Bounded law of the iterated logarithms for stationary random fields

Davide Giraudo

Ruhr-Universität Bochum (Germany)

Be'er Sheva, 18th September 2019

Law of the iterated logarithms for i.i.d. sequences

Let $(X_i)_{i\geqslant 1}$ be an i.i.d. centered sequence on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- If $\mathbb{E}\left[|X_1|^p\right]$ is finite for some $p \in [1,2)$, then $S_n/n^{1/p} \to 0$ almost surely, where $S_n = \sum_{i=1}^n X_i$.
- If $\mathbb{E}\left[X_1^2\right]$ is finite, then $\left(S_n/\sqrt{n}\right)_{n\geqslant 1}$ converges in distribution to a centered normal law with variance $\mathbb{E}\left[X_1^2\right]$.
- If $\mathbb{E}\left[X_1^2\right]$ is finite, then the law of the iterated logarithms takes place :

$$\liminf_{n\to+\infty} \frac{1}{\sqrt{n \ln \ln n}} S_n = -\sqrt{2\mathbb{E}\left[X_1^2\right]} \text{ a.s. and}$$

$$\limsup_{n\to+\infty} \frac{1}{\sqrt{n \ln \ln n}} S_n = \sqrt{2\mathbb{E}\left[X_1^2\right]} \text{ a.s.}$$

Why $\ln\ln(n)$?

We want to find an increasing sequence $(a_n)_{n\geqslant 1}$ which goes to infinity as slowly as possible and such that $\sup_{n\geqslant 1} |S_n| / \sqrt{na_n}$ is almost surely finite.

Let $(X_i)_{i\geqslant 1}$ be i.i.d. standard normal and $Y_n := \left|\sum_{i=2^n+1}^{2^{n+1}} X_i\right| / \sqrt{na_{2^n}}$

Finiteness of $\sup_{n\geqslant 1} |S_n| / \sqrt{na_n}$ implies that of $\sup_{n\geqslant 1} Y_n$.

Since the sequence $(Y_n)_{n\geqslant 1}$ is independent, by the Borel-Cantelli lemma, there exists some $M > 0$ such that $\sum_{n\geqslant 1} \mathbb{P}\{Y_n > M\}$ is finite. Since

$$\mathbb{P}\{Y_n > M\} = \mathbb{P}\left\{|X_1| > M\sqrt{a_{2^n}}\right\},$$

the decay on the tail of a normal distribution suggests $a_{2^n} = \ln(n)$.

Bounded law of the iterated logarithms for i.i.d. sequences

Let $(X_i)_{i\geqslant 1}$ be a centered i.i.d. sequence such that $\mathbb{E}\left[X_1^2\right]$ is finite. Then the random variable

$$M := \sup_{n\geqslant 1} \frac{1}{\sqrt{nLL(n)}} |S_n|$$

is almost surely finite, where $L(x) = \max\{1, \ln(x)\}$ and $LL(x) = L(L(x))$.

It is also possible (Pisier, 1976) to control the moments of M : for all $p \in [1, 2)$,

$$\|M\|_p \leqslant C_p \|X_1\|_2 ,$$

where C_p depends only on p.

Martingales with stationary increments

Definition

We say that $(X_i)_{i \geqslant 1}$ is a strictly stationary martingale difference sequence if :

- the sequence $(X_i)_{i \geqslant 1}$ is strictly stationary, in the sense that $(X_1, \ldots, X_n)$ has the same distribution as $(X_{1+k}, \ldots, X_{n+k})$ for all $k, n \geqslant 1$ and
- there exists a non-decreasing sequence $(\mathcal{F}_i)_{i \geqslant 0}$ of sub-σ-algebras of $\mathcal{F}$ such that X_i is $\mathcal{F}_i$-measurable for all $i \geqslant 1$ and $\mathbb{E}\left[X_i \mid \mathcal{F}_{i-1}\right] = 0$.

Theorem (Cuny, 2015)

Let $(X_i)_{i \geqslant 1}$ be a strictly stationary martingale difference sequence. Then for all $1 < p < 2$,

$$\left\| \sup_{n \geqslant 1} \frac{1}{\sqrt{nLL(n)}} \left| \sum_{i=1}^{n} X_i \right| \right\|_p \leqslant C_p \left\| X_1 \right\|_2 ,$$

where C_p depends only on p.

I.i.d. random fields

Let $d \geqslant 2$ be an integer and $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ an i.i.d. centered random field. We define a partial order on $\mathbb{Z}^d$ by

$$\boldsymbol{i} \preccurlyeq \boldsymbol{j} \Leftrightarrow i_q \leqslant j_q \text{ for all } q \in \{1, \dots, d\} .$$

The partial sums on rectangles are defined by

$$S_{\boldsymbol{n}} := \sum_{\boldsymbol{1} \preccurlyeq \boldsymbol{i} \preccurlyeq \boldsymbol{n}} X_{\boldsymbol{i}}.$$

Theorem (Wichura, 1973)

If $\mathbb{E}\left[X_{\boldsymbol{0}}^2 \left(L\left(|X_{\boldsymbol{0}}|\right)\right)^{d-1} / LL\left(|X_{\boldsymbol{0}}|\right)\right] < +\infty$, *then*

$$\lim_{N \to +\infty} \sup_{N\boldsymbol{1} \preccurlyeq \boldsymbol{n}} \frac{1}{\sqrt{|\boldsymbol{n}| \, LL\left(|\boldsymbol{n}|\right)}} S_{\boldsymbol{n}} = \left\|X_{\boldsymbol{0}}\right\|_2 \sqrt{d} = - \lim_{N \to +\infty} \inf_{N\boldsymbol{1} \preccurlyeq \boldsymbol{n}} \frac{1}{\sqrt{|\boldsymbol{n}| \, LL\left(|\boldsymbol{n}|\right)}} S_{\boldsymbol{n}},$$

where $|\boldsymbol{n}| = \prod_{q=1}^{d} n_q$.

Main goals

Definition

Let $d \geqslant 1$. A random field $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ is strictly stationary if for all integer k, all $\boldsymbol{i_1}, \dots, \boldsymbol{i_k}, \boldsymbol{j} \in \mathbb{Z}^d$, the vectors $(X_{\boldsymbol{i_1}}, \dots, X_{\boldsymbol{i_k}})$ and $\left(X_{\boldsymbol{i_1}+\boldsymbol{j}}, \dots, X_{\boldsymbol{i_k}+\boldsymbol{j}}\right)$ have the same law.

Goal : establish finiteness by controlling the moments of the random variable

$$M := \sup_{\boldsymbol{n} \in \mathbb{N}^d} \frac{1}{\sqrt{|\boldsymbol{n}|\, LL(|\boldsymbol{n}|)}} |S_{\boldsymbol{n}}|, \quad S_{\boldsymbol{n}} := \sum_{\boldsymbol{1} \preccurlyeq \boldsymbol{i} \preccurlyeq \boldsymbol{n}} X_{\boldsymbol{i}}, |\boldsymbol{n}| = \prod_{q=1}^{d} n_q,$$

if possible, or with an other normalisation

$$M := \sup_{\boldsymbol{n} \in \mathbb{N}^d} \frac{1}{\sqrt{|\boldsymbol{n}|\, a_{\boldsymbol{n}}}} |S_{\boldsymbol{n}}|.$$

Martingales for lexicographic order

We endow $\mathbb{Z}^d$ with the lexicographic order : $\boldsymbol{i} \leqslant_{\mathrm{lex}} \boldsymbol{j}$ if $i_d = j_d$ or $i_d < j_d$ and there exists a $q \in \{0, \ldots, d-1\}$ such that $i_p = j_p$ for $q+1 \leqslant p \leqslant d$ et $i_q < j_q$.

Definition

We say that $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ is a martingale differences random field for the lexicographic order if there exists a collection of σ-algebras $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ such that

- if $\boldsymbol{i} \leqslant_{\mathrm{lex}} \boldsymbol{j}$, then $\mathcal{F}_{\boldsymbol{i}} \subset \mathcal{F}_{\boldsymbol{j}}$ and
- for all $\boldsymbol{i} \in \mathbb{Z}^d$, $X_{\boldsymbol{i}}$ is $\mathcal{F}_{\boldsymbol{i}}$-measurable and if $\boldsymbol{i} \leqslant_{\mathrm{lex}} \boldsymbol{j}$, $\boldsymbol{i} \neq j$, then $\mathbb{E}\left[X_{\boldsymbol{j}} \mid \mathcal{F}_{\boldsymbol{i}}\right] = 0$.

Law of the iterated logarithms for martingales for the lexicographic order

Theorem (G., 2019+)

Let $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ be a strictly stationary martingale difference random field for the lexicographic order. Then for all $p\in(1,2)$,

$$\left\|\sup_{\boldsymbol{n}\in\mathbb{N}^d}\frac{1}{\sqrt{|\boldsymbol{n}|\,LL\left(|\boldsymbol{n}|\right)}}\left|S_{\boldsymbol{n}}\right|\right\|_p\leqslant C_{p,d}\left\|X_{\mathbf{0}}\right\|_{2,d-1},$$

where $C_{p,d}$ depends only on p and d and $\left\|\cdot\right\|_{2,q}$ denotes the Orlicz norm associated to the function $x\mapsto x^2\left(\ln\left(1+x\right)\right)^q$.

- **Positive aspects :** we recover (up to the term $LL(|X_{\mathbf{0}}|)$) a similar condition as in the i.i.d case for the classical LIL, and the same result as for martingales in dimension one.
- **Negative aspect :** approximation by this kind of martingales is a difficult task.

Question : are there other kind of multidimensional martingales satisfying the following constraints :

- approximation by such martingales is simple ;
- one can establish a LIL with reasonable moment conditions.

Orthomartingales : motivation (1)

Global idea : we want to use martingale properties by summing on rectangles. We would like to use the arguments known in dimension one whatever the coordinated which respect to which we sum is.

Idea in dimension two. Let

$$S_{n_1,n_2} := \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{i,j}.$$

We have to define a filtration with two indexes $(\mathcal{F}_{i,j})_{i,j\in\mathbb{Z}}$ such that

- for n_1 fixed, $(S_{n_1,n_2})_{n_2\geqslant 1}$ is a martingale ;
- for n_2 fixed, $(S_{n_1,n_2})_{n_1\geqslant 1}$ is a martingale.

The following should hold for i and j :

$$\mathbb{E}\left[X_{i,j} \mid \mathcal{F}_{i,j-1}\right] = \mathbb{E}\left[X_{i,j} \mid \mathcal{F}_{i-1,j}\right] = 0.$$

Orthomartingales : motivation (2)

It is convenient to use truncation arguments : if $(X_i)_{i\geqslant 1}$ is a martingale differences sequence for the filtration $(\mathcal{F}_i)_{i\geqslant 0}$, then for all R, $(X_i')_{i\geqslant 1}$ and $(X_i'')_{i\geqslant 1}$ are also martingale difference sequences, where

$$X_i' := X_i \mathbf{1}\left\{|X_i| \leqslant R\right\} - \mathbb{E}\left[X_i \mathbf{1}\left\{|X_i| \leqslant R\right\} \mid \mathcal{F}_{i-1}\right]$$

$$X_i'' := X_i \mathbf{1}\left\{|X_i| > R\right\} - \mathbb{E}\left[X_i \mathbf{1}\left\{|X_i| > R\right\} \mid \mathcal{F}_{i-1}\right]$$

and $X_i = X_i' + X_i''$. We would like to use the same idea in dimension two :

$$\begin{aligned} X_{i,j}' := {} & X_{i,j} \mathbf{1}\left\{|X_{i,j}| \leqslant R\right\} - \mathbb{E}\left[X_{i,j} \mathbf{1}\left\{|X_{i,j}| \leqslant R\right\} \mid \mathcal{F}_{i,j-1}\right] \\ & - \mathbb{E}\left[X_{i,j} \mathbf{1}\left\{|X_{i,j}| \leqslant R\right\} \mid \mathcal{F}_{i-1,j}\right] + \mathbb{E}\left[X_{i,j} \mathbf{1}\left\{|X_{i,j}| \leqslant R\right\} \mid \mathcal{F}_{i-1,j-1}\right] \end{aligned}$$

but in order to guarantee that $\left(X_{i,j}'\right)_{i,j\in\mathbb{Z}}$ is a martingale differences random field like before, we need

$$\mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_{i,j-1}\right] \mid \mathcal{F}_{i-1,j}\right] = \mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_{i-1,j}\right] \mid \mathcal{F}_{i,j-1}\right].$$

Orthomartingales : constraint on the filtration

Definition

The family of σ-algebras $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is a filtration if $\mathcal{F}_{\boldsymbol{i}} \subset \mathcal{F}_{\boldsymbol{j}}$ takes place for all $\boldsymbol{i}$ and $\boldsymbol{j}$ such that $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$.

Denote by $P_{\boldsymbol{i}} \colon \mathbb{L}^1 \to \mathbb{L}^1$ the operator defined by $P_{\boldsymbol{i}}(Y) := \mathbb{E}\left[Y \mid \mathcal{F}_{\boldsymbol{i}}\right]$.

Definition

The filtration $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is commuting if for all $\boldsymbol{i}$ and $\boldsymbol{j}$,

$$P_{\boldsymbol{i}} \circ P_{\boldsymbol{j}} = P_{\boldsymbol{j}} \circ P_{\boldsymbol{i}} = P_{\min\{\boldsymbol{i},\boldsymbol{j}\}}.$$

Examples of commuting filtrations

1. Let $(\varepsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be an i.i.d. random field and $\mathcal{F}_{\mathbf{i}} := \sigma\left\{\varepsilon_{\mathbf{j}}, \mathbf{j} \preccurlyeq \mathbf{i}\right\}$. Then $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is commuting.

2. Let $\left(\varepsilon_i^q\right)_{i\in\mathbb{Z}}$ be independent copies of the i.i.d. sequence $(\varepsilon_u)_{u\in\mathbb{Z}}$. Let $\mathcal{F}_{\mathbf{i}} := \sigma\left\{\varepsilon_{j_q}^q, j_q \leqslant i_q, 1 \leqslant q \leqslant d\right\}$. Then $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is commuting.

Definition of orthomartingales

Definition

Let $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ be a commuting filtration. The random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is an orthomartingale difference random field if for all $\boldsymbol{i}\in\mathbb{Z}^d$, $X_{\boldsymbol{i}}$ is $\mathcal{F}_{\boldsymbol{i}}$-measurable and for all $q\in\{1,\dots,d\}$, $\mathbb{E}\left[X_{\boldsymbol{i}}\mid\mathcal{F}_{\boldsymbol{i}-\boldsymbol{e_q}}\right]=0$, where $\boldsymbol{e_q}$ is the q-th vector of the canonical basis of $\mathbb{R}^d$.

Examples

1. An i.i.d. centered random field is an orthomartingale difference random field.
2. Let $\left(\varepsilon_i^q\right)_{i\in\mathbb{Z}}$ be independent copies of the i.i.d. centered sequence $(\varepsilon_u)_{u\in\mathbb{Z}}$. Let $\mathcal{F}_{\boldsymbol{i}}:=\sigma\left\{\varepsilon_{j_q}^q, j_q\leqslant i_q, 1\leqslant q\leqslant d\right\}$ and $X_{\boldsymbol{i}}=\prod_{q=1}^d\varepsilon_{i_q}^q$. Then $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is an orthomartingale difference random field.

Properties of orthomartingales (1)

In dimension one : if $(D_j)_{j\geqslant 1}$ is a martingale differences sequence for the filtration $(\mathcal{F}_j)_{j\geqslant 0}$, then for all $x>0$,

$$\mathbb{P}\left\{\max_{1\leqslant k\leqslant n}|S_k|>x\right\}\leqslant\int_1^{+\infty}\mathbb{P}\left\{|S_n|>xu/2\right\}\mathrm{d}u.$$

Let $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ be an orthomartingale differences random field and $S_{\boldsymbol{n}}:=\sum_{\boldsymbol{1}\preccurlyeq\boldsymbol{i}\preccurlyeq\boldsymbol{n}}X_{\boldsymbol{i}}$. Then

$$\mathbb{P}\left\{\max_{\boldsymbol{1}\preccurlyeq\boldsymbol{n}\preccurlyeq\boldsymbol{N}}|S_{\boldsymbol{n}}|>x\right\}\leqslant\int_1^{+\infty}\mathbb{P}\left\{|S_{\boldsymbol{N}}|>xu2^{-d}\right\}(1+\log u)^{d-1}\,\mathrm{d}u.$$

In particular, for $p>1$,

$$\mathbb{E}\left[\max_{\boldsymbol{1}\preccurlyeq\boldsymbol{n}\preccurlyeq\boldsymbol{N}}|S_{\boldsymbol{n}}|^p\right]\leqslant C_{p,d}\mathbb{E}\left[|S_{\boldsymbol{N}}|^p\right].$$

Properties of orthomartingales (2)

In dimension one : if $(D_i)_{i \geqslant 1}$ is a martingale differences sequence for the filtration $(\mathcal{F}_i)_{i \geqslant 0}$, then for all $p \geqslant 2$,

$$\|S_n\|_p^2 \leqslant (p-1) \sum_{i=1}^{n} \|D_i\|_p^2 .$$

Let $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ be an orthomartingale differences random field and $S_{\boldsymbol{n}} := \sum_{\boldsymbol{1} \preccurlyeq \boldsymbol{i} \preccurlyeq \boldsymbol{n}} X_{\boldsymbol{i}}$. Then

$$\|S_{\boldsymbol{n}}\|_p^2 \leqslant (p-1)^d \sum_{\boldsymbol{1} \preccurlyeq \boldsymbol{i} \preccurlyeq \boldsymbol{n}} \|X_{\boldsymbol{i}}\|_p^2 .$$

Orthomartingale, normalisation

Let $\left(\varepsilon_i^q\right)_{i\in\mathbb{Z}}$ by independent copies of the i.i.d. centered sequence $\left(\varepsilon_u\right)_{u\in\mathbb{Z}}$, $q=1,2$. Let $X_{i,j}:=\varepsilon_i^1\varepsilon_j^2$, where ε_1^1 is bounded. For all n_1,n_2,

$$
\begin{aligned}
\frac{1}{\sqrt{n_1n_2LL(n_1n_2)}}\left|S_{n_1,n_2}\right| &= \frac{1}{\sqrt{n_1LL(n_1n_2)}}\left|\sum_{i=1}^{n_1}\varepsilon_i^1\right|\frac{1}{\sqrt{n_2}}\left|\sum_{j=1}^{n_2}\varepsilon_j^2\right| \\
&= \frac{1}{\sqrt{n_1LL(n_1)}}\left|\sum_{i=1}^{n_1}\varepsilon_i^1\right| \\
&\cdot\frac{\sqrt{LL(n_1)}}{\sqrt{LL(n_1n_2)}}\frac{1}{\sqrt{n_2}}\left|\sum_{j=1}^{n_2}\varepsilon_j^2\right|
\end{aligned}
$$

hence

$$
\limsup_{n_1\to+\infty}\frac{1}{\sqrt{n_1n_2LL(n_1n_2)}}\left|S_{n_1,n_2}\right|\geqslant\sqrt{2}\left\|\varepsilon_0^1\right\|_2\frac{1}{\sqrt{n_2}}\left|\sum_{j=1}^{n_2}\varepsilon_j^2\right|.
$$

Orthomartingales : result

The random variable

$$\sup_{n_1, n_2 \geqslant 1} \frac{1}{\sqrt{n_1 n_2 LL(n_1 n_2)}} \left| S_{n_1, n_2} \right|$$

is thus almost surely infinite. This leads to the definition

$$M := \sup_{\mathbf{n} \in \mathbb{N}^d} \frac{\left| S_{\mathbf{n}} \right|}{\left| \mathbf{n} \right|^{1/2} \prod_{i=1}^{d} LL(n_i)^{1/2}}.$$

Theorem (G., (2019+))

Let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a strictly stationary orthomartingale difference random field. For all $p \in (1, 2)$,

$$\left\| M \right\|_p \leqslant C_{p,d} \left\| X_{\mathbf{0}} \right\|_{2, 2(d-1)}.$$

Main ideas of proof

Steps :

1. Using arguments like for Doob's inequality, we reduce ourselves to the case where the supremum is restricted to the elements of $\mathbb{N}^d$ whose coordinates are dyadic.

2. Use an inequality of Fan, Grama et Liu (Statistics 51, 2017) :

$$\mathbb{P}\left(\left\{\left|\sum_{j=1}^{n} d_j\right| > x\right\} \cap \left\{\sum_{j=1}^{n} d_j^2 \leqslant y\right\}\right) \leqslant 2\exp\left(-\frac{x^2}{2y}\right). \qquad (*)$$

   - **Lexicographic order :** application of (*) by rewriting the sum on a rectangle as a sum of a martingale+ maximal ergodic theorem.
   - **Orthomartingales :** the sum of squares is treated by the maximal ergodic theorem and we are reduced to the case of dimension $d-1$.

Studied questions

1. Functionals of i.i.d. random fields : application to Volterra random fields.
2. Approximation by orthomartingales : conditions on the random variables $\mathbb{E}[X_{\mathbf{i}} \mid \mathcal{F}_{\mathbf{0}}]$.

Remaining questions

1. Optimality of the result for orthomartingales.
2. Case of infinite dimension (smooth Banach spaces). Possibility : by truncation and an inequality of Pinelis.