Some recent advances on limit theorems for stationary random fields

Davide Giraudo (Université de Strasbourg, France)

6th Cincinnati Symposium on Probability
May 1, Wednesday, 2024

Plan

Stationary random fields : definitions and examples

- A random field is a collection of random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where d is an integer.
- We say that $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is strictly stationary if for each $N \in \mathbb{N}^*$ and $\mathbf{i}_1, \dots, \mathbf{i}_N, \mathbf{j} \in \mathbb{Z}^d$, the vectors $(X_{\mathbf{i}_1}, \dots, X_{\mathbf{i}_N})$ and $(X_{\mathbf{i}_1+\mathbf{j}}, \dots, X_{\mathbf{i}_N+\mathbf{j}})$ have the same distribution (the sum is taken coordinatewise).

Example

If T is a $\mathbb{Z}^d$-measure preserving action on Ω, that is, $T^{\mathbf{i}} \colon \Omega \to \Omega$, $T^{\mathbf{i}} \circ T^{\mathbf{j}} = T^{\mathbf{i}+\mathbf{j}}$ for each $\mathbf{i}$ and $\mathbf{j}$ and for each $A \in \mathcal{F}$, $\mathbb{P}\left(T^{-\mathbf{i}}A\right) = \mathbb{P}(A)$, then for each $f \colon \Omega \to \mathbb{R}$, $\left(f \circ T^{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ is a strictly stationary random field.

Stationary random fields : examples

Example (Linear processes)

We say that the random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is a linear random field if there exists an i.i.d. centered random field $(\varepsilon_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ of square integrable random variables and a family of real numbers $(a_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ such that $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} a_{\boldsymbol{k}}^2$ is finite and

$$X_{\boldsymbol{i}} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a_{\boldsymbol{k}}\varepsilon_{\boldsymbol{i}-\boldsymbol{k}} \text{ a.s..}$$

Example (Volterra random fields of order two)

Let $(\varepsilon_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ be an i.i.d. collection of centered square integrable random variables and $(a_{\boldsymbol{u},\boldsymbol{v}})_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{Z}^d}$ be a family of real numbers such that $a_{\boldsymbol{u},\boldsymbol{v}} = 0$ if $\boldsymbol{u} = \boldsymbol{v}$ and $\sum_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{Z}^d} a_{\boldsymbol{u},\boldsymbol{v}}^2$ is finite. A Volterra random field is defined as

$$X_{\boldsymbol{i}} := \sum_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{Z}^d} a_{\boldsymbol{u},\boldsymbol{v}}\varepsilon_{\boldsymbol{i}-\boldsymbol{u}}\varepsilon_{\boldsymbol{i}-\boldsymbol{v}}.$$

Limit theorems

We are interested in the asymptotic behavior of partial sums of a strictly stationary random field, that is,

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{1} \preccurlyeq \boldsymbol{i} \preccurlyeq \boldsymbol{n}} X_{\boldsymbol{i}},$$

where $\boldsymbol{1} = (1, \ldots, 1)$ and $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$ means $i_\ell \leqslant j_\ell$ for each $1 \leqslant \ell \leqslant d$.

When $d = 2$,

$$S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j}.$$

We would like to give sufficient condition on the dependence and the moments of $(X_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{Z}^d}$ and normalizations $(a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$, $(b_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ and $(c_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ such that

- $\left(a_{\boldsymbol{n}}^{-1} S_{\boldsymbol{n}}\right)_{\boldsymbol{n} \in \mathbb{N}^d}$ converges in distribution as $\min \boldsymbol{n} \to \infty$;
- $\left(b_{\boldsymbol{n}}^{-1} S_{\boldsymbol{n}}\right)_{\boldsymbol{n} \in \mathbb{N}^d}$ converges almost surely as $\max \boldsymbol{n} \to \infty$;
- $\sup_{\boldsymbol{n} \in \mathbb{N}^d} c_{\boldsymbol{n}}^{-1} \left| S_{\boldsymbol{n}} \right|$ is bounded.

Approximation by m-dependent random fields

Suppose that the strictly stationary random field $(X_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ admits the representation $X_{\boldsymbol{i}} = f\left((\varepsilon_{\boldsymbol{i}-\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}\right)$, where $(\varepsilon_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ is i.i.d.. Suppose for instance that $\mathbb{E}\left[|X_{\mathbf{0}}|^p\right]$ is finite for some $p \geqslant 1$. Define for each fixed $\boldsymbol{i}$ and each positive integer $m \geqslant 1$ the σ-algebra $\mathcal{G}_{\boldsymbol{i},m} := \sigma\left(\varepsilon_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}^d, \|\boldsymbol{k}-\boldsymbol{i}\| \leqslant m\right)$. Then we can approximate $X_{\boldsymbol{i}}$ by $\mathbb{E}\left[X_{\boldsymbol{i}} \mid \mathcal{G}_{\boldsymbol{i},m}\right]$.

Take a random variable $\varepsilon'_{\mathbf{0}}$ independent of the random field $(\varepsilon_{\boldsymbol{k}})_{k\in\mathbb{Z}^d}$ and define the physical dependence measure

$$\delta_{\boldsymbol{i},p} := \left\|X_{\boldsymbol{i}} - X^*_{\boldsymbol{i}}\right\|_p , \tag{1}$$

where $X^*_{\boldsymbol{i}} = f\left(\left(\varepsilon^*_{\boldsymbol{i}-\boldsymbol{k}}\right)_{k\in\mathbb{Z}^d}\right)$, $\varepsilon^*_{\boldsymbol{u}} = \varepsilon_{\boldsymbol{u}}$ if $\boldsymbol{u} \neq \mathbf{0}$ and $\varepsilon^*_{\mathbf{0}} = \varepsilon'_{\mathbf{0}}$ (see El Machkouri, Volný, Wu, (2013), Biermé and Durieu (2014)).

For linear processus, $\delta_{\boldsymbol{i},p}$ is $|a_{\boldsymbol{i}}| \left\|\varepsilon_{\mathbf{0}}\right\|_p$ if $\varepsilon_{\mathbf{0}} \in \mathbb{L}^p$.

For Volterra processes and $p \geqslant 2$, $\delta_{\boldsymbol{i},p}$ can be bounded by a constant times $\sqrt{\sum_{\boldsymbol{v}\in\mathbb{Z}^d}\left(a^2_{\boldsymbol{i},v} + a^2_{v,\boldsymbol{i}}\right)}$.

Martingale approximation, $d = 1$ (1)

When $d = 1$, a strategy is to use a martingale approximation. Here $T\colon \Omega \to \Omega$ is bi-measurable and measure preserving. Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$ such that $T\mathcal{F}_0 \subset \mathcal{F}_0$.

We say that $\left(D \circ T^i\right)_{i \geqslant 0}$ is a martingale difference sequence if D is $\mathcal{F}_0$-measurable and $\mathbb{E}\left[D \mid T\mathcal{F}_0\right] = 0$.

For a centered $\mathcal{F}_0$-measurable and square integrable $f \in \mathbb{L}^2$, if we can find a martingale difference sequence $\left(D \circ T^i\right)_{i \geqslant 0}$ such that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \left\| \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} f \circ T^i - \sum_{i=1}^{j} D \circ T^i \right| \right\|_2 = 0,$$

then we can deduce a functional central limit theorem for $n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} f \circ T^i$, where $\lfloor x \rfloor = \max \left\{ n \in \mathbb{Z}, n \leqslant x \right\}$.

Martingale approximation, $d = 1$ (2)

Gordin-Peligrad (2011) found a necessary and sufficient condition for the existence of such a martingale approximation, which is satisfied when

$$\textbf{Hannan's condition :}\ f \text{ is } \bigvee_{i \in \mathbb{Z}} \mathcal{F}_i - \text{measurable}, \mathbb{E}\left[f \mid \bigcap_{i \in \mathbb{Z}} \mathcal{F}_i \right] = 0 \text{ and}$$

$$\sum_{i=0}^{\infty} \left\| \mathbb{E}\left[f \circ \mid \mathcal{F}_{-i} \right] - \mathbb{E}\left[f \mid \mathcal{F}_{-i-1} \right] \right\|_2 = \sum_{i=0}^{\infty} \left\| \mathbb{E}\left[f \circ T^i \mid \mathcal{F}_0 \right] - \mathbb{E}\left[f \circ T^i \mid \mathcal{F}_{-1} \right] \right\|_2 < \infty,$$

where $\mathcal{F}_i = T^{-i} \mathcal{F}_0$, and also when

$$\textbf{Maxwell and Woodroofe condition :}\quad \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \left\| \mathbb{E}\left[\sum_{i=1}^{n} f \circ T^i \mid \mathcal{F}_0 \right] \right\|_2 < \infty$$

takes place.

Commuting filtrations

We would like to find an analog of the previous martingale approximation. The first step is to define multi-indexed martingales and the corresponding filtrations.

Definition

We say that the the collection $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is a completely commuting filtration if for each integrable random variable Y,

$$\mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_{\boldsymbol{i}}\right] \mid \mathcal{F}_{\boldsymbol{j}}\right] = \mathbb{E}\left[\mathbb{E}\left[Y \mid \mathcal{F}_{\boldsymbol{j}}\right] \mid \mathcal{F}_{\boldsymbol{i}}\right] = \mathbb{E}\left[Y \mid \mathcal{F}_{\min\{\boldsymbol{i},\boldsymbol{j}\}}\right],$$

where $\min\{\boldsymbol{i},\boldsymbol{j}\} = (\min\{i_\ell, j_\ell\})_{\ell\in[\![1,d]\!]}$.

Example

Let $\mathcal{F}_{\boldsymbol{i}} = \sigma\left(\varepsilon_{\boldsymbol{k}}, \boldsymbol{k} \preccurlyeq \boldsymbol{i}\right)$, where $(\varepsilon_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ is i.i.d. ; then $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is completely commuting.

Example

Let $\left(\mathcal{F}_i^{(1)}\right)_{i\in\mathbb{Z}}$ and $\left(\mathcal{F}_j^{(2)}\right)_{j\in\mathbb{Z}}$ be filtrations such that for each i and j, $\mathcal{F}_i^{(1)}$ is independent of $\mathcal{F}_j^{(2)}$. Then $(\mathcal{F}_{i,j})_{i,j\in\mathbb{Z}}$ is commuting, where $\mathcal{F}_{i,j} = \mathcal{F}_i^{(1)} \vee \mathcal{F}_j^{(2)}$.

Orthomartingales

Definition (Orthomartingale difference random field)

We say that $(D_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is an orthomartingale difference random field with respect to the completely commuting filtration $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ if for each $\boldsymbol{i}\in\mathbb{Z}^d$, $D_{\boldsymbol{i}}$ is integrable, $\mathcal{F}_{\boldsymbol{i}}$-measurable and for each $1\leqslant \ell\leqslant d$, $\mathbb{E}\left[D_{\boldsymbol{i}}\mid \mathcal{F}_{\boldsymbol{i}-\boldsymbol{e}_{\ell}}\right]=0$, where $\boldsymbol{e}_{\ell}$ is the ℓ-th element of the canonical basis of $\mathbb{R}^d$.

When $d=2$, this reads as $D_{i,j}$ is $\mathcal{F}_{i,j}$-measurable and

$$\mathbb{E}\left[D_{i,j}\mid \mathcal{F}_{i-1,j}\right]=0=\mathbb{E}\left[D_{i,j}\mid \mathcal{F}_{i,j-1}\right].$$

Observe that

- $\left(\sum_{i=1}^m D_{i,j}\right)_{j\geqslant 1}$ is a martingale difference sequence with respect to $\left(\mathcal{F}_{m,j}\right)_{j\geqslant 0}$
- $\left(\sum_{j=1}^n D_{i,j}\right)_{i\geqslant 1}$ is a martingale difference sequence with respect to $\left(\mathcal{F}_{i,n}\right)_{i\geqslant 0}$.

Therefore, martingale property in each coordinate can be used.
In the sequel, we will assume that the filtration $(\mathcal{F}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathbb{Z}^d}$ is of the form $\left(T^{-\boldsymbol{i}}\mathcal{F}_{\boldsymbol{0}}\right)_{\boldsymbol{i}\in\mathbb{Z}^d}$ and $X_{\boldsymbol{i}}=X_{\boldsymbol{0}}\circ T^{\boldsymbol{i}}$.

Plan

A partial sum process

Let

$$W_{\boldsymbol{n}}(f,\boldsymbol{t}) := \frac{1}{\sqrt{|\boldsymbol{n}|}} \sum_{\boldsymbol{1}\preccurlyeq \boldsymbol{i} \preccurlyeq \lfloor \boldsymbol{n}\cdot\boldsymbol{t}\rfloor} f\circ T^{\boldsymbol{i}}, \boldsymbol{t}\in[0,1]^d,$$

where $\lfloor \boldsymbol{x}\rfloor = (\lfloor x_1\rfloor, \dots, \lfloor x_d\rfloor)$, for $x\in\mathbb{R}$, $\lfloor x\rfloor$ is the unique integer for which $\lfloor x\rfloor \leqslant x < \lfloor x\rfloor + 1$ and $|\boldsymbol{n}| = \prod_{\ell=1}^d n_\ell$.

When $d=1$,

$$W_n(f,t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt\rfloor} f\circ T^i.$$

If $\left(D\circ T^i\right)_{i\geqslant 1}$ is a martingale difference sequence, then

$$W_n(D,\cdot) \to \left(\mathbb{E}\left[D^2 \mid \mathcal{I}\right]\right)^{1/2} B(\cdot) \text{ in distribution in } D([0,1]),$$

where $\mathcal{I}$ is the σ-algebra of T invariant sets and B is a standard Brownian motion independent of $\mathbb{E}\left[D^2 \mid \mathcal{I}\right]$.

Functional central limit theorem for orthomartingale difference random fields

Let $d=2$ and $D_{i,j}=\varepsilon_i\varepsilon'_j$, where the sequences $(\varepsilon_i)_{i\in\mathbb{Z}}$ and $\left(\varepsilon'_j\right)_{j\in\mathbb{Z}}$ are mutually independent, both i.i.d. and ε_i, ε'_j take the values -1 and 1 with probability 1/2. Then $(D_{i,j})_{i,j\in\mathbb{Z}}$ is a stationary orthomartingale difference random field and $(W_{m,n}(D_{0,0},s,t))_{m,n\geqslant 1}$ converges in distribution to $B_sB'_t$, where $(B_s)_{s\in[0,1]}$ and $(B'_t)_{t\in[0,1]}$ are two independent standard Brownian motions (Wang and Woodroofe (2013)).

Theorem (Volný (2015,2019))

Let T be a $\mathbb{Z}^d$-measure preserving action on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and let $\left(T^{-\mathbf{i}}\mathcal{F}_{\mathbf{0}}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ be a completely commuting filtration. Let $\left(D_{\mathbf{0}}\circ T^{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ be a strictly stationary orthomartingale difference random field such that $\mathbb{E}\left[D_{\mathbf{0}}^2\right]$ is finite.

1. *The net $(W_{\mathbf{n}}(D_{\mathbf{0}},\cdot))_{\mathbf{n}\succcurlyeq\mathbf{1}}$ converges in $D\left([0,1]^d\right)$ as $\min\mathbf{n}\to\infty$.*
2. *If moreover one of the maps $T^{\mathbf{e}_\ell}$ is ergodic, then $(W_{\mathbf{n}}(D_{\mathbf{0}},\cdot))_{\mathbf{n}\succcurlyeq\mathbf{1}}$ converges to $\|D_{\mathbf{0}}\|_2\,W(\cdot)$ in $D\left([0,1]^d\right)$ as $\min\mathbf{n}\to\infty$, where $\left(W(\mathbf{t}),\mathbf{t}\in[0,1]^d\right)$ is a standard Brownian sheet.*

Orthomartingale approximation

In dimension one : let $f_M = M^{-1}\sum_{j=1}^{M}\mathbb{E}\left[f\circ T^j \mid \mathcal{F}_0\right]$. Then $f - f_M = D_M + G_M - G_M\circ T$, where $\left(D_M\circ T^i\right)_{i\geqslant 1}$ is a martingale difference sequence. If

$$\lim_{M\to\infty}\limsup_{n\to\infty}\frac{1}{\sqrt{n}}\left\|\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} f_M\circ T^i\right|\right\|_2 = 0,$$

then $(D_M)_{M\geqslant 1}$ converges in $\mathbb{L}^2$ to some D and

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left\|\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} f\circ T^i - \sum_{i=1}^{j} D\circ T^i\right|\right\|_2 = 0.$$

In dimension $d = 2$:

$$f - f_M = f - \frac{1}{M}\sum_{i=1}^{M}\mathbb{E}\left[f\circ T^{i,0}\mid\mathcal{F}_{0,0}\right] - \frac{1}{M}\sum_{j=1}^{M}\mathbb{E}\left[f\circ T^{0,j}\mid\mathcal{F}_{0,0}\right] + \frac{1}{M^2}\sum_{i,j=1}^{M}\mathbb{E}\left[f\circ T^{i,j}\mid\mathcal{F}_{0,0}\right],$$

which can be decomposed into 4 terms, which are difference martingale in some directions and coboundaries in others.

Projective conditions

One has to find condition which guarantee that the contribution of f_M defined as previously is negligible, usuall via moment inequalities involving $\mathbb{E}\left[f \circ T^{\mathbf{i}} \mid \mathcal{F}_{\mathbf{0}}\right]$.
We will state results in the case $d = 2$ and when f is $\mathcal{F}_{0,0}$-measurable, the general case is addressed in the corresponding papers. Volný and Wang (2014) showed that if

$$\lim_{\ell\to\infty} \left\|\mathbb{E}\left[f \mid \mathcal{F}_{-\ell,0}\right]\right\|_2 = 0 = \lim_{\ell\to\infty} \left\|\mathbb{E}\left[f \mid \mathcal{F}_{0,-\ell}\right]\right\|_2 , \tag{2}$$

$$\sum_{i,j\leqslant 0} \left\|\mathbb{E}\left[f \mid \mathcal{F}_{i,j}\right] - \mathbb{E}\left[f \mid \mathcal{F}_{i-1,j}\right] - \mathbb{E}\left[f \mid \mathcal{F}_{i,j-1}\right] + \mathbb{E}\left[f \mid \mathcal{F}_{i-1,j-1}\right]\right\|_2 < \infty \tag{3}$$

and one of the maps $T^{1,0}$ or $T^{0,1}$ is ergodic, then $W_{m,n}(f,\cdot)$ converges to a Brownian sheet as $\min\{m,n\} \to \infty$. The same conclusion holds (G., 2018) if (2) and (3) are replaced by

$$\sum_{m,n=1}^{\infty} \frac{1}{m^{3/2} n^{3/2}} \left\|\mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} f \circ T^{i,j} \mid \mathcal{F}_{0,0}\right]\right\|_2 < \infty.$$

Necessary and sufficient condition for orthomartingale approximation were obtained by Peligrad and Zhang (2018). For the CLT, martingale coboundary decomposition were studied by Lin, Merlevède and Volný (2022).

Quenched functional central limit theorem (1)

Denote by μ_ω a version of the regular conditional probability $\mathbb{P}(\cdot \mid \mathcal{F}_{\mathbf{0}})$.

Definition

We say that a random field $(Y_{\boldsymbol{n}})_{\boldsymbol{n} \succcurlyeq \mathbf{1}}$ satisfies the quenched invariance principle on squares (respectively on rectangles) if there exist a real number σ and a set Ω' of probability one such that for each $\omega \in \Omega'$,

$$\frac{1}{n^{d/2}} Y_{\lfloor n\boldsymbol{t} \rfloor} \to \sigma W(\boldsymbol{t}) \text{ in distribution in } D\left([0,1]^d\right) \text{ under } \mu_\omega,$$

(respectively, if

$$\frac{1}{\sqrt{|\boldsymbol{n}|}} Y_{\lfloor \boldsymbol{n} \cdot \boldsymbol{t} \rfloor} \to \sigma W(\boldsymbol{t}) \text{ in distribution in } D\left([0,1]^d\right) \text{ under } \mu_\omega \text{ as } \min \boldsymbol{n} \to \infty).$$

Quenched functional central limit theorem (2)

Peligrad and Volný (2020) considered orthomartingale difference random fields $\left(D_{\mathbf{0}} \circ T^{\mathbf{i}}\right)_{\mathbf{i} \in \mathbb{Z}^d}$ such that one of the shift maps $T^{\mathbf{e}_\ell}$ is ergodic and got the following results :

1. If $\mathbb{E}\left[D_{\mathbf{0}}^2\right] < \infty$, then the quenched invariance principle on squares takes place for $\left(S_n\left(D_{\mathbf{0}}\right)\right)_{n \succcurlyeq 1}$ with $\sigma = \left\|D_{\mathbf{0}}\right\|_2$.
2. If we furthermore assume that

$$\mathbb{E}\left[D_{\mathbf{0}}^2\left(\log\left(1+\left|D_{\mathbf{0}}\right|\right)\right)^{d-1}\right] < \infty, \tag{4}$$

   then the quenched functional central limit theorem on rectangles takes place for $\left(S_n\left(D_{\mathbf{0}}\right)\right)_{n \succcurlyeq 1}$ with $\sigma = \left\|D_{\mathbf{0}}\right\|_2$.
3. The assumption (4) is optimal.

Peligrad, Reding and Zhang (2020) obtained quenched invariance principle over squares and rectangles under Hannan type conditions for the appropriately centered partial sums of a stationary random field.

Plan

Orthomartingale case

Theorem (G.,2022+)

Let $\left(D\circ T^{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ *be a stationary orthomartingale difference random field and let* $1<p<2$.

1. *Suppose that* $\mathbb{E}\left[|D|^p\right]<\infty$. *Then*

$$\lim_{n\to\infty}\frac{1}{n^{d/p}}\sum_{\mathbf{1}\preccurlyeq\mathbf{i}\preccurlyeq n\mathbf{1}}D\circ T^{\mathbf{i}}=0 \text{ a.s.}$$

2. *If we moreover assume that*

$$\mathbb{E}\left[|D|^p\left(\log\left(1+|D|\right)\right)^{d-1}\right]<\infty,$$

then

$$\lim_{N\to\infty}\sup_{\max\mathbf{n}\geqslant N}\frac{1}{|\mathbf{n}|^{1/p}}\left|\sum_{\mathbf{1}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n}}D\circ T^{\mathbf{i}}\right|=0 \text{ a.s.}$$

The result is valid in the vector valued case under some conditions on the smoothness of the Banach space.

Random fields expressable as a function of an i.i.d. random field are also considered.

Plan

I.i.d. case

When $d=1$, it is known that if $(X_i)_{i\geqslant 1}$ is i.i.d., centered and has a finite variance, then

$$\limsup_{n\to\infty}\frac{1}{\sqrt{nLL(n)}}\sum_{i=1}^{n}X_i=\sqrt{2}\left\|X_0\right\|_2=-\liminf_{n\to\infty}\frac{1}{\sqrt{nLL(n)}}\sum_{i=1}^{n}X_i$$

where $L(x)=\max\{1,\ln x\}$ and $LL(x)=L\circ L(x)$.

When $\left(f\circ T^{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^d}$ is an i.i.d. random field and $d>1$, it has been shown by Wichura (1973) that

$$\mathbb{E}\left[f^2\left(L\left(|f|\right)\right)^{d-1}/LL\left(|f|\right)\right]<\infty$$
$$\Leftrightarrow \limsup_{\mathbf{n}\to\infty}\frac{1}{\sqrt{|\mathbf{n}|\,LL\left(|\mathbf{n}|\right)}}S_{\mathbf{n}}(f)=\|f\|_2\sqrt{d}=-\liminf_{\mathbf{n}\to\infty}\frac{1}{\sqrt{|\mathbf{n}|\,LL\left(|\mathbf{n}|\right)}}S_{\mathbf{n}}(f),$$

where for a family of numbers $(x_{\mathbf{n}})_{\mathbf{n}\succcurlyeq\mathbf{1}}$, $\limsup_{\mathbf{n}\to+\infty}x_{\mathbf{n}}:=\lim_{N\to\infty}\sup_{\mathbf{n}\succcurlyeq N\mathbf{1}}x_{\mathbf{n}}$ and similarly for $\liminf$.

Function of i.i.d. of independent random fields

Theorem (G., 2021)

Let $(X_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ be a centered random field such that there exist an i.i.d. collection of random variables $\left\{\varepsilon_{\mathbf{u}}, \mathbf{u}\in\mathbb{Z}^d\right\}$ and a measurable function $f:\mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ such that $X_{\mathbf{i}} = f\left(\left(\varepsilon_{\mathbf{i}-\mathbf{j}}\right)_{\mathbf{j}\in\mathbb{Z}^d}\right)$. For all $1<p<2$, the following inequality holds :

$$\left\|\sup_{\mathbf{n}\in\mathbb{N}^d}\frac{1}{\sqrt{|\mathbf{n}|\,LL\left(|\mathbf{n}|\right)}}\left|\sum_{\mathbf{1}\preccurlyeq\mathbf{i}\preccurlyeq\mathbf{n}}X_{\mathbf{i}}\right|\right\|_p \leqslant c_{p,d}\sum_{j=0}^{\infty}(j+1)^{d/2}\left\|X_{\mathbf{0},j}\right\|_{2,d-1},$$

where $L(x)=\max\{1,\log x\}$, $c_{p,d}$ *depends only on* p *and* d, $\|\cdot\|_{2,d-1}$ *is the Orlicz norm associated to the function* $t\mapsto t^2\left(\log(1+t)\right)^{d-1}$,

$$X_{\mathbf{0},j}=\mathbb{E}\left[X_{\mathbf{0}}\mid\sigma\left\{\varepsilon_{\mathbf{u}},\|\mathbf{u}\|_\infty\leqslant j\right\}\right]-\mathbb{E}\left[X_{\mathbf{0}}\mid\sigma\left\{\varepsilon_{\mathbf{u}},\|\mathbf{u}\|_\infty\leqslant j-1\right\}\right],\quad j\geqslant 1;$$

$$X_{\mathbf{0},0}:=\mathbb{E}\left[X_{\mathbf{0}}\mid\sigma\left\{\varepsilon_{\mathbf{0}}\right\}\right].$$

This result applies to Hölder continuous functions of a linear random field, Volterra processes, function of Gaussian linear processes. This rests on the use of physical dependence measure introduced in Wu (2005) (norm of the difference between $X_{\mathbf{i}}$ and a coupled version with $\varepsilon_{\mathbf{0}}$ replaced by $\varepsilon'_{\mathbf{0}}$, independent of $(\varepsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$).

Orthomartingale case (1)

Let $d = 2$ and $D_{i,j} = \varepsilon_i \varepsilon'_j$, where the sequences $(\varepsilon_i)_{i\in\mathbb{Z}}$ and $\left(\varepsilon'_j\right)_{j\in\mathbb{Z}}$ are mutually independent, both i.i.d. and ε_i, ε'_j take the values -1 and 1 with probability 1/2. Then $(D_{i,j})_{i,j\in\mathbb{Z}}$ is a stationary orthomartingale difference random field and

$$\sup_{m,n\geqslant 1} \frac{1}{\sqrt{mnLL(mn)}} \left| \sum_{i=1}^{m} \sum_{j=1}^{n} D_{i,j} \right| \geqslant \sup_{m\geqslant 1} \frac{1}{\sqrt{m}} \left| \sum_{i=1}^{m} \varepsilon_i \right| \limsup_{n\to\infty} \sqrt{\frac{LL(n)}{LL(mn)}} \frac{1}{\sqrt{nLL(n)}} \left| \sum_{j=1}^{n} \varepsilon'_j \right| \geqslant \sqrt{2} \sup_{m\geqslant 1} \frac{1}{\sqrt{m}} \left| \sum_{i=1}^{m} \varepsilon_i \right|,$$

which is almost surely infinite.

Therefore, a normalization compatible with the product structure has to be taken.

Orthomartingale case (2)

Let

$$M(f) = \sup_{\mathbf{n} \succcurlyeq \mathbf{1}} \prod_{\ell=1}^{d} \frac{1}{\sqrt{n_\ell LL(n_\ell)}} \left| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} f \circ T^{\mathbf{i}} \right| .$$

Let $\|\cdot\|_{2,q}$ be the Orlicz norm associated to the function $t \mapsto t^2 (\log(1+t))^q$.

Theorem (G., 2021)

Let $d \geqslant 1$ be an integer. For all $1 \leqslant p < 2$, there exists a constant $C_{p,d}$ depending only on p and d such that for all strictly stationary orthomartingale difference random field $\left(D_{\mathbf{0}} \circ T^{\mathbf{i}}\right)_{\mathbf{i} \in \mathbb{Z}^d}$, the following inequality holds :

$$\|M(D_{\mathbf{0}})\|_p \leqslant C_{p,d} \|D_{\mathbf{0}}\|_{2,2(d-1)} .$$

Moreover, for all $r \geqslant 0$,

$$\|M(D_{\mathbf{0}})\|_{2,r} \leqslant C_{p,d,r} \|D_{\mathbf{0}}\|_{2,r+2d} .$$

Results in the same spirit have been obtained under similar conditions as for the functional central limit theorem, with the norm $\|\cdot\|_2$ replaced by $\|\cdot\|_{2,2(d-1)}$ (respectively $\|\cdot\|_{2,r+2d}$).

Some open questions

1. For an ergodic action of $\mathbb{Z}^d$, the limit in the central limit theorem for orthomartingale difference random fields is expressed as $\eta \cdot N$, where N is a standard normal random variable independent of η and η is the limit in distribution as $\min\{m, n\} \to \infty$ of

$$\eta_{m,n} = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} D \circ T^{i,j} \right)^2}.$$

   Is their a nice characterization of the possible laws of functions η ? What about the functional central limit theorem ?

2. Projective criterion in the sprit of Hannan in some directions and Maxwell and Woodroofe in the others.

3. A law of the iterated logarithms with a characterization of the lim sup/lim inf.