B}\mathbb{G}_a \xleftarrow{\mathsf{aff}} S^1 \tag{5.1.2}$$

in $\mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_\infty}(\mathsf{dSt}_k)$, where moreover $\overline{u}$ is an equivalence when k has characteristic zero. We now consider the canonical action on the left of $\mathsf{B}\mathbb{G}_a$ on $\mathsf{B}\mathbb{G}_a \times \widehat{S}^1$, and the induced action of S^1 on $\mathsf{B}\mathbb{G}_a \times \widehat{S}^1$ that operates as the identity on $\widehat{S}^1$.

Let now X be a derived Deligne–Mumford stack locally almost of finite presentation. Thanks to Theorem 4.9, we know that the morphism $\widehat{\mathsf{aff}}\colon S^1 \to \mathsf{B}\mathbb{G}_a \times \widehat{S}^1$ induces the following commutative

triangle:

$$\begin{array}{ccc} \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X & \xrightarrow{\widehat{\mathsf{aff}}^*} & \mathscr{L}X \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array} , \tag{5.1.3}$$

Moreover, since $\mathscr{L}X \simeq \mathbf{Map}(S^1, X)$, we see that $\mathscr{L}X$ carries a canonical $(S^1 \times X)$-action over X. Similarly, $\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X \simeq \mathrm{Map}(\mathsf{B}\mathbb{G}_a \times \widehat{S}^1, X)$ carries a canonical $(\mathsf{B}\mathbb{G}_a \times X)$-action over X, induced by the left action of $\mathsf{B}\mathbb{G}_a$ on $\mathsf{B}\mathbb{G}_a \times \widehat{S}^1$. The morphism of groups $\mathsf{aff}\colon S^1 \to \mathsf{B}\mathbb{G}_a$ allows to see $\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X$ as equipped with a $(S^1 \times X)$-action over X.

Proposition 5.6. *The equivalence $\widehat{\mathsf{aff}}^*$ supplied by Theorem 4.9 can be promoted to an equivalence of $(S^1 \times X)$-derived Deligne–Mumford stacks over X.*

Proof. Consider $\mathsf{B}\mathbb{G}_a \times \widehat{S}^1$ together with its natural left $\mathsf{B}\mathbb{G}_a$-action. The morphism of groups $\mathsf{aff}\colon S^1 \to \mathsf{B}\mathbb{G}_a$ allows to see $\mathsf{B}\mathbb{G}_a \times \widehat{S}^1$ as a left S^1-module in dSt_k, and the map $\widehat{\mathsf{aff}}\colon S^1 \to \mathsf{B}\mathbb{G}_a \times \widehat{S}^1$ becomes a morphism of left S^1-modules (where S^1 is considered as a left module over itself, with action given by the group multiplication). Since the $(S^1 \times X)$-actions on $\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X$ and on $\mathscr{L}X$ are induced by the universal properties of the mapping stacks, the conclusion follows from this analysis. □

Let us make the above statement more explicit. To begin with, we observe:

Remark 5.7. If $\mathscr{Y}$ is any ∞-topos, we can apply $(-) \otimes \mathscr{Y}$ to diagram (5.1.1) to obtain an equivalence

$$\begin{array}{ccc} S^1\text{-}\mathsf{CAlg}_k(\mathscr{Y}) & \xrightarrow[\sim]{A_\phi} & \varepsilon\text{-}\mathsf{CAlg}_k(\mathscr{Y}) \\ & {\scriptstyle U_{S^1}}\searrow \quad \swarrow{\scriptstyle U_\varepsilon} & \\ & \mathsf{CAlg}_k(\mathscr{Y}) , & \end{array}$$

where now $\mathsf{CAlg}_k(\mathscr{Y})$ and its variants denote the ∞-categories of sheaves with values in CAlg_k (or in its variants).

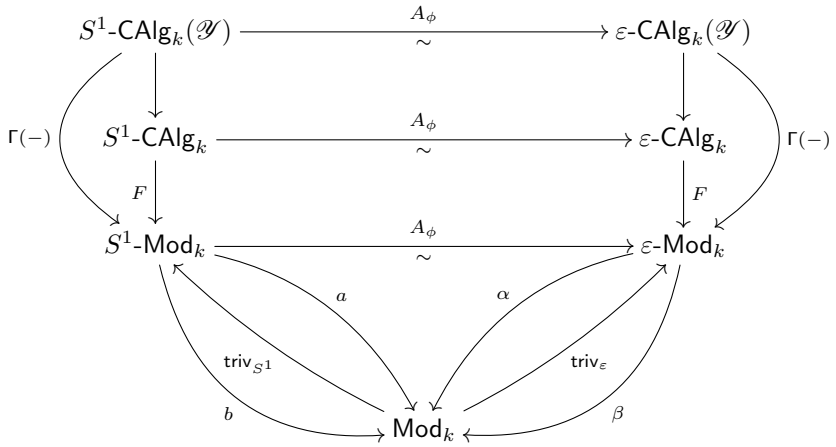
Applying Theorem 5.7 to the small étale topos $\mathsf{I}\mathscr{X}$ of $\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X$ and invoking Theorem 5.1-(b), we deduce that the $\mathsf{B}\mathbb{G}_a$-action on $\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X$ corresponds to the $(k \oplus k[-1])$-coaction on $\mathcal{O}_{\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X} \simeq \mathsf{DR}(\mathcal{O}_{\mathsf{I}^{\mathsf{DM}}X})$ given by the de Rham differential of $\mathsf{I}^{\mathsf{DM}}X$. Similarly, the S^1-action on $\mathscr{L}X$ corresponds to the free $\mathsf{C}^*_{\mathsf{sing}}(S^1; k)$-coaction on $\mathcal{O}_{\mathscr{L}X} \simeq S^1 \otimes \mathcal{O}_{\mathsf{I}^{\mathsf{DM}}X}$. In particular:

Corollary 5.8. *In the setting of diagram* (5.1.3)*, the Connes operator on $\pi_{2,*}(\mathcal{O}_{\mathscr{L}X})$ corresponds, via the equivalence of Theorem 5.7, to the action of the de Rham differential on $\pi_{1,*}(\mathcal{O}_{\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X})$.*

5.2. **Cyclic homology and variants.** Based on Corollary 5.8, we establish an HKR-type theorem for the cyclic homology, negative cyclic homology, and periodic cyclic homology of a derived DM stack X in terms of the derived de Rham complex of $\mathsf{I}^{\mathsf{DM}}X$. More precisely, the equivalence A_ϕ from diagram (5.1.1), generalized to an arbitrary base topos in Remark 5.7 (for our purpose, $\mathscr{Y}$ is the small étale topos $\mathsf{I}\mathscr{X}$ of $\mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}X$), is compatible with several natural functors as shown in the following diagram, where all functors are derived. The reference for

these facts is [27].

(5.2.1)



Here, $\Gamma(-)$ is the functor of derived global sections, F is the functor forgetting the algebra structure, the functors triv_{S^1} and $\mathsf{triv}_\varepsilon$ send a k-module to its trivial representation. The functors a and b are respectively the (homotopy) S^1-invariants and S^1-coinvariants

$$a \coloneqq (-)^{S^1} \quad \text{and} \quad b \coloneqq (-)_{S^1} \ .$$

The functors α and β are given by the following formulas

$$\alpha \coloneqq \mathrm{Map}_{\varepsilon\text{-}\mathsf{Mod}_k}(k,-) \quad \text{and} \quad \beta \coloneqq k \otimes_{k[\varepsilon]} (-) \ .$$

We have adjunctions

$$b \dashv \mathsf{triv}_{S^1} \dashv a \quad \text{and} \quad \beta \dashv \mathsf{triv}_\varepsilon \dashv \alpha \ .$$

The top two squares are commutative on the nose. As for the bottom triangle, A_ϕ intertwines triv_{S^1} and $\mathsf{triv}_\varepsilon$:

$$A_\phi \circ \mathsf{triv}_{S^1} \simeq \mathsf{triv}_\varepsilon \ .$$

Hence A_ϕ also intertwines their left and right adjoint functors:

$$\alpha \circ A_\phi \simeq a \ \text{ and } \ \beta \circ A_\phi \simeq b \ . \tag{5.2.2}$$

Further, we have the *norm map*, which are natural transformations

$$\nu : (-)_{S^1}[1] \Rightarrow (-)^{S^1} \ \text{ and } \ \nu : k \otimes_{k[\varepsilon]} (-)[1] \Rightarrow \mathrm{Map}_{\varepsilon\text{-}\mathsf{Mod}_k}(k,-).$$

We refer to the literature for the former, and recall that the latter is induced by the following morphism of $k[\varepsilon]$-bimodules, where $k[1]$ is the 1-dimensional vector space with basis ε:

$$k[1] \to k[\varepsilon] \ , \tag{5.2.3}$$

together with adjunction:

$$\mathrm{Map}_k(k[1]\otimes_{k[\varepsilon]} -, \mathrm{Map}_{k[\varepsilon]}(k,-)) \simeq \mathrm{Map}_{k[\varepsilon]}(k\otimes_k k[1]\otimes_{k[\varepsilon]} -, -) \simeq \mathrm{Map}_{k[\varepsilon]}(k[1]\otimes_{k[\varepsilon]} -, k[\varepsilon]\otimes_{k[\varepsilon]} -).$$

Definition 5.9. Let X be a derived stack. The *cyclic homology* of X is defined to be the homotopy orbits of the natural S^1-action on $\mathsf{HH}_*(X)$:

$$\mathsf{HC}(X) \coloneqq (\mathsf{HH}_*(X))_{S^1} \ .$$

The *negative cyclic homology* of X is defined to be the homotopy fixed points for this action:

$$\mathsf{HN}(X) \coloneqq (\mathsf{HH}_*(X))^{S^1} \ .$$

The *periodic cyclic homology* of X is defined to be the Tate fixed points of this action, i.e. the homotopy cofiber of the norm map:

$$\mathsf{HP}(X) \coloneqq (\mathsf{HH}_*(X))^{tS^1} \coloneqq \mathrm{cofib}(\mathsf{HC}(X)[1] \xrightarrow{\nu} \mathsf{HN}(X)) \ .$$

It is shown in [27] that the above definitions coincide with the classical ones, which can be found for example in [30].

Given a derived DM stack X, we define the derived de Rham complex of $\mathsf{I}^{\mathsf{DM}}X$ as the totalization of the following mixed graded algebra:

$$\mathsf{DR}(\mathsf{I}^{\mathsf{DM}}X) \coloneqq \left[\mathcal{O}_{\mathsf{I}^{\mathsf{DM}}X} \xrightarrow{\mathsf{d}_{\mathrm{dR}}} \mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X} \xrightarrow{\mathsf{d}_{\mathrm{dR}}} \bigwedge^2 \mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X} \xrightarrow{\mathsf{d}_{\mathrm{dR}}} \cdots\right], \tag{5.2.4}$$

where $\mathcal{O}_{\mathsf{I}^{\mathsf{DM}}X}$ is placed in cohomological degree 0, $\bigwedge^i \mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}$ stands for the i-th derived exterior power the cotangent complex of $\mathsf{I}^{\mathsf{DM}}X$, and the mixed structure is given by the de Rham differential operator d_{dR}. The de Rham complex (5.2.4) is endowed with a natural Hodge filtration given as follows: for $i \geq 0$,

$$\mathrm{Fil}^i_{\mathsf{Hdg}}\,\mathsf{DR}(\mathsf{I}^{\mathsf{DM}}X) \coloneqq \left[\cdots \to 0 \to \bigwedge^i \mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X} \xrightarrow{\mathsf{d}_{\mathrm{dR}}} \bigwedge^{i+1} \mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X} \xrightarrow{\mathsf{d}_{\mathrm{dR}}} \cdots\right], \tag{5.2.5}$$

and $\mathrm{Fil}^i_{\mathsf{Hdg}}\,\mathsf{DR}(\mathsf{I}^{\mathsf{DM}}X) = \mathsf{DR}(\mathsf{I}^{\mathsf{DM}}X)$ for $i \leq 0$.

Corollary 5.10. *Let X be a derived DM stack locally almost of finite presentation. Denote $\mathsf{DR} \coloneqq \mathsf{DR}(\mathsf{I}^{\mathsf{DM}}X)$ the derived de Rham complex of $\mathsf{I}^{\mathsf{DM}}X$. For any integer i, let $\mathsf{DR}^{\geq i} \coloneqq \mathrm{Fil}^i_{\mathsf{Hdg}}\,\mathsf{DR}$ and $\mathsf{DR}^{\leq i} \coloneqq \mathrm{cofib}(\mathsf{DR}^{\geq i} \to \mathsf{DR})$. There are canonical isomorphisms of graded vector spaces:*

$$\begin{aligned}
\mathsf{HC}(X) &\simeq \bigoplus_{i\geq 0} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}^{\leq i}[2i]\right) \ ; \\
\mathsf{HN}(X) &\simeq \prod_{i\in\mathbb{Z}} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}^{\geq i}[2i]\right) \simeq \prod_{i\leq 0} H^*_{\mathrm{dR}}(\mathsf{I}^{\mathsf{DM}}X)[2i] \times \prod_{i=1}^{\infty} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}^{\geq i}[2i]\right) \ ; \\
\mathsf{HP}(X) &\simeq \prod_{i\in\mathbb{Z}} \Gamma\left(\mathsf{I}^{\mathsf{DM}}X, \mathsf{DR}[2i]\right) \simeq \prod_{i\in\mathbb{Z}} H^*_{\mathrm{dR}}(\mathsf{I}^{\mathsf{DM}}X)[2i] \ .
\end{aligned}$$

Proof. By the commutative diagram (5.2.1), especially (5.2.2), we have

$$\begin{aligned}
\mathsf{HC}(X) &\simeq b(\Gamma(\mathcal{O}_{\mathscr{L}X})) \simeq \beta(A_\phi(\Gamma(\mathcal{O}_{\mathscr{L}X}))) \simeq k \otimes_{k[\varepsilon]} \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1])) \ ; \\
\mathsf{HN}(X) &\simeq a(\Gamma(\mathcal{O}_{\mathscr{L}X})) \simeq \alpha(A_\phi(\Gamma(\mathcal{O}_{\mathscr{L}X}))) \simeq \mathrm{Map}_{k[\varepsilon]}(k, \Gamma(\mathsf{I}^{\mathsf{DM}}X, \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]))) \ .
\end{aligned} \tag{5.2.6}$$

Here, the $k[\varepsilon]$-structure on $\Gamma(\mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]))$ is given by the de Rham differential operator

$$\mathsf{d}_{\mathrm{dR}} \colon \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1]) \to \mathrm{Sym}(\mathbb{L}_{\mathsf{I}^{\mathsf{DM}}X}[1])[-1].$$

Resolving the object $k \in \varepsilon\text{-}\mathsf{Mod}_k$ by the mixed module

$$\cdots \to k[\varepsilon][2] \xrightarrow{\cdot\varepsilon} k[\varepsilon][1] \xrightarrow{\cdot\varepsilon} k[\varepsilon],$$

and combining with (5.2.6), we obtain the claimed formulas for $\mathsf{HC}(X)$ and $\mathsf{HN}(X)$.

Combining the description of the norm map in (5.2.3) with the above resolution of k, one obtains that the norm map $\mathsf{HC}(X)[1] \to \mathsf{HN}(X)$ is induced by:

$$\bigoplus_{i>0} \mathsf{DR}^{\leq i-1}[2i-1] \xrightarrow{\mathsf{d}_{\mathrm{dR}}} \prod_{i\in\mathbb{Z}} \mathsf{DR}^{\geq i}[2i] \tag{5.2.7}$$

whose cofiber is $\prod_{i\in\mathbb{Z}} \mathsf{DR}[2i]$, since for $i \leq 0$, $\mathsf{DR}^{\geq i}[2i] = \mathsf{DR}[2i]$. The formula for $\mathsf{HP}(X)$ follows. □

The next result is an immediate consequence of Corollary 5.10. We record it as an independent statement as it provides an even more explicit description of the periodic cyclic homology of X, which might be of independent interest.

Corollary 5.11. *Let X be a derived DM stack locally almost of finite presentation. Let $\mathsf{lt}_0(X)$ be the classical inertia stack of $\mathrm{t}_0(X)$, and let $|\mathsf{lt}_0(X)|$ be its coarse moduli space. Then there are equivalences*

$$\mathsf{HP}(X) \simeq \mathsf{HP}(\mathrm{t}_0(X)) \simeq \prod_{i\in\mathbb{Z}} H^*_{\mathrm{dR}}(|\mathsf{lt}_0(X)|)[2i].$$

Proof. Note that periodic cyclic homology, as de Rham cohomology, is nil-invariant. This implies the first equivalence. Next, by the last equivalence of Corollary 5.10, we can write

$$\mathsf{HP}(\mathrm{t}_0(X)) \simeq \prod_{i\in\mathbb{Z}} H^*_{\mathrm{dR}}(\mathsf{lt}_0(X))[2i]$$

So the only thing left to prove, is that the latter is equivalent to the $\mathbb{Z}_2$-periodic de Rham cohomology of the coarse moduli space. This is a general property of the de Rham cohomology of DM stacks in characteristic 0. Let us sketch the argument for a general DM stack Y, which we can assume underived. By étale descent for de Rham cohomology we reduce to the global quotient case, $Y = [S/G]$, where S is a scheme and G is a finite group. Let us denote $|Y|$ the coarse moduli space of Y. Using again étale descent, we see that the de Rham cohomology of Y is equivalent to the G-invariant part of the de Rham cohomology of S. But the latter is equivalent to the de Rham cohomology of the coarse moduli space $|Y|$. See for instance Proposition 36 of [8] for a similar argument. □

Remark 5.12. We can give also an alternative, Koszul dual, description of $\mathsf{HC}(X)$, $\mathsf{HN}(X)$ and $\mathsf{HP}(X)$. Again, this follows easily from the commutative diagram (5.2.1), but we shall not include all the details. Let $k[u] \simeq k^{S^1}$ be the free commutative algebra on a generator u of degree two. Then we have isomorphisms

$$\mathsf{HC}(X) \simeq \left(\Gamma(\mathrm{Sym}(\mathbb{L}_{\mathsf{l}^{\mathrm{DM}}X}[1])) \otimes_k (k(u)/u\cdot k[u]), u\cdot d_{\mathrm{dR}}\right),$$

$$\mathsf{HN}(X) \simeq \left(\Gamma(\mathrm{Sym}(\mathbb{L}_{\mathsf{l}^{\mathrm{DM}}X}[1])) \otimes_k k[u], u\cdot d_{\mathrm{dR}}\right).$$

Periodic cyclic homology is equivalent to the Tate fixed points of the S^1-action on $\mathsf{HH}(X)$,

$$\mathsf{HP}(X) \simeq (\mathsf{HH}(X))^{S^1} \otimes_{k[u]} k[u,u^{-1}]$$

and in particular we have that

$$\mathsf{HP}(X) \simeq \left(\Gamma(\mathrm{Sym}(\mathbb{L}_{\mathsf{l}^{\mathrm{DM}}X}[1])) \otimes_k k(u), u\cdot d_{\mathrm{dR}}\right).$$

6. Examples and applications

In this section, we investigate some interesting classes of derived DM stacks arising naturally in geometry, where our main results can be applied to compute the free loop space and the Hochschild (co)homology. We first study the global quotient of a derived scheme by a finite group, recovering and generalizing the main results of Arinkin–Căldăraru–Hablicsek [7]. Then to illustrate the usefulness of the generality of Theorem 4.12, we compute the Hochschild (co)homology of some DM stacks that are not accessible with the results in [7], for example weighted projective lines, root stacks, quotient stacks by algebraic groups, etc.

6.1. **Global quotients: possibly derived and singular.** We compute the orbifold inertia of the global quotient of a derived scheme by a finite group, in terms of the (derived) fixed loci of this action. Fixed loci for classical stacks have been studied by Romagny [46]. The derived structure leads to some subtleties in the definition of fixed locus. Let us first briefly clarify this. We refer to [6, Appendix A] for more details (but beware we denote by Y^G what is denoted by Y^{hG} in *loc. cit*).

Definition 6.1 (Derived fixed loci)**.** Let Y be a derived stack equipped with an action of an algebraic group G. Hence we have a structural morphism $[Y/G] \to \mathsf{B}G$. The *derived fixed locus* of the G-action on Y is defined as the following *section stack*:

$$Y^G \coloneqq \mathbf{Sect}_{\mathsf{B}G}([Y/G]) \coloneqq \mathrm{Spec}(k) \times_{\mathbf{Map}(\mathsf{B}G, \mathsf{B}G)} \mathbf{Map}(\mathsf{B}G, [Y/G]) \ ,$$

where $\mathrm{Spec}(k) \to \mathbf{Map}(\mathsf{B}G, \mathsf{B}G)$ is given by the identity of $\mathsf{B}G$, and the mapping stacks are computed in the category of derived stacks.

Let g be a finite-order automorphism of a derived stack Y. Let $\langle g \rangle$ be the finite cyclic group generated by g. Then the *derived fixed locus* of g on Y, denote by Y^g, is defined to be the derived fixed locus of the action of $\langle g \rangle$ on Y:

$$Y^g \coloneqq Y^{\langle g \rangle} = \mathbf{Sect}_{\mathsf{B}\langle g \rangle}([Y/\langle g \rangle]).$$

Remark 6.2 (Weil restriction)**.** Let $p\colon \mathsf{B}G \to \mathrm{Spec}(k)$ be the structural morphism. The base-change functor $p^*\colon \mathsf{dSt}_k \to \mathsf{dSt}_{\mathsf{B}G}$ admits a right adjoint functor $p_*\colon \mathsf{dSt}_{\mathsf{B}G} \to \mathsf{dSt}_k$, called the *Weil restriction* ([34, §19.1.2]). For any derived stack Y equipped with an action of G, we get an object $[Y/G] \in \mathsf{dSt}_{\mathsf{B}G}$. By [34, Proposition 19.1.2.2 & Construction 19.1.2.3], the derived fixed loci Y^G can be equivalently interpreted via the Weil restriction:

$$Y^G \simeq p_*([Y/G]). \tag{6.1.1}$$

Using atlas of $\mathsf{B}G$ and of $[Y/G]$, one can show that

$$Y^G \simeq \lim_{n \in \mathbf{\Delta}} \mathbf{Map}(G^n, Y) \ . \tag{6.1.2}$$

Remark 6.3 (Residual action by centralizer)**.** Let the notation be as in Theorem 6.1. For any $g \in G$, let $Z(g) \coloneqq Z_G(g)$ be the centralizer of g in G. Then the derived fixed locus Y^g admits a canonical residual action of $Z(g)$. Indeed, the action of $Z(g)$ on Y clearly descends to an action on $[Y/\langle g \rangle]$ that respects the structural morphism $[Y/\langle g \rangle] \to \mathsf{B}\langle g \rangle$. Hence $\mathbf{Map}(\mathsf{B}\langle g \rangle, [Y/\langle g \rangle])$ admits a $Z(g)$-action that respects the structural morphism to $\mathbf{Map}(\mathsf{B}\langle g \rangle, \mathsf{B}\langle g \rangle)$. By definition of Y^g as the section stack, we get a $Z(g)$-action on it.

Remark 6.4 (Conjugation action)**.** Let G be a finite group acting on a derived stack Y. For any $g, h \in G$, it is clear from the definition that we have a canonical isomorphism

$$h.\colon Y^g \xrightarrow{\simeq} Y^{hgh^{-1}}, \tag{6.1.3}$$

which is given by $x \mapsto h.x$ on the level of functor of points. These isomorphisms assemble into the so-called *conjugation* action of G on $\bigsqcup_{g \in G} Y^g$, where G acts on the indexing set by conjugation.

Proposition 6.5. *Let Y be a finitely presented derived scheme equipped with an action of a finite group G. Then we have isomorphisms*

$$\mathsf{I}^{\mathsf{DM}}[Y/G] \simeq \coprod_{[g] \in G/G} [Y^g/Z(g)] \simeq \left[\left(\coprod_{g \in G} Y^g \right) /G \right],$$

where G/G denotes the set of conjugacy classes of G, and in the last stack, the G-action is the conjugation action of Theorem 6.4.

Proof. The second isomorphism is clear, let us prove the first isomorphism. Let $g \in G$ be an element of order r. We have pull-back diagrams

$$\begin{array}{ccccc} Y^g & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g\rangle, [Y/\langle g\rangle]) & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g\rangle, [Y/Z(g)]) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{id}} & \mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}\langle g\rangle) & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}Z(g)) \end{array} \tag{6.1.4}$$

The left square is a pull-back square is by definition. The right square is a pull-back square since it is obtained by applying $\mathbf{Map}(\mathsf{B}\langle g\rangle, -)$ to the following pull-back square:

$$\begin{array}{ccc} [Y/\langle g\rangle] & \longrightarrow & [Y/Z(g)] \\ \downarrow & & \downarrow \\ \mathsf{B}\langle g\rangle & \longrightarrow & \mathsf{B}Z(g) \end{array} \tag{6.1.5}$$

Now since the cotangent complex of $\mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}Z(g))$ is trivial, the bottom horizontal map

$$\mathrm{Spec}(k) \to \mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}Z(g))$$

is étale. Therefore the upper horizontal map $Y^g \to \mathbf{Map}(\mathsf{B}\langle g\rangle, [Y/Z(g)])$ is also étale.

Now note that the top map factors through the quotient $[Y^g/Z(g)]$. This follows from the fact that $\mathrm{pt} \to \mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}Z(g))$ factors through $\mathsf{B}Z(g)$. Hence

$$[Y^g/Z(g)] \to \mathbf{Map}(\mathsf{B}\langle g\rangle, [Y/Z(g)]) \to \mathsf{I}^{(r)}[Y/G]$$

is also étale, where $r = |g|$. In order to conclude the proof we only have to show that the above morphisms induce an isomorphism between the truncation of $\mathsf{I}^{\mathsf{DM}}[Y/G]$ and $\bigsqcup_{[g]\in G/G}[\mathrm{t}_0(Y)^g/Z(g)]$. This follows from the following computation

$$\mathrm{t}_0(\mathsf{I}^{\mathsf{DM}}[Y/G]) \simeq [\mathrm{t}_0 Y/G] \times_{[\mathrm{t}_0 Y/G]\times[\mathrm{t}_0 Y/G]} [\mathrm{t}_0 Y/G] \simeq \mathsf{I}[\mathrm{t}_0 Y/G] \simeq \coprod_{[g]\in G/G} [\mathrm{t}_0(Y)^g/Z(g)],$$

where we used that the fiber products commute with truncation functor t_0. □

The graded k-module $\bigoplus_{i\in\mathbb{Z}} \mathsf{HH}_{-i}([Y/G])$, obtained by taking the cohomology of the algebra object $\mathsf{HH}_*([Y/G])$, inherits a natural structure of graded k-algebra. The following corollary recovers and generalizes [7, Corollary 1.7].

Corollary 6.6. *Let Y be a finitely presented derived scheme equipped with an action of a finite group G. Then*

(1) We have an isomorphism of graded algebras

$$\begin{aligned} \bigoplus_{i\in\mathbb{Z}} \mathsf{HH}_{-i}([Y/G]) &\simeq \bigoplus_{i\in\mathbb{Z}} \left(\bigoplus_{g\in G} \bigoplus_{q-p=i} H^q(Y^g, \Omega^p_{Y^g}) \right)^G \\ &\simeq \bigoplus_{i\in\mathbb{Z}} \bigoplus_{[g]\in G/G} \bigoplus_{q-p=i} H^q(Y^g, \Omega^p_{Y^g})^{Z(g)}. \end{aligned}$$

where $\Omega^p := \bigwedge^p \mathbb{L}$ stands for the p-th term of the derived *de Rham complex.*

(2) If moreover Y is lci, we have an isomorphism of vector spaces:

$$\mathsf{HH}^i([Y/G]) \simeq \bigoplus_{[g]\in G/G} \bigoplus_{q+p=i} H^q(Y^g, \bigwedge^p \mathsf{T}_{Y^g} \otimes \det(\mathsf{N}_g)[-c_g])^{Z(g)},$$

where N_g is the normal bundle $\mathsf{N}_{Y^g/Y}$ and c_g is its rank; in other words, $\mathsf{N}_g[-1]$ denotes the relative tangent complex of the canonical morphism $Y^g \to Y$.

Proof. We apply Theorem 4.12 and Theorem 4.15 (or Theorem 4.16) to $X = [Y/G]$ and combine with Theorem 6.5. □

We have the following somewhat surprising observation: even for an underived global quotient DM stack X, when X is singular, its orbifold inertia $\mathsf{I}^{\mathsf{DM}}X$ can carry a non-trivial derived structure. Thus, even in the classical setting, the failure of being an isomorphism for the natural map $\mathsf{I}X \to \mathsf{I}^{\mathsf{DM}}X$ contains information on the singularities of the stack. We give here a simple example to illustrate this point:

Example 6.7. Consider the underived affine scheme $Y \coloneqq \mathrm{Spec}(A)$ where

$$A \coloneqq k[x,y]/(xy) \ .$$

Let $G \coloneqq \mathbb{Z}/2\mathbb{Z} = \{1, \varphi\}$ and consider the action on Y given by $\varphi \cdot \overline{x} \coloneqq -\overline{x}$ and $\varphi \cdot \overline{y} \coloneqq -\overline{y}$. We claim that Y^G has a non-trivial derived structure. In virtue of Theorem 6.5, $\mathsf{I}^{\mathsf{DM}}([Y/G])$ also has a non-trivial derived structure.

Recall that since Y is affine, the truncation of $\mathrm{t}_0(Y^G)$ coincides with the spectrum of A_G, the (classical) G-coinvariants of A. A simple computation shows that, as $\mathrm{char}(k) \neq 2$,

$$A_G = A/(\overline{x} - (-\overline{x}), \overline{y} - (-\overline{y})) = A/(\overline{x}, \overline{y}) \simeq k \ .$$

Hence $\mathrm{t}_0(Y^G)$ is isomorphic to $\mathrm{Spec}(k)$, which corresponds to the unique (reduced) singular point p_0 of Y. We are going to show that Y^G is not equivalent to $\mathrm{t}_0(Y^G)$, hence it must be derived.

Notice that the following square, where $f(x,y) \coloneqq xy$,

$$\begin{array}{ccc} Y & \xrightarrow{\ j\ } & \mathbb{A}^2_k \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{\ 0\ } & \mathbb{A}^1_k \end{array}$$

is a derived pullback square by the flatness of f. In particular,

$$\mathbb{L}_Y \simeq \mathrm{cofib}(\delta \colon j^*f^*\mathbb{L}_{\mathbb{A}^1_k} \to j^*\mathbb{L}_{\mathbb{A}^2_k}) \ .$$

Now, $j^*f^*\mathbb{L}_{\mathbb{A}^1_k} \simeq \mathcal{O}_Y \mathsf{d}t$ and $j^*\mathbb{L}_{\mathbb{A}^2_k} \simeq \mathcal{O}_Y \mathsf{d}x \oplus \mathcal{O}_X \mathsf{d}y$, where t denotes the affine coordinate on $\mathbb{A}^1_k$, and x and y denote the affine coordinates in $\mathbb{A}^2_k$. The map δ between them is induced by $\mathsf{d}t \mapsto \mathsf{d}(xy) = \overline{x}\mathsf{d}y + \overline{y}dx$. Notice that δ is injective, but its cofiber is not locally free. The fiber at the singular point p_0 is the zero map

$$k\mathsf{d}t \longrightarrow k\mathsf{d}x \oplus k\mathsf{d}y \ .$$

Thus,

$$\pi_1(p_0^*\mathbb{L}_Y) \simeq k\mathsf{d}t \ .$$

Observe now that letting G act on $\mathbb{A}^2$ by multiplication by -1 on both coordinates and trivially on $\mathbb{A}^1$, the above diagram becomes G-equivariant. In particular, $\mathbb{L}_Y$ acquires a canonical G-equivariant structure. It follows from [6, Corollaries A.27 and A.30] that

$$\mathbb{L}_{Y^G} \simeq (\mathbb{L}_Y|_{Y^G})_G \ .$$

Since G acts trivially on $\mathsf{d}t$ by construction, the above computation allows to conclude that

$$\pi_1(\mathbb{L}_{Y^G}) \simeq k\mathsf{d}t \ ,$$

whereas $\mathbb{L}_{\mathrm{t}_0(Y^G)} = 0$. In particular, Y^G has a non-trivial derived structure.

In addition to Theorem 6.1, the following notion of fixed locus is also frequently used in derived algebraic geometry :

Definition 6.8 (Naive derived fixed loci)**.** Let Y be a derived stack and g an automorphism of Y. We define the *naive derived fixed locus* $Y^{\mathbb{R}g}$ via the (derived) pull-back diagram

$$\begin{array}{ccc} Y^{\mathbb{R}g} & \longrightarrow & Y \\ \downarrow & & \downarrow \Delta \\ Y & \xrightarrow{\Delta_g} & Y \times Y \end{array} \tag{6.1.6}$$

where Δ_g is the g-twisted diagonal map (i.e. the graph of the automorphism of Y given by g), which is given by the following formula on functor of points:

$$\Delta_g : Y \to Y \times Y, \quad \Delta_g(y) \coloneqq (y, gy). \tag{6.1.7}$$

The following lemma clarifies the relation between the derived fixed loci (Theorem 6.1) and the naive derived fixed loci (Theorem 6.8) :

Lemma 6.9. *Let Y de a finitely presented derived algebraic space equipped with a finite-order automorphism g. Then the derived fixed locus Y^g is a derived algebraic space, and we have*

$$\mathsf{T}[-1]Y^g \simeq \mathscr{L}Y^g \simeq Y^{\mathbb{R}g}.$$

Proof. We give here a proof that relies on our main result Theorem 4.9. A direct proof would certainly be desirable. Let r be the order of g, and let $G = \langle g \rangle$ be the cyclic group generated by g acting naturally on Y. By Theorem 4.9 and Theorem 6.5, we have

$$\mathscr{L}[Y/G] \simeq \mathsf{T}[-1]\mathsf{I}^{\mathsf{DM}}[Y/G] \simeq \mathsf{T}[-1] \coprod_{k=1}^{r} [Y^{g^k}/G] \simeq \coprod_{k=1}^{r} [\mathsf{T}[-1]Y^{g^k}/G]$$

On the other hand,

$$\mathscr{L}[Y/G] \simeq [Y/G] \times_{[Y/G]\times[Y/G]} [Y/G] \simeq \coprod_{k=1}^{r} [Y^{\mathbb{R}g^k}/G].$$

Since the HKR isomorphism induces identity on the classical truncation, we obtain an equivalence

$$[\mathsf{T}[-1]Y^g/G] \simeq [Y^{\mathbb{R}g}/G]. \tag{6.1.8}$$

As G is generated by g, hence the G-action on Y^g is trivial. It follows that $\mathsf{T}[-1]Y^g \simeq Y^{\mathbb{R}g}$.

Since $Y^{\mathbb{R}g}$ is a derived algebraic space by construction, $\mathsf{T}[-1]Y^g$ is also a derived algebraic space. Therefore, Y^g, as a closed subscheme of $\mathsf{T}[-1]Y^g$, must be a derived scheme. Finally, applying Theorem 4.9 to Y^g, we obtain $\mathsf{T}[-1]Y^g \simeq \mathscr{L}Y^g$. □

6.2. **Mapping stacks.** Let us consider a pair given by a stack S and a derived DM stack Y such that $X \coloneqq \mathbf{Map}(S, Y)$ is still a derived DM stack. Then we have the following simple formula for its derived orbifold inertia stack.

Lemma 6.10. *Let X be as above. Then there is a natural equivalence*

$$\mathsf{I}^{\mathsf{DM}}X \simeq \mathbf{Map}(S, \mathsf{I}^{\mathsf{DM}}Y). \tag{6.2.1}$$

Proof. By definition we have that

$$\mathsf{I}^{\mathsf{DM}}X = \operatorname*{colim}_r \mathbf{Map}(\mathsf{B}\mathsf{C}_r, X) \simeq \operatorname*{colim}_r \mathbf{Map}(\mathsf{B}\mathsf{C}_r, \mathbf{Map}(S, Y)).$$

Using adjunction we can rewrite the latter, as desired, as

$$\mathbf{Map}(S, \operatorname*{colim}_r \mathbf{Map}(\mathsf{B}\mathsf{C}_r, Y)) \simeq \mathbf{Map}(S, \mathsf{I}^{\mathsf{DM}}Y) \ .$$

□

There are many interesting examples of derived DM stacks X of this kind, such as for instance:

(1) We can take S to be S^1. Then $X = \mathscr{L}Y$ is also a derived DM stack. More generally, we can take S to be the Betti stack of any finite CW complex.
(2) We can take S to be $\mathsf{B}\mathbb{G}_a$. In this case $X = \mathbf{Map}(\mathsf{B}\mathbb{G}_a, Y) \simeq \mathsf{T}[-1]Y$, which is still a derived DM stack.
(3) Examples where the source S is an algebraic curve can also be given, on condition of introducing stability conditions, or restricting to selected components of the full mapping stack (such as the sub-stack of almost constant maps considered in [52]).

In these cases, Lemma 6.10 often allows us to compute explicitly the derived orbifold inertia stack of a *derived* DM stack. We illustrate this point with an example. Let $Y = [V/G]$ be a global quotient DM stack, such that V is smooth (hence underived) and G is a finite group. We set $S = B\mathbb{G}_a$. Then we have that $X \simeq \mathsf{T}[-1][V/G]$ is equivalent to $[\mathsf{T}[-1]V/G]$, where the action of G on $\mathsf{T}[-1]V$ is induced from the G-action on V. Proposition 6.5 gives a formula for $\mathsf{I}^{\mathsf{DM}}X$, but it requires to calculate the derived fixed loci from first principles, a subtle task in the case of actions on derived schemes. Thanks to Lemma 6.10 however we can write

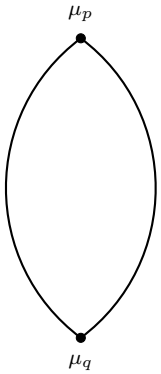
$$\mathsf{I}^{\mathsf{DM}}X \simeq \mathsf{T}[-1](\mathsf{I}^{\mathsf{DM}}[V/G]) \simeq \mathsf{T}[-1](\mathsf{I}[V/G]) \simeq \coprod_{[g]\in G/G} \mathsf{T}[-1][V^g/Z(g)]$$

where the second equivalence comes from the fact that the derived orbifold inertia of a smooth classical DM stack coincides with the classical inertia stack. So in particular we can describe $\mathsf{I}^{\mathsf{DM}}X$ in terms of the *classical* fixed loci of the G-action V^g, which are readily computable.

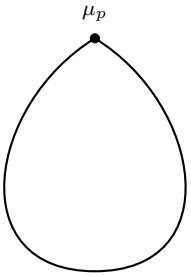
6.3. **Beyond global quotients (I) : weighted projective lines and root stacks.** Already in the category of underived classical stacks, most DM stacks are not of the form $[Y/G]$ with G a finite group acting on an algebraic space Y. Indeed, by [43, Proposition 6], any complex variety with only quotient singularities (i.e. *orbifold* in the classical sense as in [49]) admits a canonical smooth DM stack structure with only non-trivial stabilizers in codimension ≥ 2, and certainly most varieties with quotient singularities are not quotients of smooth varieties by finite groups.

Such non-global quotient examples arise naturally. Perhaps the best-known examples are Thurston's football and teardop, which also happen to be the examples that led historically to the notion of orbifold.

Example 6.11 (Thurston's football and teardrop)**.** Given two positive integers p, q that are coprime to each other, let $X = \mathbb{P}^1(p,q) = \mathrm{Proj}(k[x,y])$ be the weighted projective line with $\deg(x) = p, \deg(y) = q$. The underlying orbifold is often referred to as *Thurston's football* (or *teardrop* when $q = 1$); see the pictures below. By [9, Proposition 5.1 and Corollary 5.9], X is *not* of the form of a global quotient of a scheme by a finite group.



Football $\mathbb{P}^1(p,q)$



Teardrop $\mathbb{P}^1(p,1)$

By Theorem 3.4,

$$\mathsf{I}^{\mathsf{DM}}X \simeq \mathsf{I}X \simeq X \cup (\mathsf{I}(\mathsf{BC}_p \sqcup \mathsf{BC}_q)) \simeq X \cup ([\mathsf{C}_p/\mathsf{C}_p] \sqcup [\mathsf{C}_q/\mathsf{C}_q]) \simeq X \sqcup \bigsqcup^{p-1} \mathsf{BC}_p \sqcup \bigsqcup^{q-1} \mathsf{BC}_q \tag{6.3.1}$$

Consequently, by Theorem 4.12, we have

$$\mathsf{HH}_0(X) \simeq H^0(X, \mathcal{O}_X) \oplus H^1(X, \mathbb{L}_X) \oplus H^0(\mathsf{BC}_p, \mathcal{O}_{\mathsf{BC}_p})^{p-1} \oplus H^0(\mathsf{BC}_q, \mathcal{O}_{\mathsf{BC}_q})^{q-1} \simeq k^{p+q},$$

and $\mathsf{HH}_i(X) = 0$ for any $i \neq 0$ since $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X, \mathbb{L}_X) = 0$.

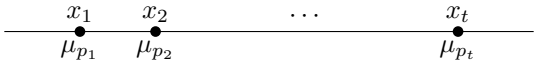
Similarly, one can compute the Hochschild cohomology using Theorem 4.16: since $\omega_X|_{\mathsf{BC}_p}$ is χ_{ζ_p}, the 1-dimensional representation of C_p given by ζ_p, we have:

$$\mathsf{HH}^*(X) \simeq H^*(X, \mathcal{O}_X) \oplus H^*(X, \mathbb{T}_X[-1]) \oplus H^*(\mathsf{BC}_p, \chi_{\zeta_p^{-1}})^{p-1} \oplus H^*(\mathsf{BC}_q, \chi_{\zeta_q^{-1}})^{q-1}.$$

As we are in characteristic zero, the last two summands vanish and $H^*(X, \mathcal{O}_X) \simeq H^*(\mathbb{P}^1, \mathcal{O})$. Hence,

$$\begin{aligned}
\mathsf{HH}^0(X) &\simeq H^0(X, \mathcal{O}_X) \simeq k \\
\mathsf{HH}^1(X) &\simeq H^1(X, \mathcal{O}_X) \oplus H^0(X, \mathbb{T}_X) \simeq k \\
\mathsf{HH}^2(X) &\simeq H^1(X, \mathbb{T}_X) = 0 \\
\mathsf{HH}^3(X) &= 0.
\end{aligned}$$

Example 6.12 (Weighted projective line)**.** A natural generalization of Theorem 6.11 is the so-called *weighted projective line*, introduced by Geigle and Lenzing in [21]. Given a collection of points $\underline{x} \coloneqq (x_1, \cdots, x_t)$ with $x_i \in \mathbb{P}^1$, and a collection of weights $\underline{p} \coloneqq (p_1, \cdots, p_t)$ with $p_i \in \mathbb{Z}_{\geq 2}$, the weighted projective line $X \coloneqq \mathbb{X}(\underline{x}, \underline{p})$, defined in [21], is obtained by attaching weight p_i to the point x_i; we depict it as follows:



Weighted projective line

As in Theorem 6.11, by Theorem 3.4, we have

$$\mathsf{I}^{\mathsf{DM}}X \simeq X \sqcup \bigsqcup_{i=1}^{t} \bigsqcup^{p_i-1} \mathsf{BC}_{p_i}. \tag{6.3.2}$$

Therefore,

$$\mathsf{HH}_0(X) = H^0(X, \mathcal{O}_X) \oplus H^1(X, \mathbb{L}_X) \oplus \bigoplus_{i=1}^{t} H^0(\mathsf{BC}_{p_i}, \mathcal{O}_{\mathsf{BC}_{p_i}})^{\oplus(p_i-1)} = k^{(\sum_i p_i)-t+2},$$

and $\mathsf{HH}_i(X) = 0$ for any $i \neq 0$.

Similar to Theorem 6.12, the Hochschild cohomology X can be computed using Theorem 4.16 as follows:

$$\mathsf{HH}^*(X) \simeq H^*(X, \mathcal{O}_X) \oplus H^*(X, \mathbb{T}_X[-1]) = \begin{cases} k \oplus k^2[-1] & \text{if } t = 1 \\ k \oplus k[-1] & \text{if } t = 2 \\ k & \text{if } t = 3 \\ k \oplus k^{t-3}[-2] & \text{if } t \geq 4 \end{cases} \tag{6.3.3}$$

The computation (6.3.3) was previously obtained by Happel [25] using piecewise hereditary algebras. Our geometric method is apparently different, and can be easily adapted beyond the case of piecewise hereditary algebras, e.g. to compute the Hochschild homology and cohomology

of stacky curves (see [51] for an indirect approach using [7]), or more generally, to root stacks, as we now turn to.

Example 6.13 (Root stacks)**.** Theorem 6.11 and Theorem 6.12 are one-dimensional instances of a general construction called *root stack*. Root stacks are DM stacks carrying universal roots of Cartier divisors (and more generally log structures) on schemes. These objects have been first introduced by Cadman [16], and have been much studied since; see [55] and references therein for an overview of the subject. For concreteness, let us restrict to the case of the root stack of a scheme X equipped with a single smooth Cartier divisor D. This is the same as the datum of a line bundle $\mathcal{O}_X(D)$ on X and a section σ, and is therefore classified by a morphism

$$X \to [\mathbb{A}^1/\mathbb{G}_m]$$

The n-th root stack of the pair (X, D) is denoted $\sqrt[n]{(X, D)}$ and is defined by the pullback square

$$\begin{array}{ccc} \sqrt[n]{(X,D)} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle(-)^n} \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m], \end{array}$$

where the right vertical arrow is given by $z \mapsto z^n$. The stack $\sqrt[n]{(X, D)}$ can also be defined via a universal property. Indeed, the root stack is universal among stacks over X carrying an n-th root of $\mathcal{O}(D)$ and of the section σ: we denote these universal roots by

$$\mathcal{O}(D_n) \quad \text{and} \quad \sqrt[n]{\sigma} \in \mathcal{O}(D_n)$$

The zero-locus of $\sqrt[n]{\sigma}$ is the stacky divisor D_n, which fits in the following diagram

$$\begin{array}{ccccc} D_n & \longrightarrow & \widetilde{D_n} & \longrightarrow & \sqrt[n]{(X,D)} \\ & \searrow & \downarrow & & \downarrow \\ & & D & \longrightarrow & X \end{array}$$

The square on the right is a fiber product; the divisor D_n is the reduction of $\widetilde{D_n}$, which is the pull-back of D to $\sqrt[n]{(X, D)}$; the map $D_n \to D$ is a μ_n-gerbe. If the line bundle $\mathcal{O}_X(D)$ admits an n-th root, then $\sqrt[n]{(X, D)}$ can be presented as a global μ_n-quotient but this fails in general. The teardrop $\mathbb{P}^1(p, 1)$ is equivalent to the root stack $\sqrt[p]{(X, D)}$ when $X = \mathbb{P}^1$ and D is equal to a point.

The inertia stack of $\sqrt[n]{(X, D)}$ can be computed as

$$\mathsf{I}^{\mathsf{DM}} \sqrt[n]{(X, D)} \simeq \sqrt[n]{(X, D)} \sqcup \Big(\coprod_{i=1}^{n-1} D_n \Big). \tag{6.3.4}$$

Thanks to Theorem 4.9, we obtain a decomposition

$$\mathscr{L} \sqrt[n]{(X, D)} \simeq \mathsf{T}[-1] \sqrt[n]{(X, D)} \sqcup \Big(\coprod_{i=1}^{n-1} \mathsf{T}[-1] D_n \Big). \tag{6.3.5}$$

Proposition 6.14. *Let D be a smooth divisor on a smooth scheme X. Let n be a positive integer. We have canonical isomorphisms:*

(1) $\Gamma(\mathsf{T}[-1] \sqrt[n]{(X, D)}, \mathcal{O}) \simeq \Gamma(\mathsf{T}[-1]X, \mathcal{O})$.
(2) $\Gamma(\mathsf{T}[-1]D_n, \mathcal{O}) \simeq \Gamma(\mathsf{T}[-1]D, \mathcal{O})$.

Proof. By the functoriality of the (shifted) tangent, $\sqrt[n]{(X,D)} \to X$ induces a morphism $\pi\colon \mathsf{T}[-1]\sqrt[n]{(X,D)} \to \mathsf{T}[-1]X$, thus a natural morphism

$$\mathcal{O}_{\mathsf{T}[-1]X} \to \pi_*(\mathcal{O}_{\mathsf{T}[-1]\sqrt[n]{(X,D)}}). \tag{6.3.6}$$

To prove (1), it suffices to show that (6.3.6) is an isomorphism. We will give two proofs.

Our first proof is via local computation: it suffices to check that (6.3.6) is an isomorphism étale locally on X. Hence we can assume that the pair (X,D) is of the form

$$(X,D) = (\mathbb{A}^1 \times S, \{0\} \times S),$$

where S is a smooth variety of dimension $\dim(X)-1$. In this case, the root stack $\sqrt[n]{(X,D)}$ is isomorphic to a stack of the form

$$\sqrt[n]{(\mathbb{A}^1,\{0\})} \times S \simeq [\mathbb{A}^1/\mu_n] \times S$$

where μ_n acts by multiplication of roots of unity.

As $\mathsf{T}[-1]([\mathbb{A}^1/\mu_n] \times S) \simeq \mathsf{T}[-1][\mathbb{A}^1/\mu_n] \times \mathsf{T}[-1]S$, we are reduced to the case where

$$(X,D) = (\mathbb{A}^1,\{0\})\ .$$

Note that

$$\mathsf{T}[-1][\mathbb{A}^1/\mu_n] \simeq [\mathsf{T}[-1]\mathbb{A}^1/\mu_n]$$

where μ_n acts diagonally. Let t be the variable of $\mathbb{A}^1$ and z be the variable of $\mathbb{A}^1 := \mathbb{A}^1/\mu_n$; $z = t^n$. Then we conclude by the following computation:

$$\begin{aligned}
\Gamma([\mathsf{T}[-1]\mathbb{A}^1/\mu_n],\mathcal{O}) &\simeq \Gamma(\mathbb{A}^1, \mathrm{Sym}(\mathbb{L}_{\mathbb{A}^1}[1]))^{\mu_n} \\
&\simeq \Gamma(\mathbb{A}^1,\mathcal{O})^{\mu_n} \oplus \Gamma(\mathbb{A}^1,\mathbb{L}_{\mathbb{A}^1}[1])^{\mu_n} \\
&\simeq k[t^n] \oplus k[t^n]\mathsf{d}(t^n) \\
&\simeq k[z] \oplus k[z]\mathsf{d}z \\
&\simeq \Gamma(\mathbb{A}^1,\mathcal{O}) \oplus \Gamma(\mathbb{A}^1,\mathbb{L}_{\mathbb{A}^1}[1]) \\
&\simeq \Gamma(\mathsf{T}[-1]\mathbb{A}^1,\mathcal{O}).
\end{aligned}$$

Now we give an alternative proof that (6.3.6) is an equivalence. By the definition of the root stack, and the fact that the shifted tangent preserves fiber products, we obtain a pull-back square

$$\begin{array}{ccc}
\mathsf{T}[-1]\sqrt[n]{(X,D)} & \longrightarrow & \mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m] \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\
\mathsf{T}[-1]X & \longrightarrow & \mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m]
\end{array}$$

By base change, it is sufficient to show that the push-forward along the map

$$p : \mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m] \to \mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m]$$

sends the structure sheaf to the structure sheaf.

It is not difficult to describe explicitly the stack $\mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m]$. Indeed, if Y is a scheme with an action of an algebraic group G, with Lie algebra $\mathfrak{g}$, it is always possible to construct a global atlas for $T[-1][Y/G]$. Namely, let $\mathsf{T}[-1]_G Y$ be the following fiber product

$$\begin{array}{ccc}
\mathsf{T}[-1]_G Y & \longrightarrow & \mathsf{Y} \\
\downarrow & & \downarrow \\
Y \times \mathfrak{g} & \longrightarrow & \mathsf{T}Y
\end{array}$$

where the right vertical arrow is the zero section, and the bottom arrow is given by the infinitesimal action. Then $\mathsf{T}[-1]_G Y$ carries a natural G-action, and we have an equivalence[3]

$$[\mathsf{T}[-1]_G Y/G] \simeq \mathsf{T}[-1][Y/G]$$

It is easy to work out this construction in the case of the multiplicative action of $\mathbb{G}_m$ on $\mathbb{A}^1$. Consider the variety

$$C = \{xy = 0\} \subset \mathbb{A}^1_x \times \mathbb{A}^1_y$$

and equip it with a $\mathbb{G}_m$-action which scales the x-direction with weight 1, and acts trivially on the y-coordinate. One can see that

$$\mathsf{T}[-1]_{\mathbb{G}_m}\mathbb{A}^1 \simeq C \quad \text{and} \quad \mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m] \simeq [C/\mathbb{G}_m]$$

In particular, let

$$\iota : \mathsf{T}[-1]B\mathbb{G}_m \simeq \mathbb{A}^1 \times \mathsf{B}\mathbb{G}_m \to \mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m]$$

be the inclusion. Then the map

$$\iota^* : \mathcal{O}(\mathsf{T}[-1][\mathbb{A}^1/\mathbb{G}_m]) \to \mathcal{O}(\mathsf{T}[-1]\mathsf{B}\mathbb{G}_m)$$

is an equivalence. By applying again base change, we reduce to prove that the derived push-forward along the map

$$q = p|_{\mathsf{T}[-1]\mathsf{B}\mathbb{G}_m} : \mathsf{T}[-1]\mathsf{B}\mathbb{G}_m \simeq \mathbb{A}^1 \times \mathsf{B}\mathbb{G}_m \to \mathsf{T}[-1]\mathsf{B}\mathbb{G}_m \simeq \mathbb{A}^1 \times \mathsf{B}\mathbb{G}_m$$

sends the structure sheaf to the structure sheaf. Note that q is the map induced at the level shifted tangent bundles by the map

$$(-)^n : \mathsf{B}\mathbb{G}_m \to \mathsf{B}\mathbb{G}_m$$

which is formally étale. Thus its differential, i.e. the induced map between the (shifted) tangents, is an equivalence. That is, q is an equivalence when restricted to the linear factor

$$\mathbb{A}^1 \simeq \mathsf{T}[-1]_*\mathsf{B}\mathbb{G}_m \subset \mathsf{T}[-1]\mathsf{B}\mathbb{G}_m \simeq \mathbb{A}^1 \times \mathsf{B}\mathbb{G}_m$$

This implies in particular that, as claimed, there is an equivalence

$$q_*(\mathcal{O}_{(\mathsf{T}[-1]\mathsf{B}\mathbb{G}_m)}) \simeq \mathcal{O}_{(\mathsf{T}[-1]\mathsf{B}\mathbb{G}_m)}$$

which is what we needed to show.

To show (2), it suffices to notice that

$$\mathsf{T}[-1]D_n \to \mathsf{T}[-1]D$$

is a μ_n-gerbe hence the derived push-forwards sends the structure sheaf to structure sheaf. □

Combining (6.3.5) and Theorem 6.14, we obtain the following consequence:

Corollary 6.15. *Let D be a smooth divisor in a smooth scheme X. Let n be a positive integer. We have a canonical isomorphism*

$$\mathsf{HH}_*(\sqrt[n]{(X,D)}) \simeq H^*(X, \mathrm{Sym}(\mathbb{L}_X[1])) \oplus H^*(D, \mathrm{Sym}(\mathbb{L}_D[1]))^{\oplus n-1}. \tag{6.3.7}$$

In other words, we have a natural isomorphism of Hochschild homology groups that is compatible with HKR decompositions:

$$\mathsf{HH}_*(\sqrt[n]{(X,D)}) \simeq \mathsf{HH}_*(X) \oplus \bigoplus_{i=1}^{n-1} \mathsf{HH}_*(D). \tag{6.3.8}$$

[3]This is a small variation on the well-known construction of the atlas of the loop space of a global quotient stack which is well documented in the literature, see for instance Proposition 2.1.8 in [17]; the details of the construction in the shifted tangent case will be spelled out in the forthcoming [48].

Remark 6.16. The decomposition (6.3.8) of the Hochschild homology in Theorem 6.15, regardless of the compatibility with HKR decompositions, was already known through non-commutative methods. Indeed, Hochschild homology being additive with respect to semi-orthogonal decompositions, (6.3.8) follows from the natural semi-orthogonal decomposition of the category of perfect complexes on the root stack $\sqrt[n]{(X,D)}$:

$$\mathrm{Perf}(\sqrt[n]{(X,D)}) \simeq \langle \mathrm{Perf}(X), \underbrace{\mathrm{Perf}(D), \cdots, \mathrm{Perf}(D)}_{n-1 \text{ copies}} \rangle. \tag{6.3.9}$$

See [28], [13], [15] and [50] for details.

Note that computation of Hochschild cohomology is often much more challenging than that of Hochschild cohomology, since it is not functorial and not additive with respect to semi-orthogonal decompositions. Nevertheless, our main theorem also allows us to compute the Hochschild cohomology of root stacks. We stress that the following result is *not* a consequence of (6.3.9).

Proposition 6.17 (Hochschild cohomology of root stacks)**.** *Let D be a smooth divisor on a smooth scheme X. Let n be a positive integer.*

$$\mathsf{HH}^*(\sqrt[n]{(X,D)}) \simeq \Gamma(\sqrt[n]{(X,D)}, \mathrm{Sym}(\mathbb{T}_{\sqrt[n]{(X,D)}}[-1])) \simeq \Gamma(\mathsf{T}^*[1]\sqrt[n]{(X,D)}, \mathcal{O}). \tag{6.3.10}$$

Moreover, if $n = 2$,

$$\mathsf{HH}^*(\sqrt{(X,D)}) \simeq \mathsf{Fib}(\Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-1])) \to \Gamma(D, \mathrm{Sym}(\mathbb{T}_D(D)[-1])[-1])). \tag{6.3.11}$$

Proof. By Theorem 4.15 and (6.3.4), we have a decomposition:

$$\mathsf{HH}^*(\sqrt[n]{(X,D)}) \simeq \Gamma(\sqrt[n]{(X,D)}, \mathrm{Sym}(\mathbb{T}_{\sqrt[n]{(X,D)}}[-1])) \oplus \Gamma(D_n, \mathrm{Sym}(\mathbb{T}_{D_n}[-1]) \otimes \mathsf{N}[-1]), \tag{6.3.12}$$

where N is the normal bundle of D_n in $\sqrt[n]{(X,D)}$.

To show (6.3.10), it suffices to prove the vanishing of the sumand $\Gamma(D_n, \mathrm{Sym}(\mathbb{T}_{D_n}[-1]) \otimes \mathsf{N}[-1])$, for which it is enough to show that $f_*(\mathrm{Sym}(\mathbb{T}_{D_n}[-1]) \otimes \mathsf{N}) = 0$ on D, where $f\colon D_n \to D$ is the natural μ_n-gerbe map. Since this vanishing can be checked étale locally on D, we are reduced to the case where the line bundle $\mathcal{O}_X(D)$ has an n-th root L. In this case, let $\pi\colon \widetilde{X} \to X$ be the n-fold cyclic cover of X branched along D and let R be the ramification divisor, we have $\sqrt[n]{(X,D)} \simeq [\widetilde{X}/\mu_n]$, $D_n \simeq [R/\mu_n]$ and π restricts to an isomorphism $\phi\colon R \to D$. Therefore

$$f_*(\mathrm{Sym}(\mathbb{T}_{D_n}[-1]) \otimes \mathsf{N}) \simeq \phi_*(\mathrm{Sym}(\mathbb{T}_R[-1]) \otimes \mathsf{N}_{R/\widetilde{X}})^{\mu_n} \simeq \mathrm{Sym}(\mathbb{T}_D[-1]) \otimes (\phi_*\mathsf{N}_{R/\widetilde{X}})^{\mu_n}.$$

However, $\phi_*\mathsf{N}_{R/\widetilde{X}} \simeq L|_D$ and the μ_n-action is by multiplication by roots of unity. Consequently, $\phi_*\mathsf{N}_{R/\widetilde{X}}^{\mu_n} = 0$.

Having proved (6.3.10), in order to show (6.3.11), it suffices to establish that when $n = 2$, for any non-negative integer p, the following fiber sequence in $\mathsf{QCoh}(X)$ (it is in fact a short exact sequence of coherent sheaves on X) :

$$0 \to \pi_*\left(\bigwedge^p \mathbb{T}_{\sqrt{(X,D)}}\right) \to \bigwedge^p \mathbb{T}_X \to i_*\left(\bigwedge^{p-1} \mathbb{T}_D\right) \otimes \mathcal{O}_X(D) \to 0. \tag{6.3.13}$$

where $i\colon D \to X$ is the closed immersion. Since all the morphisms in (6.3.13) are naturally defined, it suffices to check the exactness étale locally, hence we are reduced to the case where where the line bundle $\mathcal{O}_X(D)$ has an n-th root L. Let $\pi\colon \widetilde{X} \to X$ be the associated double cover and $\phi\colon R \to D$ be the isomorphism as before. A local computation yields for any $p \geq 1$ the following short exact sequence of coherent sheaves on $\widetilde{X}$:

$$0 \to \pi^*\Omega_X^p \to \Omega_{\widetilde{X}}^p \to i'_*\Omega_R^{p-1} \otimes \mathcal{O}_{\widetilde{X}}(-R) \to 0, \tag{6.3.14}$$

where $i' \colon R \to \widetilde{X}$ is the closed immersion. Applying the functor of derived dual to (6.3.14) and using the Grothendieck–Serre duality, we get the following short exact sequence on $\widetilde{X}$

$$0 \to \bigwedge^p \mathbb{T}_{\widetilde{X}} \to \pi^* \left(\bigwedge^p \mathbb{T}_X \right) \to i'_* \left(\bigwedge^{p-1} \mathbb{T}_R \right) \otimes \mathcal{O}_{\widetilde{X}}(2R) \to 0. \tag{6.3.15}$$

Then (6.3.13) can be deduced from (6.3.15) by applying π_* and taking μ_2-invariants. □

6.4. **Beyond global quotients (II) : quotients by algebraic groups.** Although most DM stacks are not global quotients by finite groups, in practice, by working with quotient stacks by linear algebraic groups, we can already deal with a fairly large class of DM stacks. For instance, the weighted projective line $\mathbb{P}^1(p,q)$ in Theorem 6.12 is equivalent to the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ where the 2-dimensional representation of $\mathbb{G}_m$ is of weight p, q. More generally, in [20, Theorem 2.18], it is shown that any smooth DM stack with trivial generic stabilizer is a quotient stack of the form $[Y/G]$ with G a linear algebraic group acting on an algebraic space Y. Motivated by this consideration, we compute the inertia stack as well as the Hochschild homology of derived DM stacks that are quotient stacks by linear algebraic groups.

Proposition 6.18. *Let Y be a finitely presented derived Deligne–Mumford stack equipped with an action of a linear algebraic group G. Assume that the quotient stack $[Y/G]$ is a separated derived Deligne–Mumford stack. Then there are only finitely many conjugacy classes in G with non-empty fixed loci and they are semi-simple; we denote this finite set by $\mathscr{C}$. Then we have an isomorphism*

$$\mathsf{I}^{\mathsf{DM}}[Y/G] \simeq \coprod_{[g] \in \mathscr{C}} [Y^g/Z(g)] \ , \tag{6.4.1}$$

where Y^g is the derived fixed locus and $Z(g)$ is the centralizer of g in G.
Consequently,

$$\mathscr{L}[Y/G] \simeq \coprod_{[g] \in \mathscr{C}} \mathsf{T}[-1][Y^g/Z(g)] \ . \tag{6.4.2}$$

Proof. Since $X \coloneqq [Y/G]$ is a separated Deligne–Mumford stack and G is affine, the morphism $Y \times G \to Y \times Y$ is a finite morphism. Therefore, any $[g] \in \mathscr{C}$ must be of finite order, hence semi-simple (as we are in characteristic zero). By Theorem 3.3, the classical truncation of $\mathsf{I}^{\mathsf{DM}}X$ is nothing but $\mathsf{I}[\mathrm{t}_0(Y)/G] \simeq [\mathsf{I}_G \mathrm{t}_0(Y)/G]$, where $\mathsf{I}_G \mathrm{t}_0(Y) = \mathrm{t}_0(Y \times_{Y \times Y} (Y \times G))$ is the classical universal stabilizer (a.k.a. the inertia scheme) which is finite over Y. By [19, Lemma 2.10 and Proposition 2.11] the set $\mathscr{C}$ is finite and consists of conjugacy classes of semi-simple elements.

For any $[g] \in \mathscr{C}$, i.e. Y^g is non-empty. Let r be the order of g. In the commutative diagram,

$$\begin{array}{ccccc} Y^g & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g\rangle, [Y/\langle g\rangle]) & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g\rangle, [Y/G]) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\mathrm{id}} & \mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}\langle g\rangle) & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g\rangle, \mathsf{B}G) \end{array} \tag{6.4.3}$$

the left square is a pull-back square by definition, and the right square is a pull-back square since it is obtained by applying $\mathbf{Map}(\mathsf{B}\langle g\rangle, -)$ to the following pull-back square:

$$\begin{array}{ccc} [Y/\langle g\rangle] & \longrightarrow & [Y/G] \\ \downarrow & & \downarrow \\ \mathsf{B}\langle g\rangle & \longrightarrow & \mathsf{B}G \end{array}$$

Note that the multiplication map $Z(g) \times \langle g \rangle \to G$ is a morphism of group by the definition of centralizer. This induces a morphism $\mathsf{B}Z(g) \to \mathbf{Map}(\mathsf{B}\langle g \rangle, \mathsf{B}G)$ through which the composition of the bottom row of (6.4.3) factors. As a result, we have a pull-back square:

$$\begin{array}{ccc} [Y^g/Z(g)] & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g \rangle, [Y/G]) \simeq \mathsf{I}^{(r)}[Y/G] \\ \downarrow & & \downarrow \\ \mathsf{B}Z(g) & \longrightarrow & \mathbf{Map}(\mathsf{B}\langle g \rangle, \mathsf{B}G) \end{array} \tag{6.4.4}$$

We claim that the bottom arrow (hence also the top arrow) is étale. Indeed,

$$\mathbb{L}_{\mathsf{B}Z(g)} \simeq \mathsf{Lie}(Z(g))^\vee[-1] \ ,$$

as the adjoint $Z(g)$-module, and

$$\mathbb{L}_{\mathbf{Map}(\mathsf{B}\langle g \rangle, \mathsf{B}G)}|_{\mathsf{B}Z(g)} \simeq \Gamma(\mathsf{B}\langle g \rangle, \mathsf{Lie}(G)^\vee[-1]) \simeq (\mathsf{Lie}(G)^g)^\vee[-1]$$

as $\mathsf{B}\langle g \rangle$ has no higher cohomology (we are in characteristic zero). Here g acts on $\mathsf{Lie}(G)$ by conjugation. Now the canonical isomorphism of $Z(g)$-modules

$$\mathsf{Lie}(G)^g \simeq \mathsf{Lie}(Z(g))$$

identifies the above two cotangent complexes and we conclude that the top morphism in (6.4.4) is étale.

When $[g]$ runs through the finite set $\mathscr{C}$, we get an étale morphism

$$\bigsqcup_{[g] \in \mathscr{C}} [Y^g/Z(g)] \to \mathsf{I}^{\mathsf{DM}}[Y/G]. \tag{6.4.5}$$

To check that it is an equivalence, it suffices to check that it induces an isomorphism on the classical truncation. To this end, it is enough to observe that that the following morphism is an isomorphism (see also [29, Lemma 2.3]):

$$\begin{aligned} \bigsqcup_{[g] \in \mathscr{C}} G \times^{Z(g)} \mathrm{t}_0(Y)^g &\to \mathsf{I}_G \mathrm{t}_0(Y) \\ (h, y) &\mapsto (hgh^{-1}, hy) \end{aligned}$$

where $Z(g)$ acts on $G \times Y^g$ via $t.(h, y) \coloneqq (ht^{-1}, ty)$. The last assertion in the statement follows from Theorem 4.9. □

Corollary 6.19. *Let Y be a finitely presented derived algebraic space equipped with an action of a linear algebraic group G. Assume that the quotient stack $[Y/G]$ is a separated Deligne–Mumford stack. Then there is an isomorphism of graded algebras*

$$\bigoplus_{i \in \mathbb{Z}} \mathsf{HH}_{-i}([Y/G]) \simeq \bigoplus_{i \in \mathbb{Z}} \bigoplus_{[g] \in \mathscr{C}} \bigoplus_{q-p=i} H^q([Y^g/Z(g)], \Omega^p_{[Y^g/Z(g)]}) \ ,$$

where $\Omega^p \coloneqq \bigwedge^p \mathbb{L}$ stands for the p-th term of the derived *de Rham complex, and Y^g is the* derived *fixed locus.*

Proof. It follows from the combination of Theorem 4.12 and Theorem 6.18. □

Appendix A. Recollection on derived fixed points

We collect in this appendix some basic facts about fixed points for group actions in the setting of derived algebraic geometry. All the results are standard. See [6, Appendix A].

A.1. **Weil restrictions.** Let $f\colon X \to Y$ be a morphism of derived stacks. Fiber product along f induces a functor

$$f^*\colon (\mathsf{dSt}_k)_{/Y} \longrightarrow (\mathsf{dSt}_k)_{/X} \ ,$$

which admits a left adjoint $f_!$ (given by composition with f). Furthermore, since in an ∞-topos colimits are universal, f^* admits a right adjoint f_* as well. Unraveling the definition, we see that for every $Z \to X$ and every $S \to Y$ one has

$$\mathrm{Map}_{/Y}(S, f_*(Z)) \simeq \mathrm{Map}_{/X}(X \times_Y S, Z) \ .$$

Sometimes, $f_*(Z)$ is referred to as the *Weil restriction of Z along f* (see for instance [34, §19.1.2]). In particular, one obtains the following:

Proposition A.1. *Let $f\colon X \to Y$ be a morphism in dSt_k. For every $Z \to X$, $f_*(Z) \in (\mathsf{dSt}_k)_{/Y}$ fits in the following pullback square:*

$$\begin{array}{ccc} f_*(Z) & \longrightarrow & \mathbf{Map}_{/Y}(X, Z) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathbf{Map}_{/Y}(X, X) \ , \end{array}$$

where $\mathbf{Map}_{/Y}$ denotes the internal mapping stack in $(\mathsf{dSt}_k)_{/Y}$, and where the bottom map corresponds to the identity map of $X \simeq X \times_Y Y$.

Proof. It follows unraveling the definitions. See also [34, Proposition 19.1.2.2 & Construction 19.1.2.3]. □

In the special case where $Y = \mathrm{Spec}(k)$, we introduce the following special notation:

Definition A.2. Let $X \in \mathsf{dSt}_k$ be a derived stack. For $Z \in (\mathsf{dSt}_k)_{/X}$, we write

$$\mathbf{Sect}_X(Z) \coloneqq f_*(Z) \ ,$$

and we refer to $\mathbf{Sect}_X(Z)$ as the *stack of sections of Z over X*.

A.2. **Groups and actions.** Let now $G \in \mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_1}(\mathsf{dSt}_k)$. Recall from [33, Proposition 4.1.2.10], that there is an equivalence

$$\mathsf{Mon}_{\mathbb{E}_1}(\mathsf{dSt}_k) \simeq \text{1-}\mathsf{Seg}_*(\mathsf{dSt}_k) \ ,$$

where the right-hand side denotes the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{dSt}_k)$ spanned by simplicial objects satisfying the pointed 1-Segal condition. Concretely, G is sent to the diagram

$$G^\bullet \coloneqq \cdots \quad \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \quad G \times G \quad \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} \quad G \quad \substack{\longrightarrow\\ \longleftarrow} \quad \mathrm{Spec}(k) \tag{A.2.1}$$

See also [39, §II.1.1] for a review of this equivalence. We denote the colimit of (A.2.1) computed in dSt_k by $\mathsf{B}G$, and we refer to the canonical map $\mathrm{Spec}(k) \to \mathsf{B}G$ as the *atlas*. Since G was taken to be a group and not just a monoid, $G^\bullet$ is a *groupoid object* in dSt_k and not just a pointed 1-Segal object. In particular, since all groupoids in an ∞-topos are effective, we see that $G^\bullet$ is identified with the Čech nerve of the atlas map. This construction yields an equivalence [33, Theorem 5.2.6.15]

$$\mathsf{Mon}^{\mathsf{gp}}_{\mathbb{E}_1}(\mathsf{dSt}_k) \simeq \mathsf{dSt}_*^{\geqslant 1} \ ,$$

where we set $\mathsf{dSt}_* \coloneqq (\mathsf{dSt}_k)_{\mathrm{Spec}(k)/}$.

We define the ∞-category of *derived G-stacks* (or *derived stacks equipped with a G-action*, as the ∞-category

$$G\text{-}\mathsf{dSt}_k \coloneqq (\mathsf{dSt}_k)_{/\mathsf{B}G} \ .$$

Given a derived G-stack $\mathcal{X} \to \mathsf{B}G$, write

$$X \coloneqq \mathcal{X} \times_{\mathsf{B}G} \mathrm{Spec}(k) \ ,$$

where $\mathrm{Spec}(k) \to \mathsf{B}G$ is the atlas, and we refer to this as the *derived stack underlying* $\mathcal{X}$.

Remark A.3. Let $\check{\mathcal{C}}^\bullet(G;X)$ be the Čech nerve of the canonical map $X \to \mathcal{X}$. Then $\check{\mathcal{C}}^\bullet(G;X)$ is a groupoid by [31, Proposition 6.1.2.11]. Furthermore:

(1) the geometric realization $|\check{\mathcal{C}}_\bullet(G;X)|$ is canonically equivalent to $\mathcal{X}$;

(2) the canonical morphism

$$\check{\mathcal{C}}^\bullet(G;X) \longrightarrow G^\bullet$$

satisfies the relative left 1-Segal condition (see [39, Definition II.1.24]).

In other words, $\check{\mathcal{C}}^\bullet(G;X)$ equips X with the structure of a left representation of G (in the sense e.g. of [33, Definition 4.2.2.2]). Combining property (1) with the fact that groupoids are effective in any ∞-topos, we deduce that vice-versa every left G-representation arises in this way.

Notation A.4. If X carries the structure of a left G-representation, encoded in a relative left 1-Segal object $\mathsf{R}^\bullet(G;X) \to G^\bullet$, we write

$$[X/G] \coloneqq |\mathsf{R}^\bullet(G;X)| \in (\mathsf{dSt}_k)_{/\mathsf{B}G}$$

for its geometric realization. The previous remark implies then that $\mathsf{R}^\bullet(G;X) \simeq \check{\mathcal{C}}^\bullet(G;[X/G])$, and vice-versa that every $\mathcal{X} \in (\mathsf{dSt}_k)_{/\mathsf{B}G}$ can be uniquely presented in this form.

Definition A.5. Let $[X/G] \in G\text{-}\mathsf{dSt}_k$. Its derived fixed locus is defined by

$$X^G \coloneqq \mathbf{Sect}_{\mathsf{B}G}([X/G]) \ .$$

We can give a more explicit formula for X^G in terms of Čech descent over $\mathsf{B}G$.

Lemma A.6. *Let $[X/G] \in G\text{-}\mathsf{dSt}_k$. Then*

$$X^G \simeq \lim_{n \in \mathbf{\Delta}} \mathbf{Map}(G^n, X) \ .$$

Proof. Since dSt_k is an ∞-topos, we see that

$$(\mathsf{dSt}_k)_{/\mathsf{B}G} \simeq \lim_{[n] \in \mathbf{\Delta}} (\mathsf{dSt}_k)_{/G^n} \ .$$

Write $p\colon \mathsf{B}G \to \mathrm{Spec}(k)$ and $p_n\colon G^n \to \mathrm{Spec}(k)$ for the structural maps. By [40, Appendix B], we can therefore write

$$p_*([X/G]) \simeq \lim_{[n] \in \mathbf{\Delta}} p_{n,*}(G^n \times_{\mathsf{B}G} [X/G]) \ .$$

Since $\check{\mathcal{C}}^\bullet(G;[X/G]) \simeq G^\bullet \times_{\mathsf{B}G} [X/G]$, it follows that

$$G^n \times_{\mathsf{B}G} [X/G] \simeq G^n \times X \ .$$

Then the formula for $p_{n,*}$ given in Theorem A.1 immediately implies that

$$p_{n,*}(G^n \times X) \simeq \mathbf{Map}(G^n, X) \ ,$$

whence the conclusion. □

We now provide a formula for the cotangent complex of X^G.

Construction A.7. Let $[X/G] \in G\text{-}\mathsf{dSt}_k$. We denote by $j\colon X^G \times \mathsf{B}G \to [X/G]$ the composition

$$X^G \times \mathsf{B}G \longrightarrow \mathbf{Map}(\mathsf{B}G, [X/G]) \times \mathsf{B}G \xrightarrow{\mathrm{ev}} [X/G] \ ,$$

where the first map is induced by the upper horizontal morphism in the cartesian square provided by Theorem A.1, and the second map is the evaluation. Notice that this is a morphism in $G\text{-}\mathsf{dSt}_k$. Pulling back along the atlas $\mathrm{Spec}(k) \to \mathsf{B}G$, we obtain a canonical morphism

$$\widetilde{j}\colon X^G \longrightarrow X \ .$$

Lemma A.8. *Write $\pi\colon X^G \times \mathsf{B}G \to X^G$ for the projection on the first factor. Assume that:*

(1) $\mathsf{B}G$ is $\otimes$-universal and categorically perfect;

(2) $\mathsf{B}G$ admits a global cotangent complex and is infinitesimally cohesive;

(3) $[X/G]$ admits a global cotangent complex and is infinitesimally cohesive.

Then X^G admits a global cotangent complex given by the formula

$$\mathbb{L}_{X^G} \simeq \pi_+(j^*(\mathbb{L}_{[X/G]})) \ .$$

Furthermore, if in addition G is a reductive algebraic group and the tor-amplitude of $\mathbb{L}_{[X/G]}$ is contained within $[a, b]$, then the same applies to $\mathbb{L}_{X^G}$.

Proof. The assumptions allow us to invoke Theorem 2.7 to deduce that both $\mathbf{Map}(\mathsf{B}G, [X/G])$ and $\mathbf{Map}(\mathsf{B}G, \mathsf{B}G)$ admit a global cotangent complex. Thus, the same applies to the morphism $\mathbf{Map}(\mathsf{B}G, [X/G]) \to \mathbf{Map}(\mathsf{B}G, \mathsf{B}G)$. The first half of the lemma now follows from the base-change supplied by Theorem 2.4. The second half, is a consequence of reductivity, as in this case π_+ is t-exact. □

Corollary A.9. *Let G be a reductive group and let $[X/G] \in G\text{-}\mathsf{dSt}_k$. Assume that X is a smooth (resp. derived lci) higher Artin stack. Then X^G is a smooth (resp. derived lci) higher Artin stack.*

Proof. The fact that X^G is again a higher Artin stack follows combining Theorem A.1 with [24, Proposition 4.3.4 & Theorem 5.11]. The second half of the conclusion now is implied directly by Theorem A.8. □

Remark A.10. After unraveling the definitions, we see that Theorem A.9 improves the classical result of Graber and Pandharipande [22, Proposition 1]. More specifically, if one assumes, in their notation, that the perfect obstruction theory of X arises from a G-equivariant derived lci enhancement $\widetilde{G}$, then it follows from Theorem A.9 that $\widetilde{X}^G$ is a derived lci enhancement of X^G, and from the formula supplied by Theorem A.8 one verifies that the induced perfect obstruction theory on X^G coincides with the one provided by [22, Proposition 1].

References

1. D. Abramovich, T. Graber, and A. Vistoli, *Algebraic orbifold quantum products*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 1–24. MR 1950940
2. ———, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398. MR 2450211
3. M. André, *Homologie des algèbres commutatives*, Die Grundlehren der mathematischen Wissenschaften, Band 206, Springer-Verlag, Berlin-New York, 1974. MR 352220
4. Benjamin Antieau, Bhargav Bhatt, and Akhil Mathew, *Counterexamples to Hochschild-Kostant-Rosenberg in characteristic p*, Forum of Mathematics, Sigma, vol. 9, Cambridge University Press, 2021, p. e49.
5. Benjamin Antieau and Gabriele Vezzosi, *A remark on the Hochschild-Kostant-Rosenberg theorem in characteristic p*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **20** (2020), no. 3, 1135–1145. MR 4166802
6. D. Aranha, A. A. Khan, A. Latyntsev, H. Park, and C. Ravi, *Virtual localization revisited*, Adv. Math. **479** (2025), no. part A, Paper No. 110434, 53. MR 4933128

7. D. Arinkin, A. Căldăraru, and M. Hablicsek, *Formality of derived intersections and the orbifold HKR isomorphism*, J. Algebra **540** (2019), 100–120. MR 4003476
8. Kai Behrend, *Cohomology of stacks*, Intersection theory and moduli, ICTP Lect. Notes **19** (2004), 249–294.
9. Kai Behrend and Behrang Noohi, *Uniformization of Deligne-Mumford curves*, J. Reine Angew. Math. **599** (2006), 111–153. MR 2279100
10. D. Ben-Zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966. MR 2669705
11. D. Ben-Zvi and D. Nadler, *Loop spaces and connections*, J. Topol. **5** (2012), no. 2, 377–430. MR 2928082
12. D. Bergh and O. M. Schnürer, *Decompositions of derived categories of gerbes and of families of Brauer-Severi varieties*, Doc. Math. **26** (2021), 1465–1500. MR 4334847
13. Daniel Bergh, Valery A Lunts, and Olaf M Schnürer, *Geometricity for derived categories of algebraic stacks*, Selecta Mathematica **22** (2016), no. 4, 2535–2568.
14. F. Binda and M. Porta, *GAGA problems for the Brauer group via derived geometry*, arXiv preprint arXiv:2107.03914 (2021).
15. Agnieszka Bodzenta and Will Donovan, *Root stacks and periodic decompositions*, manuscripta mathematica **175** (2024), no. 1, 53–73.
16. Charles Cadman, *Using stacks to impose tangency conditions on curves*, American journal of mathematics **129** (2007), no. 2, 405–427.
17. Harrison Chen, *Equivariant localization and completion in cyclic homology and derived loop spaces*, Advances in Mathematics **364** (2020), 107005.
18. Alain Connes, *Non-commutative differential geometry*, Publications Mathematiques de l'IHES **62** (1985), 41–144.
19. D. Edidin, T. J. Jarvis, and T. Kimura, *Logarithmic trace and orbifold products*, Duke Math. J. **153** (2010), no. 3, 427–473. MR 2667422
20. Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. MR 1844577
21. Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297. MR 915180
22. T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518. MR 1666787
23. Paul Hacking, *The moduli space of curves is rigid*, Algebra Number Theory **2** (2008), no. 7, 809–818. MR 2460695
24. D. Halpern-Leistner and A. Preygel, *Mapping stacks and categorical notions of properness*, Compos. Math. **159** (2023), no. 3, 530–589. MR 4560539
25. Dieter Happel, *Hochschild cohomology of piecewise hereditary algebras*, Colloq. Math. **78** (1998), no. 2, 261–266. MR 1659132
26. G. Hochschild, B. Kostant, and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408. MR 142598
27. Marc Hoyois, *The homotopy fixed points of the circle action on Hochschild homology*, arXiv preprint arXiv:1506.07123 (2015).
28. Akira Ishii and Kazushi Ueda, *The special mckay correspondence and exceptional collections*, Tohoku Mathematical Journal, Second Series **67** (2015), no. 4, 585–609.
29. T. Kinjo, *Multiplicative dimensional reduction*, arXiv preprint arXiv:2511.16342 (2025).
30. Jean-Louis Loday, *Cyclic homology*, vol. 301, Springer Science & Business Media, 2013.
31. J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
32. ———, *Derived algebraic geometry VIII. Quasi-coherent sheaves and Tannaka duality theorems*, `math.ias.edu/~lurie/papers/DAG-VIII.pdf`, May 2011.
33. ———, *Higher algebra*, `math.ias.edu/~lurie/papers/HA.pdf`, September 2017.
34. ———, *Spectral algebraic geometry*, `math.ias.edu/~lurie/papers/SAG-rootfile.pdf`, February 2018.
35. Tasos Moulinos, Marco Robalo, and Bertrand Toën, *A universal Hochschild-Kostant-Rosenberg theorem*, Geom. Topol. **26** (2022), no. 2, 777–874. MR 4444269
36. F. Naef and P. Safronov, *Simple homotopy invariance of the loop coproduct*, arXiv preprint arXiv:2406.19326 (2024).
37. Shinnosuke Okawa and Taro Sano, *On the noncommutative rigidity of the moduli stack of stable pointed curves*, (2019).
38. M. Porta, *Comparison results for derived Deligne-Mumford stacks*, Pacific J. Math. **287** (2017), no. 1, 177–197. MR 3613438

39. ———, *Derived methods in moduli theory*, HAL, tel-04984799 (2025).
40. M. Porta and T. Y. Yu, *Higher analytic stacks and GAGA theorems*, Adv. Math. **302** (2016), 351–409. MR 3545934
41. ———, *Derived Hom spaces in rigid analytic geometry*, Publ. Res. Inst. Math. Sci. **57** (2021), no. 3-4, 921–958. MR 4322003
42. ———, *Non-archimedean quantum K-invariants*, Ann. Sci. Éc. Norm. Supér. (4) **57** (2024), no. 3, 713–786. MR 4773296
43. David Prill, *Local classification of quotients of complex manifolds by discontinuous groups*, Duke Math. J. **34** (1967), 375–386. MR 210944
44. D. Quillen, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Proc. Sympos. Pure Math., XVII, Amer. Math. Soc., Providence, RI, 1970, pp. 65–87. MR 257068
45. Marco Robalo, *Choices of HKR isomorphisms*, Math. Res. Lett, to appear (2025).
46. Matthieu Romagny, *Group actions on stacks and applications*, Michigan Math. J. **53** (2005), no. 1, 209–236. MR 2125542
47. Nick Rozenblyum, *Connections on moduli spaces and infinitesimal hecke modifications*, arXiv preprint arXiv:2108.07745 (2021).
48. Kamil Rychlewicz, Sarah Scherotzke, Nicolò Sibilla, and Paolo Tomasini, *In preparation.*
49. I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. MR 79769
50. Sarah Scherotzke, Nicolo Sibilla, and Mattia Talpo, *Parabolic semi-orthogonal decompositions and kummer flat invariants of log schemes*, Documenta Mathematica **25** (2020), 955–1009.
51. Felix Schremmer, *Weighted projective lines and Hochschild cohomology*, Master thesis, arXiv: 2512.08414 (2025).
52. Nicolo Sibilla and Paolo Tomasini, *Equivariant elliptic cohomology and mapping stacks i*, arXiv preprint arXiv:2303.10146 (2023).
53. The Stacks project authors, *The stacks project*, `https://stacks.math.columbia.edu`, 2025.
54. Richard G. Swan, *Hochschild cohomology of quasiprojective schemes*, J. Pure Appl. Algebra **110** (1996), no. 1, 57–80. MR 1390671
55. Mattia Talpo and Angelo Vistoli, *Infinite root stacks and quasi-coherent sheaves on logarithmic schemes*, Proceedings of the London Mathematical Society **116** (2018), no. 5, 1187–1243.
56. B. Toën, *Champs affines*, Selecta Math. (N.S.) **12** (2006), no. 1, 39–135. MR 2244263
57. B. Toën and G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224. MR MR2394633
58. B. Toën and G. Vezzosi, *Algèbres simpliciales S^1-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs*, Compos. Math. **147** (2011), no. 6, 1979–2000. MR 2862069
59. Boris L Tsygan, *The homology of matrix lie algebras over rings and the hochschild homology*, Russian Mathematical Surveys **38** (1983), no. 2, 198.
60. Charles A. Weibel and Susan C. Geller, *étale descent for Hochschild and cyclic homology*, Comment. Math. Helv. **66** (1991), no. 3, 368–388. MR 1120653
61. Amnon Yekutieli, *The continuous Hochschild cochain complex of a scheme*, Canadian Journal of Mathematics **54** (2002), no. 6, 1319–1337.

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