SEMICLAPP SUMMER SCHOOL
SEMICLASSICAL SCATTERING AND RESOLVENT ESTIMATES

MARTIN VOGEL

Contents

The present text is intended as lecture notes of the course on semiclassical scattering and resolvent estimates at the summer school *Semiclapp: Semiclassical Analysis and Applications*[1], held at the Université de Côte d'Azur in Mai 2024. The presented material is mostly based on the book [6] by S. Dyatlov and M. Zworski, and on the lecture notes [10] by J. Sjöstrand. For further reading we refer also to the monographs [9] by R. B. Melrose and [11, 12] by D. R. Yafaev. The reader looking for a compact introduction to the subject and a survey of some recent results may also want to consult the review [13] by M. Zworski.

The present lecture is proceeded by courses on *Tools of Semiclassical Analysis* by M. Tacy[2], on *WKB method, Propagation of Singularities* by J. Wunsch[3], and *Introduction to Scattering Theory* by M. Ingremeau[4]. We will freely make use of the materials and notions discussed in these lectures[1].

The exercise classes for the present course will be held by D. Lafontaine[5]. The corresponding exercise sheets can be found on the summer school homepage[1].

1. Introduction and Motivation

We have seen in the introductory lecture by Maxime Ingremeau that the term «scattering» may refer to various problems and settings. One example is that scattering resonances are the rates of decay and of oscillation of solutions to the wave equation with compactly supported bounded potentials $V \in L^\infty_c$ and compact initial data,

$$\begin{cases} (\partial_t^2 - \Delta_x + V(x))u = 0, \\ u(0,x) = u_1(x) \in H^1_{\mathrm{comp}}, \\ \partial_t u(0,x) = u_2(x) \in L^2_{\mathrm{comp}}. \end{cases}$$

Date: May 14, 2024.

[1]The website https://semiclapp.sciencesconf.org/ contains more relevant information

[2]University of Auckland, https://profiles.auckland.ac.nz/melissa-tacy
[3]Northwestern University, http://math.northwestern.edu/~jwunsch/
[4]Université de Côte d'Azur, https://math.univ-cotedazur.fr/~ingremeau/
[5]CNRS & Institut de Mathématiques de Toulouse, https://www.math.univ-toulouse.fr/~dlafonta/

Roughly speaking, we have the *resonance expansion*

$$u(t,x) = \sum_{\operatorname{Im}\lambda_j > -A} \mathrm{e}^{-i\lambda_j t} a_j(x) + \mathcal{O}_{H^2(K)}(\mathrm{e}^{-tA}), \quad x \in K \Subset \mathbb{R}^n. \tag{1.1}$$

Here $\lambda_j \in \mathbb{C}$ are the *resonances* (assumed to be simple here) of

$$P = -\Delta + V(x), \tag{1.2}$$

i.e. the poles of the corresponding meromorphically continued resolvent $(P-\lambda^2)^{-1}$ seen as an operator $L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$. The functions a_j are the corresponding *resonance states* thus solving $(P-\lambda^2)a_j = 0$. See for instance [6, Theorem 3.11] for a precise statement.

From (1.1) we see that $\operatorname{Re}\lambda_j$ is the *rate of oscillation* and $-\operatorname{Im}\lambda_j$ the *rate of decay* of the resonance λ_j.

Exercise 1.1. *Compare* (1.1) *with the corresponding expansion on the compact manifold* $\mathbb{R}/\mathbb{Z}$.

In view of (1.1), we can ask a couple of natural questions

(1) Are there any resonances at all?
(2) Are there any real resonances?
(3) How close to the real axis can the resonances be?
(4) How many resonances are there? (We will not tackle this question here. The interested reader my look for instance at [6, 10])

The first question can already be quite subtle. Indeed, in dimension $n = 1$, any complex-valued non-zero potential $V \in L^\infty_c(R;\mathbb{C})$ always gives rise to infinitely many resonances, see [6, Theorem 2.16]. In contrast, in higher dimensions, there exists non-trivial complex-valued potentials $V \in L^\infty_c(R^n;\mathbb{C})$ having no resonances, see [6, Theorem 3.29]. However, real-valued potentials $V \in L^\infty_c(R^n;\mathbb{R})$ always have infinitely many resonances. Until further notice, we will restrict ourselves to the case of real-valued potentials.

The answer to the second question was already provided in the course of Maxime Ingremeau.

Proposition 1.2. *Let* $V \in L^\infty_c(\mathbb{R}^n;\mathbb{R})$*. Then, the Schrödinger operator* (1.2) *has no non-zero real resonances, i.e., the meromorphically continued resolvent* $(-\Delta + V - \lambda^2)^{-1}$ *has no poles for* $\lambda \in \mathbb{R}\backslash\{0\}$*.*

The third question, however heavily depend on further assumptions on the potential. This is numerically illustrated in Figures 1 and 2. The aim of this course is to study this question under dynamical conditions on the potential V.

Notation. We will use the following notations and conventions freely throughout this text. We write $\chi_1 \succ \chi_2$ for two compactly supported functions taking values in $[0,1]$ and $\operatorname{supp}\chi_2 \subset \complement \operatorname{supp}(1-\chi_1)$. We will denote generic constants by $C > 0$ which may change from line to line without us stating this explicitly. When a constant depends on some parameter r, we will write C_r.

2. The free resolvent

For the sake of simplicity, in particular to avoid discussions of the logarithmic covering of $\mathbb{C}^*$, we will restrict ourselves to the case of $n = 3$.

We begin by recalling some facts about the "free" Laplacian

$$P_0 = -\Delta = -\sum_1^n \partial^2_{x_i}$$

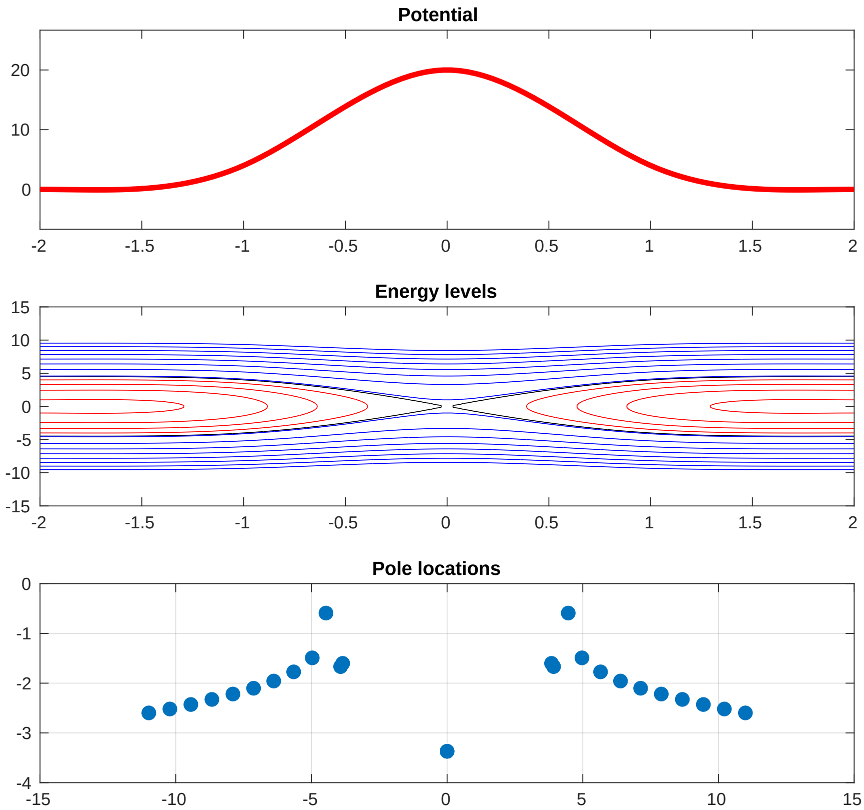

Figure 1. The bottom picture shows this first (according to size of the imaginary part) numerically computed resonances of the bump potential V depicted on the top. The middle panel shows the level sets of the Hamiltonian $p(x,\xi)=\xi^2+V(x)$. The first and last picture have been produced with the MATLAB code[6] `splinepot.m` by D. Bindel [1].

and its resolvent $R_0(z)$. We see P_0 as an unbounded operator $L(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ with domain given by the standard Sobolev space $H^2(\mathbb{R}^n)$. Using the Fourier transform, we see that its spectrum is given by $[0,+\infty[$ and it is purely absolutely continuous. For $\operatorname{Im}\lambda>0$, we see that $\lambda^2\notin[0,+\infty[$, so $P_0-\lambda^2 : H^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is bijective with bounded inverse $R_0(\lambda): L^2(\mathbb{R}^n)\to H^2(\mathbb{R}^n)$. It depends holmorphically on λ in the complex upper half-plane $\{\operatorname{Im}\lambda>0\}$ and, by the spectral theorem, satisfies the estimate

$$\|R_0(\lambda)\|_{L^2\to L^2}=\frac{1}{\operatorname{dist}(\lambda^2,[0,+\infty[)}\leq\frac{C}{|\lambda|\operatorname{Im}\lambda}.$$

Here, the last inequality follows by studying the cases $|\operatorname{Re}\lambda|>\operatorname{Im}\lambda$ and $|\operatorname{Re}\lambda|\leq\operatorname{Im}\lambda$ separately.

We have seen in the course of Maxime Ingremeau the following result

Theorem 2.1. *The free resolvent $R_0(\lambda): L^2_{\mathrm{comp}}(\mathbb{R}^3)\to H^2_{\mathrm{loc}}(\mathbb{R}^3)$ admits a holmorphic continuation from $\{\operatorname{Im}\lambda>0\}$ to $\mathbb{C}$. Moreover, its Schwartz kernel is given by*

$$R_0(\lambda,x,y)=\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}\tag{2.1}$$

Remark 2.2. *1. Since neither L^2_{comp} nor H^2_{loc} are Banach spaces, let us recall that the conclusion means that for every cut-off function $\chi\in C_c^\infty(\mathbb{R}^n)$, the cut-off resolvent*

[6]Available as part of the MatScat package by D. Bindel at https://www.cs.cornell.edu/~bindel/cims/matscat/

$\chi R_0(\lambda)\chi : L^2 \to H^2$ admits a holmorphic continuation from $\{\operatorname{Im}\lambda > 0\}$ to $\mathbb{C}$.
2. Notice that

$$R_0(\lambda, w) = \frac{e^{i\lambda|w|}}{4\pi|w|}$$

is locally integrable since the singularty $|z|^{-1}$ is integrable in $\mathbb{R}^3$ (pass to polar coordinates).

Our first goal is to estimate the norm of the free resolvent. We will do this using the strong Huygens principle. Indeed, we will prove such an estimate by relating the free resolvent $R_0(\lambda)$ to the propagator of the wave equation[7]

$$U(t) := \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}.$$

This is defined via the Fourier transform $\mathcal{F}$ by

$$U(t)u = \mathcal{F}^{-1}\frac{\sin(t|\xi|)}{|\xi|}\mathcal{F}u, \quad u \in \mathcal{S}(\mathbb{R}^3).$$

Since $\sup_{\lambda\in\mathbb{R}} |\sin(t|\xi|)|/|\xi| = |t|$, this extends to a bounded operator $L^2 \to L^2$. A version of the *strong Huygens principle* then states

$$\phi \in C_c^\infty(B(0,R)) \quad \Longrightarrow \quad B(0,R) \cap \operatorname{supp} U(t)\phi = \emptyset, \text{ when } t > 2R. \tag{2.2}$$

Exercise 2.3. *Prove the strong Huygens principle (2.2). For this prove that*

$$U(t)\phi(t) = \frac{1}{4\pi t}\int_{\partial B(0,R)} \phi(y)dS(y), \quad t > 0.$$

Let $\operatorname{Im}\lambda > 0$. Then the free resolvent $R_0(\lambda)$ can be expressed as

$$R_0(\lambda) = \int_0^\infty \mathrm{e}^{i\lambda t}U(t)dt, \tag{2.3}$$

where the integral converges in operator norm.

Exercise 2.4. *Prove (2.3) by taking the Fourier transform and using a contour deformation.*

Theorem 2.5. *For $R > 0$ and $\chi \in C_c^\infty(B(0,R))$ we have*

$$\|\chi R_0(\lambda)\chi\|_{L^2\to H^s} \leq C\langle\lambda\rangle^{s-1}\mathrm{e}^{2R(\operatorname{Im}\lambda)_-}, \quad s = 0,1,2. \tag{2.4}$$

Remark 2.6. *The estimates provided by Theorem 2.5 can be interprated as a* high energy estimate *as it provides quantitative control for $|\operatorname{Im}\lambda| = \mathcal{O}(1)$ as $|\operatorname{Re}\lambda| \to \infty$. This is a regime where semiclassical methods are very useful. Indeed, put $z = \lambda^2 h^2$, with $h = |\operatorname{Re}\lambda|^{-1}$, then $|\operatorname{Im} z| = \mathcal{O}(h)$, $h \to 0$. Here, since we are in odd dimensions, z is seen as an element of the z^2-covering Λ of $\mathbb{C}$, given by the graph of $\mathbb{C} \ni \lambda \mapsto \lambda^2$. The estimate (2.4) then gives that for $\operatorname{Re} z \in K \Subset \mathbb{R}_+^*$ and $|\operatorname{Im} z| \leq Ch$*

$$\|\chi(-h^2\Delta - z)^{-1}\chi\|_{L^2\to L^2} \leq \frac{C}{h}. \tag{2.5}$$

Proof. 1. Using the strong Huygens principle (2.2) shows that $\chi U(t)\chi = 0$ for $t > 2R$. So, in view of (2.3), for $\operatorname{Im}\lambda > 0$

$$\chi R_0(t)\chi = \int_0^{2R} \mathrm{e}^{i\lambda t}\chi U(t)\chi dt \ : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3). \tag{2.6}$$

[7]Given $\phi \in L^2(\mathbb{R}^3)$, the function $u(t,x) = U(t)\phi$ solves the $(\partial_t^2 - \Delta)u = 0$ with initial data $u(0,x) = 0$ and $\partial_t u(0,x) = \phi(x)$.

By analytic continuation, this also holds, and is holomorphic, for all $\lambda \in \mathbb{C}$. Notice that

$$\begin{aligned} \|U(t)\|_{L^2\to H^1} &\le C\|U(t)\|_{L^2\to L^2} + C\|\sqrt{-\Delta}U(t)\|_{L^2\to L^2} \\ &\le C(1+|t|). \end{aligned} \tag{2.7}$$

Here, we used that

$$\begin{aligned} \|U(t)\|_{L^2\to L^2} &= \sup_{\lambda\in\mathbb{R}} \left|\frac{\sin t\lambda}{\lambda}\right| = |t|, \\ \|\sqrt{-\Delta}U(t)\|_{L^2\to L^2} &= \sup_{\lambda\in\mathbb{R}} |\sin t\lambda| = 1. \end{aligned} \tag{2.8}$$

Combining (2.6), (2.7) yields

$$\|\chi R_0(\lambda)\chi\|_{L^2\to H^1} \le C\mathrm{e}^{2R(\operatorname{Im}\lambda)_-}.$$

Note that the constant $C>0$ depends on χ.

2. For the case $s=0$, combine first (2.6), (2.7), to get

$$\|\chi R_0(\lambda)\chi\|_{L^2\to L^2} \le C\mathrm{e}^{2R(\operatorname{Im}\lambda)_-}. \tag{2.9}$$

To improve this estimate, write

$$\lambda\chi R_0(\lambda)\chi = \int_0^{2R} D_t(\mathrm{e}^{i\lambda t})\chi U(t)\chi dt = -i\chi(\mathrm{e}^{2i\lambda R}U(2R) - U(0))\chi - \int_0^{2R} \mathrm{e}^{i\lambda t}\chi D_t U(t)\chi dt.$$

Notice that $D_t U(t) = -i\cos t\sqrt{-\Delta}$. Using (2.8), and making a similar estimate, we get that

$$|\lambda|\|\chi R_0(\lambda)\chi\|_{L^2\to L^2} \le C\mathrm{e}^{2R(\operatorname{Im}\lambda)_-}, \tag{2.10}$$

which gives the result when $s=0$, upon combining (2.9), (2.10) and dividing by $1+|\lambda|$.

3. Finally, for $s=2$, consider a $\chi_1 \in C_c^\infty(B(0,R))$ with $\chi_1 = 1$ near $\operatorname{supp}\chi$. Since $(-\Delta-\lambda^2)R_0(\lambda)=1$, as an operator on $L^2(\mathbb{R}^3)$ when $\operatorname{Im}\lambda>0$, so by analytic continuation for all $\lambda\in\mathbb{C}$, after applying cut-off functions. Therefore,

$$\begin{aligned} \|\chi R_0(\lambda)\chi\|_{L^2\to H^2} &\le C\|\Delta\chi R_0(\lambda)\chi\|_{L^2\to L^2} + C\|\chi R_0(\lambda)\chi\|_{L^2\to L^2} \\ &\le C\|\chi\Delta R_0(\lambda)\chi\|_{L^2\to L^2} + C\|[\Delta,\chi](\chi_1 R_0(\lambda)\chi_1)\chi\|_{L^2\to L^2} \\ &\quad + C\|\chi R_0(\lambda)\chi\|_{L^2\to L^2} \\ &\le C(1+|\lambda|)\|\chi R_0(\lambda)\chi\|_{L^2\to L^2} + C\|(\chi_1 R_0(\lambda)\chi_1)\chi\|_{L^2\to H^1} \\ &\le C\langle\lambda\rangle\mathrm{e}^{2R(\operatorname{Im}\lambda)_-}, \end{aligned}$$

concluding the proof. □

Remark 2.7. *Another consequence of the strong Huygens principle is the analytic continuation of $R_0(\lambda)$ from $\operatorname{Im}\lambda>0$ to $\mathbb{C}$.*

3. Resolvent bounds for non-trapping potentials

In this section we will show bounds on the cut-off resolvent of semiclassical Schrödinger operators

$$P = -h^2\Delta + V, \quad V\in C_c^\infty(\mathbb{R}^n;\mathbb{R}). \tag{3.1}$$

Note that P_h has semiclassical principal symbol $p(x,\xi)=|\xi|^2+V(x)$. Let P_h be as in (3.1) and write

$$R(\lambda,h) = (P-\lambda^2)^{-1}.$$

We will write $P_0 = -h^2\Delta$, and the free resolvent as

$$R_0(\lambda,h) = (P_0-\lambda^2)^{-1}.$$

Recall from Maxime Ingremeau's course that $R(\lambda, h)$ admits a meromorphic continuation from $\mathrm{Im}\,\lambda > 0$ to $\mathbb{C}$ as an operator $L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$. When $z = \lambda^2$, we will abuse notation and write $R(z,h) = (P-z)^{-1}$ and $R_0(z,h) = (P_0 - z)^{-1}$.

The aim of this section is to show cut-off resolvent estimates near energies $E > 0$ where the potential V is *non-trapping*, that is, energies for which all trajectories of the Hamilton flow in the energy shell $p^{-1}(E)$ escape to infinity (both in the past and in the future).

Proposition 3.1. *The Schrödinger operator* (3.1) *has no non-zero real resonances, i.e., the meromorphically continued resolvent* $R(\lambda, h)$ *has no poles for* $\lambda \in \mathbb{R}\backslash\{0\}$.

Proof. We leave the proof as an exercise. Alternatively, see [6, Theorem 3.33] or [10, Section 2.4]. □

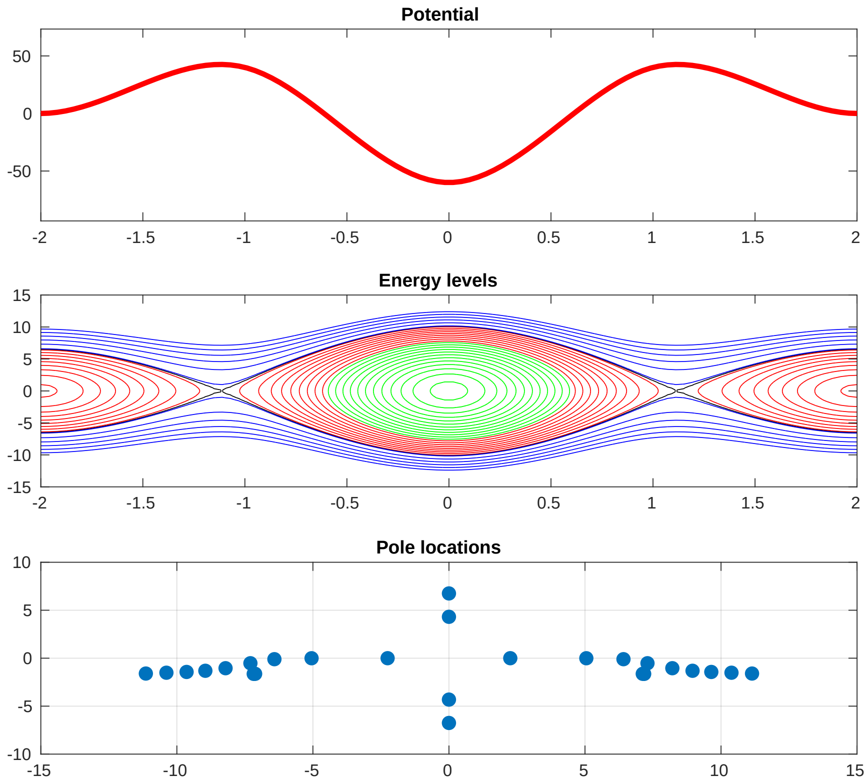

Figure 2. The bottom picture shows this first (according to size of the imaginary part) numerically computed resonances of the bump potential V depicted on the top. The middle panel shows the level sets of the Hamiltonian $p(x,\xi) = \xi^2 + V(x)$. The first and last picture have been produced with the MATLAB code[6] `splinepot.m` by D. Bindel [1].

Theorem 3.2. *(No dynamical assumptions) Let* $V, \partial_r V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d;\mathbb{R})$*, let* $I \Subset \mathbb{R}^*_+$ *be a compact interval and let* $E \in I$*. Then, there exists a* $C > 0$ *such that for every* $\chi \in C^\infty_c$ *there exists a* $C_1 > 0$ *such that*

$$\|\chi R(E,h)\chi\|_{L^2\to L^2} \leq C_1 \mathrm{e}^{C/h}.$$

This result due to [5] goes back to N. Bruq [2, 3] for smooth potentials. In dimension $d = 1$, we do not need to assume a bounded radial derivate, see the exercises. We refer the interested reader to [6, Section 6.6] for references to further developments.

3.1. **Geometry of trapping.** Recall from Jared Wunsch's lecture that p induces the Hamilton vector field

$$H_p = \sum_{j=1}^{n} \partial_{\xi_j} p \, \partial_{x_j} - \partial_{x_j} p \, \partial_{\xi_j}$$

and the Hamilton flow

$$\Phi_t = \exp(tH_p) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n, \quad t \in \mathbb{R}.$$

so that the classical trajectories $\rho(t) = (x(t), \xi(t)) = \Phi_t(\rho_0)$ solve Hamilton's equations

$$\frac{d}{dt}\rho(t) = H_p(\rho(t)), \ \rho(0) = \rho_0 \in T^*\mathbb{R}^n. \tag{3.2}$$

Trajectories of the Hamilton flow corresponding to a certain potential are illustrated in Figure 3.

Exercise 3.3. *Check that the Hamilton flow* $\exp(tH_p)$ *is a global flow.*

Since V has compact support, there exists an $R_0 > 0$ such that $\operatorname{supp} V \subset B(0, R)$. Consequently, for $|x| \geq R_0$ we have $p(x, \xi) = |\xi|^2$ and

$$\frac{d}{dt}x(t) = 2\xi(t), \quad \frac{d}{dt}\xi(t) = 0. \tag{3.3}$$

This means that outside the support of V, the classical trajectories are straight lines.

Next, we discuss notions of *trapped* trajectories $\rho(t)$, see (3.2), of the Hamilton flow. The introduced notions are illustrated in Figure 3 below.

Definition 3.4. *1. We say that a point* ρ_0 escapes to infinity *as* $t \to +\infty$ *(respectively as* $t \to -\infty$*) if for* $\rho(t) = \Phi_t(\rho_0)$

$$|x(t)| \to \infty \quad \text{as } t \to +\infty \text{ (respectively as } t \to -\infty). \tag{3.4}$$

2. We define the incoming tail Γ^- *and the outgoing tail* Γ^+ *to be the sets of points* ρ_0 *which do not escape to infinity as* $t \to +\infty$ *and as* $t \to -\infty$*, respectively. In other words*

$$\Gamma^\pm = \{\rho_0 \in T^*\mathbb{R}^d; |x(t)| \not\to \infty, \ t \to \mp\infty\}.$$

3. The trapped set $K \subset T^*\mathbb{R}^n$ *is the set of points which do not escape to infinity in either time direction*

$$K := \Gamma^+ \cap \Gamma^-$$

4. Given a set $J \subset \mathbb{R}$*, we define the trapped set at energies* J *as* $K_J := K \cap p^{-1}(J)$*. Specifically, when* $J = \{E\}$*, then* $K_E := K_{\{E\}}$*.*

Remark 3.5. *1. By* (3.3) *we see that condition* (3.4) *is equivalent to* $|\Phi_t(\rho_0)| \to \infty$ *as* $t \to +\infty$ *(respectively as* $t \to -\infty$*).*

2. All non-positive energies are trapping, i.e.

$$K_{]-\infty,0]} = p^{-1}(]-\infty, 0]).$$

This is illustrated by the green energy levels in Figure 2, whereas the red level sets correspond to energies where trapping occurs. See also Figure 3.

3. If $K_E = \emptyset$*, then* $\Gamma^\pm \cap p^{-1}(E) = \emptyset$*.*

Exercise 3.6. *Show that if* $E_0 > 0$ *is a non-trapping energy for* V*, then every energy* E *in a sufficiently small neighborhood of* E_0 *is also non-trapping.*

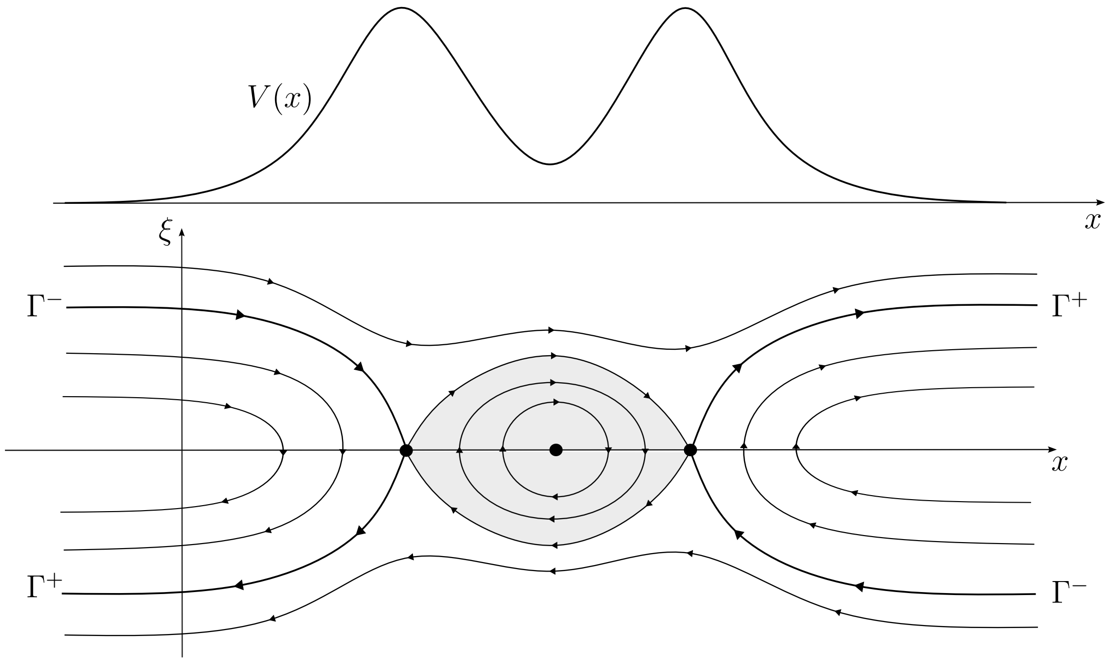

Figure 3. The second panel illustrates the Hamiltonian dynamics for the symbol $p(x, \xi) = \xi^2 + V(x)$ for the potential $V(x)$ in the first panel. The grey area show trapped set. See Figure 2 for a corresponding numerical simulation. The thick black lines thick black lines show incoming and outgoing tails.

3.2. **Resolvent estimate at non-trapping energies.** The aim here is to prove

Theorem 3.7. *Suppose that $E > 0$ is a non-trapping energy for V, i.e. $K_E = \emptyset$. Then, for any $\chi \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ there exist $C, h_0 > 0$ such that for $0 < h \leq h_0$*

$$\|\chi R(E,h)\chi\|_{L^2 \to L^2} \leq \frac{C}{h} \tag{3.5}$$

Remark 3.8. *1. The estimate (3.5) is natural since the free resolvent (2.5) satisfies it. In [4] N. Burq showed the following stronger result: Consider a compact set $J \in \mathbb{R}_+^*$ such that $K_J = \emptyset$. Then, for any $\chi \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$, any $C_0 > 0$ there exist $C, h_0 > 0$ such that for $0 < h \leq h_0$*

$$\|\chi R(z,h)\chi\|_{L^2 \to L^2} \leq \frac{C}{h} \tag{3.6}$$

for any $z \in K + i[-C_0 h, C_0 h]$ ($z \in \Lambda$ as in Remark 2.6). In particular, (3.6) implies that P has no resonances in $K + i[-C_0 h, C_0 h]$.

2. For a recent study concerning the sharpness of the constant $C > 0$ (3.5), we refer the reader to [7].

3. In view of Exercise 3.6, it is straightforward to obtain an estimate of the form (3.5) uniform in energies E in a small compact neighborhood I of a non-trapping energy. Adapting the proof of [6, Theorem 6.26], one may then deduce that P has no resonances in the strip $I + i[-Ch, Ch]$, for some $C > 0$.

Before we turn to the proof of Theorem 3.7, we need some results in preparation.

Lemma 3.9. *Let $1_{\operatorname{supp} V} \prec \psi \in C_c^\infty(\mathbb{R}^3; [0,1])$. Let $\phi \in L^2_{\mathrm{comp}}$ and $E > 0$, then*

$$(1-\psi)R(E,h)\phi = R_0(E,h)(1-\psi)\phi - R_0(E,h)[P_0,\psi]R(E,h)\phi. \tag{3.7}$$

Proof. 1. Let $1_{\operatorname{supp} V} \prec \psi \in C_c^\infty(\mathbb{R}^3; [0,1])$, $z = \lambda^2$ with $\operatorname{Im}\lambda > 0$ and $z \notin \operatorname{Spec}(P_h)$. Then, for $\phi \in L^2_{\mathrm{comp}}$

$$(P_0 - z)(1-\psi)(P-z)^{-1}\phi = (1-\psi)\phi - [P_0, \psi](P-z)^{-1}\phi.$$

Applying $(P_0 - z)^{-1}$ gives

$$(1-\psi)(P-z)^{-1}\phi = (P_0 - z)^{-1}(1-\psi)\phi - (P_0-z)^{-1}[P_0,\psi](P-z)^{-1}\phi.$$

This relation continues by analytic continuation to the non-physical sheets, and we get (3.7). ∎

Proof of Theorem 3.7. 1. We follow the strategy of proof of N. Burq [4] (a similar strategy in the context of the damped wave equation has been employed by G. Lebeau [8]) and prove the result by contradiction: suppose that (3.5) fails to hold. Then there exists a subsequence $\{\phi_n\}_{n\in\mathbb{N}} \subset L^2$ and $\{h_n\}_{n\in\mathbb{N}}$ such that

$$\chi\phi_n = \phi_n, \quad h_n \to 0, \quad \|\chi R(E,h)\phi_n\|_{L^2} > \frac{n}{h_n}\|\phi_n\|_{L^2}.$$

We can and will assume that $\chi \in C_c^\infty(\mathbb{R}^3;[0,1])$ is equal to 1 near the support of V. Put

$$u_n := R(E,h)\phi_n \in H^2_{\mathrm{loc}}(\mathbb{R}^3),$$

so that $(P_h - E)u_n = \phi_n$. We rescale u_n and ϕ_n as follows: Define $\widetilde{\phi}_n := \|\chi u_n\|^{-1}\phi_n$ and put $\widetilde{u}_n := R(E,h)\widetilde{\phi}_n$. Now drop the tilde and note that we have

$$u_n := R(E,h)\phi_n, \quad \|\chi u_n\|_{L^2} = 1, \quad \chi\phi_n = \phi_n, \quad \|\phi_n\|_{L^2} = o(h_n). \tag{3.8}$$

2. Next, we show that $u_n \in L^2_{\mathrm{loc}}$ uniformly. Let $\rho \in C_c^\infty$, let $\psi \in C_c^\infty$ as in Lemma 3.9, and let $\psi \prec \chi_0 \prec \chi$. Then, using Lemma 3.9,

$$\begin{aligned} \rho u_n &= \rho\psi u_n + \rho(1-\psi)u_n \\ &= \rho\psi u_n + \rho R_0(E,h)(1-\psi)\phi_n - \rho R_0(E,h)[P_0,\psi]u_n. \end{aligned} \tag{3.9}$$

By (3.8) we know that

$$\|\rho\psi u_n\|_{L^2} \le \|\rho\psi\chi u_n\|_{L^2} \le C. \tag{3.10}$$

Let $\chi \prec \chi_1 \in C_c^\infty(\mathbb{R}^3;[0,1])$ be such that $\chi_1 = 1$ near $\operatorname{supp}\rho$. Using (2.5) we get

$$\|\rho R_0(E,h)(1-\psi)\phi_n\|_{L^2} = \|\rho\chi_1 R_0(E,h)\chi_1(1-\psi)\phi_n\|_{L^2} \le Ch^{-1}. \tag{3.11}$$

Using elliptic regularity we can show that

$$\begin{aligned} \|\chi_0 u_n\|_{H_h^2} &\le C\|\chi(P-E)u_n\|_{L^2} + C\|\chi u_n\|_{L^2} \\ &\le C\|\chi\phi_n\|_{L^2} + C\|\chi u_n\|_{L^2} \\ &\le C. \end{aligned} \tag{3.12}$$

Here, we used that $(P-E)u_n = \phi_n$. The constant $C > 0$ here only depends on χ_0 and χ.

Exercise 3.10. *Prove the estimate in the first line of (3.12). Use either a direct integration by parts argument or elliptic regularity.*

So, calling again upon (2.5) and using (3.12),

$$\begin{aligned} \|\rho R_0(E,h)[P_0,\psi]R(E,h)\phi_n\|_{L^2} &= \|\rho\chi_1 R_0(E,h)\chi_1[P_0,\psi]u_n\|_{L^2} \\ &\le C\|\rho\chi_1 R_0(E,h)\chi_1\|_{L^2\to L^2}\|[P_0,\psi]\chi_0 u_n\|_{L^2} \\ &\le Ch^{-1}\|[P_0,\psi]\|_{H_h^1\to L^2}\|\chi_0 u_n\|_{H_h^1} \\ &\le C \end{aligned} \tag{3.13}$$

Here, we used as well that $[P_0,\psi]=\mathcal{O}(h):H^1_h\to L^2$ since the symbol of $[P_0,\psi]$ is given by $\frac{h}{i}\{\xi^2,\psi\}+\mathcal{O}_{S(1)}(h^2)$. Summing up, for every $\rho\in C_c^\infty$, there exists a $C_\rho>0$ such that for all $n\in\mathbb{N}$

$$\|\rho u_n\|_{L^2}\leq C_\rho. \tag{3.14}$$

Notice that the constant only depends on the support of ρ and on $\|\rho\|_{L^\infty}$.

Remark 3.11. *Notice by using (3.14) and by repeating the argument (3.12) with different cut-off functions, we get that for any $\eta\in C_c^\infty$ there exists a $C>0$ such that*

$$\|\eta u_n\|_{H^2_h}\leq C. \tag{3.15}$$

3. Next, we study the outgoing behavior of the free resolvent. Let $\chi_1,\chi_2\in C_c^\infty$ have disjoint supports. By Theorem 2.1 (and a rescaling as in Remark 2.6) we know that the Schwartz kernel of $\chi_1R_0(E,h)\chi_2$ is given by

$$K_0(x,y)=\mathrm{e}^{i\frac{\sqrt{E}}{h}|x-y|}\frac{\chi_1(x)\chi_2(y)}{4\pi h^2|x-y|}$$

Applying the partial semiclassical Fourier transform $(\mathcal{F}_h)_{x\to\xi}$, we get that the Schwartz kernel of $(\mathcal{F}_h)_{x\to\xi}\circ\chi_1R_0(E,h)\chi_2$ is given by

$$\check{K}_0(\xi,y)=\frac{1}{4\pi h^2}\int_{\mathbb{R}^3}\mathrm{e}^{\frac{i}{h}\sqrt{E}|x-y|-\frac{i}{h}\xi\cdot x}\frac{\chi_1(x)\chi_2(y)}{|x-y|}dx.$$

Let $\chi_0\in C_c^\infty$. Since $\chi_0(hD_x)=\mathcal{F}_h^{-1}\chi_0\mathcal{F}_h$, we get that the Schwartz kernel of $(\mathcal{F}_h)_{x\to\xi}\circ\chi_0(hD_x)\chi_1R_0(E,h)\chi_2$ is given by

$$\check{K}(\xi,y)=\frac{1}{4\pi h^2}\int_{\mathbb{R}^3}\mathrm{e}^{\frac{i}{h}\sqrt{E}|x-y|-\frac{i}{h}\xi\cdot x}\frac{\chi_0(\xi)\chi_1(x)\chi_2(y)}{|x-y|}dx. \tag{3.16}$$

Strengthen the assumptions on χ_0 to

$$\operatorname{supp}\chi_0\cap\left\{\sqrt{E}\frac{x-y}{|x-y|};\ x\in\operatorname{supp}\chi_1,\ y\in\operatorname{supp}\chi_2\right\}. \tag{3.17}$$

Consider, formally at first, the differential operator

$$L:=\left|\sqrt{E}\frac{(x-y)}{|x-y|}-\xi\right|^{-2}\left(\sqrt{E}\frac{(x-y)}{|x-y|}-\xi\right)\cdot hD_x.$$

On the support of the integrand in (3.16), L is a differential operator with smooth, uniformly bounded coefficients. Furthermore, there

$$L\mathrm{e}^{\frac{i}{h}\sqrt{E}|x-y|-\frac{i}{h}\xi\cdot x}=\mathrm{e}^{\frac{i}{h}\sqrt{E}|x-y|-\frac{i}{h}\xi\cdot x}.$$

Performing repeated integration by parts shows that

$$\check{K}(\xi,y)=\chi_0(\xi)\chi_2(y)\mathcal{O}_{C^\infty(\mathbb{R}^3_\xi\times\mathbb{R}^3_y)}(h^\infty).$$

So

$$(\mathcal{F}_h)_{x\to\xi}\circ\chi_0(hD_x)\chi_1R_0(E,h)\chi_2=\mathcal{O}(h^\infty):L^2(\mathbb{R}^3_y)\to\mathcal{S}(\mathbb{R}^3_\xi)$$

Taking the inverse of the partial Fourier transform gives that for every $\chi_1,\chi_2\in C_c^\infty$ with disjoint supports, every $\chi_0\in C_c^\infty$ satisfying (3.17), we get that

$$\chi_0(hD_x)\circ\chi_1R_0(E,h)\chi_2=\mathcal{O}(h^\infty):L^2(\mathbb{R}^3)\to\mathcal{S}(\mathbb{R}^3). \tag{3.18}$$

Exercise 3.12. *Under the above assumptions show that for $u\in L^2$*

$$\mathrm{WF}_h(\chi_0(hD_x)\circ\chi_1R_0(E,h)\chi_2u)\subset\left\{\sqrt{E}\frac{x-y}{|x-y|};\ x\in\operatorname{supp}\chi_1,\ y\in\operatorname{supp}\chi_2\right\}.$$

4. With all preparations done, let us now turn to the proof of Theorem 3.7. We may associate to the sequence u_n given in (3.8) a semiclassical defect measure μ. We define this here in a slightly modified way: for every $a \in C_c^\infty$ and $\eta \in C_c^\infty$ equal to 1 near the x-projection of the support of a, we consider the sequence

$$(\mathrm{Op}_h(a)\eta u_n | u_n).$$

By the symbolic calculus, we note that the operator $\mathrm{Op}_h(a)\eta$ is, up to operators $L^2 \to L^2$ bounded by $\mathcal{O}(h^\infty)$, independent of the choice of the function η.

Similar as in the lecture of Jared Wunsch, we can show that there exists a subsequence u_{n_j} and a positive Radon measure μ on $T^*\mathbb{R}^3$ such that

$$\lim_{j\to\infty} (\mathrm{Op}_h(a)\eta u_{n_j} | u_{n_j}) = \langle \mu, a\rangle.$$

Furthermore, since u_n is a quasimode for $(P-E)$, i.e. $(P-E)u_n = \chi\phi_n = o(h_n)$, see (3.8), we have that

$$\operatorname{supp}\mu \subset p^{-1}(E). \tag{3.19}$$

and that μ is invariant under the action of the Hamilton flow Φ_t. That is $(\Phi_t)_*\mu = \mu$.

5. We now show that

$$\langle \mu, \chi^2\rangle = 1. \tag{3.20}$$

Take $\psi \in C_c^\infty$ equal to 1 near 0, and let $\chi \prec \chi_1 \in C_c^\infty(\mathbb{R}^3; [0,1])$. Then, for $\varepsilon > 0$,

$$\begin{aligned}(P-E)(1-\psi(\varepsilon h_n D_x))\chi u_n &= [P_0, (1-\psi(\varepsilon h_n D_x))]\chi u_n + (1-\psi(\varepsilon h_n D_x))(P-E)\chi u_n \\ &= \mathcal{O}_{L^2}(h_n).\end{aligned}$$

Here, we used (3.8) and that $[P_0, (1-\psi(\varepsilon h D_x))], [P_0, \chi] = \mathcal{O}(h) : H_h^1 \to L^2$ in combination with the same argument as in (3.12) combined in (3.14). For $|\xi| > 0$ large enough, $P - E$ is elliptic in the semiclassical sense, i.e. $p(x,\xi) - E \geq \xi^2/2$. Thus, for $\varepsilon > 0$ small enough, we apply a suitable parametrix and get that

$$(1-\psi(\varepsilon h_n D_x))\chi u_n = \mathcal{O}_{L^2}(h_n).$$

Let $\chi \prec \eta$, then

$$\begin{aligned}1 &= (\chi^2 u_n | u_n) \\ &= (\psi(\varepsilon h_n D_x)\chi u_n | \psi(\varepsilon h_n D_x)\chi u_n) + 2\mathrm{Re}\,((1-\psi(\varepsilon h_n D_x))\chi u_n | \psi(\varepsilon h_n D_x)\chi u_n) \\ &\quad + ((1-\psi(\varepsilon h_n D_x))\chi u_n | (1-\psi(\varepsilon h_n D_x))\chi u_n) \\ &= (\chi\psi(\varepsilon h_n D_x)^2 \chi u_n | u_n) + \mathcal{O}(h_n) \\ &= (\mathrm{Op}_h(\chi^2\psi(\varepsilon\cdot)^2)\eta u_n | u_n) + \mathcal{O}(h_n).\end{aligned}$$

Taking, first the limit $h_n \to 0$ gives that $\langle \mu, \chi^2\psi(\varepsilon\cdot)^2\rangle$. We may choose a ψ such that $\psi(\varepsilon\xi) \nearrow 1$, as $\varepsilon \to 0$, so we conclude (3.20) by monotone convergence.

6. Given (3.20), if we can prove that $\mu \equiv 0$ then we have shown a contradiction. To this end we will show that μ is equal to 0 in a neighborhood of any trajectory of H_p, and so it is identically 0.

Let $R_0, R_1 > 0$ be such that $\operatorname{supp} V \subset B(0, R_0)$ and let $R \gg R_1 > R_0$. Consider a trajectory starting at $\rho_0 = (x_0, \xi_0) \in p^{-1}(E)$. Since $|x(t)| \to \infty$, as $t \to \pm\infty$. There exists a $t_1 < 0$ such that for all $t < t_0$ we have that $|x(t)| > R$. Thus, for $t < t_1$ the trajectory is $\rho(t) = (x(t), \xi(t)) = (2\xi_1 t, \xi_1)$, where $\xi_1 = \xi(t_1)$. Since $\rho(t) \in p^{-1}(E)$ for all t, it follows that $|\xi_1| = \sqrt{E}$, and so for all $t < t_1$

$$\frac{\xi(t)\cdot x(t)}{|x(t)|} = -\sqrt{E} < 0. \tag{3.21}$$

Choose a t such that $|x(t)| > R$. Let $\chi_1 \in C_c^\infty$ be supported in a sufficiently small neighborhood of $x(t)$ and let $\chi_0 \in C_c^\infty(\mathbb{R}^3;[0,1])$ be supported in a sufficiently small neighborhood of $\xi(t)$. Let $\chi_2 \in C_c^\infty(B(0,R_1);[0,1])$. Then, by (3.21), for $R > 0$ sufficiently large, we have that for all $x \in \operatorname{supp}\chi_1$, $y \in \operatorname{supp}\chi_2$ and $\xi \in \operatorname{supp}\chi_0$ that

$$\frac{\xi\cdot(x-y)}{|(x-y)|} = -\sqrt{E} + \mathcal{O}(R^{-1}) < 0. \tag{3.22}$$

Upon potentially increasing $R > 0$, we may take a $\chi \prec \psi$ and we may strengthen our assumptions on χ_2 by assuming that $\chi_2 \succ 1_{\operatorname{supp}\nabla\psi}$. Then, by (3.7)

$$\chi_0(hD_x)\chi_1 u_n = -\chi_0(hD_x)\chi_1 R_0(E,h)\chi_2[P_0,\psi]u_n. \tag{3.23}$$

Thanks to (3.22) the assumption (3.16) holds, and we get by (3.18) and (3.15) that

$$\begin{aligned}\|\chi_0(hD_x)\chi_1 u_n\|_{L^2} &\leq \|\chi_0(hD_x)\chi_1 R_0(E,h)\chi_2\|_{L^2\to L^2}\|[P_0,\psi]\chi_2 u_n\|_{L^2}\\ &\leq \mathcal{O}(h^\infty)\|[P_0,\psi]\|_{H_h^1\to L^2}\|\chi_2 u_n\|_{H_h^2}\\ &= \mathcal{O}(h^\infty).\end{aligned} \tag{3.24}$$

But this implies that $(x(t),\xi(t)) \notin \operatorname{supp}\mu$. Since μ is invariant under the the action of the Hamilton flow Φ_t, it follows that $\Phi_t(\rho_0) \notin \operatorname{supp}\mu$ for all $t \in \mathbb{R}$. Since $\rho \in p^{-1}(E)$ was arbitrary, it follows in view of (3.19) that $\mu \equiv 0$, giving a contradiction to (3.20) and completing the proof. □

References

1. D. Bindel and M. Zworski, *Theory and computation of resonances in* 1*d scattering*, https://www.cs.cornell.edu/~bindel/cims/resonant1d/.
2. N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel*, Acta Mathématica **180** (1998), 1–29.
3. ———, *Lower bounds for shape resonances widths of long range Schrödinger operators*, Amer. J. Math. (2002), no. 4, 677–735.
4. N. Burq, *Semi-classical estimates for the resolvent in nontrapping geometries*, International Mathematics Research Notices (2002), no. 5.
5. Kiril Datchev, *Quantitative limiting absorption principal in the semiclassical limit*, Geom. Funct. Anal. **24** (2014), 740–747.
6. S. Dyatlov and M. Zworski, *Mathematical Theory of Scattering Resonances*, American Mathematical Society, 2019.
7. J. Galkowski, E. A. Spence, and J. Wunsch, *Optimal constants in nontrapping resolvent estimates and applications in numerical analysis*, Pure and Applied Analysis **2** (2020), no. 1.
8. G. Lebeau, *Équation des ondes amorties*, Algebraic and Geometric Methods in Mathematical Physics (Kaciveli, 1993), Math. Phys. Stud. 19, Kluwer Acad. Publ., Dordrecht (1996).
9. R. B. Melrose, *Geometric Scattering Theory*, Cambridge University Press, 1995.
10. J. Sjöstrand, *Lectures on resonances*, https://sjostrand.perso.math.cnrs.fr/, 2002.
11. D. R. Yafaev, *Mathematical scattering theory: General theory*, Translations of mathematical monographs, vol. 105, American Mathematical Society, 1992.
12. ———, *Mathematical scattering theory: Analytic theory*, Mathematical Surveys and Monographs, vol. 158, American Mathematical Society, 2010.
13. M. Zworski, *Mathematical study of scattering resonances*, Bull. Math. Sci. **7** (2017), 1–85.

(Martin Vogel) Institut de Recherche Mathématique Avancée - UMR 7501, CNRS and Université de Strasbourg, 7 rue René-Descartes, 67084 Strasbourg Cedex, France.
Email address: vogel@math.unistra.fr