Models of Shimura varieties in mixed characteristic

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1 Introduction

1.1 In the study of integral models of Shimura varieties, results are largely limited to Shimura varieties of PEL-type. These are Shimura varieties serving as moduli spaces of abelian varieties endowed with a polarization, a given ring acting as endomorphisms and a level structure. An example of Mumford, see [Mu2, §4], shows that not all Shimura varieties constructed from subgroups of the symplectic group are obtained in this way. It can however be shown that, over $\mathbf{C}$, all such Shimura varieties parametrize polarized abelian varieties with level structure and given spaces of Hodge classes. In this paper, we will attempt to use this fact to study models in mixed characteristic of these Shimura varieties.

To this end, a "local p-adic analogue" of a Shimura variety is introduced in the following way. We start with a principally polarized abelian scheme X over the ring W of Witt vectors of the algebraic closure $\overline{\mathbf{F}}_p$ of a finite field. We fix a tensor product $\mathrm{T}_{\mathrm{DR}}(X/W)$ of DeRham cohomology groups of X and their duals, twisted to make the weight equal to 0, and we choose a class $\xi \in \mathrm{Fil}^0 \mathrm{T}_{\mathrm{DR}}(X/W)$. By the comparison theorem between the crystalline and the DeRham cohomology of X, there is a semi-linear Frobenius endomorphism on $\mathrm{T}_{\mathrm{DR}}(X/W) \otimes \mathbf{Q}$. We will assume that ξ is invariant under this endomorphism, or in other words, that ξ is a *Tate class*.

In Section 2, we will investigate the subspace $\mathcal{N}$ of the formal deformation space $\mathcal{M}$ of X_0, the special fibre of X, parametrizing principally polarized liftings Y/W of X_0 such that the image of ξ in $\mathrm{T}_{\mathrm{DR}}(Y/W)$ is a Tate class lying in $\mathrm{Fil}^0 \mathrm{T}_{\mathrm{DR}}(Y/W)$. We have to make the assumption that the reduction $X_0/\overline{\mathbf{F}}_p$ of X is *ordinary* to obtain any results. Recall that X_0 is ordinary if and only if $\mathrm{rk}_{\mathbf{Z}_p} T_p X_0 = \dim X_0$. In that case we can show that we can find an $\mathcal{N}$ with properties analogous to those of a Shimura variety. In fact, it will be the largest connected formal subscheme of $\mathcal{M}$ such that the class ξ can be extended to a horizontal section $\tilde{\xi} \in \mathrm{Fil}^0 \mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N})$ on the formal deformation of X_0 over $\mathcal{N}$ and such that the polarization of X_0 can be lifted to a polarization of $\mathcal{X}_{\mathcal{N}}$. We will prove that $\mathcal{N}$ is formally smooth. We can in fact say more about it. Because we assume X_0 to be ordinary, its deformation space $\mathcal{M}$ is canonically endowed with the structure of a toroidal formal group, i. e. a formal group which is isomorphic to $\widehat{\mathbf{G}}_m^d$ for some integer d. We will find that $\mathcal{N}$ is a formal subtorus of $\mathcal{M}$. This implies in particular that, if it has positive dimension, it contains infinitely many torsion points. Because these torsion points correspond to abelian varieties of CM-type, this provides us with another similarity with Shimura varieties, see [Mu2] or [De1].

The results of Section 2 are actually somewhat more general that those described above, because the abelian schemes we consider are defined over the integral closure $\bar{W}$ of W in the algebraic closure of its fraction field. In this more general case we can still construct $\mathcal{N}$ as above, the only difference being that it will no longer be a subtorus of $\mathcal{M}$, but the translate of such a subtorus over a $\bar{W}$-valued torsion point of $\mathcal{M}$. The statement about the CM-points remains true. The reader is referred to Theorem 2.8 for the precise statement.

In Section 3, the results of Section 2 will be applied to the study of models in mixed characteristic of Shimura varieties, let us say over the ring $\bar{W}$ considered above. For the lack of a better way of obtaining such a model of a Shimura variety, we will construct it by taking the Zariski closure of the canonical model of the Shimura variety inside an appropriate moduli space of abelian schemes over $\bar{W}$. In order to apply the results about Tate classes to this situation, we make use of a theorem of Blasius, which states that if ξ is a Hodge class on an abelian variety over a number field, then it is a Tate class at all primes where the abelian variety has good reduction. As we noted above, a Shimura variety serving as moduli space of principally polarized abelian varieties parametrizes abelian varieties endowed with given spaces of Hodge classes. Therefore, the results of Section 2 can be used to study the completion of our model in a point in characteristic p corresponding to an ordinary abelian variety X_0. We find that all components of this completion are translates of formal subtori of the formal deformation space $\mathcal{M}$ of X_0. The exact result is given in Theorem 3.7.

Acknowledgements. This paper is a generalization of a part of my thesis which was written under the direction of F. Oort. His encouragement and support have been of great value to me. Among the many comments on my thesis and on earlier versions of this paper, I would like to mention those of L. Illusie and of J.-P. Wintenberger. This paper owes much to both of them and I thank them most heartily.

1.2 Notation. Let us fix some notation to be in force for the rest of the paper. We choose an embedding of $\bar{\mathbf{Q}}$ into $\mathbf{C}$ and by this embedding we will consider all number fields as subfields of $\mathbf{C}$. Throughout the paper, p will be a prime number, $\mathbf{F}_p$ will denote the finite field with p elements and $\overline{\mathbf{F}}_p$ an algebraic closure of $\mathbf{F}_p$. The ring of infinite Witt vectors of $\overline{\mathbf{F}}_p$ will be denoted by W and its fraction field by B. We will write $\bar{B}$ for an algebraic closure of B and $\bar{W}$ for the integral closure of W in $\bar{B}$. The Frobenius mapping on W will be denoted by σ.

We will use the symbol T to denote tensor constructions of cohomology groups, roughly in the way as in [De2]. If X/k is an algebraic variety, then

$$\mathrm{T}_{\mathrm{DR}}(X/k) = \bigotimes_{\alpha\in I} \mathrm{H}^{n_\alpha}_{\mathrm{DR}}(X/k)(m) = \bigotimes_{\alpha\in I} \mathrm{H}^{n_\alpha}_{\mathrm{DR}}(X/k) \otimes k(m),$$

where I is a finite set of indices and $n_\alpha, m \in \mathbf{Z}$. We will always assume that $\sum_\alpha n_\alpha = 2m$. In the above expression, $k(m) = k(1)^{\otimes m}$ denotes a *Tate twist*, which, in the case of DeRham cohomology, means that $k(1) = k$ and

$$\mathrm{Fil}^i\, k(1) = \begin{cases} k(1) & \text{if } i \leq -1 \\ 0 & \text{if } i \geq 0. \end{cases}$$

If $i < 0$, then $\mathrm{H}^i_{\mathrm{DR}}(X/k)$ is defined to be the dual of $\mathrm{H}^{-i}_{\mathrm{DR}}(X/k)$. If X is an S-scheme for some scheme S, we will write $\mathrm{H}^i_{\mathrm{DR}}(X/S)$ and $\mathrm{T}_{\mathrm{DR}}(X/S)$ for the relative DeRham cohomology groups and tensor products of such groups respectively. A Tate twist only affects the Hodge filtration, just as in the absolute case.

In the case of crystalline cohomology we write $\mathrm{T}_{\mathrm{cris}}(X_0/B)$ for a tensor product of the crystalline cohomology groups of a smooth algebraic variety $X_0/\overline{\mathbf{F}}_p$ and their duals. In this case, the Tate twist is defined by $B(1) = B$ on which Frobenius acts as $p^{-1}\sigma$.

When considering the Betti cohomology of a smooth variety $X/\mathbf{C}$, the cohomology groups will be denoted by $\mathrm{H}^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$, and $\mathrm{T}_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ will be a tensor product of these groups and their duals. Here the Tate twist is given by $\mathbf{Q}(1) = 2\pi i\mathbf{Q} \subset \mathbf{C}$. It only has an effect when T_{B} is compared with the corresponding tensor in the DeRham cohomology. If we have a family

$$X \xrightarrow{f} S$$

over $\mathbf{C}$, we write $\mathrm{T}f^{\mathrm{an}}_*\mathbf{Q}$ for a tensor product of groups $\mathrm{R}^i f^{\mathrm{an}}_*\mathbf{Q}$.

2 Local moduli of Tate classes

2.1 In this section we will study the formal deformation space of an ordinary abelian variety over a finite field endowed with a number of Tate classes. Let us first recall some well-known facts about deformations of ordinary abelian varieties.

An abelian variety $X_0/\overline{\mathbf{F}}_p$ is called *ordinary* if the Tate module T_pX_0 is a $\mathbf{Z}_p$-module of rank $g = \dim X_0$. If X_0 is ordinary, then the formal deformation space $\mathcal{M}/W$ of X_0 is canonically isomorphic to

$$\mathrm{Hom}(T_pX_0 \otimes T_pX_0^t, \widehat{\mathbf{G}}_m),$$

see [Ka, 2.1] for the proof. This means in particular that $\mathcal{M}$ is canonically endowed with the structure of a formal group. We will refer to this structure as the *Serre–Tate group structure* on $\mathcal{M}$.

Let $\mathcal{X}_{\mathcal{M}}/\mathcal{M}$ be the universal formal deformation of X_0. We write $R = W[[t_1, \ldots, t_{g^2}]]$ and $R_0 = \overline{\mathbf{F}}_p[[t_1, \ldots, t_{g^2}]] = R/pR$ and choose an isomorphism $\mathcal{M} \cong \mathrm{Spf}(R)$. These notations will be fixed for the rest of the section. In the proof of our theorem, we will need a description of the cohomology group

$$H = \mathrm{H}^1_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \cong \mathrm{H}^1_{\mathrm{cris}}(\mathcal{X}_{\mathcal{M}} \otimes R_0/R).$$

Because X_0 is ordinary, this is an "ordinary Hodge-F-crystal on R_0 of level ≤ 1". This means that it is provided with the following data:

– A Hodge filtration of length at most 1, denoted by Fil.

– An integrable connection $\nabla\colon H \longrightarrow H \otimes_R \Omega_{R/W}$.

– A φ-linear Frobenius mapping $F(\varphi)\varphi^*$ for each lifting φ of the Frobenius of R_0 to R that is compatible with σ on W.

These data satisfy a number of conditions for which the reader is referred to [DI, 1.4]. All we need is a theorem of Deligne and Illusie, [DI, 1.4.2], which provides us with a detailed description of such a crystal. They prove the following theorem.

2.2 Theorem (Deligne, Illusie). *Suppose that H is an ordinary Hodge F-crystal on R_0 of level ≤ 1. Then the following statements hold.*

1. *There is a unique $U \subset H$ such that for every lifting φ of Frobenius, the map*

$$F(\varphi)\varphi^*|_U\colon U \to U$$

is an isomorphism and such that $H = U \oplus \mathrm{Fil}^1 H$.

2. *There are R-bases $a_1, \ldots, a_r$ of U and $b_1, \ldots, b_s$ of $\mathrm{Fil}^1 H$ with the properties that for every φ as above*

$$\nabla a_i = 0 \textit{ for } 1 \leq i \leq r,$$

$$\textit{there exist } \eta_{ij} \in \Omega^1_{R/W} \textit{ such that } \nabla b_i = \sum_{j=1}^{r} a_j \otimes \eta_{ij} \textit{ for } 1 \leq i \leq s,$$

$$F(\varphi)\varphi^* a_i = a_i \quad \textit{for } 1 \leq i \leq r, \textit{ and}$$

$$\textit{there are } u_{ij}(\varphi) \in R \textit{ such that } F(\varphi)\varphi^* b_i = pb_i + p\sum_{j=1}^{r} u_{ij}(\varphi) a_j \textit{ for } 1 \leq i \leq s.$$

Each η_{ij} is a closed form satisfying

$$\varphi^*\eta_{ij} = p\eta_{ij} + p\mathrm{d}u_{ij}(\varphi).$$

3. *There exist formal power series $\tau_{ij} \in B[[t_1, \ldots, t_{g^2}]]$ such that we have $\eta_{ij} = \mathrm{d}\tau_{ij}$ for all i and j and*

$$\varphi^*\tau_{ij} - p\tau_{ij} - pu_{ij}(\varphi) = 0$$

for all i, j and φ. We have $\tau_{ij}(0) \in pW$ and if $p \neq 2$, the series $q_{ij} = \exp(\tau_{ij})$ are defined, lie in R and satisfy $q_{ij}(0) \equiv 1 \bmod pW$.

4. *Suppose that $\mathcal{M} = \mathrm{Spf}(R)$ is the formal deformation space of an ordinary abelian variety X_0, that $\mathcal{X}_{\mathcal{M}}/\mathcal{M}$ is the universal family and that $H = \mathrm{H}^1_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M})$. Then $r = s = g = \dim X_0$ and 2.2.3 also holds for $p = 2$. Moreover, p and the elements $q_{ij} - 1$ form a regular system of parameters in $R = W[[t_1, \ldots, t_{g^2}]]$. If φ is the lifting of Frobenius for which $\varphi(q_{ij}) = q_{ij}^p$, we have*

$$F(\varphi)\varphi^* b_i = pb_i$$

for all $1 \leq i \leq g$.

Proof. Statement 2.2.1 follows from [DI, 1.3.2]. The statements 2.2.2 and 2.2.3 can be found in [DI, 1.4.2]. In the case that $p \neq 2$, statement 2.2.4 follows from [DI, 1.4.7] because the map

$$\mathrm{gr}\,\nabla\colon T_{R/W} \longrightarrow \mathrm{Hom}_R(\mathrm{Fil}^1 H, U)$$

induced by ∇ is the Kodaira–Spencer mapping, which is bijective because $\mathcal{X}_{\mathcal{M}}/\mathcal{M}$ is the universal family, see [DI, 2.1.3]. The case $p = 2$ follows from the appendix by Katz to [DI]. $\square$

2.3 The $q_{ij} \in R$ provided by the above theorem give "canonical coordinates" on $\mathcal{M}$ and hence an isomorphism

$$\mathcal{M} \xrightarrow{\cong} (\widehat{\mathbf{G}}_m)^{g^2},$$

see [DI, A3], thus inducing a new group structure on $\mathcal{M}$. Note that although the above isomorphism depends on the choice of the a_i and b_i, the group structure does not. We will call it the *Dwork group structure* on $\mathcal{M}$. The relation with the Serre–Tate group structure is given by the following theorem, which can be found in [DI, A3.1].

2.4 Theorem (Katz). *The Serre–Tate group structure and the Dwork group structure on $\mathcal{M}$ are the same.*

2.5 We come to the definition of a Tate class. It is shown in [BO, 2.4], that if X is a smooth proper $\bar{W}$-scheme, there is a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{DR}}(X/\bar{W}) \otimes \bar{B} \cong \mathrm{H}^i_{\mathrm{cris}}(X_0/W) \otimes \bar{B}$$

for each nonnegative integer i. Here X_0 denotes the special fibre of $X/\bar{W}$. This isomorphism induces an isomorphism

$$\mathrm{T}_{\mathrm{DR}}(X/\bar{W}) \otimes \bar{B} \cong \mathrm{T}_{\mathrm{cris}}(X_0/B) \otimes \bar{B}$$

of tensor constructions in the DeRham and crystalline theories. A cohomology class $\xi \in \mathrm{T}_{\mathrm{DR}}(X/\bar{W}) \otimes \bar{B}$ is called a *Tate class* if this isomorphism maps it to a Frobenius invariant element of $\mathrm{T}_{\mathrm{cris}}(X_0/B) \subset \mathrm{T}_{\mathrm{cris}}(X_0/B) \otimes \bar{B}$.

2.6 We are now ready to describe the main result of this section. It provides an analogue to the construction of Shimura varieties in the following way. Given an abelian variety $X/\bar{W}$ with ordinary reduction and a Tate class $\xi \in \mathrm{Fil}^0 \mathrm{T}_{\mathrm{DR}}(X/\bar{W})$, we will construct the largest irreducible formal subscheme $\mathcal{N} \subset \mathcal{M}$ such that ξ can be continued to a horizontal section over $\mathcal{N}$, lying in the middle step of the Hodge filtration everywhere. The proof of this theorem is an explicit calculation using the description 2.2 of the DeRham cohomology of the universal deformation of X. Theorem 2.2 will allow us to give an explicit description of a formal class $\tilde{\xi}$ providing a horizontal continuation of ξ. With *formal class* we mean to say that the power series expressing this class are allowed to have coefficients in $\bar{B}$ rather than in $\bar{W}$. Evidently, this class will not necessarily lie in Fil^0 at every point of $\mathcal{M}$.

The theorem of Deligne and Illusie provides us with a splitting of the Hodge filtration on $\mathrm{H}^1_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M})$ and hence with a decomposition $\oplus T^{p,q}$ of the tensor $\mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M})$, respecting the Hodge filtration. It is a rather surprising observation that the vanishing of the component in $T^{-1,1}$ of $\tilde{\xi}$ at a point of $\mathcal{M}$ implies that the value of $\tilde{\xi}$ at that point lies in Fil^0. This leads us to consider the formal subscheme $\mathcal{N}$ of $\mathcal{M}$ defined by the vanishing of this component. We will see that on $\mathcal{N}$, the class $\tilde{\xi}$ gives a real, and not just a formal, section of T_{DR}. At that point, it will also be evident that the vanishing of the said component of $\tilde{\xi}$ gives rise to a system of linear equations in the $\log(q_{ij})$ with coefficients in $\bar{W}$. The fact that ξ is a Tate class will be used to show that these coefficients actually lie in $\mathbf{Z}_p$. This suffices to show that $\mathcal{N}$ is the subscheme we are looking for and that it is the translate of a formal subtorus of $\mathcal{M}$ over a torsion point.

We need a lemma before we can state the promised theorem.

2.7 Lemma. *Choose an isomorphism $\mathcal{M} \cong \mathrm{Spf}(W[[t_1, \ldots, t_{g^2}]])$ such that x has coordinates $t_1 = \cdots = t_{g^2} = 0$. If $\eta \in \mathrm{T}_{\mathrm{DR}}(X/\bar{W})$, then there is a unique element*

$$\tilde{\eta} \in \mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \otimes_{W[[t_1,\ldots,t_{g^2}]]} \bar{B}[[t_1, \ldots, t_{g^2}]] =: \mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} \bar{B}$$

such that $\nabla\tilde{\eta} = 0$ and $\tilde{\eta}_X = \eta$. We will refer to $\tilde{\eta}$ as the formal horizontal continuation *of η.*

Proof. It follows from 2.2 that $\mathrm{H}^1_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} \bar{B}$ has a basis of horizontal sections, namely the basis consisting of the elements a_i and $b_i - \sum_j \tau_{ij} a_j$. Because $\tau_{ij} = \log(q_{ij})$ and since the $q_{ij} - 1$ lie in the maximal ideal of $\bar{W}[[t_1, \ldots, t_{g^2}]]$, this basis gives us a horizontal basis $\tilde{\beta}$ of $\mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} \bar{B}$. We can specialize the basis $\tilde{\beta}$ to obtain a basis β of $\mathrm{T}_{\mathrm{DR}}(X/\bar{W}) \otimes \bar{B}$. The element $\eta \in \mathrm{T}_{\mathrm{DR}}(X/\bar{W})$ can be written in a unique way as a linear combination of the elements of β and the same linear combination of the corresponding elements of $\tilde{\beta}$ is the promised horizontal continuation of η. The unicity of $\tilde{\eta}$ follows from the fact that in a power series ring an element with zero derivative is constant. □

2.8 Theorem. *Let X be a principally polarized abelian scheme over $\bar{W}$ with ordinary special fibre $X_0/\overline{\mathbf{F}}_p$. Suppose that we are given a finite number of tensor constructions $\mathrm{T}^{(k)}_{\mathrm{DR}}(X/\bar{W})$ and Tate classes $\xi_k \in \mathrm{T}^{(k)}_{\mathrm{DR}}(X/\bar{W})$. Then there exists a largest formal subscheme $\mathcal{N} \subset \mathcal{M}_{\bar{W}}$ with the properties that*

A1- the point $x \in \mathcal{M}(\bar{W})$ corresponding to X lies in $\mathcal{N}(\bar{W})$,

A2- $\mathcal{N}$ is irreducible,

A3- after replacing the classes ξ_k by suitable multiples, the formal horizontal continuations $\widetilde{\xi_k}$ of the classes ξ_k restrict to elements $\chi_k \in \mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N})$ and

A4- the polarization on X extends to a polarization on $\mathcal{X}_{\mathcal{N}}$.

In A3 and A4, $\mathcal{X}_{\mathcal{N}}$ denotes the pull back of the universal deformation $\mathcal{X}_{\mathcal{M}} \to \mathcal{M}$ to $\mathcal{N} \subset \mathcal{M}$. The formal subscheme $\mathcal{N}$ has the properties that

B1- $\mathcal{N}$ is the translate of a formal subtorus of $\mathcal{M}$ over a torsion point and

B2- the relative dimension of $\mathcal{N}$ over $\bar{W}$ is equal to the dimension of the subvariety of $\mathrm{Gr}(g, \mathrm{H}^1_{\mathrm{DR}}(X_{\bar{B}}/\bar{B}))$ parametrizing the $V \subset \mathrm{H}^1_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$ with the properties that

- *V is totally isotropic for the pairing induced by the polarization and*
- *for all k, the class $\xi_k \in \mathrm{T}^{(k)}_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$ lies in the middle step of the filtration induced by V.*

2.9 Remark. Condition A3 in the theorem may require some explanation. Recall that we fixed coordinates $t_1, \ldots, t_{g^2}$ such that x corresponds to the origin. The condition says that the image of $\widetilde{\xi_k}$ in $\mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N}) \mathbin{\widehat{\otimes}}_x \bar{B}$ is equal to the image of $\chi_k \in \mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N})$ in this space. The phrasing of A3 is justified by the fact that the map

$$\mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N}) \longrightarrow \mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N}) \otimes \bar{B}[[t_1, \ldots, t_{g^2}]] = \mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N}) \mathbin{\widehat{\otimes}}_x \bar{B}$$

is injective if $\mathcal{N}$ satisfies the conditions of the theorem. To see why this is the case, it suffices to show that the prime ideal $I \subset \bar{W}[[t_1, \ldots, t_{g^2}]]$ defining $\mathcal{N}$ satisfies

$$\bar{B}[[t_1, \ldots, t_{g^2}]]I \cap \bar{W}[[t_1, \ldots, t_{g^2}]] = I.$$

This equality holds because the map $\bar{W}[[t_1, \ldots, t_{g^2}]] \hookrightarrow \bar{B}[[t_1, \ldots, t_{g^2}]]$, being flat, satisfies the going down condition and the ideal $\bar{B}[[t_1, \ldots, t_{g^2}]]I$ is contained in the maximal ideal of $\bar{B}[[t_1, \ldots, t_{g^2}]]$ since $x \in \mathcal{N}(\bar{W})$.

2.10 We will first introduce some notation which will be used in the proof of this theorem.

Let A be a commutative ring with 1 of characteristic 0 and V a free A-module. If we have elements $z_1, \ldots, z_g, w_1, \ldots, w_g \in V$, if $\underline{i} = (i_1, \ldots, i_n)$, $\underline{j} = (j_1, \ldots, j_n)$ is a pair of multi-indices with $1 \le i_\ell, j_\ell \le g$ and if $\pi \subset \{1, \ldots, 2n\}$ is a subset with $|\pi| = n$, we define

$$\underline{z}^{\otimes \underline{i}} := z_{i_1} \otimes \cdots \otimes z_{i_n} \in V^{\otimes n}$$

and

$$(\underline{z}^{\otimes \underline{i}} \otimes \underline{w}^{\otimes \underline{j}})^\pi = (z_{i_1} \otimes \cdots \otimes z_{i_n} \otimes w_{j_1} \otimes \cdots \otimes w_{j_n})^\pi := x_1 \otimes \cdots \otimes x_{2n} \in V^{\otimes 2n},$$

where the x_i are defined as follows. Let $m_1, \dots, m_n$ and $m'_1, \dots, m'_n$ be the unique ascending sequences of integers such that we have $\pi = \{m_1, \dots, m_n\}$ and $\{1, \dots, 2n\} = \{m_1, \dots, m_n\} \cup \{m'_1, \dots, m'_n\}$. We put

$$x_{m_\ell} := z_{i_\ell},$$

$$x_{m'_\ell} := w_{j_\ell}.$$

Let $R = A[[X_{ij}]]_{i,j=1,\dots,g}$ and suppose that $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ is a basis of V. We identify V with the submodule $1 \otimes V \subset R \otimes_A V$ and we put

$$c_i = b_i - \sum_j X_{ij} a_j \in R \otimes_A V.$$

Suppose that elements $\gamma^\pi_{\underline{i},\underline{j}} \in A$ are given and let

$$\tilde{\xi} = \sum_{\underline{i},\underline{j},\pi} \gamma^\pi_{\underline{i},\underline{j}} (\underline{a}^{\otimes \underline{i}} \otimes \underline{c}^{\otimes \underline{j}})^\pi \in R \otimes_A V^{\otimes 2n}, \tag{2.10.1}$$

where the summation runs over all n-tuples $\underline{i}$ and $\underline{j}$ and all $\pi \subset \{1, \dots, 2n\}$ as above. We expand the right hand side of 2.10.1 and write it as $\Sigma_0(\underline{X}) + \cdots + \Sigma_n(\underline{X})$, where Σ_ℓ is a linear combination, with coefficients in R, of tensors of ℓ b_i and $2n - \ell$ a_i.

To make this precise, let $J = \{1, \dots, n\}$ and, for each $\Delta \subset J$ and every multi index $\underline{i}$, put

$$e_{\Delta,\underline{i}}(\underline{z}, \underline{w}) = x_1 \otimes \cdots \otimes x_n$$

where

$$x_\ell = \begin{cases} z_{i_\ell} & \text{if } \ \ell \in \Delta \\ w_{i_\ell} & \text{if } \ \ell \notin \Delta. \end{cases}$$

We then have

$$\Sigma_\ell(\underline{X}) = \sum_{\substack{\Delta \subset J \\ |\Delta| = \ell}} \sum_{\underline{i},\underline{j},\pi} \gamma^\pi_{\underline{i},\underline{j}} (\underline{a}^{\otimes \underline{i}} \otimes e_{\Delta,\underline{j}}(\underline{b}, -\sum_j X_{ij} a_j))^\pi.$$

2.11 Proof of Theorem 2.8 Since the abelian scheme X is principally polarized, the spaces $\mathrm{H}^i_{\mathrm{DR}}(X/\bar{W})$ are isomorphic to their duals. Because of condition A4, this isomorphism will extend over all of $\mathcal{N}$ if it exists, so we can assume that all the $\mathrm{T}^{(k)}_{\mathrm{DR}}(X/\bar{W})$ are tensor products of $\mathrm{H}^i_{\mathrm{DR}}(X/\bar{W})^{\otimes j}$ with $i, j > 0$. As

$$\mathrm{H}^i_{\mathrm{DR}}(X/\bar{W}) = \bigwedge^i \mathrm{H}^1_{\mathrm{DR}}(X/\bar{W}) \hookrightarrow \mathrm{H}^1_{\mathrm{DR}}(X/\bar{W})^{\otimes i},$$

we can even assume that, for every k, $\mathrm{T}^{(k)}_{\mathrm{DR}}(X/\bar{W}) = \mathrm{H}^1_{\mathrm{DR}}(X/\bar{W})^{\otimes 2j_k}(j_k)$ with $j_k > 0$.

Only for the proof of statement B2 we have to consider all the classes ξ_k simultaneously, the rest of the theorem follows as soon as we prove it in the case where there is only one single class. This is what we will do first.

Let ξ be one of the classes ξ_k and let $\mathrm{T}_{\mathrm{DR}}(X/\bar{W}) = \mathrm{T}_{\mathrm{DR}}^{(k)}(X/\bar{W})$. We resort to the notations of Deligne and Illusie in [DI], cf. 2.2. Because $\tau_{ij} = \log q_{ij}$, the τ_{ij} can be evaluated in the point $x \in \mathcal{M}(\bar{W})$, giving $\tau_{ij}(x) \in \bar{B}$. It follows that

$$c_i = b_i - \sum_j \tau_{ij} a_j \in \mathrm{H}^1_{\mathrm{DR}}(X_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} B$$

and that these elements can be evaluated in x to give elements $c_i(x) \in \mathrm{H}^1_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$. The set $\{a_i(x), c_i(x)\}_{i=1,\ldots,g}$ is a basis of $\mathrm{H}^1_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$ because $\{a_i(x), b_i(x)\}_{i=1,\ldots,g}$ is one.

Suppose that A/W is the canonical lifting of X_0 and let $a \in \mathcal{M}(W)$ be the corresponding point. Under the canonical isomorphism

$$\mathrm{H}^1_{\mathrm{DR}}(X_{\bar{B}}/\bar{B}) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{cris}}(X_0/W) \otimes \bar{B} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{DR}}(A_{\bar{B}}/\bar{B}), \tag{2.11.1}$$

the class $a_i(x)$ is mapped to $a_i(a)$ and $c_i(x)$ is mapped to $c_i(a) = b_i(a)$, see [BO, 2.10]. By the definition of a Tate class in $\mathrm{T}_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$, the isomorphism

$$\mathrm{T}_{\mathrm{DR}}(X_{\bar{B}}/\bar{B}) \xrightarrow{\sim} \mathrm{T}_{\mathrm{DR}}(A_{\bar{B}}/\bar{B})$$

deduced from 2.11.1 maps Tate classes to Tate classes. The tensors in the $a_i(a)$ and the $b_i(a)$ form a basis of the space $\mathrm{T}_{\mathrm{DR}}(A_{\bar{B}}/\bar{B})$ and the Tate classes in this space are precisely the classes that can be written as a linear combination with coefficients in $\mathbf{Q}_p$ of tensors consisting of the same numbers of $a_i(a)$ and $b_i(a)$. It follows that the Tate class ξ can be written as $\tilde{\xi}(x)$, where

$$\tilde{\xi} = \sum_{\underline{i},\underline{j},\pi} \gamma^{\pi}_{\underline{i},\underline{j}} (\underline{a}^{\otimes \underline{i}} \otimes \underline{c}^{\otimes \underline{j}})^{\pi} \in \mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} B, \tag{2.11.2}$$

with $\gamma^{\pi}_{\underline{i},\underline{j}} \in \mathbf{Q}_p$. Here we use the notations introduced in 2.10. Because the classes a_i $(i = 1, \ldots, g)$ and the formal classes c_i $(i = 1, \ldots, g)$ are horizontal, it follows that $\nabla\tilde{\xi} = 0$. Therefore, $\tilde{\xi}$ is the formal horizontal continuation of ξ which was introduced in 2.7. By replacing ξ by a multiple, we can make sure that $\gamma^{\pi}_{\underline{i},\underline{j}} \in \mathbf{Z}_p$ for each $\underline{i}$, $\underline{j}$ and π. From now on we will assume that such a replacement has taken place.

We adopt the notations of 2.10 and we write $\tilde{\xi} = \Sigma_0(\underline{\tau}) + \cdots + \Sigma_n(\underline{\tau})$. Recall that the $\Sigma_\ell(\underline{\tau})$ are elements of $\mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} \bar{B}$, in fact even of $\mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \mathbin{\widehat{\otimes}_x} B$. For every $\mathcal{N}$ with property A3 of the theorem, $\Sigma_{n-1}(\underline{\tau})$ must vanish on $\mathcal{N}$. In the expression of this element with respect to the basis formed by the tensors in the a_i and the b_j, the coefficients are homogeneous linear polynomials in the τ_{ij} with coefficients in $\mathbf{Z}_p$. Let $\underline{\Pi}_{n-1}(\underline{X})$ be the vector of expressions provided by these coefficients, written multiplicatively. This means that $\log(\underline{\Pi}_{n-1}(\underline{q}))$ is the vector of coefficients of $\Sigma_{n-1}(\underline{\tau})$. The condition that $\Sigma_{n-1}(\underline{\tau})$ should vanish on the $\mathcal{N}$ we are looking for implies that $\log(\underline{\Pi}_{n-1}(\underline{q}))$ should vanish on $\mathcal{N}$

and hence that every $\mathcal{N}$ satisfying A1–A3 is contained in the formal subscheme of $\mathcal{M}$ given by the equations

$$\underline{\Pi}_{n-1}(\underline{q}) = \underline{\zeta}, \tag{2.11.3}$$

where $\underline{\zeta}$ denotes a sequence of p-power roots of unity. The condition A1, that x should be a point of $\mathcal{N}$, completely determines $\underline{\zeta}$. Gaussian elimination shows that the formal subscheme of $\mathcal{M}$ described by the equations in 2.11.3 is the union of translates of a fixed formal subtorus of $\mathcal{M}$ over various torsion points. Two such translates are either equal or the sets of their $\bar{W}$-valued points are disjoint. Let $\mathcal{N}'$ be the unique component of the formal subscheme described by 2.11.3 containing the point x. It follows from the above that every $\mathcal{N} \subset \mathcal{M}$ satisfying A1–A3 is contained in $\mathcal{N}'$. As we saw, $\mathcal{N}'$ is the translate of a formal subtorus of $\mathcal{M}$ and hence it is smooth and irreducible. It is therefore evident that $\mathcal{N}'$ satisfies A1 and A2, and we will show that it also verifies condition A3.

Consider the class

$$\chi = \sum_{\underline{i},\underline{j},\pi} \gamma^{\pi}_{\underline{i},\underline{j}} (\underline{a}^{\otimes \underline{i}} \otimes \underline{b}^{\otimes \underline{j}})^{\pi} \in \mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}),$$

where the $\gamma^{\pi}_{\underline{i},\underline{j}} \in \mathbf{Z}_p$ are the same as in Expression 2.11.2 for $\tilde{\xi}$. With the notations of 2.10, we have that $\chi = \Sigma_n(\underline{\tau})$. The structure theorem of Deligne and Illusie shows that

$$\nabla\chi = -\Sigma_{n-1}(\underline{\eta}) \in \mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{M}}/\mathcal{M}) \otimes \Omega_{\mathcal{M}/W}.$$

Since the coefficients of the tensors in the a_i and b_j in $\Sigma_{n-1}(\underline{X})$ are linear polynomials in the X_{ij}, the expression $\Sigma_{n-1}(\underline{\eta})$ has an obvious meaning. As $\Sigma_{n-1}(\underline{\tau})$ vanishes on $\mathcal{N}'$ and $\underline{\eta} = \mathrm{d}\underline{\tau}$, it follows that $\Sigma_{n-1}(\underline{\eta}) = 0$, so χ is horizontal on $\mathcal{N}'$. The fact that

$$(\Sigma_n(\underline{\tau}) + \Sigma_{n-1}(\underline{\tau}) + \ldots + \Sigma_0(\underline{\tau}))(x) = \tilde{\xi}(x) \in \mathrm{Fil}^0\, \mathrm{T}_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$$

implies that

$$\Sigma_{n-1}(\underline{\tau})(x) = \Sigma_{n-2}(\underline{\tau})(x) = \ldots = \Sigma_0(\underline{\tau})(x) = 0,$$

and hence that $\tilde{\xi}(x) = \Sigma_n(\underline{\tau})(x) = \chi(x)$. Because $\mathcal{N}'$ is smooth and both classes are horizontal on $\mathcal{N}'$, it follows that the image of $\tilde{\xi}$ in $\mathrm{T}_{\mathrm{DR}}(\mathcal{X}_{\mathcal{N}}/\mathcal{N})\,\widehat{\otimes}_x\,\bar{B}$ is equal to the image of χ in this module. This shows that $\mathcal{N}'$ satisfies A1–A3, and since we already saw that any formal subscheme satisfying these conditions is contained in it, it follows that $\mathcal{N}'$ is the largest such subscheme.

It is a well-known fact, see for example [dJN, 3.7], that the condition A4 of the theorem determines a formal subtorus $\mathcal{N}'' \subset \mathcal{M}$. The intersection $\mathcal{N}$ of $\mathcal{N}'$ with $\mathcal{N}''$ is the translate of a formal subtorus of $\mathcal{M}$ over a torsion point and it has the properties A1–A4. By what was said above, it is also clear that $\mathcal{N}$ is the largest formal subscheme of $\mathcal{M}$ satisfying A1–A4. With the exception of statement B2, the theorem has now been proved.

For the proof of B2, we write $H = \mathrm{H}^1_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$, $T^{(k)} = \mathrm{T}^{(k)}_{\mathrm{DR}}(X_{\bar{B}}/\bar{B})$, $\bar{a}_i = a_i(x)$ and $\bar{c}_i = c_i(x)$, for $i = 1, \dots, g$. Recall that $\{\bar{a}_1, \dots, \bar{a}_g, \bar{c}_1, \dots, \bar{c}_g\}$ is a basis of H. The g^2-dimensional affine space $U/\bar{B}$ with coordinates X_{ij} $(i, j = 1, \dots, g)$ can be identified with an open subset of the Grassmann variety $\mathrm{Gr}(g, H)$ parametrizing g-dimensional linear subspaces of H by letting the subspace generated by

$$\left\{ \bar{c}_i + \sum_j X_{ij} \bar{a}_j \right\}_{i=1,\dots,g}$$

correspond to the point (X_{ij}). In analogy with the above we consider the vector bundle $H \times U \to U$ with constant basis $\{a_1, \dots, a_g, c_1, \dots, c_g\}$. The sections $b_1, \dots, b_g$ with $b_i = c_i + \sum_j X_{ij} a_j$ generate the subbundle $\mathrm{Fil} \subset H \times U$ which gives a g-dimensional subspace in each fibre. Writing

$$\xi_k = \sum_{\underline{i}, \underline{j}, \pi} (\gamma^{\pi}_{\underline{i}, \underline{j}})_k (\underline{a}^{\otimes \underline{i}} \otimes \underline{c}^{\otimes \underline{j}})^{\pi}(x),$$

we see that the intersection of U with the subvariety of $\mathrm{Gr}(g, H)$ determined by the conditions in B2 consists of the points y for which

- $\mathrm{Fil}_y \subset H$ is totally isotropic for the pairing induced by the polarization and
- for each k, the class

  $$\sum_{\underline{i}, \underline{j}, \pi} (\gamma^{\pi}_{\underline{i}, \underline{j}})_k (\underline{a}^{\otimes \underline{i}} \otimes \underline{c}^{\otimes \underline{j}})^{\pi}(y)$$

  lies in the middle step of the filtration on $T^{(k)}$ induced by $\mathrm{Fil}_y \subset H$.

Application of the same arguments as we used above shows that this intersection is given by the same linear equations we found for $\mathcal{N}$. $\square$

2.12 Corollary. *If $Y = \mathcal{X}_y$ for some point $y \in \mathcal{N}(\bar{W})$, then the image of each of the classes ξ_k under the canonical isomorphism*

$$\mathrm{T}^{(k)}_{\mathrm{DR}}(X_{\bar{B}}/\bar{B}) \xrightarrow{\sim} \mathrm{T}^{(k)}_{\mathrm{cris}}(X_0/B) \otimes \bar{B} \xrightarrow{\sim} \mathrm{T}^{(k)}_{\mathrm{DR}}(Y_{\bar{B}}/\bar{B})$$

is a Tate class lying in $\mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(Y_{\bar{B}}/\bar{B})$.

Proof. This follows immediately from Theorem 2.8 and its proof, because the image of $\xi_k = (\widetilde{\xi_k})_x$ under the above isomorphism is equal to $(\widetilde{\xi_k})_y$, see Equation 2.11.1 and the remarks following it. $\square$

3 Application to Shimura varieties

3.1 In this section, the results of Section 2 will be applied to the study of Shimura varieties. We will first recall some generalities on Shimura varieties and then we will construct a model $\widetilde{M}$ in mixed characteristic of a Shimura variety M by embedding it in a moduli space of principally polarized abelian varieties and taking the Zariski closure of the image. Although this is a rather ad hoc method, we will nevertheless be able to investigate the completion of $\widetilde{M}$ at a point x_0 in positive characteristic corresponding to an ordinary abelian variety.

For this investigation we will use Theorem 2.8, which we can apply because a Shimura variety can be considered as the "largest" moduli space of abelian varieties over which a given Hodge class can be continued horizontally inside Fil^0. Since a Hodge class is also a Tate class, this interpretation shows that the components of the completion of $\widetilde{M}$ at x_0 are formal subschemes of the formal deformation space of the ordinary abelian variety corresponding to x_0 to which we can apply Theorem 2.8. This implies that all these components are translates of formal subtori, so in particular that they are smooth. It also shows that the normalization of $\widetilde{M}$ is smooth at x_0.

3.2 Let Y be a principally polarized abelian variety of dimension g over $\mathbf{C}$ and let $V = \mathrm{H}^1_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Q})$ and $V_{\mathbf{Z}} = \mathrm{H}^1_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Z})$. Let $G \subset \mathrm{GL}(V)$ be the Mumford–Tate group of Y and $K \subset G(\mathbf{A}^f)$ a compact open subgroup. The Hodge structure on V is determined by a map $h\colon S := R_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m \to G_{\mathbf{R}}$. Let $K_\infty \subset G(\mathbf{R})$ be the centralizer of h. In this situation, we can form the Shimura variety $M_K(\mathbf{C}) = (K_\infty \times K)\backslash G(\mathbf{A})/G(\mathbf{Q})$ as in [De1]. By loc. cit., $M_K(\mathbf{C})$ is the set of complex points of a $\mathbf{C}$-scheme M_K, which, again by [De1], can be defined over a number field E. In the sequel, we will always consider weakly canonical models of the Shimura varieties we encounter, see [De1, 3.13] for the definition and [De1, 5.9] for the proof that such a model exists in this case. If we choose K sufficiently small, then M_K is smooth and there exists a universal family of abelian varieties $f_K\colon \mathcal{X}_K \longrightarrow M_K$. This family can be defined over a number field.

On the abelian variety Y there are various spaces of Hodge classes. These will be denoted $\mathrm{C}^{(k)}_{\mathrm{B}}(Y, \mathbf{Q}) \subset \mathrm{T}^{(k)}_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Q})$ for the Betti cohomology and $\mathrm{C}^{(k)}_{\mathrm{DR}}(Y/\mathbf{C}) \subset \mathrm{T}^{(k)}_{\mathrm{DR}}(Y/\mathbf{C})$ for the DeRham cohomology. The spaces $\mathrm{C}^{(k)}_{\mathrm{DR}}$ are obtained from the $\mathrm{C}^{(k)}_{\mathrm{B}}$ by tensoring with $\mathbf{C}$ and applying the appropriate comparison theorem. The $\mathrm{C}^{(k)}_{\mathrm{B}}$ and hence the $\mathrm{C}^{(k)}_{\mathrm{DR}}$ are determined by the Mumford–Tate group G. Since it follows from [De2, 3.1] that G is the stabilizer of a one-dimensional space of Hodge classes in a finite direct sum $\bigoplus_k \mathrm{T}^{(k)}_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Q})$, the reader will convince himself that in what follows it suffices to consider a finite number of tensors an a finite number of Hodge classes $\xi_k \in \mathrm{C}^{(k)}_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Q}) \subset \mathrm{T}^{(k)}_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Q})$. For each k, we will write $\Xi^{(k)} \subset \mathrm{T}^{(k)}_{\mathrm{B}}(Y(\mathbf{C}), \mathbf{Q})$ for the subspace generated by ξ_k.

If K is chosen sufficiently small, the $\mathrm{C}^{(k)}_{\mathrm{B}}(Y, \mathbf{Q})$ can be extended to spaces of global sections

$$\mathrm{C}^{(k)}_{\mathrm{B}}(\mathcal{X}_K/M_K) \subset \mathrm{H}^0(M_K(\mathbf{C}), \mathrm{T}^{(k)}(f^{\mathrm{an}}_K)_*(\mathbf{Q})),$$

and the $\mathrm{C}_{\mathrm{DR}}^{(k)}(Y/\mathbf{C})$ can be extended to spaces of horizontal sections

$$\mathrm{C}_{\mathrm{DR}}^{(k)}(\mathcal{X}_K/M_K) \subset \mathrm{H}^0(M_K, \mathrm{Fil}^0\, \mathrm{T}_{\mathrm{DR}}^{(k)}(\mathcal{X}_K/M_K)).$$

In fact, $M_K(\mathbf{C})$ is locally determined by the second property, or by the equivalent property for the $\Xi^{(k)}$, cf. [Mu1]. The spaces $\mathrm{C}_{\mathrm{DR}}^{(k)}(\mathcal{X}_K/M_K)$ can be defined over a number field.

If we consider the algebraic group $\mathrm{CSp}(V)$, and if $K' \subset \mathrm{CSp}(V)(\mathbf{A}^f)$ is a compact open subgroup, the Shimura varieties we obtain are the moduli spaces $A_{K'}/\mathbf{Q}$ of principally polarized abelian varieties with level K' structure. If K' is sufficiently small, $A_{K'}$ is a fine moduli space and there exists a universal family over it. Choosing K' still smaller, we can also make sure that for all $K^{(3)} \subset K'' \subset K'$ the transition maps $A_{K^{(3)}} \to A_{K''}$ are finite étale. From now on, we will assume that all our compact open subgroups are small enough for this to be the case. For each K' there is a natural number n such that $K'_n \subset K'$, where K'_n denotes the congruence subgroup of $\mathrm{CSp}(V_{\mathbf{Z}})(\hat{\mathbf{Z}}) \subset \mathrm{CSp}(V)(\mathbf{A}^f)$ of the elements congruent to 1 modulo n. For such a number n there is a map $A_n := A_{K'_n} \twoheadrightarrow A_{K'}$. The Shimura variety A_n is the moduli space of principally polarized abelian varieties with symplectic level n structure, so by [MF], Appendix to Chapter 7, A, it admits a canonical model over $\mathbf{Q}$, which in turn has a smooth model $\mathcal{A}_n$ over $\mathbf{Z}[1/n]$. Given a $K \subset G(\mathbf{A}^f)$ as above and a number n such that $K \subset K'_n$, there is a map $M_K \to A_n$. Because we stick with our convention to use weakly canonical models, this map can be defined over E. In particular, such a map exists if we take K to be the compact open subgroup $K_n = K'_n \cap G(\mathbf{A}^f)$ of $G(\mathbf{A}^f)$, in which case we will denote it by φ_n. From now on, we will write M_n for M_{K_n} and $f_n\colon \mathcal{X}_n \to M_n$ for the universal family $\mathcal{X}_{K_n}$.

3.3 Lemma. *For each prime number p, there exists a number n, prime to p, such that the map $\varphi_n\colon M_n \to A_n$ is a closed immersion.*

Proof. For the duration of this proof, we will write $\mathrm{T}^{(k)}(V) := \mathrm{T}_{\mathrm{B}}^{(k)}(Y(\mathbf{C}), \mathbf{Q})$. Recall that we are given one-dimensional subspaces $\Xi^{(k)} \subset \mathrm{T}^{(k)}(V)$ for each of a finite number of k. We will need the following description of the points of $M_n(\mathbf{C})$ cf. [De1, 1.12].

The points of $A_n(\mathbf{C})$ correspond to abelian varieties X over $\mathbf{C}$ endowed with a principal polarization and a full symplectic level n structure, i.e. a symplectic isomorphism $X[n] \cong V_{\mathbf{Z}}/nV_{\mathbf{Z}}$. A point of $M_n(\mathbf{C})$ corresponds to a principally polarized abelian variety $X/\mathbf{C}$ with level n structure and endowed with a $G(\mathbf{Q})$-class $\bar{\iota}$ of isomorphisms $V \xrightarrow{\cong} \mathrm{H}_{\mathrm{B}}^1(X(\mathbf{C}), \mathbf{Q})$ with the properties that

- letting h_X and h_Y be the maps determining the Hodge structures on $\mathrm{H}_{\mathrm{B}}^1(X(\mathbf{C}), \mathbf{Q})$ and V respectively, the maps $\iota^{-1} h_X \iota$ and $h_Y\colon S \to G_{\mathbf{R}}$, are conjugate by an element of $G(\mathbf{R})$ for any $\iota \in \bar{\iota}$, and
- the level structure lifts to an isomorphism $k\colon \mathrm{H}_{\mathrm{B}}^1(X(\mathbf{C}), \mathbf{A}^f) \to V \otimes \mathbf{A}^f$ such that the induced map

$$V \otimes \mathbf{A}^f \xrightarrow{\iota_{\mathbf{A}^f}} \mathrm{H}_{\mathrm{B}}^1(X(\mathbf{C}), \mathbf{A}^f) \xrightarrow{k} V \otimes \mathbf{A}^f$$

belongs to $G(\mathbf{A}^f)$.

The first condition implies that for any element $\iota \in \bar{\iota}$, the isomorphism $\mathrm{T}^{(k)}(\iota)\colon \mathrm{T}^{(k)}(V) \xrightarrow{\cong} \mathrm{T}^{(k)}_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ induced by ι maps $\Xi^{(k)}$ into $\mathrm{C}^{(k)}_{\mathrm{B}}(X, \mathbf{Q})$. The image only depends on $\bar{\iota}$. It follows that such a class $\bar{\iota}$ can be determined by giving a 1-dimensional subspace of $\mathrm{C}^{(k)}_{\mathrm{B}}(X, \mathbf{Q})$ for each k.

It follows from [De1, 1.15] that the map φ_n is finite for n sufficiently large. Let $\Delta_n \subset M_n \times M_n$ be the diagonal and let $\Gamma_n \subset M_n \times M_n$ be the graph of the relation $\varphi_n(x) = \varphi_n(y)$. We will consider the first projection R_n of $\overline{\Gamma_n \backslash \Delta_n}$. For each n, this is a closed subscheme of M_n and it follows from the above description of the points of $M_n(\mathbf{C})$ that $R_{n'}$ maps into R_n if $n' \geq n$. We have to show that, unless R_n is empty, we can increase its codimension by increasing n.

Suppose that R is an irreducible component of R_n of maximal dimension and let $x \in R(\mathbf{C})$ be a generic point. The fact that $x \in R_n(\mathbf{C})$ implies that, on the corresponding abelian variety X, there are several one dimensional subspaces of Hodge classes in some $\mathrm{C}^{(k)}_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ endowing X with the structure of a point of M_n. Since φ_n is finite, there are only finitely many such subspaces. If we choose $n' > n$, prime to p, so large that the intersections of all these spaces with $\mathrm{T}^{(k)}_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Z})$ are different modulo n', then $R_{n'}$ does not have any irreducible components mapping onto R. Repeating this for all components of R_n, we find a number n' such that the codimension of $R_{n'}$ in $M_{n'}$ is strictly larger than that of R_n. $\square$

3.4 We intend to study the reduction of our Shimura variety at a prime of which we now fix the residue characteristic $p > 0$. By the lemma, there is an integer n, prime to p, such that M_n admits a closed immersion into the moduli space of principally polarized abelian varieties with symplectic level n structure. We fix such an n and a field E such that M_n has a weakly canonical model over $\mathrm{Spec}(E)$. We will assume that E is large enough for $\mathcal{X}_n$, the level structure and the spaces $\mathrm{C}^{(k)}_{\mathrm{DR}}(\mathcal{X}_n/M_n)$ to be defined over E. As we noted above, the moduli space A_n has a model $\mathcal{A}_n$ over $\mathrm{Spec}(\mathbf{Z}[1/n])$. After the necessary base changes, this provides us with the diagram

$$\begin{array}{ccccc} M_n & \hookrightarrow & A_n & \hookrightarrow & \mathcal{A}_n \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(E) & = & \mathrm{Spec}(E) & \hookrightarrow & \mathrm{Spec}\left(\mathcal{O}_E[\tfrac{1}{n}]\right). \end{array}$$

Note that $f_n\colon \mathcal{X}_n \to M_n$ is the pull back to M_n of the universal family on A_n. After replacing E by an algebraic extension and M_n by a component, we can assume that M_n is geometrically connected. From now on, we will assume this to be the case.

We define $\widetilde{M}$ to be the Zariski closure of M_n in $\mathcal{A}_n$. By pulling back the universal family on $\mathcal{A}_n$ to $\widetilde{M}$, we obtain an abelian scheme

$$\tilde{f}\colon \widetilde{\mathcal{X}} \longrightarrow \widetilde{M}.$$

The pull back of $\widetilde{\mathcal{X}}$ to the generic fibre $M_n = \widetilde{M}_E$ is canonically isomorphic to $\mathcal{X}_n$.

3.5 Lemma. *There are unique subspaces*

$$\widetilde{\mathrm{C}}^{(k)}_{\mathrm{DR}} \subset \mathrm{H}^0(\widetilde{M}, \mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(\widetilde{\mathcal{X}}/\widetilde{M}))$$

restricting to $\mathrm{C}^{(k)}_{\mathrm{DR}}(\mathcal{X}_n/M_n) \cap \mathrm{T}^{(k)}_{\mathrm{DR}}(\widetilde{\mathcal{X}}/\widetilde{M})$ *on* M_n.

Proof. Any section of $\mathrm{C}^{(k)}_{\mathrm{DR}}(\mathcal{X}_n/M_n)$ has a multiple that can be extended to a section of $\mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(\widetilde{\mathcal{X}}/\widetilde{M})$ because $\mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_n/M_n)$, $\mathrm{T}^{(k)}_{\mathrm{DR}}(\widetilde{\mathcal{X}}/\widetilde{M})$, $\mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_n/M_n)$ and $\mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(\widetilde{\mathcal{X}}/\widetilde{M})$ are locally free and their formation commutes with base change. The space $\widetilde{\mathrm{C}}^{(k)}_{\mathrm{DR}}$ is unique for the same reason. □

3.6 We are now in a position to apply Theorem 2.8 to the variety M_n and its model $\widetilde{M}$. Assume that $F \supset E$ is a number field with ring of integers $\mathcal{O}$. Suppose that $\mathfrak{p}$ is a prime ideal of $\mathcal{O}$ and that the residue field $\mathbf{F}_q = \mathcal{O}/\mathfrak{p}$ has characteristic p. Assume that we are given a point $x \in \widetilde{M}(\mathcal{O}_\mathfrak{p})$ corresponding to an abelian scheme $X/\mathcal{O}_\mathfrak{p}$ with ordinary reduction $X_0/\mathbf{F}_q$. The reduction of x modulo $\mathfrak{p}$ will be denoted by $x_0 \in \widetilde{M}(\mathbf{F}_q)$. We will use the notations W, $\bar{W}$, B and $\bar{B}$ from 1.2 and choose an embedding of $\mathcal{O}_\mathfrak{p}$ in $\bar{W}$.

3.7 Theorem. *Let* $\widehat{M}/\bar{W}$ *be the base change to* $\bar{W}$ *of the completion of* $\widetilde{M}$ *in the point* $x_0 \in \widetilde{M}(\mathbf{F}_q)$. *There is a unique irreducible component*

$$\mathcal{C} \subset \widehat{M} \subset \mathcal{M}_{\bar{W}} \cong \mathrm{Hom}(T_pX_0 \otimes T_pX_0^t, \widehat{\mathbf{G}}_m)$$

containing x. *Moreover,* $\mathcal{C}$ *is a translate of a formal subtorus of* $\mathcal{M}$ *over a torsion point.*

Proof. Let us first note that $\widehat{M}$ has a finite number of irreducible components. To show this, one has to prove that there is a finite extension R of $\widehat{\mathcal{O}}_\mathfrak{p}$ such that the irreducible components over R of the completion of $\widetilde{M}$ are irreducible over $\bar{W}$. By the flatness of $\widetilde{M}$, it suffices to show this for the generic fibre, and that case follows from [GD, 4.5.10].

For each k in our finite set of indices, let $\widetilde{\mathrm{C}}^{(k)}_{\mathrm{DR}} \subset \mathrm{H}^0(\widetilde{M}, \mathrm{T}^{(k)}_{\mathrm{DR}}(\widetilde{\mathcal{X}}/\widetilde{M}))$ be as in Lemma 3.5. We will first show that every formal subscheme $\mathcal{C} \subset \mathcal{M}_{\bar{W}}$ satisfying the conditions of the theorem has the properties that

1. X corresponds to a point of $\mathcal{C}$,
2. $\mathcal{C}$ is irreducible,
3. $\dim_{\bar{W}} \mathcal{C} = \dim M_n$,
4. the polarization on X extends to a polarization on $\mathcal{X}_\mathcal{C}$ and
5. every class in each $C^{(k)} := (\widetilde{\mathrm{C}}^{(k)}_{\mathrm{DR}})_{x_F} \subset \mathrm{T}^{(k)}_{\mathrm{DR}}(X_F/F)$ has a multiple ξ such that the formal horizontal continuation $\tilde{\xi}$ of ξ is defined on $\mathcal{C}$ and lies in $\mathrm{Fil}^0\, \mathrm{T}^{(k)}_{\mathrm{DR}}(\mathcal{X}_\mathcal{C}/\mathcal{C})$.

In 3.7.4 and 3.7.5, $\mathcal{X}_{\mathcal{C}}$ denotes the pull back to $\mathcal{C}$ of the universal formal deformation $\mathcal{X}_{\mathcal{M}}/\mathcal{M}$ of X_0. See Remark 2.9 on Property 3.7.5. The above properties will place us in a situation similar to that of theorem 2.8, and we will finish the proof by showing that $\mathcal{C}$ is equal to the translated subtorus $\mathcal{N}$ constructed there.

It is clear from the construction of $\mathcal{C}$ that it satisfies 3.7.1 and 3.7.2. Property 3.7.3 follows from the fact that $\widetilde{M}$ is flat over $\mathcal{O}_{\mathfrak{p}}$ and $\dim M_n = \dim_{\bar{W}} \widetilde{M}_{\bar{W}} = \dim_{\bar{W}} \mathcal{C}$. Property 3.7.4 holds because the polarization on X extends to $\widetilde{\mathcal{X}}/\widetilde{M}$. This leaves us with point 3.7.5.

Let $\xi \in C^{(k)}$. It is the value at x_F of a horizontal section $\xi_n \in \mathrm{Fil}^0\, \mathrm{T}_{\mathrm{DR}}^{(k)}(\mathcal{X}_n/M_n)$. It follows from Lemma 3.5 that after replacing ξ and ξ_n by multiples, there exists a class $\eta \in \mathrm{H}^0(\widetilde{M}, \mathrm{Fil}^0\, \mathrm{T}_{\mathrm{DR}}^{(k)}(\widetilde{\mathcal{X}}/\widetilde{M}))$ restricting to ξ_n on M_n. By base change, η gives an element of $\mathrm{Fil}^0\, \mathrm{T}_{\mathrm{DR}}^{(k)}(\mathcal{X}_{\mathcal{C}}/\mathcal{C})$, which, by abuse of notation, we continue to denote by η. Let $\eta_{\bar{B}}$ be the image of η in $\mathrm{T}_{\mathrm{DR}}^{(k)}(\mathcal{X}_{\mathcal{C}}/\mathcal{C}) \mathbin{\widehat{\otimes}_x} \bar{B}$. We will show that $\eta_{\bar{B}}$ is the restriction to $\mathcal{C}_{\bar{B}}$ of the formal horizontal continuation $\tilde{\xi}$ of ξ, cf. 2.9.

We will write S for the local ring of $\mathcal{A}_n$ at x_0 and $T = S/I$ for the local ring of $\widetilde{M}$ at this point. We have $\hat{S}_{\bar{W}} \cong \bar{W}[[t_1, \ldots, t_{g(g+1)/2}]]$, this ring is a quotient of the coordinate ring $\bar{W}[[t_1, \ldots, t_{g^2}]]$ of $\mathcal{M}_{\bar{W}}$. Let $S_{\bar{B}}$ and $T_{\bar{B}}$ be the local rings at $x_{\bar{B}}$ of $(A_n)_{\bar{B}}$ and $(M_n)_{\bar{B}}$ respectively. We have isomorphisms $\widehat{S_{\bar{B}}} \cong \bar{B}[[t_1, \ldots, t_{g(g+1)/2}]]$ and

$$\widehat{T_{\bar{B}}} = \widehat{S_{\bar{B}}}/\widehat{S_{\bar{B}}}I = \hat{T} \otimes_{\hat{S}} \widehat{S_{\bar{B}}} \cong \hat{T}_{\bar{W}} \otimes_{\bar{W}[[t_1,\ldots,t_{g^2}]]} \bar{B}[[t_1, \ldots, t_{g^2}]] = \hat{T}_{\bar{W}} \mathbin{\widehat{\otimes}_x} \bar{B}.$$

Since M_n is smooth, $\widehat{T_{\bar{B}}}$ is isomorphic to $\bar{B}[[s_1, \ldots, s_d]]$, where $d = \dim M_n$.

The class

$$\eta_{\bar{B}} \in \mathrm{T}_{\mathrm{DR}}^{(k)}(\mathcal{X}_{\mathcal{C}}/\mathcal{C}) \mathbin{\widehat{\otimes}_x} \bar{B} = \mathrm{T}_{\mathrm{DR}}^{(k)}(\widetilde{\mathcal{X}}_{\hat{T}}/\hat{T}) \mathbin{\widehat{\otimes}_x} \bar{B} \cong \mathrm{T}_{\mathrm{DR}}^{(k)}(\widetilde{\mathcal{X}}_{\widehat{T_{\bar{B}}}}/\widehat{T_{\bar{B}}})$$

is horizontal. At $x_{\bar{B}}$, it specializes to ξ, so the fact that $\widehat{T_{\bar{B}}}$ is a power series ring implies that $\eta_{\bar{B}}$ is equal to the image of $\tilde{\xi}$ in $\mathrm{T}_{\mathrm{DR}}^{(k)}(\mathcal{X}_{\mathcal{C}}/\mathcal{C}) \mathbin{\widehat{\otimes}_x} \bar{B}$. This completes the proof of the fact that $\mathcal{C}$ possesses the five properties listed above.

Once we show that $\mathcal{C}$ is determined by the above properties and that such a $\mathcal{C}$ is the translate of a subtorus over a torsion point, we have proved Theorem 3.7. Note that property 3.7.5 holds for every Hodge class on X, but by the remark made in 3.2, it will suffice to consider only the finite set of tensors $T^{(k)}$ and the classes ξ_k generating the $\Xi^{(k)} \subset T^{(k)}$.

The classes $\xi_k \in \mathrm{T}_{\mathrm{DR}}^{(k)}(X_{\bar{W}}/\bar{W})$ are Hodge classes, so by a theorem of Blasius, see [Og, 4.1], they are Tate classes. We can therefore apply Theorem 2.8, which provides us with a translated subtorus $\mathcal{N}$ which contains $\mathcal{C}$. Because $\dim_{\bar{W}} \mathcal{N} = \dim_{\bar{W}} \mathcal{C}$, and since both $\mathcal{N}$ and $\mathcal{C}$ are irreducible, it follows that $\mathcal{N} = \mathcal{C}$. This proves Theorem 3.7. □

3.8 Corollary. *The components over $\bar{W}$ of the completion of $\widetilde{M}$ at a closed point corresponding to an ordinary abelian variety are smooth. The normalization of $\widetilde{M}$ at such a point is smooth.*

3.9 Corollary. *Let S be the local ring of $\widetilde{M}$ at x_0 and let S' be the normalization of S. The point $x_{\bar{W}} \in \mathrm{Spec}(S)(\bar{W})$ extends to a point $x'_{\bar{W}} \in \mathrm{Spec}(S')(\bar{W})$. There exists a point $y \in \mathrm{Spec}(S')(\bar{W})$ with the properties that*

– *the reduction $y_0 \in \mathrm{Spec}(S')(\overline{\mathbf{F}}_p)$ is equal to the reduction of $x'_{\bar{W}}$ and*

– *the abelian variety corresponding to the image of y in $\widetilde{M}(\bar{W})$ is of CM-type.*

Proofs. According to [GD, 7.8.3(vii)], S' is a semi-local ring and its completion $\widehat{S'}$ is equal to the normalization of $\hat{S}$. This proves Corollary 3.8. There is a one to one correspondence between the closed points of $\mathrm{Spec}(S')$ and the components of $\mathrm{Spec}(\hat{S})$. To find the promised point y, it therefore suffices to find a point $y \in \widehat{M}(\bar{W})$, on the same component as $x_{\bar{W}}$, such that the abelian variety corresponding to y is of CM-type.

By Theorem 3.7, the component of $\widehat{M}$ containing x is the translate of a formal subtorus of $\mathcal{M}$ over a torsion point, so it contains a torsion point $y \in \mathcal{M}(\bar{W})$. By [dJN, 3.5], the corresponding abelian variety is of CM-type. □

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