Hecke Orbit Conjecture

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Shimura variety

Shimura datum. A *Shimura datum* is a pair (G, X) where G is a reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of a morphism of algebraic groups $h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ which satisfies the following properties:

- for all $h \in X$, $\mathrm{Ad} \circ h$ defines a Hodge structure of type $\{(-1,1),(0,0),(1,-1)\}$ on $\mathrm{Lie}(G_{\mathbb{R}})$;
- for all $h \in X$, $\mathrm{ad}\, h(i)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$;
- G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

where G^{ad} is the quotient of G by its center $Z(G)$.

Shimura variety. Let (G, X) be a Shimura datum. Let $\mathbb{A}_f$ denote the finite adele ring of $\mathbb{Q}$. For a compact open subgroup $K \subset G(\mathbb{A}_f)$, we consider the double coset space

$$\mathrm{Sh}_K(G,X)(\mathbb{C}) \overset{\text{def}}{=} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

Here, $G(\mathbb{Q})$ acts on X by conjugation and on $G(\mathbb{A}_f)$ by left multiplication, and K acts on $G(\mathbb{A}_f)$ by right multiplication (and trivially on X):

$$q(x,a)k = (qx, qak), \quad \forall q \in G(\mathbb{Q}),\ x \in X,\ a \in G(\mathbb{A}_f),\ k \in K.$$

If K is sufficiently small, the theory of canonical models tells us that $\mathrm{Sh}_K(G,X)(\mathbb{C})$ is the $\mathbb{C}$-points of a quasi-projective algebraic variety $\mathrm{Sh}_K(G,X)$ defined over a number field $E(G,X) \subset \mathbb{C}$, called the *reflex field*. We call it the *Shimura variety attached to the Shimura datum (G,X) of level K*. For open compact subgroups $K' \subset K$ of $G(\mathbb{A}_f)$, there is a canonical transition map

$$\mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$$

Among all Shimura varieties, there is an important class called Shimura varieties of PEL-type, which can be interpreted as the moduli spaces of polarized abelian varieties equipped with some extra endomorphisms and level structures. See the next page for some basic examples.

Statement of the Hecke orbit conjecture

Let $\mathrm{Sh}_K(G,X)$ be a Shimura variety of PEL-type defined over its reflex field E. Let v be a finite place of E of residue characteristic $p > 0$. Suppose that K is of the form $K = K^pK_p$, where $K^p \subset G(\mathbb{A}_f^p)$ is an open compact subgroup of the prime-to-p adelic points of G, and K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. Using the moduli interpretation of $\mathrm{Sh}_K(G,X)$, Kottwitz defined a canonical integral model $\mathcal{M}_K$ of $\mathrm{Sh}_K(G,X)$ that is smooth over $\mathcal{O}_{E,v}$. We denote by $\mathcal{M}_{K,0}$ the special fiber of $\mathcal{M}_K$. We fix a point $x_0 \in \mathcal{M}_{K,0}(\overline{\mathbb{F}}_p)$.

Central leaves. The locus of all points of $\mathcal{M}_{K,0}$ having "the same p-adic invariants as x_0" is a smooth locally closed algebraic subvariety $\mathcal{C}(x_0)$ of $\mathcal{M}_{K,0}$, call the *central leaf* in $\mathcal{M}_{K,0}$ passing through x_0. More precisely, $\mathcal{C}(x_0)$ is characterized by the following property: for every algebraically closed extension field Ω of $\overline{\mathbb{F}}_p$ and every geometric point $y \in \mathcal{M}_{K,0}(\Omega)$, the p-divisible groups with imposed polarization and endomorphism structure attached to the fibers at y and x_0 of the universal abelian scheme are isomorphic over Ω. For instance, if x_0 lies in the ordinary locus of $\mathcal{M}_{K,0}$, then $\mathcal{C}(x_0)$ is the whole ordinary locus of $\mathcal{M}_{K,0}$.

Prime-to-p Hecke orbit. To define the prime-to-p Hecke orbit of x_0, we consider $\mathcal{M}_{K_p,0} := \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathcal{M}_{K^pK_p,0}$. Then there is a natrual group action of $G(\mathbb{A}_f^p)$ on $\mathcal{M}_{K_p,0}$. Let $\tilde{x}_0 \in \mathcal{M}_{K_p,0}(\overline{\mathbb{F}}_p)$ be a lifting of x_0. The prime-to-p Hecke orbit of x_0, denote by $\mathcal{H}^p(x_0)$, is defined to be the image of the orbit $G(\mathbb{A}_f^p) \cdot \tilde{x}_0$ under the natural projection $\mathcal{M}_{K_p,0} \to \mathcal{M}_{K,0}$. It is easy to see every point $y \in \mathcal{H}^p(x_0)$ has isomorphic associated p-divisible group (with additional polarization and endomorphisms) as x_0. Therefore, one has $\mathcal{H}^p(x_0) \subset \mathcal{C}_{\mathcal{M}}(x_0)$. The Hecke orbit conjecture predicts that $\mathbb{H}^p(x_0)$ is actually Zariski dense in $\mathbb{C}(x_0)$.

Conjecture 0.1. Suppose that x_0 is not in the basic locus. Then the prime-to-p Hecke orbit $\mathcal{H}^p(x_0)$ is Zariski dense in the central leaf $\mathcal{C}_{(x_0)}$. In particular, if x_0 lies in the μ-ordinary locus of $\mathcal{M}_{K,0}$, then $\mathcal{H}^p(x_0)$ is Zariski dense in $\mathcal{M}_{K,0}$.

The proof of the conjecture in the case of Hilbert modular varieties will be sketched in the following sections.

Examples of Shimura varieties of PEL-type

- Let $G = \mathrm{GL}_2$ and X be the conjugacy class of the homomorphism $h : \mathbb{S} \to \mathrm{GL}_{2,\mathbb{R}}$ defined by $h(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$. For an integer $N \geq 4$, let
$$K_1(N) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \mod N \right\}.$$
Then $\mathrm{Sh}_{K_1(N)}(G,X)$ is the usual modular curve over $\mathbb{Q}$ of level $\Gamma_1(N)$ which parametrizes elliptic curves together with a point of order N.
- Let $G = \mathrm{GSp}_{2g}$ be the symplectic similitude group of dimension $g \geq 1$, and X be the union of upper and lower Siegel upper half planes of genus g. For a sufficiently small open compact subgroup $K \subset G(\mathbb{A}_f)$, $\mathrm{Sh}_K(G,X)$ is the Siegel modular variety that parametrizes principally polarized abelian varieties of dimension g equipped with a K-level structure.
- Let F be a totally real number field of degree $g = [F : \mathbb{Q}]$, and $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$, and X is the conjugacy class of the homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}} = \prod_{\tau : F \hookrightarrow \mathbb{C}} \mathrm{GL}_{2,\mathbb{R}}$ given by $h(a+bi) = \prod_{\tau : F \hookrightarrow \mathbb{C}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$. For a sufficiently small open compact subgroup $K \subset G(\mathbb{A}_f)$, the associated Shimura variety $\mathrm{Sh}_K(G,X)$ is the Hilbert modular variety that paramerizes abelian varieties of dimension g equipped with real multiplication by the ring of integers of F and a level structure corresponding to K.

Proof in the case of Hilbert modular variety

- Using Serre-Tate theory to perform explicit calculation on the formal completion of a smooth ordinary point in the modular variety. The proof is due to Chai[1].
Serre-Tate theory tells us the formal completion $\mathscr{M}_{\bar{y}}$ of $\mathscr{M}$ at a smooth ordinary point $\bar{y} = (A_1, \iota_1) \in \mathscr{M}(k)$ represents the deformation functor Def_{A_1}, and is endowed with a formal group structure. This severely limits the possible forms of the formal completion $\tilde{Z}_{\bar{y}}$ of the Zariski closure $\tilde{Z}$ of reduced prime-to-p Hecke orbit of the smooth ordinary point $\bar{y}$. The result can be generalized globally by faithfully flat descent[1]:

  Proposition 0.2. Let $\{\mathfrak{p}_i\}_{i \in \omega}$ be the set of primes of the ring of integers $\mathscr{O}_F$ in F over p. For each irreducible component W of the smooth locus $\tilde{Z}_{\mathrm{sm}}$ of $\tilde{Z}$ there exists a subset $\omega \subseteq \{1, \dots, r\}$ such that the tangent sheaf $T_W \subseteq \bigoplus_{i \in \omega} T_{\mathscr{M}_F}(\mathfrak{p}_i) \otimes_{\mathscr{O}_{\mathscr{M}_F}} \mathscr{O}_W$.

- Study the formal completion of the related supersingular point with Serre-Tate theory.
The existence of a supersingular point s in $\tilde{Z}$ is guaranteed by the existence of finite strata. The action of the stablizer group of the Hecke action at s decomposes into products, rendering the tangent space of one irreducible component of the Zariski closure of the smooth ordinary locus large enough to contain the whole tangent space.

Current results and future research

The Hecke orbit conjecture was first proved in the case when the central leaf is the ordinary locus of the Siegel moduli space $\mathscr{A}_g$ of g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}}_p$ by Chai[1]. Subsequently it was shown for all central leaves in $\mathscr{A}_g$; see Chai[1, 2]. The method used for $\mathscr{A}_g$ uses a special property of Siegel modular varieties and extends to some modular varieties of PEL type C which also has this property. The same idea with an alternation in the analysis of the endomorphism algebra applies to the case for quarternion Shimura variety, see Zhou[3]. The Hecke orbit conjecture for general PEL-type Shimura varieties remains open. In the ongoing research we aim at studying the case of Picard modular surface, or some global rigidity statement related to Galois representations.

References

[1] Ching-Li Chai. Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli. *Inventiones mathematicae*, 121(1):439–479, Dec 1995.

[2] Ching-Li Chai. *Hecke orbits on Siegel modular varieties*, pages 71–107. Birkhäuser Boston, Boston, MA, 2005.

[3] Rong Zhou. Motivic cohomology of quaternionic shimura varieties and level raising, 2019.