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The Canonical Embedding of Stable Curves

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Introduction

In this paper we will examine the canonical map of a stable curve.
Our main reference for definitions, conventions and general geometrical theorems will be [Ha].
We will assume throughout this paper:
(i) k is an algebraically closed field.
(ii) A curve is a reduced, connected scheme of dimension 1, that is proper over k.
(iii) A smooth curve is a curve that is regular in every point.
(iv) A semi-stable curve is a curve that has only ordinary double points as singularities.
(v) A semi-stable curve X is called a stable curve if it satisfies the extra condition that every non singular rational component of X meets the other components of X in at least 3 points.
The main importance of stable curves is that the moduli space of smooth curves of genus $g\geq 2$ can be completed by adding stable curves (see [D,M]).

On a smooth curve C we have a canonical dualising sheaf $\omega_{C/k}=\Omega_{C/k}$. The global sections of the line bundle ω_C give the canonical morphism $\phi:C \longrightarrow \mathbb{P}^{g-1}$. ω_C is very ample if $g(C)\geq 2$ and in this case it is well known that ϕ is a closed immersion if and only if C is not hyperelliptic (see eg. [Ha]). In this case Petri has proven (see [Pe], [A,C,G,H] or [S-D]) that the homogeneous ideal of $\phi(C)$ is generated by elements of degree 2 and 3. Furthermore he gives precise conditions for when only elements of degree 2 are needed.
On a stable curve X we also have a dualising sheaf ω_X (see 1.2). We define the arithmetic genus of X to be
$\pi=\dim \Gamma(X,\omega_X)$.
Under certain conditions (see sections 1 and 2) the sheaf ω_X gives a birational morphism $\phi:X\longrightarrow \mathbb{P}^{\pi-1}$. In section 3 we will

prove, under some extra conditions, that the canonical ideal of such X is generated by elements of degree 2 and 3. (see section 3, especially theorem 3.18).
This generalises the first part of Petri's theorem.

1. Generators of $\Gamma(X,\omega_X)$

In this section we will look at some general properties of ω_X. We will exhibit a system of generators for $\Gamma(X,\omega_X)$, find the base points of this system and determine which components of X are mapped to a point by the canonical map. Most of these results can also be deduced from [Ca], especially from theorem D.
Finally we will reduce our problem to the case where X does not have any of these "bad" points or components.

1.1 Let X be a semi-stable curve over k of genus π and let $f:X'\to X$ be the normalisation of X. Let $C'_1\ldots C'_N$ be the irreducible components of X' let $g_i=g(C'_i)$ and $C_i=f(C'_i)$. Finally let $x_j,y_j\in X'$ (j=1..M) be the points such that $z_j:=f(x_j)=f(y_j)$ are the double points of X.
This notation will remain fixed throughout this paper.

1.2 The canonical invertible sheaf $\omega_{X/k}$ is equal to f_* of the sheaf of 1-forms on X' with at most simple poles in the points x_i and y_i satisfying $\mathrm{Res}_{x_i}\eta+\mathrm{Res}_{y_i}\eta=0$ for all sections η near z_i.—
So we see that we can regard $\Gamma(C'_i,\Omega_{C'_i})$ as a submodule of $\Gamma(X,\omega_X)$ (for i=1..N) and $\bigoplus_{n\geq 0}\Gamma(C'_i,\Omega_{C'_i}^{\otimes n})$ as a subalgebra of $\bigoplus_{n\geq 0}\Gamma(X,\omega_X^{\otimes n})$.
It follows that we can choose linearly independent $\omega_{i1},..,\omega_{ig_i}$ (i=1..N) in $\Gamma(X,\omega_X)$ such that $\{\omega_{i1},..,\omega_{ig_i}\}\subset\Gamma(C'_i,\Omega_{C'_i})$ is a basis.
If $p\in C'_i$ and $f(p)\in X$ is a singular point, we will speak of the pole of $\omega\in\Gamma(X,\omega_X)$ at p, meaning the pole of $f^*(\omega)$ at p.

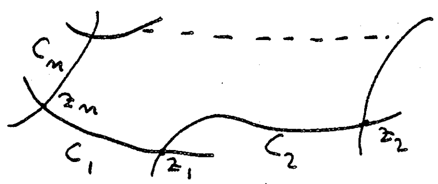
1.3 To X we will associate a graph G(X) such that the vertices of G(X) correspond to the irreducible components of X and two vertices are joined by one edge for every intersection point of the corresponding components. We will use the name C_i and z_j for the components and double points of X as well as for the vertices and edges of G(X).

Similar to the above we shall associate a subgraph $G(\omega)$ of G(X) to every $\omega \in \Gamma(X,\omega_X)$ consisting of the vertices C_i such that ω doesn't vanish on the component C_i of X and of the edges z_i such that ω has a pole in $z_i \in X$.

First we will determine for which double points z of X there exist sections in $\Gamma(X,\omega_X)$ with a pole in z. This will enable us to determine the base points of $\Gamma(X,\omega_X)$.

1.4 Lemma: *For every simple cycle* H *in* G(X) *there is an* $\omega \in \Gamma(X,\omega_X)$ *with* $G(\omega)=H$.

Proof: The situation looks like this:



Let $x_i \in C_i'$ with $f(x_i)=z_i$. There is a differential form ω_1 on C_1' with poles in x_1 and y_n (Riemann-Roch).

$\mathrm{Res}_{x_1}\omega_1 + \mathrm{Res}_{y_n}\omega_1 = 0$. On C_2' there is a differential form ω' with $\mathrm{Res}_{x_1}\omega_1 + \mathrm{Res}_{y_1}\omega' = 0$. Thus we get a section ω_2 of ω_X over $C_1 \cup C_2$. $\mathrm{Res}_{x_2}\omega_2 + \mathrm{Res}_{y_n}\omega_2 = 0$. Continuing this way we can extend to ω_n on $\bigcup_{i=1}^{n} C_i$. We have $\mathrm{Res}_{x_n}\omega_n + \mathrm{Res}_{y_n}\omega_n = 0$ so ω_n is a section in ω_X over $\bigcup_{i=1}^{n} C_i$. Extending by 0 gives the promised $\omega \in \Gamma(X,\omega_X)$.

For every cycle H of G we will choose a corresponding differential form η_H.

1.5 Lemma: *If* $\omega \in \Gamma(X,\omega_X)$ *has at least one pole and* $G(\omega)$ *is connected then* $G(\omega)$ *is not a tree.*

Proof: On any component of X where ω doesn't vanish ω must have at least two poles. But a graph every vertex of which is incident with two or more edges can not be a tree.

1.6 <u>Proposition:</u> *Let* $\omega \in \Gamma(X,\omega_X)$. *Then* ω *is a linear combination of the* $\omega_{i,j}$ *and the* η_H *where* H *is a simple cycle of* $G(\omega)$.
Proof: By induction on n, the number of poles of ω. If n=0 then ω is a linear combination of the $\omega_{i,j}$. If n>0 then choose a cycle H of $G(\omega)$ ($G(\omega)$ is not a tree) and an edge z that occurs in H. Since both ω and η_H have a pole in z there exists a $\lambda \in k$ such that $\omega-\lambda\eta_H$ doesn't have a pole in z. H and therefore $G(\omega-\lambda\eta_H)$ is a subgraph of $G(\omega)$, so $\omega-\lambda\eta_H$ has only poles in points where ω has poles. It follows that $\omega-\lambda\eta_H$ has at least one pole less than ω. By the induction hypothesis $\omega-\lambda\eta_H$ is a linear combination of the $\omega_{i,j}$ and the η_H for the cycles H of $G(\omega-\lambda\eta_H)$ and these H are also cycles of $G(\omega)$.

1.7 <u>Proposition:</u> $\Gamma(X,\omega_X)$ *is generated by the* $\omega_{i,j}$ *(*$i=1..N, j=1..g_i$*) and the* η_H *for the simple cycles* H *of* G(X).

1.8 <u>Note:</u> This may not be a basis: consider this curve:

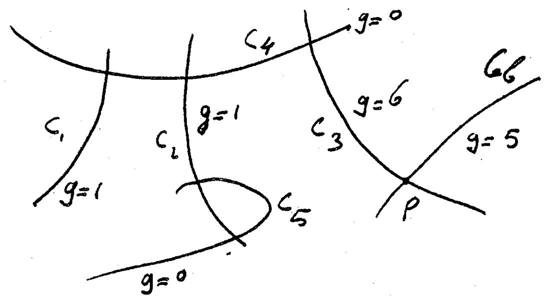
It is worth noting that $H_1(G(X),\mathbb{Z})$ is a free group, so we can choose a $\mathbb{Z}$-basis $H_1,..H_m$ of it. Then it is not difficult to show that $\eta_{H_1},..,\eta_{H_m}$ are linearly independent modulo $\bigoplus_{i=1}^{N} \Gamma(C_i', \Omega_{C_i'})$. So $\pi = \sum_{i=1}^{N} g_i + \operatorname{rank} H_1(G(X),\mathbb{Z})$.

1.9 <u>Proposition:</u> *Let X be a semi-stable curve, then the base locus of* $\Gamma(X,\omega_X)$ *consists of:*
(i) *The double points* $p \in X$ *such that the corresponding edge of* G(X) *is not contained in any cycle.*
(ii) *The rational components* C *of* X *such that all the singular points on* C *satisfy* (i).
Proof: (i) If p is a double point then p is not a base point if and only if there is a section in $\Gamma(X,\omega_X)$ with a pole in

p. According to Proposition 1.7 this is the case if and only if p is contained in a cycle of $G(X)$.

(ii) If C is a smooth curve, $g(C)>0$ then $\Gamma(C,\omega_C)$ is base-point free, so a smooth point of X can only be a base point if it lies on a rational component C_i of X. But if there is a point on C_i that is contained in a cycle H of $G(X)$ then η_H restricts to a differential form on C_i with two poles and no zeros, so p can't be a base point.

1.10 <u>Example:</u> Let X be the following curve:



The base locus is $\{p\} \cup C_4$

1.11 If the base locus of $\Gamma(X,\omega_X)$ is nonempty, then the canonical map is not defined everywhere on X. This problem can be solved in the following way: Let Y be the base locus of $\Gamma(X,\omega_X)$ and $U=X\setminus Y$. Then $\Gamma(X,\omega_X)$ gives a morphism $U \longrightarrow \mathbb{P}^{\pi-1}$. The closure of the image of this map will be the canonical image of X. Obviously the result is not changed by omitting all rational components of X as in 1.9 (ii), so we may assume that X has no such components. In this case $\dim Y=0$. Let $\mathcal{J}$ be the sheaf of ideals of $\mathcal{O}_X$ locally defined by the image of $\Gamma(X,\omega_X)$ via the map $\Gamma(X,\omega_X) \longrightarrow \Gamma(V,\omega_X) \xrightarrow{\sim} \Gamma(V,\mathcal{O}_X)$ and let $\tilde{X}$ be the blowing-up of X in $\mathcal{J}$. Then we obtain a map $\tilde{X} \longrightarrow \mathbb{P}^{\pi-1}$ (see [Ha II,§7]). The restriction of this map to U is equal to the canonical map $U \longrightarrow \mathbb{P}^{\pi-1}$. If p is a base point, then $\mathcal{J}_p$ is generated by the two local parameters at p. It follows that the fibre over p of the map $\tilde{X} \longrightarrow X$ has two points. Therefore the image of the map $\tilde{X} \longrightarrow \mathbb{P}^{\pi-1}$ is precisely the canonical image of X. We also see that $\tilde{X}$ has the same components as X and that X' is the normalisation of $\tilde{X}$. So $\tilde{X}$ is a semi-stable curve, the canonical system of $\tilde{X}$ is base-point free and since we can examine the connected components of $\tilde{X}$ separately we can assume that $\tilde{X}$ is connected. Note that $\tilde{X}$ may not be a stable curve, even if X is.

In the sequel we will, without loss of generality, assume that:

1.12 X is a semi-stable curve, $\pi \geq 2$ and for every edge of G(x) there is a (simple) cycle containing it.
As we have seen this implies that the canonical system is base-point free. It also implies that if X is not smooth then on every C_i' there are at least two points that are mapped to a singular point of X.

Let $\phi : X \longrightarrow \mathbb{P}^{\pi-1}$ be the map determined by $\Gamma(X, \omega_X)$. We will now examine which components of X are mapped to a point by ϕ.

1.13 Remark: All semi-stable curves that satisfy 1.12 but have $\pi=1$ are mapped to a point by ϕ, so this is not a very interesting case. The only such curves are:
(i) Smooth elliptic curves.
(ii) Semi-stable curves X such that G(X) is a cycle and all g_i are 0. These curves are not stable.

1.14 Proposition: *Let* X *be a semi-stable curve that satisfies* 1.12. *Then the only only components of* X *that are mapped to a point by* ϕ *are the rational components* C_i *of* X *such that only two points of* C_i' *are mapped to a double point of* X.
Proof: Because all smooth curves of genus ≥2 are not mapped to a point by their canonical system and on open subsets the canonical system of X is equal to the canonical system of a smooth curve we only need to consider the components C_i of X with $g_i=0$ or 1. If C_i is such a component of X, then X is not smooth, since $\pi \geq 2$. This means that there are two points $p, q \in C_i'$ such that f(p) and f(q) are double points of X (possibly the same). Because X satisfies the conditions of 1.12 we can choose p and q such that there is a differential form ω on X with poles in p and q.
If $g_i=1$ then we also have a differential form ω_{i1} on X, and

the restrictions to C_i of ω and ω_{i1} are independent, so C_i isn't mapped to a point.

If $g_i=0$ and there are no other points $r \in C_i'$ with $f(r)$ a singular point of X then

codim $\{\omega \in \Gamma(X,\omega_X) \mid \omega$ vanishes on $C_i\}=1$ so C_i is mapped to a point. In the case such an r does exist we have an $\omega' \in \Gamma(X,\omega_X)$ with a pole in r, so ω and ω' are independent on C_i, which implies that C_i is not mapped to a point by ϕ.

1.15 If X as in 1.12 has a rational component C_1 as in 1.14 then we proceed as follows:

Because $\pi \geq 2$ there must be two different double points on this component. If C_1 intersects other rational components of X that satisfy 1.14 then let C be the longest chain of such rational curves in X containing C_1. Then ϕ maps this entire chain to one point. Only the 'outer' curves of C intersect the rest of X, each in one point. Therefore the canonical model is the same if we replace C by one rational curve intersecting $\overline{X \setminus C}$ in these two points. So now we have reduced to the case that C_1 intersects only components of X that are not as in 1.14, say in p and q. Now we can make a curve X_1 by omitting C_1 and identifying p and q to an ordinary double point r. So $X \setminus C_1 \cong X_1 \setminus \{r\}$. Note that X_1 also satisfies 1.12. Any $\omega \in \Gamma(X,\omega_X)$ without poles in p and q corresponds (by the isomorphism $X \setminus C_1 \cong X_1 \setminus \{r\}$) to an $\omega' \in \Gamma(X_1,\omega_{X_1})$. An $\omega \in \Gamma(X,\omega_X)$ that does have poles in p and q gives a section in $\Gamma(X_1 \setminus \{r\},\omega_{X_1})$ that can be extended to X_1 by giving it a pole in r. This can also be reversed, so it gives an isomorphism $\Gamma(X,\omega_X) \cong \Gamma(X_1,\omega_{X_1})$ such that the restriction of $\omega \in \Gamma(X,\omega_X)$ to $X \setminus C_1$ is the same as the restriction of its image to $X_1 \setminus \{r\}$. From this it follows that the maps $X \setminus C_1 \rightarrow \mathbb{P}^{\pi-1}$ given by $\Gamma(X,\omega_X)$ and $X_1 \rightarrow \mathbb{P}^{\pi-1}$ given by $\Gamma(X_1,\omega_{X_1})$ are the same. So the canonical image of X is the same as that of X_1.

In view of 1.15 we can restrict our attention to curves satisfying 1.12 without the rational components of 1.14. So from now on we will assume:

1.16 X is a stable curve of genus $\pi \geq 2$ that satisfies the conditions of 1.12.

Note that demanding that a curve X as in 1.12 is stable is the same as demanding that it has no components as in 1.14.

Summarizing we have:

1.17 If X satisfies the conditions of 1.16 then the canonical map $\phi : X \longrightarrow \mathbb{P}^{\pi-1}$ is defined everywhere on X and doesn't map any components of X to a point. Moreover, for every semi-stable curve X_1 there is an X as in 1.16 with the same canonical image.

2. Hyperelliptic behaviour

In this section we will determine when the canonical map ϕ is birational and show that if ϕ is not birational then $\phi(X)$ contains a rational normal curve such that ϕ gives a double covering of this curve by one or two components of X. The main results are 2.4, 2.7, 2.9 and 2.16.
Throughout this section X will be a stable curve that satisfies the conditions of 1.16. $\phi:X \longrightarrow \mathbb{P}^{\pi-1}$ will be the canonical map.

First we will examine when ϕ maps a component C_i of X birationally to its image (with reduced induced structure) and what is the image of C_i if this is not the case.

2.1 Let C be a smooth curve, $\mathcal{L}$ be a line bundle on C and $D \in \text{div } C$, $D>0$. If $W \subset \Gamma(C,\mathcal{L})$ is a subspace then we will denote: $W(-D):=W \cap \Gamma(C,\mathcal{L}(-D))$.
We will frequently make use of the following easy lemma:
<u>Lemma:</u> *Let* C *be a smooth curve,* $\mathcal{L}$ *a line bundle on* C *and* D *an effective divisor. Let* W *and* W' *be vectorspaces such that:*

$$\begin{array}{ccc} W' & \subset & \Gamma(C,\mathcal{L}) \\ \cap & & \cap \\ W & \subset & \Gamma(C,\mathcal{L}(D)) \end{array}$$

Then: If $p,q \in C\setminus \text{Supp } D$ *such that* $\dim W'(-p-q) = \dim W'-2$ *then* $\dim W(-p-q) = \dim W -2$.

2.2 We put the following relation $\mathcal{R}_i$ on the points $x \in C_i'$ for which f(x) is a double point of X: $x\ \mathcal{R}_i\ y$ if and only if there is a simple cycle in G(X) containing f(x) and f(y).
<u>Lemma:</u> $\mathcal{R}_i$ *is an equivalence relation.*
Proof: The reflexivity and symmetry are clear, so let's prove the transitivity. Suppose $x,y,z \in C_i'$ are different points such that $x\ \mathcal{R}_i\ y$ and $y\ \mathcal{R}_i\ z$. We must show that $x\ \mathcal{R}_i\ z$ (if two are equal then there is nothing to prove). There exist cycles H containing f(x) and f(y) and H' containing f(y) and f(z). Because both η_H and $\eta_{H'}$ have a pole in y there are $\lambda,\mu \in k$ such that $\lambda\eta_H+\mu\eta_{H'}$ doesn't have a pole in y. It does however

have poles in x and z. It follows from 1.6 that there must be a simple cycle containing f(x) and f(z).

2.3 Let $D_{i\alpha} \in \operatorname{div} C_i'$ ($\alpha=1..K_i$) such that supp $D_{i\alpha}$ are the equivalence classes of $\mathcal{R}_i$. Then by 1.7 the restriction of $f^*\Gamma(X,\omega_X)$ to C_i' can be identified with
$W= \sum_\alpha \Gamma(C_i',\Omega_{C_i'}(D_{i\alpha})) \subset \Gamma(C_i',\Omega_{C_i'}(\Sigma D_{i\alpha}))$. Let L=dim W.
Next we remark that ϕ maps the component C_i of X into the linear subspace
$\mathbb{P}V= \mathbb{P}(\Gamma(X,\omega_X)/\{\omega \in \Gamma(X,\omega_X) \mid \omega|_{C_i}=0\})^* \subset \mathbb{P}\Gamma(X,\omega_X)^* = \mathbb{P}^{\pi-1}$.
We see that the the map $C_i \longrightarrow \mathbb{P}V$ given by ϕ is the same as the map $\phi_i : C_i' \longrightarrow \mathbb{P}^{L-1}$ given by W (more precisely they are equal on the open part where f gives an isomorphism of C_i' with C_i).
<u>Remark:</u> Because X satisfies 1.16 all $D_{i\alpha}$ have degree ≥ 2 and therefore all $\Gamma(C_i',\Omega_{C_i'}(D_{i\alpha}))$ are base-point free.

2.4 <u>Proposition:</u> *Let* i *be an index such that* $g_i \geq 3$ *and* C_i' *is not hyperelliptic. Then* ϕ *maps* C_i *birationally to its image.*
Proof: We have to show that ϕ_i maps C_i' birationally to its image. Let $p,q \in C_i' \setminus \bigcup$ supp $D_{i\alpha}$ then it suffices to show that dim W(-p-q)=L-2. To do this we use Lemma 2.1 with $\mathcal{L}=\Omega_{C_i'}$, $W'=\Gamma(C_i',\Omega_{C_i'})$ and $D=\Sigma D_{i\alpha}$.

Now we investigate the behaviour of ϕ on the hyperelliptic components of X.

2.5 <u>Lemma:</u> *Let* C *be a hyperelliptic curve of genus* $g \geq 2$ *and* $x,y \in C$ *such that* x+y *doesn't belong to the* g_2^1 *on* C. *Then the linear system given by* $\Gamma(C,\Omega_C(x+y))$ *separates points on* $C\setminus\{x,y\}$ *and separates tangent vectors at those points.*
Proof: We have to show that if $p,q \in C\setminus\{x,y\}$ then $l(K+x+y-p-q)=g-1$. Riemann-Roch gives:
$l(K+x+y-p-q)=g-1+l(p+q-x-y)$. So we have to show that $l(p+q-x-y)=0$. But $l(p+q-x-y)>0$ if and only if $p+q \sim x+y$ and x+y and p+q belong to the g_2^1, so this is not the case.

2.6 Again let C be a hyperelliptic curve of genus $g\geq 2$. Let E_α ($\alpha=1..K$) be divisors belonging to the g^1_2. We can conclude from the proof of 2.5 that the map given by $\Gamma(C,\Omega_C(E_\alpha))$ maps p and q to the same image if and only if $p+q$ belongs to the g^1_2. Therefore if $W=\sum_\alpha \Gamma(C,\Omega_C(E_\alpha))$ every $s \in W(-p)$ vanishes in q if and only if $p+q$ belongs to the g^1_2, so the map given by W also maps p and q to the same image if and only if $p+q$ belongs to the g^1_2.
In the same way as in [Ha,IV 5.3] we can prove that the image is a rational normal curve.

Returning to the situation of 2.4 and combining the results of 2.5 and 2.6 we have proven:

2.7 <u>Proposition:</u> *If* C_i' *is hyperelliptic then either:*
(i) *All the* $D_{i\alpha}$ *belong to the* g^1_2. *In this case* $\phi|_{C_i}$ *gives a double covering by* C_i *of a rational normal curve in a linear subspace* $\mathbb{P}V \subset \mathbb{P}^{\pi-1}$.
(ii) *For some* α $D_{i\alpha}$ *does not belong to the* g^1_2 *and* ϕ *maps* C_i *birationally to* $\phi(C_i)$.

Next we examine the elliptic components of X.

2.8 Let C be an elliptic curve. If $E \in \text{div } C$ is effective and $\deg E \geq 3$ then $\Omega_C(E)$ is very ample, so for every linear system W such that $\Gamma(C,\Omega_C(E)) \subset W \subset \Gamma(C,\Omega_C(E'))$ (for some effective divisor $E'>E$) according to Lemma 2.1 the map determined by W is birational.
Now let E_α ($\alpha=1..K$) be effective divisors of degree 2. As before in the hyperelliptic case we wish to examine $W=\sum_\alpha \Gamma(C,\Omega_C(E_\alpha))$. We see that if $p,q \in C\setminus\bigcup \text{Supp } E_\alpha$ such that $p+q \not\sim E_\beta$ for some β then $\dim \Gamma(C,\Omega_C(E_\alpha-p-q))=0$, so in this case $\dim W(-p-q)=\dim W -2$ by Lemma 2.1. It follows that if $E_\alpha \not\sim E_\beta$ for some α and β then W separates points and tangent vectors on $C\setminus\bigcup \text{Supp } E_\alpha$. On the other hand, if $E_\alpha \sim E_\beta$ for all α and β then $\dim W(-p-q)=\dim W -1$ if and only if $p+q \sim E_\alpha$, so in this case ϕ maps C_i 2 to 1 to $\phi(C_i)$. As before it is not difficult to show that $\phi(C_i)$ is a rational normal curve in

$\mathbb{P}V$.

This proves the following:

2.9 <u>Proposition:</u> *If* C_i' *is an elliptic curve then either:*

(i) *All* $D_{i\alpha}$ *belong to the same linear equivalence class. In this case* $\phi|_{C_i}: C_i \longrightarrow \phi(C_i)$ *is a double covering of a rational normal curve.*

(ii) $D_{i\alpha} \sim D_{i\beta}$ *for some* α *and* β. *Then* ϕ *maps* C_i *birationally to its image.*

Finally we come to the rational components of X.

2.10 Let C be a rational curve. As in 2.8 we see that if E is an effective divisor of degree ≥3 on C then $\Omega_C(E)$ is very ample, so if $W \supset \Gamma(C,\Omega_C(E))$ is a linear system then it determines a birational morphism (by Lemma 2.1).

So let E_α (α=2..K) be effective divisors of degree 2. We are going to examine $W=\bigoplus_\alpha \Gamma(C,\Omega_C(E_\alpha))$. We see that W is the sum of $W_\alpha=\Gamma(C,\Omega_C(E_1))\oplus\Gamma(C,\Omega_C(E_\alpha))$ for α=2..K. Every W_α is a base-point free g_2^1 and obviously if p+q doesn't belong to W_α then dim W(-p-q)=0. It follows that if $p+q \in C\setminus\bigcup \operatorname{Supp} E_\alpha$ and there exists an α such that p+q does not belong to W_α then dim W(-p-q)=dim W -2 by lemma 2.1. Therefore dim W(-p-q)=dim W -1 if and only if p+q belongs to W_α for α=2..K.

Because every W_α is a one dimensional linear subspace of the complete g_2^2 on C it is uniquely determined by two divisors contained in it. Since every W_α contains E_1 we have that $W_\alpha=W_\beta$ if and only if they have one more common divisor. It follows that if $p,q \in C\setminus\bigcup \operatorname{Supp} E_\alpha$ and dim W(-p-q)=dim W -1 then all the W_α determine the same g_2^1, so W gives a double covering by C of its image curve in a projective space. Like before we can prove that the image is a rational normal curve.

This proves (with the notation of 2.3):

2.11 <u>Proposition:</u> *If* C_i' *is a rational curve then either:*

(i) $\Gamma(C_i', \Omega_{C_i'}(D_{i1})) \oplus \Gamma(C_i', \Omega_{C_i'}(D_{i\alpha}))$ *determine the same* g_2^1 *for* $\alpha=2..K_i$. *In this case* ϕ *maps* C_i *2 to 1 to a rational normal curve.*

(ii) *In all other cases* ϕ *gives a birational morphism of* C_i *to* $\phi(C_i)$.

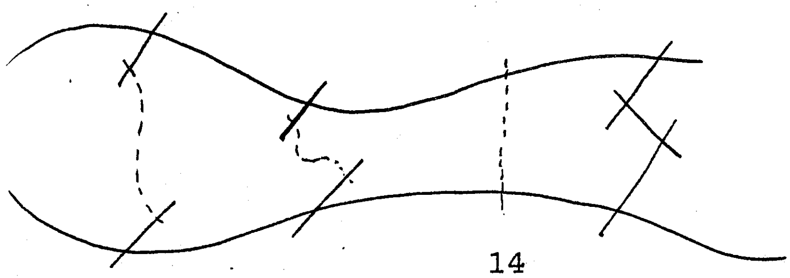
Now that we have fully investigated ϕ componentwise, the only problem left is to determine whether it is possible that two components of X have the same image.

2.12 <u>Lemma:</u> *If* $p,q \in X$ *are smooth points,* $p \in C_i$, $q \in C_{i'}$ *for* $i \neq i'$ *then* $\phi(p)=\phi(q)$ *implies* $g_i=g_{i'}=0$.

Proof: If $g_i>0$ then for some $j \leq g_i$ ω_{ij} (see 1.2) doesn't vanish in p, but it does in q, so this contradicts the fact that $\phi(p)=\phi(q)$. Therefore $g_i=0$. The same argument proves that $g_{i'}=0$.

2.13 Let C_i and C_j be two rational components of X. We want to determine exactly when C_i and C_j have smooth points that are mapped to the same image by ϕ. Obviously, for this to be the case there sould not be any differential form $\omega \in \Gamma(X,\omega_X)$ that vanishes on C_i but doesn't vanish on C_j. Therefore, if z is a double point on C_i, every simple cycle in G(X) that contains the edge corresponding to z (there must be such cycles), passes through C_j. So for every double point $z \in C_i$ there must be a simple path from C_i to C_j beginning with z. If $z_1, z_2 \in C_i$ are double points and P_1 and P_2 are simple paths from C_i to C_j such that z_i is the first edge of P_i, then P_1 and P_2 do not pass through any common vertices other then C_i and C_j: if they do then there are cycles in G(X) containing C_i, but not C_j.

So in case C_i and C_j contain smooth points with the same image, the situation looks like this:



It is easily seen from this figure that there cannot exist a k, $i\neq k\neq j$ such that there is a smooth point of C_k that is mapped to the same image as a smooth point of C_i. For in that case we have a similar situation as above for C_i and C_k, and we see that there must be a cycle in G(X) passing through C_i, but not through C_j.

2.14 If we are in the situation of the figure above, then we can choose coordinates w_i and w_j on C_i' and C_j' respectively, such that $f(0_i)$ and $f(0_j)$, $f(1_i)$ and $f(1_j)$ and $f(\infty_i)$ and $f(\infty_j)$ are double points of X that are joined by a path in G(X). Let $\lambda_4^{(i)},..,\lambda_K^{(i)}$ be the other points of C_i' such that $f(\lambda_\alpha^{(i)})$ is a singular point of X. Choose $\lambda_4^{(j)},..,\lambda_K^{(j)} \in C_j'$ such that $f(\lambda_\alpha^{(j)})$ are double points of X such that there is a simple path from $f(\lambda_\alpha^{(i)})$ to $f(\lambda_\alpha^{(j)})$. We remark that $0,1,\infty$, $\lambda_4^{(j)},..,\lambda_K^{(j)}$ are precisely the points on C_j' that are mapped to a double point of X by f. Let $\lambda_1^{(i)}=\lambda_1^{(j)}=\infty$, $\lambda_2^{(i)}=\lambda_2^{(j)}=1$ and $\lambda_3^{(i)}=\lambda_3^{(j)}=0$.
We can complete $\{\omega_{11},..,\omega_{Ng_N}\}$ with $\{\eta_1,..\eta_M\}$ $(M=\pi-\Sigma g_i)$ to a basis of $\Gamma(X,\omega_X)$, such that for $\alpha=2..K$ the restriction of $f^*(\eta_\alpha)$ to C_i' and C_j' is equal to $(w_i-\lambda_\alpha^{(i)})^{-1}dw_i$ and $(w_j-\lambda_\alpha^{(j)})^{-1}dw_j$ respectively.
Now it follows from Lemmas 2.15 and 2.1 that there exist smooth points $p \in C_i$ and $q \in C_j$ with $\phi(p)=\phi(q)$ if and only if $\lambda_\alpha^{(i)}=\lambda_\alpha^{(j)}$ (viewed as elements of k) for $\alpha=4..K$. In this case ϕ maps C_i and C_j to the same rational curve in $\mathbb{P}^{\pi-1}$. Because this curve lies in a M-dimensional linear subspace and it is the image of $\mathbb{P}^1$ under the map given by $\Gamma(\mathbb{P}^1,\mathcal{O}(M))$ it is a rational normal curve.

2.15 <u>Lemma:</u> *Let* $\lambda \in k$. $\Gamma(\mathbb{P}^1,\Omega(0+1+\lambda+\infty))$ *gives a map* $\psi_\lambda:\mathbb{P}^1\to\mathbb{P}^2$. *If we choose the base* $\{z^{-1}dz,(z-1)^{-1}dz,(z-\lambda)^{-1}dz\}$ *of* $\Gamma(\mathbb{P}^1,\Omega(0+1+\lambda+\infty))$ *then* ψ_λ *maps* $\mathbb{P}^1$ *to the curve with (affine) equation:* $x_1x_2+(1-\lambda)x_2-\lambda x_1=0$. *If* $\lambda\neq\lambda'$ *then* $\psi_\lambda(\mathbb{P}^1)\cap\psi_{\lambda'}(\mathbb{P}^1) = \{\psi_\lambda(0),\psi_\lambda(1),\psi_\lambda(\lambda),\psi_\lambda(\infty)\}$
The proof is an easy calculation.

2.16 <u>Proposition:</u> *If* $i \neq j$ *and* $p \in C_i$ *and* $q \in C_j$ *are smooth points of* X *then* $\phi(p)=\phi(q)$ *only in the following case:*

(i) $g_i=g_j=0$.

(ii) *We can choose coordinates* w_i *and* w_j *on* C_i *and* C_j *respectively such that* $0,1,\infty,\lambda_4,..,\lambda_K \in \mathbb{P}^1$ *are precisely the points of* C_i' *and* C_j' *that are mapped to the double points of* X.

(iii) *There exist disjoint simple paths* P_α *such that* P_α *starts with* $f(\lambda_\alpha^{(i)})$ *and ends with* $f(\lambda_\alpha^{(j)})$.

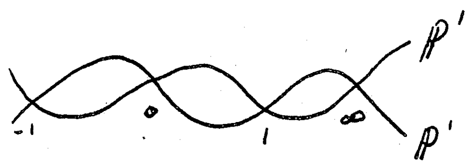
(iv) *The cycles formed by the* P_α *are the only cycles of* G(X) *passing through* C_i *or* C_j.

In this case ϕ *maps* C_i *and* C_j *birationally to the same rational normal curve.*

To conclude this section something different:

2.17 Suppose char $k \neq 2$.

The curve:



Is the subset of $\mathbb{P}^1 x\ \mathbb{P}^1$ (with coordinates $((\lambda:\mu),(\sigma:\tau))$) defined by the equation: $\lambda^2\tau^2(\tau+\sigma)^2=\mu^2\sigma^2(\sigma-\tau)^2$.

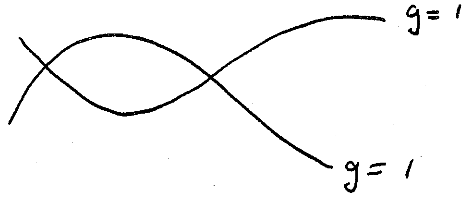
It is mapped 2 to 1 to a quadric in $\mathbb{P}^2$, so it behaves in some way like a hyperelliptic curve. We can find a flat family of smooth hyperelliptic curves in wich this curve occurs:

Consider in $\mathbb{A}^1 x\ \mathbb{P}^1 x\ \mathbb{P}^1$ (with coordinates $(a,(\lambda:\mu),(\sigma:\tau))$) the subset V defined by:

$$\lambda^2\tau(\tau+a\sigma)(\tau+\sigma)(\tau+(a+1)\sigma)=\mu^2\sigma(\sigma+a\tau)(\sigma-\tau)(\sigma-(a+1)\tau).$$

The projection $V \longrightarrow \mathbb{A}^1$ is a flat morphism, the fibre of 0 of which is our stable curve.

Another such example: Fix $\lambda,\lambda' \in k$.



The subset of $\mathbb{P}^2 x\ \mathbb{P}^1$ (with coordinates $((\xi_1:\xi_2:\xi_3),(\sigma:\tau))$) defined by: $\xi_1\xi_2=0$,

$\sigma^2(\xi_1(\xi_1-\lambda\xi_2)+\xi_2(\xi_2-\lambda'\xi_3))=$

$$\tau^2(\xi_1\xi_3+\xi_2\xi_3-\xi_3^2).$$

The canonical map maps this curve 2 to 1 to the curve defined by $\xi_1\xi_2=0$ in $\mathbb{P}^2$.

Let $W \subset \mathbb{A}^1 x\ \mathbb{P}^2 x\ \mathbb{P}^1$ (with coordinates $(a,(\xi_1:\xi_2:\xi_3),(\sigma:\tau))$) be defined by ($\lambda,\lambda' \in k$ fixed):

$$\xi_1\xi_2=a\xi_3\ ,\ \sigma^2(\xi_1(\xi_1-\lambda\xi_2)+\xi_2(\xi_2-\lambda'\xi_3))=\tau^2(\xi_1\xi_3+\xi_2\xi_3-\xi_3^2).$$

As before the projection $W \longrightarrow \mathbb{A}^1$ is a flat morphism. The fibre of 0 is the curve we started with.

3. Generators of the canonical ideal

In this section we will show that if X satisfies certain conditions (3.1) the the homogeneous ideal of the canonical image of X is generated by elements of degree 2 and 3. This generalizes in a way the theorem of Petri ([Pe]). The proof of Petri's theorem can also be found in [A,C,G,H] for $k=\mathbb{C}$ and in [S-D] for arbitrary algebraically closed fields.

Recall the notation of 1.1, 1.2 and 1.3.
For the rest of this section we will assume that:

3.1 X is a stable curve that satisfies 1.16, $g_i>2$ for i=1..N and none of the C_i' is hyperelliptic.
In this case it follows from 2.4 and 2.12 that ϕ maps X birationally to $\phi(X)$ (with reduced induced structure).

For the rest of this section we fix the following notation:

3.2 Let $M=\pi-\Sigma g_i$. It follows from 1.2 and 1.7 that we can choose $\omega_1,..,\omega_M \in \Gamma(X,\omega_X)$ such that $G(\omega_\alpha)$ is a simple cycle for $\alpha=1,..,M$ and $\{\omega_{11},..,\omega_{Ng_N},\omega_1,..,\omega_M\}$ is a basis of $\Gamma(X,\omega_X)$.
For i=1,..,N let $A_i=k[X_{i1},..,X_{ig_i}]$; let $A=\underset{i}{\otimes} A_i$ and $B=A[X_1,..,X_M]$.
Let S^*V denote the symmetric algebra on a k-vectorspace V. The identification $X_{ij}\longmapsto \omega_{ij}$, $X_i\longmapsto \omega_i$ gives isomorphisms $A_i\cong S^*\Gamma(C_i',\Omega_{C_i'})$ and $B \cong S^*\Gamma(X,\omega_X)$.
For $P \in B$ let deg P denote the degree of P considered as a polynomial over k and $\deg_A P$ its degree as a polynomial in $A[X_1,..,X_M]$.
Let $R=\underset{n\geq 0}{\oplus} \Gamma(X,\omega_X^{\otimes n})$ then we have a homomorphism of k-algebras $\theta:B\longrightarrow R$. The kernel I of this map is the homogeneous ideal of the canonical image of X.
Also, for i=1..N we have maps $\theta_i:A_i\longrightarrow R_i=\underset{n\geq 0}{\otimes} \Gamma(C_i',\Omega_{C_i'}^{\otimes n})$. Let $I_i=\ker\theta_i$. Obviously I_i is the canonical ideal of C_i'.

3.3 For $i=1,..,N$ we have an exact, commuting diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & R \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & I_i & \longrightarrow & A_i & \longrightarrow & R_i \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where the rightmost vertical arrow $R_i \hookrightarrow R$ is the inclusion of 1.2. From this diagram we see: $I_i = I \cap A_i$.

We recall:

3.4 <u>Max Noether's Theorem:</u> *The maps* θ_i *are surjective.*
The proof of this theorem can be found in [A,C,G,H] (for $k=\mathbb{C}$) and in [S-D] (k arbitrary).

3.5 <u>Lemma:</u> *Let* $n>0$ *and* $s \in \Gamma(X,\omega_X^{\otimes n})$ *be such that* s *only has poles of order* $\leq r<n$. *Then there exists a* $Q \in B$, *homogeneous of degree* n, *such that* $\deg_A Q \leq r$ *and* $s=\theta(Q)$.
Proof: By induction on r. If $r=0$ then the Lemma follows from Noether's Theorem.
So let $r>0$ and let m be the number of poles of order r that s has on X'. By decreasing induction on m we will show, that there is a $Q' \in B$ such that $s-\theta(Q')$ has only poles of order $<r$.
Let $x \in X'$ such that s has a pole in x. By 3.1 and 1.17 there must be a β such that ω_β has a pole in x. Suppose $x \in C_i'$ then there must be precisely one other point $y \in C_i'$ such that ω_β also has a pole in y (because $G(\omega_\beta)$ is a simple cycle). Now there is a $Q'' \in A_i$, homogeneous of degree $n-r>0$ such that $\theta_i(Q'')$ has a zero in y, but not in x. Therefore $\theta(Q''X_\beta^r)$ vanishes on all components but C_i, it has a pole of order r in x, a pole of order $<r$ in y and no other poles. So there is a $\lambda \in k$ such that $s-\lambda\theta(Q''X_\beta^r))$ has a pole of order $<r$ in x and poles of order r exactly in the other points where s has poles of order r. Therefore $s-\theta(Q''X_\beta^r)$ has m-1 poles of order r.

3.6 For $s \in \Gamma(X,\omega_X^{\otimes n})$, $\mathcal{P}(s)$ will denote the collection of points x of X' such that s has a pole of order n in x. For such points f(x) must be a double point.
On the collection of such points of X' we have the following relation: $x \simeq y$ if and only if $x \in \mathcal{P}(\omega)$ $(\omega \in \Gamma(X,\omega_X)) \Rightarrow$ $y \in \mathcal{P}(\omega)$.
Of course we have $x \simeq y$ if and only if every (simple) cycle of G(X) that contains f(x) also contains f(y). We remark that $\mathcal{P}(\omega) \cap \mathcal{P}(\omega') = \mathcal{P}(\omega\omega')$.

3.7 <u>Lemma:</u> $\simeq$ *is an equivalence relation.*
Proof: The reflexivity and transitivity are obvious, we have to prove the symmetry. So suppose $x \not\simeq y$ then there exists an $\omega \in \Gamma(X,\omega_X)$ such that $x \in \mathcal{P}(\omega)$, $y \notin \mathcal{P}(\omega)$. $\Gamma(X,\omega_X)$ is base-point free, so there is an $\omega' \in \Gamma(X,\omega_X)$ such that $y \in \mathcal{P}(\omega')$. If $x \notin \mathcal{P}(\omega')$ then $y \not\simeq x$ and we have what we want. If $x \in \mathcal{P}(\omega')$ then there is a $\lambda \in k$ such that $\omega'+\lambda\omega$ has no pole in x. But it has a pole in y, so $y \not\simeq x$.
Note that if K is an equivalence class for $\simeq$ then for every i it contains at most two points of C_i'.

3.8 <u>Lemma:</u> *Suppose* $x \simeq y$, *let* $Q, Q' \in B$ *homogeneous of degree* n, *such that* $x,y \in \mathcal{P}(\theta(Q)) \cap \mathcal{P}(\theta(Q'))$. *If* $\lambda, \mu \in k$ *such that* $x \notin \mathcal{P}(\theta(\lambda Q+\mu Q'))$ *then* $y \notin \mathcal{P}(\theta(\lambda Q+\mu Q'))$.
Proof: Choose local parameters t_x at x and t_y at y. We consider the ratios $(\mathrm{Res}_x f^*\omega : \mathrm{Res}_y f^*\omega)$ for all $\omega \in \Gamma(X,\omega_X)$ that have a pole at x (and since $x \simeq y$ also at y). This ratio is independent of ω, because if it is different for ω and ω', then there is a linear combination of ω and ω' with a pole in x but not in y, contradicting the fact that $x \simeq y$. Let (a:b) be this ratio.
It follows that if $S \in B$ is a monomial of degree n such that $x \in \mathcal{P}(\theta(S))$, then locally at x we have:
$f^*\theta(S)=(\alpha a^n t_x^{-n}$ +higher order terms$)(dt_x)^n$ and at y:
$f^*\theta(S)=(\alpha b^n t_y^{-n}$ +h.o.t.$)(dt_y)^n$, for some $0 \neq \alpha \in k$. Since a polynomial is the sum of monomials this is also true for polynomials, so we see that if $x \notin \mathcal{P}(\theta(\lambda Q+\mu Q'))$ then $\alpha=0$ so

$y \notin \mathcal{P}(\theta(\lambda Q+\mu Q'))$.

3.9 Lemma: *Let* K *be an equivalence class of* $\approx$. *Then there exist* $\omega_K, \omega_K' \in \Gamma(X,\omega_X)$ *such that* $G(\omega_K)$ *and* $G(\omega_K')$ *are simple cycles and* $\mathcal{P}(\omega_K\omega_K')=K$.

Proof: We have $K=\bigcap_{\omega:K\subset\mathcal{P}(\omega)} \mathcal{P}(\omega)$. This intersection only has to be take over the ω such that $G(\omega)$ is a simple cycle.
Therefore it suffices to prove:
Let G be a graph and C_1, C_2 and C_3 simple cycles in G, such that $E(C_1)\cap E(C_2)\cap E(C_3) \ni e$ for some edge e of G. Then there are simple cycles C and C' such that
$e \in E(C)\cap E(C') \subset E(C_1)\cap E(C_2)\cap E(C_3)$. The proof of this graphtheoretical lemma can be found in the appendix.
Let $U_K, U_K' \in B$ such that $\theta(U_K)=\omega_K$ and $\theta(U_K')=\omega_K'$.

3.10 Lemma: *Let* $\alpha,\beta \in \{1,..,M\}$, *then there are* $C_{\alpha\beta} \in B$, *homogeneous of degree* 2, $\deg_A C_{\alpha\beta} \le 1$ *and for every* $\approx$*-class* K *there is a* $\lambda_{\alpha\beta K} \in k$ *such that:*

(3.10.1) $\omega_\alpha\omega_\beta + \sum_K \lambda_{\alpha\beta K}\omega_K\omega_K' + \theta(C_{\alpha\beta}) = 0.$

Proof: $\mathcal{P}(\omega_\alpha\omega_\beta)=\mathcal{P}(\omega_\alpha)\cap\mathcal{P}(\omega_\beta)$ is the union of $\approx$-classes. If $K \not\subset \mathcal{P}(\omega_\alpha\omega_\beta)$ then we take $\lambda_{\alpha\beta K}=0$. If $K \subset \mathcal{P}(\omega_\alpha\omega_\beta)$ then we choose $x \in K$ and take $\lambda_{\alpha\beta K}$ such that $\omega_\alpha\omega_\beta + \lambda_{\alpha\beta K}\omega_K\omega_K'$ has a pole of order <2 at x. By Lemma 3.8, $\omega_\alpha\omega_\beta + \lambda_{\alpha\beta K}\omega_K\omega_K'$ has a pole of order <2 at all points of K. By 3.9 it has a pole of order 2 precisely at the other points where $\omega_\alpha\omega_\beta$ has a pole of order 2. So we have chosen the $\lambda_{\alpha\beta K}$ such that $\omega_\alpha\omega_\beta + \sum_K \lambda_{\alpha\beta K}\omega_K\omega_K'$ only has poles of order ≤ 1. The existence of the promised $C_{\alpha\beta}$ now follows from 3.5.

3.11 Lemma: *Let* $\alpha \in \{1,..,M\}$ *then for every* $\approx$*-class* K *there are* $D_{\alpha K} \in B$, *homogeneous of degree* 3, $\deg_A D_{\alpha K} \le 2$ *and* $\mu_{\alpha K} \in k$ *such that:*

(3.11.1) $\omega_\alpha\omega_K\omega_K' + \mu_{\alpha K}\omega_K^2\omega_K' + \theta(D_{\alpha K}) = 0.$

Proof: $\mathcal{P}(\omega_\alpha\omega_K\omega_K')=\mathcal{P}(\omega_\alpha)\cap K$ so it is either $\emptyset$ or equal to K. If $\mathcal{P}(\omega_\alpha)\cap K=\emptyset$ then we take $\mu_{\alpha K}=0$. If $\mathcal{P}(\omega_\alpha)\cap K=K$ then we choose $x \in K$ and take $\mu_{\alpha K}$ such that $\omega_\alpha\omega_K\omega_K' + \mu_{\alpha K}\omega_K^2\omega_K'$ does not have a pole of order 3 at x. By 3.8 it has poles of order ≤ 2 at all points of K, so $\mathcal{P}(\omega_\alpha \omega_K \omega_K' + \mu_{\alpha K}\omega_K^2\omega_K')=\emptyset$. The existence of $D_{\alpha K}$ now follows from 3.5.

Next we recall the relations $\mathcal{R}_i$ and the divisors $D_{i\alpha}$ from 2.2 and 2.3.

3.12 Let $K_{i\alpha}=\deg D_{i\alpha}-1$. Then there are points $x_{i\alpha}, y_{i\alpha\beta} \in C_i'$ ($\alpha=1..K_i, \beta=1..K_{i\alpha}$) such that $D_{i\alpha} = x_{i\alpha} + \sum_\beta y_{i\alpha\beta}$. Choose $\eta'_{i\alpha\beta} \in \Gamma(C_i', \Omega_{C_i'}(x_{i\alpha}+y_{i\alpha\beta}))\setminus\Gamma(C_i',\Omega)$ (for $\beta=1,..,K_{i\alpha}$).
Then, together with $f^*\omega_{i1},..,f^*\omega_{ig_i}$ the $\eta'_{i\alpha\beta}$ form a basis of $W_i:=\sum_\alpha \Gamma((C_i',\Omega_{C_i'}(D_{i\alpha}))$. W_i is, as we have seen in 2.3, the restriction of $f^*\Gamma(X,\omega_X)$ to C_i'.
Because there is a simple cycle H in G(X) that contains $f(x_{i\alpha})$ and $f(y_{i\alpha\beta})$ there exist $\eta_{i\alpha\beta} \in \Gamma(X,\omega_X)$ such that the restriction of $f^*\eta_{i\alpha\beta}$ to C_i' is equal to $\eta'_{i\alpha\beta}$.
Let $V_{i\alpha\beta} \in B$ such that $\theta(V_{i\alpha\beta})=\eta_{i\alpha\beta}$.

3.13 <u>Lemma:</u> *Let* $\omega \in \Gamma(X,\omega_X)$ *then there are* $\nu_{i\alpha\beta}, \nu_{ij} \in k$ *such that* $\omega + \sum_{\alpha,\beta} \nu_{i\alpha\beta}\eta_{i\alpha\beta} + \sum_j \nu_{ij}\omega_{ij}$ *vanishes on* C_i'.
Proof: This immediately follows from the fact that the $f^*\omega_{ij}$ ($j=1..g_i$) and the $\eta'_{i\alpha\beta}$ ($\alpha=1..K_i, \beta=1..K_{i\alpha}$) form a basis of the restriction to C_i' of $f^*\Gamma(X,\omega_X)$.

3.14 For every $i=1..N$ and $x,y \in C_i'$ with $x\neq y$ there are $N_{ixy\gamma} \in A_i$ ($\gamma=1..g_i$) such that:
(i) $\{\theta(N_{ixy\gamma})\}_{\gamma=1..g_i} \subset \Gamma(C_i',\Omega_{C_i'})$ is a basis.
(ii) $\theta(N_{ixy1}) \in \Gamma(C_i',\Omega_{C_i'}(-y))\setminus\Gamma(C_i',\Omega_{C_i'}(-x))$
$\theta(N_{ixy2}) \in \Gamma(C_i',\Omega_{C_i'}(-x))\setminus\Gamma(C_i',\Omega_{C_i'}(-y))$
$\theta(N_{ixy\gamma}) \in \Gamma(C_i',\Omega_{C_i'}(-x-y))$ ($\gamma=3,..,g_i$).
We will write $N_{i\alpha\beta\gamma}$ for $N_{ix_{i\alpha}y_{i\alpha\beta}\gamma}$.

3.15 <u>Lemma:</u> *Let* $P \in A_i$ *be homogeneous of degree* d *and* $x,y \in C_i'$. *Then:*

(i) *If* $\theta(P) \in \Gamma(C_i', \Omega_{C_i'}(-x))$ *then there are* $P_2,..,P_{g_i} \in A_i$, *homogeneous of degree* d-1 *such that* $P = \sum_{\gamma\geq 2} P_\gamma N_{ixy\gamma}$.

(ii) *If* $\theta(P) \in \Gamma(C_i', \Omega_{C_i'}(-x-y))$ *then there are* $P', P_3,..,P_{g_i} \in A_i$ *homogeneous,* deg P'=d-2, deg P_γ=d-1 *(*γ*=3..*g_i*), such that* $P = P'N_{ixy1}N_{ixy2} + \sum_{\gamma\geq 3} P_\gamma N_{ixy\gamma}$.

The proof of this is elementary algebra.

3.16 We list the following elements of I:

If $i \neq j$ then obviously we have: $\omega_{ik}\omega_{jl}=0$, so:

(A) $X_{ik}X_{jl} \in I$ $(i \neq j)$.

Let K be an equivalence class for $\approx$.

If i is an index such that C_i' does not contain any points of K, then $\omega_K\omega_K'$ vanishes on C_i'. Therefore we have:

(B) $X_{ij}U_KU_K' \in I$ (for all $\approx$-classes K such that K does not contain points of C_i')

If i is an index such that C_i' contains one point of K, then choose $y \in C_i'$ arbitrarily.

For $\gamma=2,..,g_i$ $N_{ixy\gamma}\omega_K\omega_K'$ vanishes outside C_i' and has no poles of order 2 on C_i', so by 3.5 there are $E_{ixK\gamma} \in B$ ($\gamma=2..g_i$), homogeneous of degree 3, $\deg_A E_{ijK\gamma} \leq 2$, such that:

(C) $N_{ixy\gamma}U_KU_K' + E_{ixK\gamma} \in I$ (x the only point of C_i' in K, $\gamma=2..g_i$)

If K contains two points $x,y \in C_i'$, then the only points of C_i' where ω_K has poles are x and y (because $G(\omega_K)$ is a simple cycle). It follows that there are $F_{ixy\gamma} \in A$ ($\gamma=3..g_i$), homogeneous of degree 2, such that:

(D) $N_{ixy\gamma}U_K + F_{ixy\gamma} \in I$ $(x,y \in C_i' \cap K,\ \gamma=3..g_i)$

And similarly there are $F_{ixy} \in A$, homogeneous of degree 3 such that:

(E) $N_{ixy1}N_{ixy2}U_K + F_{ixy} \in I$ $(x,y \in C_i' \cap K)$

In exactly the same way as above we see that there are $G_{i\alpha\beta\gamma} \in A$ ($\gamma=3..g_i$), homogeneous of degree 2, such that:

(F) $N_{i\alpha\beta\gamma}V_{i\alpha\beta} + G_{i\alpha\beta\gamma} \in I \quad (\gamma=3..g_i)$

And there are $G_{i\alpha\beta} \in A$, homogeneous of degree 3, such that:

(G) $N_{i\alpha\beta 1}N_{i\alpha\beta 2}V_{i\alpha\beta} + G_{i\alpha\beta} \in I$

Because $\theta(N_{i\alpha\beta 1})\eta_{i\alpha\beta}$ and $\theta(N_{i\alpha\gamma 1})\eta_{i\alpha\gamma}$ both have a pole of order 1 at $x_{i\alpha}$ and no other poles, there are $\rho_{i\alpha\beta\gamma} \in k$ and $H_{i\alpha\beta\gamma} \in A$, homogeneous of degree 2, such that:

(H) $\rho_{i\alpha\beta\gamma}N_{i\alpha\beta 1}V_{i\alpha\beta} + \rho_{i\alpha\gamma\beta}N_{i\alpha\gamma 1}V_{i\alpha\gamma} + H_{i\alpha\beta\gamma} \in I$

We recall from 3.10.1 that:

(I) $X_\alpha X_\beta + \sum_K \lambda_{\alpha\beta K}U_K U'_K + C_{\alpha\beta} \in I$

and from 3.11.1:

(J) $X_\alpha U_K U'_K + \mu_{\alpha K}U_K^2 U'_K + D_{\alpha K} \in I$

Finally we deduce from 3.13 that:

(K) $X_{ik}(X_\gamma + \sum_{\alpha,\beta} \nu_{i\alpha\beta\gamma}V_{i\alpha\beta} + \sum_j \nu_{ij\gamma}X_j) \in I \quad (k=1..g_i)$

3.17 <u>Proposition:</u> *The ideal* I *is generated by the ideals* I_i *and the polynomials listed in* (A)-(K) *above.*

Proof: Let J be the ideal generated by $I_1,..,I_N$ and the polynomials (A)-(K). Let $P \in I$. By induction on $m:=\deg_A P$ we will show that $P \in J$.

(i) If m=0 then $P \in A$. If follows from (A) that there is a polynomial $P' \equiv P \bmod J$ such that $P' = \sum_{i=1}^{N} Q_i$, with $Q_i \in A_i$. By looking at each component of X separately we see that $Q_i \in I_i$, so $P' \in J$ and therefore $P \in J$.

(ii) m=1. We will show that there is a $P' \equiv P \bmod J$ such that $P' \in A$. From (A) it follows that there exists a $P_1 \equiv P \bmod J$ such that P_1 has no terms that are divisible by $X_{ik}X_{jl}$ for $i \neq j$. We can assume that $\deg_A P_1 = 1$.

Because the ω_{ij} and the ω_α form a basis of $\Gamma(X,\omega_X)$ we must have $\deg P_1 \geq 2$. Using (K) we show that there is a $P_2 \equiv P_1 \bmod J$

such that:

$$P_2 = \sum_i \left(\sum_{\alpha,\beta} Q_{i\alpha\beta} V_{i\alpha\beta} \right) + \text{terms in } A \quad (Q_{i\alpha\beta} \in A_i).$$

Because only $\eta_{i\alpha\beta}$ has a pole in $y_{i\alpha\beta}$, $\theta(Q_{i\alpha\beta})$ vanishes in $y_{i\alpha\beta}$. It follows from Lemma 3.15 (i), (F) and (H) (with $\gamma=1$) that we can find $P_3 \equiv P_2$ mod J such that:

$$P_3 = \sum_i \left(\sum_\alpha Q_{i\alpha} V_{i\alpha 1} \right) + \text{terms in } A \quad (Q_{i\alpha} \in A_i).$$

Therefore $\theta(Q_{i\alpha})$ vanishes at $x_{i\alpha}$ and $y_{i\alpha 1}$, so from 3.15 (ii), (F) and (G) we deduce that there is a $P' \equiv P_3 \equiv P$ mod J, such that $P' \in A$.

(iii) $m \geq 2$. We will show that there is a $P' \equiv P$ mod J such that $\deg_A P' < m$.

It follows from (A) that we can find a $P_1 \equiv P$ mod J such that P_1 has no terms that are divisible by $X_{ik}X_{jl}$ for $i \neq j$. By (I) and (J) we see that there is a $P_2 \equiv P_1$ mod J such that:

$$P_2 = \sum_K \left(\sum_i Q_{iK} \right) U_K^{m-1} U_K' + \text{terms of } \deg_A < m \quad (Q_{iK} \in A_i)$$

The outer summation is taken over all $\approx$-classes K. It follows from (B) that we can assume that the inner sum is taken over those i for which C_i' contains points of K.

Let L be a $\approx$-class. Because only $\omega_L^{m-1}\omega_L'$ has poles of order m in the points of L every $\theta(Q_{iL})$ must vanish in the points of L on C_i'. For those i such that there is one such point we use 3.15 (i) and (C), for the i such that there are two such points we use 3.15 (ii), (D) and (E) to conclude that there is a $P' \equiv P_2$ mod J with $\deg_A P' < m$.

3.18 <u>Theorem:</u> *Let* X *be a stable curve that satisfies the conditions of* 3.1 *and suppose X is not a smooth curve of genus* 3. *Then the homogeneous ideal of the canonical image of* X *is generated by elements of degree* 2 *and* 3.

Proof: If X is smooth then this follows from Petri's theorem, so from now on we will assume that X is not smooth.

It also follows from Petri's theorem, that if $g_i > 3$ then I_i is generated by elements of degree 2 and 3. So it follows from proposition 3.17 that we only have to look at the i such that $g_i = 3$. In this case $I_i = (S_i)$, $S_i \in A_i$, homogeneous of degree 4. The theorem is proven if we can write S_i as a combination of polynomials of degree 2 and 3 in I.

Since X is not smooth we have $K_i \geq 1$ and $K_{i\alpha} \geq 1$, so it makes sense to speak of the points $x_{i1}, y_{i11} \in C_i'$ and of V_{i11}. From 3.16 (F) and (G) we have :

$N_{i111}N_{i112}V_{i11} + G_{i11} \in I$

$N_{i113}V_{i11} + G_{i113} \in I$

By multiplying the first by N_{i113} and the second by $N_{i111}N_{i112}$ and subtracting we find $T \in I_i$. If $T \neq 0$ then deg T=4 so $T = \lambda S$ ($\lambda \in k, \lambda \neq 0$) and we are done. So we only have to prove than $T \neq 0$.

But suppose T=0, then $N_{i111}N_{i112}G_{i113}$ is divisible by N_{i113}, so G_{i113} is divisible by N_{i113}. This is impossible because then $\eta_{i11}' \in \mathrm{Span}(\omega_{i1}, \omega_{i2}, \omega_{i3})$.

3.19 <u>Example:</u> Let X be a stable curve, $\pi=5$, such that X' is a smooth curve of genus 4. Let $x, y \in X'$ such that $f(x)=f(y)=z$, the double point of X. Now theorem 3.18 is applicable to X. We see that we can choose a basis $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ of $\Gamma(X', \Omega_{X'})$ such that ω_1 vanishes in y, not in x, ω_2 vanishes in x, not in y and ω_3 and ω_4 vanish in both x and y. Let $\omega_5 \in \Gamma(X', \Omega_{X'}(x+y)) \backslash \Gamma(X', \Omega_{X'})$, so $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ is a basis of $\Gamma(X, \omega_X)$.

For X we have: N=1, $K_1=1$, $K_{1\alpha}=1$, $N_{111\gamma}=X_\gamma$ ($\gamma=1..4$) and we can take $\eta_{111}= \omega_5$. There is only one $\approx$-class: $K=\{x,y\}$, so we can take $\omega_K = \omega_K' = \omega_5$. We observe that 3.16 (B) and (C) do not occur and that in (D) we can take $N_{1xy\gamma} = X_\gamma$ ($\gamma=1..4$).

Let I' be the canonical ideal of X'. We get: I, the canonical ideal of X is generated by I' and $X_3X_5 + F_3$, $X_4X_5 + F_4$ (D), $X_1X_2X_5 + F$ (E), $X_3X_5 + G_3$, $X_4X_5 + G_4$ (F), $X_1X_2X_5 + G$ (G). (H)-(K) are not needed (they are trivial, for example (I) gives $X_5^2 - X_5^2 + C \in I$ for some C). Finally we observe that (D) and (E) only differ from (F) and (G) by elements of I' so I is generated by I', $P_1 = X_3X_5 + F_3$, $P_2 = X_4X_5 + F_4$ and $P_3 = X_1X_2X_5 + F$.

3.20 We continue example 3.19. It follows from Petri's work that I' is generated by polynomials $P, Q \in A$ with deg P=2 and deg Q=3. If we suppose $\{\omega_3, \omega_4, \omega_1, \omega_2\}$ is a cleverly chosen basis of $\Gamma(X',\Omega_{X'})$ (in the sequel we will call such a basis a Petri-basis, in 3.21 we will return to this point) then Petri gives explicit formulas for P and Q.

In this case: $P=X_1X_2-\lambda_1X_1X_3-\lambda_2X_2X_3-\mu_1X_1X_4-\mu_2X_2X_4-bX_3X_4$.
$(\lambda_1,\lambda_2,\mu_1,\mu_2,b \in k)$. Let $Q'=X_4P_1-X_3P_2= X_4F_3-X_3F_4$. Then $Q' \in I'$ and, since X_3 doesn't divide F_3, $Q'\neq 0$, so $\deg Q'=3$.
Suppose that $Q'=LP$ for some linear form L. Then $L=a_3X_3+a_4X_4$ $(a_3,a_4 \in k)$ because Q' does not have any terms of the form $X_1^2X_2$ or $X_1X_2^2$. Now $LP=Q'$ implies that
$(a_3P + F_4)X_3= (-a_4P + F_3)X_4$ so X_4 divides $a_3P + F_4$ but since $X_4X_5+ a_3P + F_4 \in I$ this implies that $\omega_5 \in \Gamma(X',\Omega_{X'})$, contradiction.

Therefore $LP\neq Q'$ for any L so $Q'\in (P,P_1,P_2)$, $I'\subset (P,P_1,P_2)=:J$ and $I=(P,P_1,P_2,P_3)$.

Finally we show that I=J. Of course it suffices to prove that $P_3 \in J$. We have:
$P_3 \equiv \lambda_1X_1X_3X_5+\lambda_2X_2X_3X_5+\mu_1X_1X_4X_5+\mu_2X_2X_4X_5+bX_3X_4X_5+F\equiv P' \bmod J$,
with $P'\in A$ and therefore $P'\in I'\subset J$ so $P_3 \in J$.

Summarizing, we see that we have shown that if $\{\omega_3,\omega_4,\omega_1,\omega_2\}$ is a Petri-basis of $\Gamma(X,\omega_X)$ then $I=(P,P_1,P_2)$, so I is generated by elements of degree 2.

3.21 We will now describe when a basis $\{\omega_1',\omega_2',\omega_3',\omega_4'\}$ is a Petri-basis of $\Gamma(X',\Omega_{X'})$ (following [S-D]).

Let p_1,p_2,p_3,p_4 be points on X', in general position with respect to $\Gamma(X',\Omega_{X'})$ and let $D=p_3+p_4$. Then by [S-D], for allmost all choices of $p_1..p_4$ the system $|K-D|$ is a base-point free pencil. For each i, $V_i:=\Gamma(X',\Omega_{X'}(-\sum_{i\neq j} p_j))$ is one dimensional. Now $\{\omega_1',\omega_2',\omega_3',\omega_4'\}$ is a Petri-basis if $|K-D|$ is base-point free and if $\omega_i' \in V_i$ for each i.

Finally we examine when our basis $\{\omega_3,\omega_4,\omega_1,\omega_2\}=\{\omega_1',\omega_2',\omega_3',\omega_4'\}$ is a Petri-basis.

Let $\phi':X'\longrightarrow \mathbb{P}^3$ be the canonical embedding. Then for our basis to be a Petri-basis at least the span of $\omega_1'(=\omega_3)$ and $\omega_2'(=\omega_4)$ should be base-point free. This is the same as demanding that the line $\phi'(x),\phi'(y)$ is not a trisecant of $\phi'(X')$.

In the case it is a trisecant $\{\omega_1',\omega_2',\omega_3',\omega_4'\}$ cannot be a

Petri-basis. If it is not then we can choose $p_3=x$ and $p_4=y$ and now we only have to choose p_1 and p_2 such that p_1, p_2, p_3, p_4 are in general position, which is obviously always possible.

So we can choose $\{\omega_3, \omega_4, \omega_1, \omega_2\}$ to be a Petri-basis if and only if $\phi'(x)$ and $\phi'(y)$ are not on a trisecant of $\phi'(X')$.

Appendix

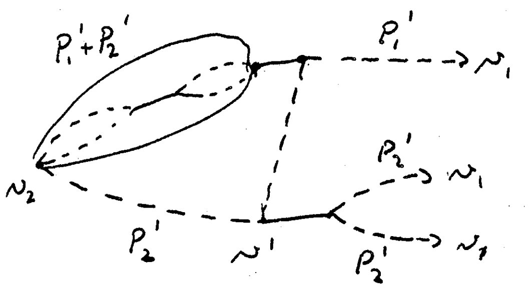
We will prove:

Lemma: *Let* G *be a graph and* C_1, C_2, C_3 *simple cycles in* G *and* e *an edge such that* $e \in E(C_1) \cap E(C_2) \cap E(C_3)$. *Then there are simple cycles* C_4 *and* C_5 *in* G *such that*
$e \in E(C_4) \cap E(C_5) \subset E(C_1) \cap E(C_2) \cap E(C_3)$.

Proof: If e is a loop then there is nothing to prove. Otherwise e connects two vertices v_1 and v_2. C_1, C_2 and C_3 give simple paths P_1, P_2 and P_3 from v_2 to v_1. We have to show that there are two paths P_4 and P_5 from v_2 to v_1 such that $E(P_4) \cap E(P_5) \subset E(P_1) \cap E(P_2) \cap E(P_3)$. If either P_1, P_2 or P_3 is an edge connecting v_2 and v_1 then we are done.

Otherwise we proceed as follows. Let G' be the graph obtained by omitting the edge e from G and identifying two vertices of G if they are connected by an edge in $E(P_1) \cap E(P_2) \cap E(P_3)$ and omitting these edges (that have now become loops). P_1, P_2 and P_3 give paths P_1', P_2' and P_3' in G' such that $E(P_1') \cap E(P_2') \cap E(P_3') = \emptyset$. If we can show that there are two paths P_4' and P_5' in G' such that $E(P_4') \cap E(P_5') = \emptyset$ then the Lemma is proven.

We will show this by induction on $\#E(P_1') \cup E(P_2') \cup E(P_3')$. If this number is 2 then we are done. Otherwise the situation looks like this:



We see that there are 2 disjoint paths from v_2 to v'. There are 3 paths from v' to v_1 such that there are no edges shared by all three of them. The induction hypothesis says that there are 2 disjoint paths from v' to v_1. We can link the paths of these two pairs together to get two disjoint paths from v_2 to v_1. (If A is used twice, then it is used in different directions, so it cancels out.)

Appendix 2

Suppose X is a stable curve of genus g, with no rational components and such that the canonical system has base points. In 1.11 a way is described to obtain a canonical image of X in spite of the base points of $\Gamma(X, \omega_X)$. The insatisfactory thing about this method is that the canonical image of a connected curve will be disconnected whenever $\Gamma(X, \omega_X)$ has base points. We propose to circumvent this problem in the following way: Instead of embedding X as it is, we choose a (one dimensional) family Y/C of curves in which X occurs as (the only) singular fibre, say over the point $x \in C$. Note that C is an open part of a curve. It easily follows from [D,M], Section 1, that we can choose Y to be a nonsingular surface over k. We choose an open part of C such that the canonical restriction $\Gamma(Y, \omega_{Y/C}) \to \Gamma(X, \omega_X)$ is an isomorphism. Then we try to map Y to a projective space (of dimension g) using the sheaf $\Gamma(Y, \omega_{Y/C})$. We will consider the image of the fibre Y_x as the 'canonical' image of X. It must be noted that a priori there is nothing canonical about this image since we have chosen Y rather arbitrarily. However, an open subset of this image is the image of the complement of the base locus of $\Gamma(X, \omega_X)$ under the map induced by the canonical system on X, and therefore the 'canonical' image we have just defined contains the canonical image in the sense of 1.11.

Of course the system $\Gamma(Y, \omega_{Y/C})$ will not be base-point free because the canonical system of X is not. We therefore proceed in the same way as in 1.11: Let $\{y_1, .., y_j\}$ be the support of the base scheme of $\Gamma(Y, \omega_{Y/C})$. We take the closure of the image of $Y \backslash \{y_1, .., y_j\}$ under the map corresponding to this system. The image of Y that is obtained in this way is equal to the image of the blowing-up of Y in the sheaf of ideals of the base scheme of $\Gamma(Y, \omega_{Y/C})$ under the map defined by the inverse image of $\Gamma(Y, \omega_{Y/C})$ (see [Ha II, §7]). We will now determine the blowing-up $\pi : \tilde{Y} \to Y$ of Y.

Because all fibres of Y/C except the one over x are non singular and of genus greater than zero, the canonical systems of the other fibres are base-point free, so the set $\{y_1, .., y_j\}$ is exactly the base locus of the canonical system of $X = Y_x$. Moreover, the stalk in y_i of the sheaf of ideals of the base scheme is the image of $\Gamma(Y, \omega_{Y/C})$ under the map

$$\Gamma(Y, \omega_{Y/C}) \to \Gamma(U, \omega_{Y/C}) \tilde{\to} \Gamma(U, \mathcal{O}_Y) \to \mathcal{O}_{Y, y_i}$$

(where U is an open subset of Y on which $\omega_{Y/C}$ is trivial), which is precisely the maximal ideal of $\mathcal{O}_{Y,y_i}$. This follows from the fact that the image of the map

$$\Gamma(X, \omega_X) \to \Gamma(V, \omega_X) \tilde{\to} \Gamma(V, \mathcal{O}_X) \to \mathcal{O}_{X,y_i}$$

(where V trivialises ω_X) is generated by the two local parameters t_1 and t_2 on X in y_i (see 1.11) and is therefore equal to the maximal ideal of $\mathcal{O}_{X,y_i}$, and from the fact that $\mathcal{O}_{X,y_i} = \mathcal{O}_{Y,y_i}/(t_1 t_2)$. It follows that the image of $\Gamma(Y, \omega_{Y/C})$ generates the maximal ideal of $\mathcal{O}_{Y,y_i}$ modulo $t_1 t_2$, so it generates the maximal ideal itself. So we see that in this case in the blowing-up the points $y_1, .., y_j$ are all replaced by a $\mathbf{P}^1$ (cf [Ha V, 3.1]).

Finally we will determine the image of $\tilde{Y}$ under the map obtained from the inverse image on $\tilde{Y}$ of $\Gamma(Y, \omega_{Y/C})$. On the $\mathbf{P}^1$ lying over y_i we can choose homogeneous coördinates $(\xi_1 : \xi_2)$ corresponding to the generators t_1 and t_2 of the maximal ideal of $\mathcal{O}_{Y,y_i}$. If we choose a basis $\{\omega_1, .., \omega_g\}$ of $\Gamma(Y, \omega_{Y/C})$ we see that there exist contstants $a_l, b_l \in k (l = 1..g)$ such that $\omega_l = a_l t_1 + b_l t_2$ mod m^2_{Y,y_i}. On the $\mathbf{P}^1$ lying over y_i we have: $\pi^* \omega_l = a_l \xi_1 + b_l \xi_2 + h.o.t.$. These sections generate $\Gamma(\mathbf{P}^1, \mathcal{O}(1))$ on this fibre, because $\omega_1, .., \omega_g$ generate m_{Y,y_i}. So we have shown that for all points y_i the fibre over y_i of $\pi : \tilde{Y} \to Y$, which is a $\mathbf{P}^1$, is mapped biregularly to a straight line.

From the description of the 'image' of Y as the closure of the image of $Y \backslash \{y_1, .., y_j\}$ it is obvious that this line containes the two points of the canonical image of X (in the sense of 1.11) that correspond to y_i. Therefore in this case the 'canonical' image image of X is like the one that was obtained in 1.11, but this time for every base point of $\Gamma(X, \omega_X)$ the two points of the canonical image of X corresponding to it are joined by a straight line. It follows that the 'canonical' image of X described above doesn't depend on the Y chosen after all.

Bibliography

[A,C,G,H] E. Arbarello, M Cornalba, P.A. Griffiths, J. Harris: Geometry of Algebraic curves, vol I. Grundlehren 267, Springer (1985).

[Ca]. F. Catanese: Pluricanonical Gorenstein Curves, in Enumerative Geometry and Classical Algebraic Geometry, 51-95. Progress in Mathematics no. 24, Birkhäuser (1982).

[D,M] P. Deligne, D. Mumford: The Irreducibility of the Space of Curves of Given Genus. Publ. Math. I.H.E.S. no. 36 (1969), 75-100.

[Ha] R. Hartshorne: Algebraic Geometry. Graduate Texts in Mathematics vol. 52. Springer (1977).

[Pe] K. Petri: Ueber die Invariante Darstellung Algebraische Funktionen einer Veränderlichen. Math. Ann. 88 (1922), 242-289.

[S-D] B. Saint-Donat: On Petri's Analysis of the Linear System of Quadrics through a Canonical Curve. Math. Ann. 206 (1973), 157-175