Abelian varieties with ℓ-adic Galois representation of Mumford's type

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Abstract

This paper is devoted to the study of 4-dimensional abelian varieties over number fields with the property that the Lie algebra of the image of some associated ℓ-adic Galois representation is $\overline{\mathbf{Q}}_\ell$-isomorphic to $\mathfrak{c} \oplus (\mathfrak{sl}_2)^3$. Such varieties will be referred to as 'abelian varieties with ℓ-adic Galois representation of Mumford's type'. We show that such abelian varieties have potentially good reduction at all prime ideals and we determine the possible Newton polygons and the possible isogeny types of these reductions.

Introduction

A Shimura variety 'of Hodge type' parametrizes a family (or families) of abelian varieties with certain properties. The fact of being of Hodge type offers some powerful tools for the study of a Shimura variety. If a Shimura variety of Hodge type is a moduli space for abelian varieties of PEL type (that is, abelian varieties characterized by a polarization, endomorphisms and a level structure), this is even more useful, since it then is the solution to a moduli problem that can be formulated in any characteristic. This can be particularly useful for the study of models in mixed characteristic.

An example due to Mumford shows that not all families of Hodge type are of PEL type. He constructs 1-dimensional families associated to representations V of algebraic groups G of the following type. One has $\mathrm{Lie}(G)_{\overline{\mathbf{Q}}} \cong \mathfrak{c}_{\overline{\mathbf{Q}}} \oplus \mathfrak{sl}^3_{2,\overline{\mathbf{Q}}}$, where $\mathfrak{c} \subset \mathrm{Lie}(G)$ denotes the 1-dimensional centre, and the induced representation of $\mathrm{Lie}(G)_{\overline{\mathbf{Q}}}$ on $V \otimes \overline{\mathbf{Q}}$ is the tensor product of the standard representations. In this paper, we will refer to such a pair (G, V) as being of Mumford's type, cf. 1.2. If $X/\mathbf{C}$ is a fibre of one of Mumford's families, then the representation of its Mumford–Tate group on $\mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ factors through a representation of Mumford's type. If the fibre in question is not of CM type, then this representation is itself of Mumford's type. Conversely, it can be shown that if $X/\mathbf{C}$ is an abelian variety such that the representation of its Mumford–Tate group on $\mathrm{H}^1_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ factors through a representation of Mumford's type, then it arises as a fibre in one of Mumford's families.

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If X is a fibre over a number field of a family of Mumford's type associated to a pair (G, V), then, for any prime number ℓ, the associated ℓ-adic Galois representation factors through $G(\mathbf{Q}_\ell)$. Unless X is of CM type, the image is Zariski dense. Conversely, let F be a number field and let X/F be an abelian variety such that Zariski closure of the image of the Galois representation on $V_\ell = \mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_\ell)$ is an algebraic group $G_\ell/\mathbf{Q}_\ell$ such that (G_ℓ, V_ℓ) is of Mumford's type. In what follows we will simply say that the ℓ-adic Galois representation associated to X is of Mumford's type. The Mumford–Tate conjecture predicts that the Mumford–Tate group of X then is of the above kind and therefore that X is a fibre in a family of the type we consider. This means that, conjecturally, the abelian varieties arising from Mumford's construction are exactly those for which the associated ℓ-adic Galois representations are of Mumford's type. Abelian varieties with a Galois representation of this kind are the examples of the smallest dimension where the Mumford–Tate conjecture is unsettled. Some evidence supporting the Mumford–Tate conjecture is provided by the fact that if X is an abelian variety over a number field such that one associated ℓ-adic representation is of Mumford's type, then all associated Galois representations are of this kind, cf. lemma 1.3.

Shimura curves of Mumford's type have been extensively studied. For the study of the reduction of such a curve and of its and zeta- or L-functions, a comparison to a Shimura curve of PEL type can be used. This has been done in work of Shimura, Morita and, more recently, Reimann. The results have some implications for the reduction of a fibre over a number field of a family of Mumford's kind.

In the present paper, we study the reduction properties of abelian varieties over number fields such that the associated Galois representations are of Mumford's type. It is therefore not unrelated to the Mumford–Tate conjecture, showing that many properties of the abelian varieties in question can be derived from their Galois representations.

We briefly sketch the contents of the article.

In section 1, we introduce the notions of a representation of Mumford's type and of an abelian variety with associated ℓ-adic Galois representation of Mumford's type. Some fundamental properties are established, notably the lemma on ℓ-independence referred to above. This lemma in fact states some well known results due to Serre and to Pink.

In section 2, we study the properties of good reduction of an abelian variety for which the associated Galois representations are of Mumford's type. It is shown in corollary 2.2 that any such abelian variety has potentially good reduction everywhere.

In section 3 we investigate the Newton polygons of certain crystalline p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p^{\mathrm{nr}})$ factoring through a representation (G_p, V_p) of Mumford's type. It is shown in proposition 3.2 that there are only four possibilities. Imposing a finer condition on the image of the Galois representation permits a refinement of the result. For example, if G_p is split, then the only possible Newton polygons are $4 \times 0, 4 \times 1$ and $8 \times 1/2$. The precise results can be found in propositions 3.5 and 3.6. These results have rather obvious implications for an abelian variety for which the associated Galois representations are of

Mumford's type, cf. 4.4.

In section 4, we investigate the possible isogeny types of a reduction of such an abelian variety. The final result is stated in corollary 4.4, which is derived from proposition 4.1. In 4.1, we use arguments going back to Serre to deduce the possible isogeny types of a reduction at a finite place from properties of an associated ℓ-adic Galois representation satisfying appropriate conditions.

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Contents

1 Representations of Mumford's type.

1.1 Notations. For any field F, we denote by $\bar{F}$ an algebraic closure of F and we write $\mathcal{G}_F = \mathrm{Gal}(\bar{F}/F)$. If $\bar{v}$ is a valuation of $\bar{F}$, then $\mathcal{I}_{F,\bar{v}} \subset \mathcal{G}_F$ is the inertia group of $\bar{v}$. In case F is a local field, we write just $\mathcal{I}_F \subset \mathcal{G}_F$ for the inertia subgroup. For a number field or local field F, we write $\mathcal{O}_F$ for the ring of integers or the valuation ring, respectively.

Let p be a prime number. Denote by $\mathbf{Q}_p$ the p-adic completion of $\mathbf{Q}$, by $\mathbf{Q}_p^{\mathrm{nr}}$ the maximal unramified extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$, by $\mathbf{Q}_{p^h}$ the unique extension of $\mathbf{Q}_p$ of degree h contained in $\mathbf{Q}_p^{\mathrm{nr}}$ and by $\mathbf{C}_p$ the completion of $\overline{\mathbf{Q}}_p$. The Frobenius automorphism of any of the fields $\mathbf{Q}_p^{\mathrm{nr}}$ and $\mathbf{Q}_{p^h}$ will be denoted by σ.

If F is a field of characteristic 0 and p is a prime number, then $\mathcal{G}_F$ acts on the set of p-power roots of unity in $\bar{F}$. This action defines the cyclotomic character $\chi_p \colon \mathcal{G}_F \to \mathbf{Z}_p^*$. When there can be no confusion as to the value of p, we write χ instead of χ_p. For $i \in \mathbf{Z}$, we write $\mathbf{Q}_p(i)$ for the one dimensional $\mathbf{Q}_p$-linear representation of $\mathcal{G}_F$ where $\mathcal{G}_F$ acts through χ_p^i.

1.2 Representations of Mumford's type. Let K be a field of characteristic 0, let G be an algebraic group over K and let V be a faithful K-linear representation of G. We will say that the pair (G, V) is *of Mumford's type* if

- $\mathrm{Lie}(G)$ has one dimensional centre $\mathfrak{c}$,
- $\mathrm{Lie}(G)_{\bar{K}} \cong \mathfrak{c}_{\bar{K}} \oplus \mathfrak{sl}_{2,\bar{K}}^3$ and
- $\mathrm{Lie}(G)_{\bar{K}}$ acts on $V_{\bar{K}}$ by the tensor product of the standard representations.

We do not require G to be connected.

If (G, V) is of Mumford's type, then there exists an alternating bilinear form on V (unique up to a scalar) which is fixed by G up to a character. If moreover G is connected, then there exists a central isogeny $N\colon \widetilde{G} \to G$ where $\widetilde{G}$ is an algebraic group over K such that $\widetilde{G} \cong \mathbf{G}_{m,K} \times \widetilde{G}^{\mathrm{ss}}$ and $\widetilde{G}^{\mathrm{ss}}_{\bar{K}} \cong \mathrm{SL}_{2,\bar{K}}^3$. The induced representation of $\widetilde{G}_{\bar{K}}$ on $V_{\bar{K}}$ is isomorphic to the tensor product of the standard representations. The Shimura varieties and families of abelian varieties constructed by Mumford in [Mum69, §4] are associated to $\mathbf{Q}$-linear representations of Mumford's type.

Let F be a field, ℓ a prime number and V_ℓ a $\mathbf{Q}_\ell$-vector space. We say that a continuous Galois representation $\rho\colon \mathcal{G}_F \to \mathrm{GL}(V_\ell)(\mathbf{Q}_\ell)$ is of Mumford's type if the Zariski closure $G_\ell \subset \mathrm{GL}(V_\ell)$ of $\rho(\mathcal{G}_F)$ has the property that (G_ℓ, V_ℓ) is of Mumford's type. If X/F is a fibre of one of the families of abelian fourfolds from [Mum69] which is not of CM type, then the associated ℓ-adic representations are of Mumford's type.

1.3 Lemma. *Let X be a 4-dimensional abelian variety over a number field F. For each prime number ℓ, let G_ℓ be the Zariski closure of the image of the ℓ-adic Galois representation of $\mathcal{G}_\ell$ on $V_\ell = \mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_\ell)$. Suppose that there exists prime number p such that (G_p, V_p) is of Mumford's type. Then (G_ℓ, V_ℓ) is of Mumford's type for all ℓ and there are infinitely many prime numbers ℓ such that $\mathrm{Lie}(G_\ell)^{\mathrm{ss}}$ is $\mathbf{Q}_\ell$-simple.*

Proof. It follows from [Ser81] that, after replacing F by a finite extension, we can assume that all G_ℓ are connected. Since

$$\mathrm{End}(X_{F'}) \otimes_{\mathbf{Z}} \mathbf{Q}_p \cong \mathrm{End}_{\mathcal{G}_{F'}}(V_p) = \mathrm{End}_{G_p}(V_p) = \mathbf{Q}_p$$

for any finite extension F' of F, one has $\mathrm{End}(X_{\bar{F}}) = \mathbf{Z}$, so $\mathrm{End}_{G_\ell}(V_\ell) = \mathrm{End}_{\mathcal{G}_F}(V_\ell) = \mathbf{Q}_\ell$ for all ℓ. For any prime number ℓ, one deduces from Faltings' theorem that G_ℓ is reductive and from the facts that X is polarizable and $\mathrm{End}(X) = \mathbf{Z}$ together with the classification of semi-simple Lie algebras that $\mathrm{Lie}(G_\ell) \cong \mathfrak{c} \oplus \mathfrak{sp}_8$ or $\mathrm{Lie}(G_\ell) \cong \mathfrak{c} \oplus \mathfrak{sl}_2^3$. In the second case, (G_ℓ, V_ℓ) is of Mumford's type and in the first case $G_\ell = \mathrm{GSp}_{8,\mathbf{Q}_\ell}$. As [Ser81] and [Chi92, 3.10] show that the rank of G_ℓ is independent of ℓ, one concludes that (G_ℓ, V_ℓ) is of Mumford's type for every ℓ.

It follows from [Pin98, 5.13] that there exist a connected reductive group G over $\mathbf{Q}$ and a representation V of G such that $(G, V) \otimes \mathbf{Q}_\ell \cong (G_\ell, V_\ell)$ for all primes ℓ belonging to a set of primes of Dirichlet density 1. Of course, $\mathrm{Lie}(G)$ has one dimensional centre $\mathfrak{c}$ and $\mathrm{Lie}(G)_{\overline{\mathbf{Q}}} \cong \mathfrak{c}_{\overline{\mathbf{Q}}} \oplus \mathfrak{sl}_{2,\overline{\mathbf{Q}}}^3$. The theorem of Pink cited above also implies that $\mathrm{Lie}(G)^{\mathrm{ss}}$ is $\mathbf{Q}$-simple. The conditions imply that for any maximal torus of G^{ss}, the group $\mathcal{G}_{\mathbf{Q}}$ acts on the associated root datum through $\{(\pm1, \pm1, \pm1)\} \rtimes \mathrm{S}_3$ and that the projection of the image to S_3 contains the 3-cycles. The Lie algebra $\mathrm{Lie}(G)^{\mathrm{ss}}_{\mathbf{Q}_\ell}$ is $\mathbf{Q}_\ell$-simple if and only if the projection to S_3 of the image of $\mathcal{G}_{\mathbf{Q}_\ell} \subset \mathcal{G}_{\mathbf{Q}}$ in $\{(\pm1, \pm1, \pm1)\} \rtimes \mathrm{S}_3$ still contains these cycles. It thus follows from the Chebotarev density theorem that the set of primes ℓ such that $G_{\mathbf{Q}_\ell}$ has the desired properties and where $(G, V) \otimes \mathbf{Q}_\ell \cong (G_\ell, V_\ell)$ has density 1/3 (if the image of $\mathcal{G}_F$ in $\{(\pm1, \pm1, \pm1)\} \rtimes \mathrm{S}_3$ projects onto S_3) or 2/3 (in the other case). □

2 Good reduction

2.1 Proposition. *Let F be a field, v a discrete valuation of F and let ℓ be a prime number such that the residue characteristic at v is different from ℓ. Suppose that X/F is an abelian variety such that the representation of $\mathcal{G}_F$ on $V = \mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_\ell)$ factors through a morphism $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_\ell)$ for an algebraic group $G \subset \mathrm{GL}(V)$ such that*

- *the pair (G, V) is of Mumford's type and*
- $\mathrm{Lie}(G)^{\mathrm{ss}}$ *is $\mathbf{Q}_\ell$-simple.*

Then X has potentially good reduction at v.

Proof. Write $\mathcal{O}_v$ for the local ring of F at v and $\mathcal{I}_{F,\bar{v}}$ for the inertia group of a valuation $\bar{v}$ of $\bar{F}$ extending v. After replacing F by a finite extension, we can assume that G is connected.

Let Y be any abelian variety over F and let $\rho_Y : \mathcal{G}_F \to \mathrm{GL}(\mathrm{H}^1_{\text{ét}}(Y_{\bar{F}}, \mathbf{Q}_\ell))$ be the associated ℓ-adic Galois representation. It follows from [GRR72, Exposé I, 3.6] that there exists a finite extension F' of F such that the inertia group $\mathcal{I}_{F',\bar{v}}$ acts unipotently and such that $(\rho_Y(\alpha) - \mathrm{id})^2 = 0$ for each $\alpha \in \mathcal{I}_{F',\bar{v}}$.

We compare this with the unipotent elements of $G(\mathbf{Q}_\ell)$. If $x \in G(\mathbf{Q}_\ell)$ is unipotent but $x \neq \mathrm{id}$, then, viewed as an element of $G(\overline{\mathbf{Q}}_\ell)$, it is the image by 'tensor product' of an element $(y_1, y_2, y_3) \in \mathrm{SL}_2(\overline{\mathbf{Q}}_\ell)^3$, with each y_i unipotent. Since the image of (y_1, y_2, y_3) is $\mathcal{G}_{\mathbf{Q}_\ell}$-invariant and since there is at least one $y_i \neq \mathrm{id}$, it follows that all $y_i \neq \mathrm{id}$. Up to conjugation, we can assume that for $i = 1, 2, 3$, one has

$$y_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$$

with $a_i \in \overline{\mathbf{Q}}_\ell$. As $a_1a_2a_3 \neq 0$, one easily computes that the index of nilpotency of $x - \mathrm{id}$ is equal to 4.

As the Galois representation associated to X factors through $G(\mathbf{Q}_\ell)$, the two above statements imply that there exists a finite extension F' of F such that $\mathcal{I}_{F',\bar{v}}$ acts trivially. It follows from [ST68] that $X_{F'}$ has good reduction. □

2.2 Corollary. *Let X be a 4-dimensional abelian variety over a number field F and assume that for some prime number ℓ the representation of $\mathcal{G}_F$ on $\mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_\ell)$ is of Mumford's type. Then X has potentially good reduction at all places F.*

Proof. Apply lemma 1.3 to find a prime number ℓ' where the conditions of proposition 2.1 are verified. □

2.3 Corollary. *Let X be an absolutely simple abelian fourfold over a number field F which does not have potentially good reduction everywhere. Then the Mumford–Tate conjecture is true for X.*

Proof. With a little work, one derives from [MZ95, table 1] that the statement is true if $\mathrm{End}(X_{\bar{F}}) \neq \mathbf{Z}$. We can therefore assume that $\mathrm{End}(X_{\bar{F}}) = \mathbf{Z}$.

Let ℓ be a prime number and let G_ℓ be the Zariski closure of the image of the Galois representation on $V_\ell = \mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_\ell)$. As in the proof of 1.3 one shows that either $G_\ell = \mathrm{GSp}_{8,\mathbf{Q}_\ell}$ or (G_ℓ, V_ℓ) is of Mumford's type. By the hypothesis that X does not have potentially good reduction everywhere and corollary 2.2, the second possibility is excluded and since the Mumford–Tate group of X is contained in GSp_8, the corollary follows. □

3 Newton polygons

3.1 Preliminaries. Following Katz (cf. [Kat78, 1.3]), we recall the definition of Newton polygons. Let $\alpha \in \mathbf{Q}$ and $r, s \in \mathbf{Z}$, $s \geq 1$, such that $\alpha = r/s$ (in lowest terms). Define a $\mathbf{Q}_p^{\mathrm{nr}}$-module $D(\alpha)$ by $D(\alpha) = \mathbf{Q}_p[T]/(T^s - p^r) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p^{\mathrm{nr}}$ and a σ-linear map $\mathrm{Fr}\colon D(\alpha) \to D(\alpha)$ by $\mathrm{Fr}(x \otimes \lambda) = (Tx) \otimes (\sigma(\lambda))$. Note that $\dim_{\mathbf{Q}_p^{\mathrm{nr}}} D(\alpha) = s$.

If D is a finite dimensional $\mathbf{Q}_p^{\mathrm{nr}}$-vector space endowed with a σ-linear map $\mathrm{Fr}\colon D \to D$, then there exist a finite sequence $r_1/s_1 \leq \cdots \leq r_k/s_k$ of rational numbers (with $r_i, s_i \in \mathbf{Z}$, $s_i \geq 1$ and r_i, s_i relatively prime) and an isomorphism of $\mathbf{Q}_p^{\mathrm{nr}}$-vector spaces with Fr-action $D \cong \oplus_{i=1}^k D(r_i/s_i)$. Let $d = \dim_{\mathbf{Q}_p^{\mathrm{nr}}} D$ and let $(\alpha_1, \ldots, \alpha_d)$ be the sequence of rational numbers

$$(r_1/s_1 \ (s_1 \text{ times}), r_2/s_2 \ (s_2 \text{ times}), \ldots, r_k/s_k \ (s_k \text{ times})).$$

The α_i are the *Newton slopes* of D and we define the *Newton polygon of D* as the piecewise linear curve in $\mathbf{R}^2$ joining the points $(0,0)$, $(1, \alpha_1)$, $(2, \alpha_1 + \alpha_2)$, $\ldots$, $(d, \alpha_1 + \cdots + \alpha_d)$ in this order. The *break points* of the Newton polygon are the points $(i, \alpha_1 + \cdots + \alpha_i)$ (for $0 < i < d$), where $\alpha_i < \alpha_{i+1}$. These break points clearly have integral coordinates.

Suppose that F is a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$. A *filtered module over* F is a finite dimensional $\mathbf{Q}_p^{\mathrm{nr}}$-vector space D endowed with a σ-linear map $\mathrm{Fr}\colon D \to D$ and an exhaustive, separated, decreasing filtration $(\mathrm{Fil}^i)_{i\in\mathbf{Z}}$ of $D \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} F$ by sub-F-vector spaces. Since D is finite dimensional, the filtration is of finite length.

As in 1.1, we write $\mathcal{G}_F = \mathrm{Gal}(\bar{F}/F) = \mathcal{I}_F$ and we assume that $\mathcal{C}$ is a tannakian subcategory of the category of continuous, finite dimensional, crystalline $\mathbf{Q}_p$-linear representations of $\mathcal{G}_F$. Recall that a $\mathbf{Q}_p$-linear representation W of $\mathcal{G}_F$ is *crystalline* if

$$\dim_{\mathbf{Q}_p} W = \dim_{\mathbf{Q}_p^{\mathrm{nr}}} (W \otimes_{\mathbf{Q}_p} B_{\mathrm{crys}})^{\mathcal{G}_F},$$

where B_{crys} denotes the crystalline period ring constructed by Fontaine, see for example [Fon94a]. We refer to [Fon94b] for more details and background. Following Fontaine, cf. [Fon78], one can define several fibre functors on $\mathcal{C}$.

First of all, one has the functor $\underline{\omega}_{\text{ét}}$ sending an object of $\mathcal{C}$ to its underlying $\mathbf{Q}_p$-vector space. If W is an object of $\mathcal{C}$, then $\mathcal{G}_F$ acts on the $\mathbf{Q}_p$-vector space $\underline{\omega}_{\text{ét}}(W)$. Let H be the automorphism group of the functor $\underline{\omega}_{\text{ét}}$, by construction an algebraic group over $\mathbf{Q}_p$. Then the action of $\mathcal{G}_F$ on the $\underline{\omega}_{\text{ét}}(W)$ gives rise to a morphism $\rho\colon \mathcal{G}_F \to H(\mathbf{Q}_p)$ with Zariski dense image.

Secondly, one defines a $\mathbf{Q}_p^{\mathrm{nr}}$-valued fibre functor $\underline{\omega}_{\mathrm{crys},\mathbf{Q}_p^{\mathrm{nr}}}$ on $\mathcal{C}$ by

$$\underline{\omega}_{\mathrm{crys},\mathbf{Q}_p^{\mathrm{nr}}}(W) = (W \otimes_{\mathbf{Q}_p} B_{\mathrm{crys}})^{\mathcal{G}_F}$$

for each object W of $\mathcal{C}$. Since W is a crystalline representation of $\mathcal{G}_F$, it follows from the definition that $\underline{\omega}_{\mathrm{crys},\mathbf{Q}_p^{\mathrm{nr}}}(W)$ is a filtered module of dimension $\dim_{\mathbf{Q}_p} W$. This implies in particular that one can speak of the the Newton polygon of $\underline{\omega}_{\mathrm{crys},\mathbf{Q}_p^{\mathrm{nr}}}(W)$. We will also refer to this Newton polygon as the Newton polygon of W. It has has break points in $\mathbf{Z}^2$.

The third definition, which can be found in [Fon78, §6] is more complicated. Let h be an integer and suppose that D is a filtered module with Newton slopes in $(1/h)\mathbf{Z}$. For each $\alpha \in (1/h)\mathbf{Z}$, put $D_\alpha = \{d \in D \mid \mathrm{Fr}^h d = p^{h\alpha} d\}$ and

$$\underline{\Lambda}_h(D) = \bigoplus_{\alpha\in(1/h)\mathbf{Z}} D_\alpha.$$

One verifies that $\underline{\Lambda}_h(D)$ is a $\mathbf{Q}_{p^h}$-vector space (with $\mathbf{Q}_{p^h}$ as in 1.1) and that the natural map $\mathbf{Q}_p^{\mathrm{nr}} \otimes_{\mathbf{Q}_{p^h}} \underline{\Lambda}_h(D) \to D$ is an isomorphism.

We return to the category $\mathcal{C}$ and assume that there exists an integer h such that each object W of $\mathcal{C}$ has all its Newton slopes in $(1/h)\mathbf{Z}$. Then $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^h}} = \underline{\Lambda}_h \circ \underline{\omega}_{\mathrm{crys},\mathbf{Q}_p^{\mathrm{nr}}}$ is a $\mathbf{Q}_{p^h}$-valued fibre functor on $\mathcal{C}$. For any object W of $\mathcal{C}$ one has

$$\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^h}}(W) = \bigoplus_{\alpha} \underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^h}}(W)_\alpha$$

for α running through the Newton slopes of W. Let H be the automorphism group of $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^h}}$, it is an algebraic group over $\mathbf{Q}_{p^h}$. If follows from [Fon78, 6.4] that there exists a cocharacter

$$\mu_{N,h}\colon \mathbf{G}_{m,\mathbf{Q}_{p^h}} \longrightarrow H,$$

called the *Newton cocharacter*, such that, for any object W of $\mathcal{C}$ and each slope $\alpha \in \mathbf{Q}$ of W, this cocharacter makes $\mathbf{G}_m$ act on $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^h}}(W)_\alpha$ through the cocharacter $\cdot^{h\alpha}$. See [Pin98, §2] for the case where F is a finite extension of $\mathbf{Q}_p$.

It follows from [Fon94b, 5.1.7] that every crystalline representation V of $\mathcal{G}_F$ is Hodge–Tate. Let us recall what this means. The group $\mathcal{G}_F$ operates on $\mathbf{C}_p$ by continuity and we endow $V \otimes \mathbf{C}_p$ with a 'twisted' $\mathcal{G}_F$ action, $g(v \otimes c) = \rho(g)(v) \otimes g(c)$. Put

$$V\{i\} = \{v \in V \otimes \mathbf{C}_p \mid g(v) = \chi^{-i}(g)v\}$$

and $V(i) = V\{i\} \otimes_{\mathbf{Q}_p} \mathbf{C}_p$ for each $i \in \mathbf{Z}$.[1] The fact that V is *Hodge–Tate* means by definition that the natural injection $\oplus_{i\in\mathbf{Z}} V(i) \to V \otimes \mathbf{C}_p$ is an isomorphism. It follows from [Fon94a] and [Fon94b] that the Hodge–Tate decomposition of $V \otimes \mathbf{C}_p$ gives the associated graded of $(D \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} F) \otimes_F \mathbf{C}_p$ for the 'filtered module filtration' on $D \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} F$.

3.2 Proposition. *Let G be an algebraic group over $\mathbf{Q}_p$ and V a $\mathbf{Q}_p$-linear representation of G such that (G,V) is of Mumford's type. Let F be a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$ and let $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ a continuous, polarizable, crystalline representation of Hodge–Tate weights* 0 *and* 1. *Then the Newton polygon of ρ is either*

- $4 \times 0, 4 \times 1$ *or*
- $2 \times 0, 4 \times 1/2, 2 \times 1$ *or*
- $0, 3 \times 1/3, 3 \times 2/3, 1$ *or*
- $8 \times 1/2$.

Proof. The Zariski closure of $\rho(\mathcal{G}_F)$ has a finite number of connected components, so we can replace F by a finite extension such that this Zariski closure is connected. We can thus assume that G is connected.

Since V has Hodge–Tate weights 0 and 1, one has $V \otimes \mathbf{C}_p = V(0) \oplus V(1)$ and the existence of a polarization implies that $\dim V(0) = \dim V(1) = 4$. The Hodge polygon of $D = (V \otimes B_{\mathrm{crys}})^{\mathcal{G}_F}$ therefore has slopes $4 \times 0, 4 \times 1$ so the fact that the Newton polygon lies above the Hodge polygon (cf. [Fon94b, 5.4]) implies in that all Newton slopes of D are ≥ 0. The fact that V is polarizable implies that if α is a Newton slope of D, then $1-\alpha$ is a Newton slope of D of the same multiplicity. It follows that all Newton slopes lie in $[0,1]$

[1]Note the sign which may cause confusion because $\mathbf{Q}_p(1)(-1) = \mathbf{Q}_p(1)$ and $\mathbf{Q}_p(1)(i) = 0$ for $i \neq -1$.

and that the sum of the Newton slopes is equal to 4. It is easily checked that the conditions that the Newton polygon has integral break points, that all Newton slopes lie in $[0,1]$ and that the slopes α and $1-\alpha$ occur with the same multiplicity imply that all Newton slopes are in $(1/12)\mathbf{Z}$.

Let $\mathcal{C}$ be the tannakian subcategory of $\mathrm{Rep}_{\mathcal{G}_F}$ generated by V and $\mathbf{Q}_p(1)$. The above arguments imply that all objects of $\mathcal{C}$ have their Newton slopes in $(1/12)\mathbf{Z}$. Let H be the automorphism group of the functor $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^{12}}}$ introduced in 3.1. We have the Newton cocharacter

$$\mu = \mu_{N,12} \colon \mathbf{G}_{m,\mathbf{Q}_{p^{12}}} \longrightarrow H$$

as in loc. cit. and since H is a Galois twist (in fact even an inner twist) of the automorphism group of $\underline{\omega}_{\text{ét}}$, we deduce a cocharacter $\mu\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to G_{\overline{\mathbf{Q}}_p}$.

As $N\colon \widetilde{G} \to G$ is a central isogeny, there exists a positive integer k such that the k-th power $\mu^k\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to G_{\overline{\mathbf{Q}}_p}$ lifts to a cocharacter $\tilde{\mu}\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to \widetilde{G}_{\overline{\mathbf{Q}}_p}$. As $\widetilde{G}_{\overline{\mathbf{Q}}_p}$ decomposes as a product, one has $\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$, where $\tilde{\mu}_0\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to \mathbf{G}_{m,\overline{\mathbf{Q}}_p}$ and $\tilde{\mu}_i\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to \mathrm{SL}_{2,\overline{\mathbf{Q}}_p}$ for $i = 1,2,3$. This implies that there exist rational numbers n_0, n_1, n_2, n_3 such that the Newton slopes of V are $n_0 \pm n_1 \pm n_2 \pm n_3$. The sum of the Newton slopes of V is equal to 4, so $n_0 = 1/2$. One can assume that $n_1 \geq n_2 \geq n_3 \geq 0$ and as the slopes of V are in $[0,1]$ one must have $n_1 + n_2 + n_3 \leq 1/2$. We distinguish 4 cases.

If $n_i = 0$ for $i = 1,2,3$, then the Newton polygon is $8 \times 1/2$.

The second possibility is that $n_1 = n_2 = n_3 > 0$. In this case, $0 \leq 1/2 - 3n_1 < 1/2$ is the smallest slope of V. It is of multiplicity 1, so it is an integer and hence $n_1 = 1/6$. The Newton polygon of V is $0, 3 \times 1/3, 3 \times 2/3, 1$.

In the third case, we assume that $n_1 = n_2 > n_3$. If $n_3 \neq 0$ then the smallest slopes of V are $0 \leq 1/2 - 2n_1 - n_3 < 1/2 - 2n_1 + n_3 < 1/2$, both of multiplicity 1, so both are 0 and hence $n_3 = 0$. It follows that $n_3 = 0$ and that $0 \leq 1/2 - 2n_1 < 1/2$ is the smallest slope of V, of multiplicity 2. We conclude that $n_1 = 1/4$ and that the Newton slopes of V are $2 \times 0, 4 \times 1/2, 2 \times 1$.

In the fourth case, we assume that $n_1 > n_2$. The representation of $\mathcal{G}_F$ on $\mathrm{End}(V)$ belongs to $\mathcal{C}$ and as $L = \mathrm{Lie}(G)^{\mathrm{ss}} \subset \mathrm{End}(V)$ is a G-stable subspace, it is an object of $\mathcal{C}$ as well. The Newton slopes of L are $\pm 2n_i$ for $i = 1,2,3$ (each with multiplicity 1) and 3×0. In this case, $0 < 2n_1 \leq 1$ is a slope of L with multiplicity 1. It follows that $n_1 = 1/2$ and since $n_1 + n_2 + n_3 \leq 1/2$ also that $n_2 = n_3 = 0$. The Newton polygon of V is therefore $4 \times 0, 4 \times 1$. □

3.3 Remark. The assumption that ρ is polarizable means that there exists a non-degenerate alternating bilinear $\mathcal{G}_F$-equivariant map $V \times V \to \mathbf{Q}_p(-1)$, where $\mathbf{Q}_p(-1)$ is as in 1.1. In our case, there is an alternating form $\langle \cdot, \cdot \rangle$ on V (unique up to scalars) which is fixed by G up to a character, cf. 1.2. If one has $\langle \rho(g)x, \rho(g)y \rangle = \chi^{-1}(g)\langle x, y \rangle$ for all $g \in \mathcal{G}_F$ and all $x, y \in V$, then V is polarizable.

3.4 Let us keep the notations and hypotheses of 3.2 and let us assume that G is connected. We will refine the results of proposition 3.2 according to the nature of the decomposition of $\mathrm{Lie}(G)^{\mathrm{ss}}$ in $\mathbf{Q}_p$-simple factors.

As explained in [Ser78, 1.4], the Hodge–Tate decomposition $V \otimes \mathbf{C}_p = V(0) \oplus V(1)$ is given by a cocharacter

$$\mu = \mu_{\mathrm{HT}} \colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p}$$

such that $V(i)$ is the subspace of $V \otimes \mathbf{C}_p$ where μ acts via $\cdot^i \colon \mathbf{G}_m \to \mathbf{G}_m$. By the theorem of Sen, see [Ser78, Théorème 2], the Zariski closure (over $\mathbf{Q}_p$) in G of the image of μ is equal to the Zariski closure in G of $\rho(\mathcal{G}_F)$.

Let $\widetilde{G}$ and $N \colon \widetilde{G} \to G$ be as in 1.2. As $N \colon \widetilde{G} \to G$ is a central isogeny, there exists a positive integer k such that the k-th power $\mu^k \colon \mathbf{G}_{m,\mathbf{C}_p} \to G_{\mathbf{C}_p}$ lifts to a cocharacter $\tilde{\mu} \colon \mathbf{G}_{m,\mathbf{C}_p} \to \widetilde{G}_{\mathbf{C}_p}$. One has $\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$, where $\tilde{\mu}_0 \colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathbf{G}_{m,\mathbf{C}_p}$ and $\tilde{\mu}_i \colon \mathbf{G}_{m,\mathbf{C}_p} \to \mathrm{SL}_{2,\mathbf{C}_p}$ for $i = 1, 2, 3$. The fact that $V_{\mathbf{C}_p}$ is the direct sum of two eigenspaces for μ and hence for μ^k implies that the tensor product of the standard representations of the factors of $\widetilde{G}_{\mathbf{C}_p}$ is the direct sum of two eigenspaces for $\tilde{\mu}$, so exactly one of the $\tilde{\mu}_i$ (for $i = 1, 2, 3$) is non-trivial. This proves the following proposition.

3.5 Proposition. *Let G be an algebraic group over $\mathbf{Q}_p$ and V a $\mathbf{Q}_p$-linear representation of G such that (G, V) is of Mumford's type. Let F be a finite extension of $\mathbf{Q}_p^{\mathrm{nr}}$ and let $\rho \colon \mathcal{G}_F \to G(\mathbf{Q}_p)$ a continuous, polarizable, crystalline representation with Hodge–Tate weights 0 and 1 as in 3.2. Assume moreover that G is connected. Then the image of the composite map*

$$\mathrm{pr}_\mu \colon \mathbf{G}_{m,\mathbf{C}_p} \xrightarrow{\ \mu\ } G_{\mathbf{C}_p} \longrightarrow G^{\mathrm{ad}}_{\mathbf{C}_p} \xrightarrow{\ \cong\ } \left(\mathrm{PSL}_{2,\mathbf{C}_p}\right)^3$$

projects non-trivially to exactly one of the factors.

If $\mathrm{Lie}(G)^{\mathrm{ss}}$ is not $\mathbf{Q}_p$-simple, then G^{ad} is not $\mathbf{Q}_p$-simple and the image of $\rho(\mathcal{G}_F)$ in $G^{\mathrm{ad}}(\mathbf{Q}_p)$ projects non-trivially to exactly one of the factors.

3.6 Proposition. *With notations and hypotheses as in proposition 3.5, let G_1^{ad} be the $\mathbf{Q}_p$-simple factor of G^{ad} such that $\rho(\mathcal{G}_F)$ projects non-trivially to $G_1^{\mathrm{ad}}(\mathbf{Q}_p)$. Then the possible Newton polygons of ρ are $8 \times 1/2$ and*

1. $4 \times 0, 4 \times 1$ *in case* $\left(G_1^{\mathrm{ad}}\right)_{\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}_{2,\overline{\mathbf{Q}}_p}$,
2. $2 \times 0, 4 \times 1/2, 2 \times 1$ *in case* $\left(G_1^{\mathrm{ad}}\right)_{\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}^2_{2,\overline{\mathbf{Q}}_p}$,
3. $0, 3 \times 1/3, 3 \times 2/3, 1$ *in case* $\left(G_1^{\mathrm{ad}}\right)_{\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}^3_{2,\overline{\mathbf{Q}}_p}$.

Proof. Let G_1 be the connected component of the inverse image of G_1^{ad} in G and let $\widetilde{G}_1$ be the connected component of its inverse image in $\widetilde{G}$. One has $\widetilde{G}_1 = \mathbf{G}_m \times \widetilde{G}_1^{\mathrm{ss}}$ and $\widetilde{G}^{\mathrm{ss}}_{1,\overline{\mathbf{Q}}_p} \cong \mathrm{SL}^n_{2,\overline{\mathbf{Q}}_p}$ with $n = 1$ in the first case, $n = 2$ in the second and $n = 3$ in the third. After replacing F by a finite extension, one can assume that ρ factors through $G_1(\mathbf{Q}_p)$.

We will prove the proposition along the lines of the proof of proposition 3.2. Let $\mathcal{C}$ be the tannakian subcategory of $\mathrm{Rep}_{\mathcal{G}_F}$ generated by V and $\mathbf{Q}_p(1)$, let $H_{\mathbf{Q}_{p^{12}}}$ be the automorphism group of the functor $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^{12}}}$ introduced in 3.1 and let

$$\mu = \mu_{N,12}\colon \mathbf{G}_{m,\mathbf{Q}_{p^{12}}} \longrightarrow H_{\mathbf{Q}_{p^{12}}}$$

be the Newton cocharacter. As in loc. cit., $H_{\mathbf{Q}_{p^{12}}}$ is a inner form of the automorphism group of $\underline{\omega}_{\mathrm{\acute{e}t}}$ and we deduce a cocharacter $\mu\colon \mathbf{G}_{m,\overline{\mathbf{Q}}_p} \to G_{1,\overline{\mathbf{Q}}_p}$.

If $G^{\mathrm{ad}}_{1,\overline{\mathbf{Q}}_p} \cong \mathrm{PSL}_{2,\overline{\mathbf{Q}}_p}$, then $G_{1,\overline{\mathbf{Q}}_p} \cong \mathrm{GL}_2$ so, with the notations of the proof of proposition 3.2, $n_2 = n_3 = 0$ and it follows that the Newton polygon is either $4\times 0, 4\times 1$ or $8\times 1/2$.

In the second case distinguished in the proposition, one shows in the same way that $n_3 = 0$, which leaves 3 possible Newton polygons, namely $4\times 0, 4\times 1$ or $2\times 0, 4\times 1/2, 2\times 1$ or $8\times 1/2$. Assume that the Newton polygon is $4\times 0, 4\times 1$, so that $n_1 = 1/2$ and $n_2 = 0$. In that case, 3.1 furnishes a $\mathbf{Q}_p$-linear fibre functor $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_p}$ whose automorphism group $H_{\mathbf{Q}_p}$ is an inner form of a subgroup of G_1. This means that there is a $\mathcal{G}_{\mathbf{Q}_p}$-equivariant inclusion

$$\mathrm{Lie}(H)^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p} \subset \mathrm{Lie}(G_1)^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p} \cong \mathfrak{sl}^2_{2,\overline{\mathbf{Q}}_p}.$$

The action of $\mathcal{G}_{\mathbf{Q}_p}$ on the right hand side does not fix any factor. The Newton cocharacter is a map $\mu_{N,1}\colon \mathbf{G}_{m,\mathbf{Q}_p} \to H_{\mathbf{Q}_p}$ and this implies (with the notations of the proof of 3.2) that $n_1 = n_2$. This is a contradiction, proving that in this case the Newton polygon is either $2\times 0, 4\times 1/2, 2\times 1$ or $8\times 1/2$.

We finally consider the third case of the proposition and assume that the Newton slopes are $4\times 0, 4\times 1$ or $2\times 0, 4\times 1/2, 2\times 1$, which implies that $n_1 > 0$ and $n_3 = 0$. The construction of 3.1 provides the fibre functor $\underline{\omega}'_{\mathrm{crys},\mathbf{Q}_{p^2}}$ over $\mathbf{Q}_{p^2}$, its automorphism group $H_{\mathbf{Q}_{p^2}}$ and the Newton cocharacter $\mu_{N,2}\colon \mathbf{G}_{m,\mathbf{Q}_{p^2}} \to H_{\mathbf{Q}_{p^2}}$. As above, $H_{\mathbf{Q}_{p^2}}$ is an inner form of a subgroup of G. Since $\mathcal{G}_{\mathbf{Q}_{p^2}}$ acts transitively on the factors of $\mathrm{Lie}(G)^{\mathrm{ss}}_{\overline{\mathbf{Q}}_p} \cong \mathfrak{sl}^3_{2,\overline{\mathbf{Q}}_p}$, this implies that $n_1 = n_2 = n_3$, contradiction. □

4 Isogeny types

4.1 Proposition. *Let p be a prime number and F a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Suppose that $X/\mathcal{O}$ is an abelian scheme such that, for some prime number ℓ, the Galois representation on $V = \mathrm{H}^1_{\mathrm{\acute{e}t}}(X_{\bar{F}}, \mathbf{Q}_\ell)$ factors through a morphism $\rho\colon \mathcal{G}_F \to G(\mathbf{Q}_\ell)$ for an algebraic group $G \subset \mathrm{GL}(V)$ such that (G, V) is of Mumford's type and such that $\mathrm{Lie}(G)^{\mathrm{ss}}$ is $\mathbf{Q}_\ell$-simple. Then either*

- *the Newton polygon of the geometric special fibre $X_{\bar{k}}$ has slopes $8\times 1/2$ and $X_{\bar{k}}$ is isogenous to $\left(X^{(1)}\right)^4$, where $X^{(1)}/\bar{k}$ is an elliptic curve, or*
- *$X_{\bar{k}}$ is isogenous to a product of an elliptic curve and a simple abelian threefold*

– *or $X_{\bar{k}}$ is simple.*

Proof. If $\ell = p$, then the proposition follows from 3.6, so we can assume that $\ell \neq p$.

After replacing F by a finite extension, we can assume that k has even degree over its prime field, that G is connected and that all elements of $\rho(\mathcal{G}_F)$ are congruent to id modulo ℓ^2. As one has $\widetilde{G}(\overline{\mathbf{Q}}_\ell) = \mathbf{G}_m(\overline{\mathbf{Q}}_\ell) \times \mathrm{SL}_2(\overline{\mathbf{Q}}_\ell)^3$, the second condition implies that any element of $\rho(\mathcal{G}_F)$ lifts uniquely to an element of $\widetilde{G}(\overline{\mathbf{Q}}_\ell)$ congruent to id modulo ℓ^2. Uniqueness of the lift implies that it actually lies in $\widetilde{G}(\mathbf{Q}_\ell)$ so ρ lifts uniquely to a map $\tilde{\rho}\colon \mathcal{G}_F \to \widetilde{G}(\mathbf{Q}_\ell)$ satisfying the same congruence condition as ρ. Again by uniqueness of the lifting, $\tilde{\rho}$ is a group homomorphism.

Let $\mathrm{Frob} \in \mathcal{G}_F$ be a geometric Frobenius element in $\mathcal{G}_F$, which means that Frob induces on $\bar{k}$ the *inverse* of the map $x \mapsto x^q$, where $q = |k|$. Put $\pi = \tilde{\rho}(\mathrm{Frob}) \in \widetilde{G}(\mathbf{Q}_\ell)$ and define $\widetilde{T} \subset \widetilde{G}$ as the Zariski closure of the subgroup of $\widetilde{G}(\mathbf{Q}_\ell)$ generated by π. The congruence condition on $\tilde{\rho}(\mathcal{G}_F)$ implies that the subgroup of $(\overline{\mathbf{Q}}_\ell)^*$ generated by the eigenvalues of π does not contain any root of unity other than 1 and hence that $\widetilde{T}$ is connected. It follows that $\widetilde{T}$ is a torus of $\widetilde{G}$. To get rid of the scalars, we perform the following construction. We have $\sqrt{q} \in \mathbf{Z}$ by assumption and let $\widetilde{T}'$ be the Zariski closure of the subgroup of $\widetilde{G}(\mathbf{Q}_\ell)$ generated by $\alpha = \pi/\sqrt{q}$. Thus, $\widetilde{T} \cong \mathbf{G}_{m,\mathbf{Q}_\ell} \times \widetilde{T}'$, where $\mathbf{G}_{m,\mathbf{Q}_\ell}$ is the group of scalars in $\widetilde{G}$ and $\widetilde{T}'$ is a torus of $\widetilde{G}^{\mathrm{ss}} \subset \widetilde{G}$. Let $\overline{T} \subset \widetilde{G}^{\mathrm{ss}}$ be a maximal torus (over $\mathbf{Q}_\ell$) containing $\widetilde{T}'$.

There exists an isomorphism $X(\overline{T}) \cong \mathbf{Z}^3$ such that the weights of the representation of $\overline{T}$ on V correspond to the elements $(\pm1, \pm1, \pm1) \in \mathbf{Z}^3$. The natural action of $\mathcal{G}_{\mathbf{Q}_\ell}$ on $X(\overline{T})$ induces an action on $\mathbf{Z}^3$ stabilizing the set of weights $\{(\pm1, \pm1, \pm1)\}$. This $\mathcal{G}_{\mathbf{Q}_\ell}$-action on $\mathbf{Z}^3$ therefore factors through the group $\{\pm1\}^3 \rtimes \mathrm{S}_3$ acting naturally on $\mathbf{Z}^3$. As $\widetilde{G}^{\mathrm{ss}}$ is $\mathbf{Q}_\ell$-simple, one can assume (possibly after multiplying by -1 on one of the coordinates) that the image of $\mathcal{G}_{\mathbf{Q}_\ell}$ in $\{\pm1\}^3 \rtimes \mathrm{S}_3$ contains a cycle of S_3 of length 3.

The element $\alpha \in \overline{T}(\mathbf{Q}_\ell)$ defines a $\mathcal{G}_{\mathbf{Q}_\ell}$-equivariant map $\mathrm{ev}\colon X(\overline{T}) \to (\overline{\mathbf{Q}}_\ell)^*$. Since the weights of the representation of $\overline{T}$ on V correspond to the elements $(\pm1, \pm1, \pm1) \in \mathbf{Z}^3$, the images $\mathrm{ev}(\pm1, \pm1, \pm1)$ are the eigenvalues of α on V. It follows that the $\mathrm{ev}(\pm1, \pm1, \pm1)$ are in $\overline{\mathbf{Q}}$ and have all complex absolute values equal to 1. Here, and just for the duration of this proof, $\overline{\mathbf{Q}}$ denotes the algebraic closure of $\mathbf{Q}$ in $\overline{\mathbf{Q}}_\ell$. It follows that the image of ev actually lies in $(\overline{\mathbf{Q}})^*$. As $\widetilde{T}'$ is defined as the Zariski closure of the subgroup of $\widetilde{G}(\mathbf{Q}_\ell)$ generated by α, it follows that $\ker(\mathrm{ev})$ is the kernel of the natural surjection $X(\overline{T}) \to X(\widetilde{T}')$. We use this surjection to identify $X(\widetilde{T}')$ with $\mathbf{Z}^3/\ker(\mathrm{ev})$. For any element $(x_1, x_2, x_3) \in \mathbf{Z}^3$, we denote its image in $X(\widetilde{T}')$ by $(x_1, x_2, x_3)'$.

The map ev induces an injective map $\mathrm{ev}'\colon X(\widetilde{T}') \to (\overline{\mathbf{Q}})^* \subset (\overline{\mathbf{Q}}_\ell)^*$ which we can use to define an action of $\mathcal{G}_{\mathbf{Q}}$ on $X(\widetilde{T}')$ extending the action of $\mathcal{G}_{\mathbf{Q}_\ell}$. This action stabilizes $\{(\pm1, \pm1, \pm1)'\} \subset X(\widetilde{T}')$. All complex absolute values of the eigenvalues of α on V are equal to 1, so it follows that $\mathrm{ev}(P)\overline{\mathrm{ev}(P)} = 1$ for every complex conjugation $\bar{\cdot}$ in $\mathcal{G}_{\mathbf{Q}}$ and for every vertex $P = (\pm1, \pm1, \pm1)$ of the cube. This implies that all complex conjugations act on $X(\widetilde{T}')$ by inversion (multiplication by -1).

In what follows, we fix a p-adic valuation v on $\overline{\mathbf{Q}}$, normalized by $v(q) = 1$. The composite $\varphi = v \circ \mathrm{ev}' \colon X(\widetilde{T}') \to \mathbf{Q}$ is $\mathbf{Z}$-linear, but in general neither $\mathcal{G}_{\mathbf{Q}}$-equivariant nor injective.

4.2 Lemma. *Let $x \in X(\widetilde{T}')$. If $\varphi(\sigma(x)) = 0$ for all $\sigma \in \mathcal{G}_{\mathbf{Q}}$, then $x = 0$.*

Proof. First assume that x lies in the subgroup M of $X(\widetilde{T}')$ generated by $\{(\pm 1, \pm 1, \pm 1)'\}$. All absolute values of $\mathrm{ev}'(x) \in (\overline{\mathbf{Q}})^*$ are equal to 1, except maybe the p-adic ones. As σ runs through $\mathcal{G}_{\mathbf{Q}}$, $\varphi(\sigma(x))$ runs through all p-adic valuations of $\mathrm{ev}'(x)$, so these are all 0. It follows that $\mathrm{ev}'(x)$ is a root of unity. Since $\mathrm{ev}'(M) \subset (\overline{\mathbf{Q}})^*$ does not contain any root of unity other than 1, one has $\mathrm{ev}'(x) = 1$. The injectivity of ev' implies that $x = 0$.

The general case follows because $X(\widetilde{T}')$ is free and $M \subset X(\widetilde{T}')$ is of finite index. □

Since $\ker(\mathrm{ev})$ is stable under the $\mathcal{G}_{\mathbf{Q}_\ell}$-action, the fact that the image of $\mathcal{G}_{\mathbf{Q}_\ell}$ in $\{\pm 1\}^3 \rtimes \mathrm{S}_3$ contains a cycle of length 3 implies that we have the following possibilities for $\ker(\mathrm{ev})$.

A. $\ker(\mathrm{ev}) = \mathbf{Z}^3 = X(\overline{T})$

B. $\ker(\mathrm{ev}) = \{(x_1, x_2, x_3) \in \mathbf{Z}^3 \mid x_1 + x_2 + x_3 = 0\}$

C. $\ker(\mathrm{ev}) = \{(x, x, x) \mid x \in \mathbf{Z}\}$

D. $\ker(\mathrm{ev}) = \{0\}$

The cases B. and C. can only occur if the image of $\mathcal{G}_{\mathbf{Q}_\ell}$ in $\{\pm 1\}^3 \rtimes \mathrm{S}_3$ is contained in $\{\pm(1,1,1)\} \times \mathrm{S}_3$. We consider the possibilities A.–D. for $\ker(\mathrm{ev})$ one by one and prove the proposition in each case.

In case A., all elements of $\{(\pm 1, \pm 1, \pm 1)\}$ are mapped to $1 \in (\overline{\mathbf{Q}})^*$, so $\pi = \sqrt{q}$, all Newton slopes are equal to $1/2$ and X is isogenous to $(X^{(1)})^4$, where $X^{(1)}$ is the elliptic curve over k corresponding to the Weil number $\sqrt{q}$.

In case B., $X(\widetilde{T}')$ is a free $\mathbf{Z}$-module of rank 1. Let $w = (1,1,1)' \in X(\widetilde{T}')$, $x = (-1,1,1)'$, $y = (-1,-1,1)'$ and $z = (-1,-1,-1)'$. One has $w - x = x - y = y - z$, so it follows that $\varphi(w - x) = \varphi(x - y) = \varphi(y - z)$ and that $\varphi(w) = -\varphi(z)$. As lemma 4.2 implies that $\varphi(w) \neq 0$, the only possibility up to sign is that $\varphi(w) = 3\lambda$, $\varphi(x) = \lambda$, $\varphi(y) = -\lambda$ and $\varphi(z) = -3\lambda$ with $\lambda \in \mathbf{Q}$, $\lambda > 0$. The Newton slopes of X_k are

$$(-3\lambda + 1/2), 3 \times (-\lambda + 1/2), 3 \times (\lambda + 1/2), (3\lambda + 1/2)$$

so the facts that the slopes are in $[0,1]$ and that the Newton polygon has integral break points imply that $\lambda = 1/6$.

It follows that π has 4 distinct eigenvalues on V. The action of $\mathcal{G}_{\mathbf{Q}}$ on the set of eigenvalues corresponds to a $\mathbf{Z}$-linear action of $\mathcal{G}_{\mathbf{Q}}$ on $X(\widetilde{T}')$ stabilizing the set $\{w, x, y, z\}$ and any complex conjugation corresponds to the inversion. This implies that there are two orbits for the action of $\mathcal{G}_{\mathbf{Q}}$, so X_k has two non-isomorphic isogeny factors (possibly with multiplicities > 1). In the classification of isogeny classes abelian varieties over $\mathbf{F}_q$ by Weil numbers, cf.

[Tat69], the first factor corresponds to $\mathrm{ev}(1,1,1)\sqrt{q} \in \overline{\mathbf{Q}}$. This is an algebraic number of degree 2 and by the formulas of [Tat69], Théorème 1, the corresponding abelian variety is an elliptic curve $X^{(1)}/k$ with Newton slopes $0, 1$. The other simple factor corresponds to $\mathrm{ev}(-1,1,1)\sqrt{q}$, so it is an abelian variety with slopes $3 \times 1/3, 3 \times 2/3$. It is therefore an absolutely simple abelian threefold $X^{(3)}/k$ and $X_k \sim X^{(1)} \times X^{(3)}$.

In case C., $X(\widetilde{T}')$ is a free $\mathbf{Z}$-module of rank 2. We see that $(1,1,1)' = (-1,-1,-1)' = 0$. The Weil number $\sqrt{q} = \mathrm{ev}(1,1,1)\sqrt{q}$ corresponds to an isogeny factor $X^{(1)}/k$ of X_k which is an elliptic curve with Newton polygon $2 \times 1/2$.

The image of $\mathcal{G}_{\mathbf{Q}}$ in $\{\pm 1\}^3 \rtimes \mathrm{S}_3$ contains a cycle of length 3 and this element induces a cyclic permutation of $\{(-1,1,1),(1,-1,1),(1,1,-1)\}$. Since complex conjugation acts by inversion, this implies that $\mathcal{G}_{\mathbf{Q}}$ acts transitively on

$$\{\mathrm{ev}(\pm(-1,1,1)), \mathrm{ev}(\pm(1,-1,1)), \mathrm{ev}(\pm(1,1,-1))\} \subset (\overline{\mathbf{Q}})^*.$$

It follows that $\mathrm{ev}(-1,1,1)\sqrt{q}$ is a Weil number of degree 6 over $\mathbf{Q}$, and since it corresponds to an isogeny factor of X_k, the corresponding abelian variety is a simple abelian threefold $X^{(3)}/k$. This proves that $X_k \sim X^{(1)} \times X^{(3)}$ in this case.

For case D., one shows in the same way as above that the orbit under $\mathcal{G}_{\mathbf{Q}}$ of $\mathrm{ev}(-1,1,1)$ contains

$$\{\mathrm{ev}(\pm(-1,1,1)), \mathrm{ev}(\pm(1,-1,1)), \mathrm{ev}(\pm(1,1,-1))\} \subset (\overline{\mathbf{Q}})^*,$$

so there are one or two orbits for the action of $\mathcal{G}_{\mathbf{Q}}$ on $\{\mathrm{ev}(\pm 1, \pm 1, \pm 1)\}$ and $\mathrm{ev}(-1,1,1)\sqrt{q}$ is a Weil number of degree 6 or 8. Therefore, either X_k is simple, or $X_k \sim X^{(1)} \times X^{(3)}$, where $X^{(i)}/k$ is simple of dimension i (for $i = 1, 3$).

This proves the proposition for the isogeny type of X. Our arguments still hold after replacing F, and hence k, by any finite extension, so the same statement is true for $X_{\bar{k}}$. □

4.3 Remark. In this remark we assume that $\ell \neq p$ and that the Newton polygon of X_k is not $8 \times 1/2$.

The above arguments imply that if X_k is simple, then each eigenvalue of its Frobenius automorphism is of degree 8 over $\mathbf{Q}$. This implies that $\mathrm{End}(X_k)$ is commutative, so $\mathrm{End}_{\mathbf{Q}}(X_k)$ is a number field of degree 8. In fact, the proof of proposition 4.1 implies that this is true for k sufficiently large, but if $\mathrm{End}_{\mathbf{Q}}(X_k)$ is a number field of degree 8, then the same thing is true for the endomorphism algebra of a model of X_k over a subfield of k.

If X_k is reducible, then $X_k \sim X^{(1)} \times X^{(3)}$ and either the Weil numbers corresponding to $X^{(1)}$ and to $X^{(3)}$ are both of degree 2 or the Weil number of $X^{(3)}$ is of degree 6 and the Weil number corresponding to $X^{(1)}$ is of degree 1 or 2. In this case, $\mathrm{End}_{\mathbf{Q}}(X_k) = D_1 \times D_2$ for division algebras $D_1 = \mathrm{End}_{\mathbf{Q}}(X^{(1)})$ and $D_2 = \mathrm{End}_{\mathbf{Q}}(X^{(3)})$. Either

- D_1 and D_2 are number fields of degrees 2 and 6 respectively or
- D_1 is a quaternion division algebra over $\mathbf{Q}$ and D_2 is a number field of degree 6 or

– D_1 is a number field of degree 2 and D_2 is a division algebra of dimension 9 over its 2-dimensional centre.

4.4 Corollary. *Let X be a 4-dimensional abelian variety over a number field F and assume that for some prime number ℓ, the representation of $\mathcal{G}_F$ on $\mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_\ell)$ is of Mumford's type. Then X has potentially good reduction at any non-archimedean place v of F. For any such v, we have the following possibilities.*

1. *The reduction of X at v is isogenous to the fourth power of a supersingular elliptic curve and the Newton polygon of this reduction is $8 \times 1/2$.*

2. *The reduction of X at v is absolutely simple or geometrically isogenous to a product of an elliptic curve and an absolutely simple abelian threefold. The Newton polygon of the reduction of X at v is either $4 \times 0, 4 \times 1$ or $2 \times 0, 4 \times 1/2, 2 \times 1$ or $0, 3 \times 1/3, 3 \times 2/3, 1$.*

Proof. Combine lemma 1.3, corollary 2.2 and propositions 3.2 and 4.1. □

4.5 Remark. Suppose that X/F is as in the corollary. If the image of the representation of $\mathcal{G}_F$ on $\mathrm{H}^1_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_p)$ is known, then a stronger result on the possible Newton polygons of the reduction of X at a place v of residue characteristic p follows from proposition 3.6.

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