# Classification of local linear p-adic differential equations

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# Linear p-adic differential equations

Let $C_p$ denote a complete and algebraically closed field containing the field of p-adic numbers $\mathbb{Q}_p$. The field $K = C_p(\{z\})$ of the convergent Laurent series is equipped with the differentiation $\delta = z\frac{d}{dz}$. Then $\widehat{K} = C_p((z))$, the field of the formal Laurent series, is the completion of $K$. A homogeneous linear differential equation over $K$ has the scalar form

$$(\delta^n + a_{n-1}\delta^{n-1} + \cdots + a_1\delta + a_0)y = 0,$$

with all $a_i$ and $f \in K$.
This equation can be rewritten in matrix form $(\delta + A)Y = 0$, where $Y$ is a vector of length $n$ and $A$ is an $n \times n$-matrix with entries in $K$.

# The basic example

A differential module $M$ over $K$ is a finite dimensional vector space over $K$, equipped with an additive operator $\delta = \delta_M : M \to M$, satisfying $\delta(fm) = f\delta(m) + \delta(f)m$. After choosing a basis of $M$ over $K$ one finds a matrix differential equation. The choice of a cyclic vector turns $M$ into a scalar equation. We want to classify the differential modules over $K$. The basic difficulty is the following example:

The equation $(\delta - \lambda)y = \frac{1}{1-z} = \sum z^n$ with $\lambda \in C_p,\ \lambda \notin \mathbb{Z}_{\geq 0}$ has in $\widehat{K}$ only one solution, namely $\sum \frac{1}{n-\lambda} z^n$. This power series is divergent if and only if $\liminf_{n\to\infty} |n-\lambda|^{1/n} = 0$. $\lambda$ is called a *p-adic Liouville number* if the above power series is divergent. (Introduced by D. Clark 1966)

# Some properties of p-adic Liouville numbers

Let $\mathcal{L}$ denote the set of the p-adic Liouville numbers. $\lambda \in \mathcal{L}$ if and only if $\lambda \notin \mathbb{Z}_{\geq 0}$ and $\lambda$ can be approximated rapidly be positive integers. Thus $\mathcal{L}$ consists of the elements $\mathbb{Z}_p \ni a = \sum a_i p^i$, all $a_i \in \{0, \dots, p-1\}$ such that there are large gaps of zeroes in the sequence $a_0, a_1, a_2, \dots$.

$\mathcal{L}$ is uncountable and has measure 0,
every element of $\mathcal{L}$ is transcendental over $\mathbb{Q}$,
$a \in \mathcal{L}$, $n, m \in \mathbb{Z}$, $m > 0$ implies $n + ma \in \mathcal{L}$,
$\mathcal{L} \cap -\mathcal{L}$ is an infinite set.

# What can one do with divergent solutions?

In complex analysis, a divergent solution, i.e., an $F \in \mathbb{C}((z))$ with $F \notin \mathbb{C}(\{z\})$, can be seen as the asymptotic expansion of a meromorphic function living on a sector at $z = 0$.

Is there a way of introducing in the p-adic 'plane' $C_p$, sectors at $z = 0$ and asymptotic expansions?

One also knows that there is a complex valued $C^\infty$-function with Taylor expansion $F$. This has some equivalent in p-adic analysis, however far from being useful for our purposes.
Does the rigid analytic theory provide 'functions' with a divergent expansion at $z = 0$? Is there an equivalent of Borel summation or multisummation?

# A quick survey of classification in the complex case

Now $K = \mathbb{C}(\{z\})$ and $\widehat{K} = \mathbb{C}((z))$. A differential module $M$ over $K$ (or $\widehat{K}$) is called *regular singular* if an associated matrix equation has the form $\delta + A$ and $A$ has entries in $\mathbb{C}\{z\}$ (or in $\mathbb{C}[[z]]$).

It can be shown that a regular singular $M$ can in fact be represented by $\delta + A$, where $A$ is a constant matrix. Further the conjugacy class of the monodromy $e^{2\pi i A}$ is the classifying object for regular singular modules.

In order to describe irregular modules, we restrict (for convenience) to 'unramified' modules. For $q \in z^{-1}\mathbb{C}[z^{-1}]$ we consider the module $E(q) = Ke$ with $\delta e = qe$ and $\widehat{E(q)} = \widehat{K}e$.

# The isotypical decomposition

An unramified differential module $\widehat{M}$ over $\widehat{K}$ has a unique decomposition (the isotypical decomposition)

$$(\widehat{E(q_1)} \otimes M_1) \oplus \cdots \oplus (\widehat{E(q_s)} \otimes M_s),$$

where $q_1, \ldots, q_s$ are distinct elements of $z^{-1}\mathbb{C}[z^{-1}]$ and $M_1, \ldots, M_s$ are regular singular modules. The $q_1, \ldots, q_s$ are called the eigenvalues of the module.

For an unramified differential module $M$ over $K$, the isotypical decomposition of the module $\widehat{M} := \widehat{K} \otimes_K M$ does, in general, not descent to $K$.

# The Stokes maps

The regular singular modules $M_i$ descent to $K$ (call these objects again $M_i$) and by definition the $\widehat{E(q_i)}$, too. However $M$ is in general not isomorphic to

$$(E(q_1) \otimes M_1) \oplus \cdots \oplus (E(q_s) \otimes M_s).$$

The obstruction is well organized into a set of Stokes maps and can be formulated as follows:

For every $i \neq j$ the element $q_i - q_j$ has a finite number of singular directions $e^{id}, 0 \leq d < 2\pi$ on the unit circle. (The function $e^{\int (q_i - q_j) \frac{dz}{z}}$ has maximal descent for $z = re^{id}, \ r > 0, r \to 0$). There is a somewhat abstract solution space $V$ of $M$ available. It is the direct sum of the solution spaces $V_i$ of $E(q_i) \otimes M_i$.

# Complex classification

For every singular direction $d$, the module $M$ produces a unipotent Stokes map $St_d$ which has the form

$$id + \sum_{d \text{ singular for } q_i - q_j} L_{i,j}, \text{ where}$$

$$L_{i,j} : V \overset{projection}{\rightarrow} V_i \overset{linear}{\rightarrow} V_j \overset{inclusion}{\rightarrow} V.$$

The differential modules $M$ over $K$ are classified by the 'formal module' $\widehat{M}$ and the collection of Stokes maps $\{St_d\}$.

Moreover for a given formal module $\widehat{M}$ and any prescribed collection of maps as above $\{St_d\}$ there is a unique module $M$ over $K$ having these data.

# Differential Galois groups

Consider a differential field $F$ of characteristic 0 with an algebraically closed field of constants $C$. For any linear differential equation over $F$, say in the scalar form $L(y) = 0$ and of degree $n$, there is an extension of differential fields $F \subset PV$, such that:

$C$ is the field of constants of $PV$;
$PV$ contains $n$ independent solutions $y_1, \ldots, y_n$ for $L(y) = 0$;
$PV$ is generated over $F$ by $y_1, \ldots, y_n$ and their derivatives.

This field $PV$ is called the Picard-Vessiot field; it is unique up to isomorphism. The differential Galois group $G$ of $L$ is the group of the automorphisms of $PV/F$ commuting with the differentiation. The group $G$ is an algebraic subgroup of $\mathrm{GL}(n, C)$.

# The differential Galois group in the complex case

*Theorem* (Ramis et al.) Let $M$ be a differential module over $K = \mathbb{C}(\{z\})$ of dimension $n$.Then the differential Galois group of $M$ is the smallest algebraic subgroup of $\mathrm{GL}(n, C)$ containing the (formal) differential Galois group of $\widehat{K} \otimes_K M$ over $\widehat{K}$ and the collection of the Stokes matrices $\{St_d\}$.

We note that from the isotypical decomposition of a differential module over $\widehat{K}$ one can easily compute the formal differential Galois group.

# Returning to the p-adic case

We start with a remarkable result:

**Theorem** 1. *Let $M$ be a differential module over $K = C_p(\{z\})$. The isotypical decomposition of $\widehat{K} \otimes M$ descends to $K$.*

In other words, the irregularity of $M$ does not introduce divergent power series. The reason is the following. In decomposing an operator $L \in K[\delta]$ one has to cope with the non commutativity of this ring. The rule $\delta \cdot z^n = z^n\delta + nz^n$ (with $n \in \mathbb{Z}$) produces divergence in the complex case, but not in the p-adic case since $|n|_p \leq 1$ for $n \in \mathbb{Z}$.

# Regular singular modules over $K = C_p(\{z\})$

We recall that a regular singular module $M$ over $K$ has on a suitable basis the matrix form $\delta + A$, where $A$ has entries in $C_p\{z\}$. Thus $A = \sum_{n\geq 0} A_n z^n$. One can arrange things such that $A(0) = A_0$ has eigenvalues $\lambda_1, \ldots, \lambda_s$ such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$. These eigenvalues are unique up to a shift over integers and we will call them the eigenvalues of $M$.
There is a unique formal transformation $F = 1 + F_1 z + F_2 z^2 + \cdots$ such that $F^{-1}(\delta + A)F = \delta + A_0$. In contrast to the complex case, where $F$ is convergent, the $F$ is in general divergent.

*The basic example.* $\delta + \begin{pmatrix} 0 & a \\ 0 & \lambda \end{pmatrix}$ with $a \in z \cdot C_p\{z\}$ and $\lambda \in \mathcal{L}$. Then $\begin{pmatrix} 1 & -y \\ 0 & 1 \end{pmatrix}\{\delta + \begin{pmatrix} 0 & a \\ 0 & \lambda \end{pmatrix}\}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \delta + \begin{pmatrix} 0 & a+(\delta-\lambda)y \\ 0 & \lambda \end{pmatrix}$ and the solution $y$ of the equation $a + (\delta - \lambda)y = 0$ is for many $a$ divergent.

# The oriented graph $E$

We associate to a regular singular differential module $M$ with eigenvalues $\lambda_1, \dots, \lambda_s$ as above the oriented graph $E$ with vertices $v_1, \dots, v_s$ and $v_i \to v_j$ is an oriented edge if and only if $i \neq j$ and $\lambda_j - \lambda_i \in \mathcal{L}$.

For the basic example, eigenvalues $\{0, \lambda\}$ with $\lambda \in \mathcal{L}$ and $-\lambda \notin \mathcal{L}$, the graph $E$ is just $v_1 \to v_2$. The situation can become rather complicated, indeed:

**Theorem** 2. *Let $E$ be a finite oriented graph such that: the two ends of every oriented edge are distinct and for vertices $a \neq b$, there is at most one oriented edge $a \to b$. Then there exists $\{\lambda_1, \dots, \lambda_s\} \subset C_p$ with oriented graph $E$.*

# Submodules of $M$

As before, $M$ is regular singular with eigenvalues $\lambda_1, \ldots, \lambda_s$. Then $\widehat{M} = \widehat{K} \otimes_K M$ has a unique decomposition $\oplus_{i=1}^{s} \widehat{M}_i$ such that each $\widehat{M}_i$ has the single eigenvalue $\lambda_i$. Further $\widehat{M}_i = \widehat{K} \otimes_K M_i$ for a unique (regular singular) differential module $M_i$ over $K$. For any subset $S \subset \{v_1, \ldots, v_s\}$ one defines $M(S) := \oplus_{v_i \in S} M_i$.

**Theorem** 3. *Let $S \subset \{v_1, \ldots, v_s\}$. Suppose that there is no oriented edge $v_i \rightarrow v_j$ in $E$ such that $v_i \notin S,\ v_j \in S$. Then $M(S)$ can be identified with a submodule $N$ of $M$ such that unique differential submodule $M(S) \subset M$ such that $\widehat{M(S)} = \oplus_{v_i \in S} \widehat{M}_i$. For 'generic' $M$, these are the only subsets $S$ such that $M(S)$ can be seen as a submodule of $M$.*

**Corollary** 4. *A regular singular module $M$ is a direct sum of submodules which have a connected oriented graph.*

# Classification

A regular singular module $M$, with eigenvalues $\lambda_1, \ldots, \lambda_s$, is, as before, represented by $\delta + A$, where $A \in \mathrm{End}(C_p\{z\} \otimes V)$ for some vector space $V = V_1 \oplus \cdots \oplus V_s$ over $C_p$. One writes $A = \sum_{1 \le i,j \le s} A_{i,j}$ where $A_{i,j} : C_p\{z\} \otimes V_i \to C_p\{z\} \otimes V_j$. $A_{i,i}$ is constant and has only $\lambda_i$ as eigenvalue, $A_{i,j}$ has for $i \neq j$ its entries in $z \cdot C_p\{z\}$. Define $\tilde{A}$ by $\tilde{A}_{i,i} = A_{i,i}$ and $\tilde{A}_{i,j} = 0$ for $i \neq j$. As before, there is a unique $F \in \mathrm{GL}(C_p[[z]] \otimes V)$ with $F(0) = 1$ and $F^{-1}\{\delta + A\}F = \delta + \tilde{A}$.

We propose to measure the divergence of $F = \sum F_{i,j}$ by the set $\{\overline{L}_{i,j}\}_{i \neq j}$ where $L_{i,j} \in C_p[[z]] \otimes_{C_p} Hom(V_i, V_j)$ is the unique solution of the equation $\delta(L_{i,j}) - L_{i,j}A_{i,i} + A_{j,j}L_{i,j} + A_{i,j} = 0$ and $\overline{L}_{i,j}$ is the image of $L_{i,j}$ in $(C_p[[z]]/C_p\{z\}) \otimes_{C_p} Hom(V_i, V_j)$.

# Conjectures

$L_{i,j}$ is an "approximation" of $F_{i,j}$. In the special case that $A_{k,l} = 0$ for all pairs $k \neq l$ and $(k,l) \neq (i,j)$, then $L_{i,j} = F_{i,j}$. If $\lambda_j - \lambda_i \notin \mathcal{L}$, then $\overline{L}_{i,j} = 0$. We want to see $\overline{L}_{i,j}$ for $i \to j \in E$ as an analogue of the Stokes matrix $St_d$ at a singular direction $d$. This leads to:

**Conjecture**. 1. *The map* $M \mapsto \{\overline{L}_{i,j}\}_{i \to j \in E}$ *induces a bijection between the isomorphy classes of the regular singular differential modules* $M$ *with prescribed* $\widehat{M}$ *(as above) and* $\prod_{i \to j \in E} (C_p[[z]]/C_p\{z\}) \otimes_{C_p} Hom(V_i, V_j)$.

It suffices to consider the case where $E$ is connected. The case $E = \{v_1 \to v_2\}$ is all right and probably $E = \{v_1 \to v_2 \to v_3\}$, too. In general, the divergence for an arrow $i \to j \in E$ is influenced by the other arrows and direct computations becomes too complicated.

# Conjectures

For $i \to j \in E$ one defines the subspace $W_{i,j} \subset Hom(V_i, V_j)$ by $W_{i,j}$ is the smallest subspace such that $\overline{L}_{i,j}$ has image 0 in $(C_p[[z]]/C_p\{z\}) \otimes_{C_p} (Hom(V_i, V_j)/W_{i,j})$.

**Conjecture** 2. *The differential Galois group of $M$, seen as an algebraic subgroup of $\mathrm{GL}(V)$, is the smallest algebraic group containing the (well known) differential Galois group of $\widehat{M}$ and such that its Lie algebra contains $W_{i,j} \subset End(V)$ for all $i \to j \in E$.*

The case $E = \{v_1 \to v_2\}$ is all right and probably $E = \{v_1 \to v_2 \to v_3\}$, too.

# Conjectures

Conjecture 2 implies that for 'generic' $M$ the differential Galois group is the smallest algebraic subgroup of $\mathrm{GL}(V)$ containing the differential Galois group of $\widehat{M}$ and such that its Lie algebra contains $Hom(V_i, V_j)$ for all $i \to j \in E$.

Another consequence of Conjecture 2 is:

*$G$ is the differential Galois group of a regular singular differential module over $K$ if and only if $G/U(G)$ is, for the Zariski topology, topologically generated by one element. Here $U(G)$ denotes the algebraic subgroup of $G$ generated by the unipotent elements of $G$.*

# Realizable subgroups of $\mathrm{SL}_2$

We give here the list of the (conjugacy classes) of the algebraic subgroups of $\mathrm{SL}_2(C_p)$ which occur as differential Galois group $G$ for regular singular differential modules $M$ of dimension 2 over $K$. The $G \subset \mathrm{SL}_2$ is equivalent to det $M$ is the trivial module.

$E = \ * \ .$ *Suppose that $M$ has only one eigenvalue $\lambda$.*
Now $p$-adic Liouville numbers play no role. $M$ can be represented by $\delta + \begin{pmatrix} \lambda & * \\ 0 & \lambda \end{pmatrix}$ with $* \in \{0, 1\}$ and $2\lambda \in \mathbb{Z}$. The possibilities for $G$ are $1, \ \{\pm 1\}, \ \mathbb{G}_a, \ \{\pm 1\} \times \mathbb{G}_a$.

*Suppose that $M$ has two eigenvalues $\lambda_1, \lambda_2$.* By assumption $\lambda_1 - \lambda_2 \notin \mathbb{Z}$ and $\lambda_1 + \lambda_2 \in \mathbb{Z}$. After a shift over an integer we may suppose that $\lambda_1 = \lambda, \ \lambda_2 = -\lambda$ and $2\lambda \notin \mathbb{Z}$.There are several cases:

(a) $E = \ * \ *$ . $2\lambda \notin \mathcal{L} \cup -\mathcal{L}$ and p-adic Liouville numbers play no role. The differential Galois group is $\mathbb{G}_m$ or finite cyclic.

(b) $E = \ * \rightarrow *$ . $2\lambda \in \mathcal{L},\ 2\lambda \notin -\mathcal{L}$ and $M$ "generic".
Since $2\lambda$ is not rational one finds that the formal differential Galois group is $\mathbb{G}_m$. This is a subgroup of $G$. Since $M$ is generic, it has only one non trivial submodule. One concludes that $G$ is (conjugated to) the Borel subgroup $B \subset \mathrm{SL}_2(C_p)$. (In the non generic case, $G = \mathbb{G}_m$).

(c) $E = \ * \leftrightarrow *$ . $2\lambda \in \mathcal{L} \cap -\mathcal{L}$ and $M$ "generic".
The formal differential Galois group is again $\mathbb{G}_m$. Since $M$ is generic it has only trivial submodules . Then $G$ can only be $\mathrm{SL}_2$ or the infinite dihedral group $D_\infty$. In the latter case $N := K(z^{1/2}) \otimes M$, as differential module over $K(z^{1/2})$, has differential Galois group $\mathbb{G}_m$. Therefore $N$ is a direct sum of two 1-dimensional subspaces.
However, the eigenvalues of $N$ are $2\lambda, -2\lambda$. Since $\pm 4\lambda \in \mathcal{L}$, the module $N$ has only trivial submodules and this excludes the possiblity $G = D_\infty$. We conclude that the differential Galois group of $M$ is $\mathrm{SL}_2$. (In the non generic case, $G = B$ or $G = \mathbb{G}_m$).

Hence the list of realizable subgroups of $\mathrm{SL}_2$ for regular singular equations is: $\mathrm{SL}_2,\ B,\ \mathbb{G}_m, \mathbb{G}_a,\ \{\pm 1\} \times \mathbb{G}_a,$ finite cyclic .

# Comparison with the complex case

If $M$ is irregular singular, then $p$-adic Liouville numbers play no role and the differential Galois groups are the groups $\mathbb{G}_m$ (unramified case) and $D_\infty$ (ramified case).

*Comparison with the realizable subgroups of* $\mathrm{SL}_2$ *for equations over* $\mathbb{C}(\{z\})$
$SL_2$, irregular and there are non trivial Stokes matrices $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
$B$, irregular, all Stokes matrices have the form $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$ or all $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
$D_\infty$, irregular and ramified, and the Stokes matrices are trivial.
$\mathbb{G}_m$, regular singular or irregular singular, unramified, trivial Stokes.
$\mathbb{G}_a,\ \{\pm 1\} \times \mathbb{G}_a,$ finite cyclic , for regular singular.

WANTED: A summation method for divergent $p$-adic power series and/or an interpretation of $C_p[[z]]/C_p\{z\}$. This would be helpful in separating the divergence coming from the oriented arrows of $E$.

# The Malgrange–Sibuya theorem

In the complex case one considers on the circle of directions $S^1$ at $z = 0$ the sheaf $\mathcal{A}$. The stalk of this sheaf at a direction $e^{id}$ consists of the germs of analytic functions, defined on a sector around this direction and having an asymptotic expansion (in $\mathbb{C}[[z]]$). Further $\mathcal{A}^o$ is the subsheaf consisting of the elements with asymptotic expansion 0.

**Theorem** *The exact sequence of sheaves*

$$0 \to \mathcal{A}^o \to \mathcal{A} \to \mathbb{C}[[z]] \to 0$$

*on $S^1$ induces an isomorphism $\mathbb{C}[[z]]/\mathbb{C}\{z\} \to H^1(S^1, \mathcal{A}^0)$.*

This result can be extended to linear algebraic groups over $\mathbb{C}$ and the latter plays an important role in the complex asymptotic theory.

# The Malgrange–Sibuya theorem

For the group $GL_n$ this theorem states that there is a canonical isomorphisms

$$\mathrm{GL}_n(\mathbb{C}[[z]])/\mathrm{GL}_n(\mathbb{C}\{z\}) \to H^1(S^1, \mathrm{GL}_n(\mathcal{A})^o).$$

The sheaf $\mathrm{GL}_n(\mathcal{A})^o$ is defined by: The stalk at a direction $e^{id}$ consists of the elements of $\mathrm{GL}_n(\mathcal{A})_d$ having the identity as asymptotic expansion.