C^1-LOCAL FLATNESS AND GEODESICS OF THE LEGENDRIAN SPECTRAL DISTANCE

SIMON ALLAIS AND PIERRE-ALEXANDRE ARLOVE

Abstract. In this article, we give an explicit computation of the order spectral selectors of a pair of C^1-close Legendrian submanifolds belonging to an orderable isotopy class. The C^1-local flatness of the spectral distance and the characterisation of its geodesics are deduced. Another consequence is the C^1-local coincidence of spectral and Shelukhin-Chekanov-Hofer distances. Similar statements are then deduced for several contactomorphism groups.

1. Introduction

Recently Nakamura [19] and the authors of this article [1] defined independently the same distance on the isotopy class of a closed Legendrian whenever it is orderable. While Nakamura showed that the topology induced by this distance is the interval topology (see [9] for a definition), the authors showed the spectrality of this distance and therefore named it the spectral distance. Both results suggest the natural character and importance of the spectral distance: on the one hand it is a powerful object to study the geometry of this infinite dimensional space, and on the other it allows to study and quantify contact dynamic phenomena. We refer directly to [19, 1] for the illustration of our previous words.

In this article we study the orderable isotopy class of a closed Legendrian submanifold endowed with the spectral distance as a metric space on its own, being much inspired by the seminal work of Bialy-Polterovich on the geodesics of Hofer's metric [6]. In particular we show that it is C^1-locally flat in the sense that C^1-locally it is isometric to a normed vector space (see Section 1.1). As a consequence we also get in this setting the C^1-local flatness of the Shelukhin-Chekanov-Hofer distance [24]. The flatness property allows us moreover to give a complete description of the geodesics of these two distances (see Section 1.2). Finally this allows us to get similar statements in some cases for universal covers of certain contactomorphism groups.

1.1. C^1-flatness.

From now on $(M, \xi = \ker\alpha)$ denotes a cooriented contact manifold endowed with a contact form α the Reeb flow of which is complete. We denote $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) the Legendrian isotopy class of some closed Legendrian submanifold of M (resp. the universal cover of $\mathcal{L}$) endowed with the C^1-topology. Assuming $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable, let $\ell^\alpha_\pm : \mathcal{L}\times\mathcal{L}\to\mathbb{R}$ (resp. $\widetilde{\mathcal{L}}\times\widetilde{\mathcal{L}}\to\mathbb{R}$) denote the order α-spectral selectors so that the α-spectral distance between two submanifolds $\Lambda_0, \Lambda_1 \in \mathcal{L}$ (resp.

Date: march 15, 2024.

2020 *Mathematics Subject Classification.* 53D10, 57R17, 58B20.

Key words and phrases. Orderability in contact geometry, spectral invariants, Legendrian isotopies, contactomorphisms, Hofer's geometry, geodesics.

$\widetilde{\mathcal{L}}$) is defined as

$$\mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda_0,\Lambda_1) := \max\left\{\ell^\alpha_+(\Lambda_0,\Lambda_1), -\ell^\alpha_-(\Lambda_0,\Lambda_1)\right\} \in [0,+\infty).$$

The map $\mathrm{d}^\alpha_{\mathrm{spec}}$ is a genuine distance on $\mathcal{L}$ and is *a priori* just a pseudo-distance on $\widetilde{\mathcal{L}}$. We refer to Section 2.3 for details.

Let $\Lambda \in \mathcal{L}$ be a closed Legendrian submanifold. According to the Weinstein neighborhood theorem, there is an open set of M containing Λ which is contactomorphic to an open set of $J^1\Lambda$ containing the zero-section, identifying Λ with the zero-section and the contact form α with the canonical contact form $\alpha_0 := \mathrm{d}z - p\cdot \mathrm{d}q$. Every Legendrian C^1-close to Λ in $\mathcal{L}$ is then identified uniquely with the 1-jet of a map $f \in C^\infty(\Lambda,\mathbb{R})$ of some C^2-neighborhood U of the zero map. Let us call the induced continuous embedding $\Phi : U \to \mathcal{L}$ an α-Weinstein parametrization of $\mathcal{L}$ centered at Λ (it is a homeomorphism between U and a C^1-neighborhood of Λ). As $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are locally homeomorphic, one naturally extends the notion of Weinstein parametrization to $\widetilde{\mathcal{L}}$.

Following the terminology introduced by Bialy-Polterovich in [6], we say that $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$, and $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ introduced in Section 2.1) endowed with a pseudo-distance d is C^1-locally flat if at any point of it there exists a C^1-neighborhood on which d is isometric to the restriction of a normed distance to some open neighborhood of a vector space. In our cases, the vector space will always be $C^\infty(\Lambda,\mathbb{R})$ endowed with the C^0-norm $f \mapsto \max|f|$ for some closed manifold Λ.

Theorem 1.1. *If $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable then endowed with the Legendrian spectral distance it is C^1-locally flat. More precisely, for every $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), and every α-Weinstein parametrization $\Phi : U \to \mathcal{L}$ (resp. $U \to \widetilde{\mathcal{L}}$) centered at Λ, there exists $U' \subset U$ a C^2-neighborhood of the zero map such that for all $f, g \in U'$*

$$\ell^\alpha_+(\Phi(f),\Phi(g)) = \max(f-g) \qquad \text{and} \qquad \ell^\alpha_-(\Phi(f),\Phi(g)) = \min(f-g),$$

in particular $\mathrm{d}^\alpha_{\mathrm{spec}}(\Phi(f),\Phi(g)) = \max|f-g|$.

Some known examples of orderable $\mathcal{L}$ or $\widetilde{\mathcal{L}}$ are the isotopy class of the zero-section of any 1-jet space over a closed manifold, the universal cover of the isotopy class of $\mathbb{R}\mathrm{P}^n$ in the standard contact $\mathbb{R}\mathrm{P}^{2n+1}$, the universal cover of the isotopy class of a fiber S^*_xX of any unit cotangent bundle (see *e.g.* [1, Examples 2.10] for references).

An analogous statement can be given for contactomorphisms when appropriate spectral selectors exist. For instance such exist when $\widetilde{\mathcal{L}}(\Delta)$ is orderable for the contact product $M \times M \times \mathbb{R}$ (see Section 5) which is the case when M is a hypertight closed contact manifold or a closed unit cotangent bundle or any contact boundary of a compact Liouville domain, the symplectic homology of which does not vanish (we also refer to [1, Examples 2.10]). Other situations in which suitable spectral selectors exist concern universal covers of contactomorphism groups of contact lens spaces [2].

Remark 1.2. Note that for the 1-jet bundle of a closed manifold X a stronger statement can be directly deduced from Corollary 5.4 of [7]. This Corollary tells us that $j^1f \preceq j^1g$ (*cf.* Section 2.1) if and only if $f \leq g$ everywhere, where $f, g : X \to \mathbb{R}$ are smooth functions and j^1f, j^1g denote the graph of their respective 1-jet. This indeed implies that $\ell^{\alpha_0}_+(j^1f, j^1g) = \max(f-g)$ and $\ell^{\alpha_0}_-(j^1f, j^1g) = \min(f-g)$, where $\alpha_0 = \mathrm{d}z - p\cdot\mathrm{d}q$ is the canonical contact form of $J^1X = T^*X \times \mathbb{R}$.

An important consequence of Theorem 1.1 is that the Shelukhin-Chekanov-Hofer (SCH) distance $\mathrm{d}^\alpha_{\mathrm{SCH}}$ agrees C^1-locally with the spectral distance $\mathrm{d}^\alpha_{\mathrm{spec}}$, and therefore is C^1-locally flat (we refer to Section 2.2 for definitions).

Corollary 1.3. *Suppose $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable. Then for every $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) there exists $\mathcal{U}$ a C^1-neighborhood of Λ such that*

$$\mathrm{d}^\alpha_{\mathrm{SCH}}(\Lambda_1, \Lambda_0) = \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda_1, \Lambda_0) \text{ for all } \Lambda_1, \Lambda_0 \in \mathcal{U}.$$

Therefore endowed with the SCH distance $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is C^1-locally flat.

Let us discuss another corollary which has motivated the writing of Theorem 1.1. Recall from [20] that a Legendrian isotopy $(\Lambda_t) \subset \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is said to be monotone if $\Lambda_t \preceq \Lambda_s$ whenever $t \leq s$ (*cf.* Section 2.1). The next corollary answers [20, Question 2.4.].

Corollary 1.4. *If $\mathcal{L}$ is orderable (resp. $\widetilde{\mathcal{L}}$ is orderable) then an isotopy $(\Lambda_t) \subset \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is monotone if and only if it is non-negative.*

Proof. Suppose by contradiction that a monotone isotopy (Λ_t) is not non-negative at some time $t_0 \in [0,1]$. One can assume $t_0 = 0$. Let $\Phi : U \to \mathcal{L}$ be a parametrization centered at Λ_0 given by Theorem 1.1. Therefore for $t > 0$ small enough Λ_t lies in $\Phi(U)$ and its corresponding function $f_t := \Phi^{-1}(\Lambda_t) \in U$ is such that $\min f_t < 0$. Thus on the one hand $\ell^\alpha_-(\Lambda_t, \Lambda_0) < 0$ by Theorem 1.1. But on the other hand $\ell^\alpha_-(\Lambda_t, \Lambda_0) \geq 0$ since $\Lambda_0 \preceq \Lambda_t$ which brings the contradiction. ☐

1.2. **Geodesics.** Note that Theorem 1.1 directly implies that "straight" paths $t \mapsto \Phi(tf)$ are minimizing geodesics for the spectral distance. More precisely recall that in a pseudo-metric space (X, d) the length induced by the pseudo-distance d of a continuous curve $\gamma : [a,b] \to X$, for some real numbers $a \leq b$ is defined as follows

$$\mathrm{Length}_d(\gamma) := \sup \left\{ \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)) \mid k \in \mathbb{N} , a = t_0 < \cdots < t_k = b \right\}. \tag{1}$$

In the following $I \subset \mathbb{R}$ will denote an interval and a path $\gamma : I \to X$ is by definition a continuous map.

Definition 1.5. A path $\gamma : I \to X$ is a *minimizing geodesic* if $d(\gamma(a), \gamma(b)) = \mathrm{Length}_d(\gamma|_{[a,b]})$ for all $a, b \in I$ such that $a < b$. A path $\gamma : I \to X$ is a *geodesic* if for all $t \in I$ there exists a neighborhood $J \subset I$ of t such that $\gamma|_J$ is a minimizing geodesic.

Remark 1.6.

1. In our situation we are interested only in the subset of paths consisting of smooth isotopies. Corollary 1.3 implies that the SCH-length functional and spectral length functional agree on smooth isotopies (see Section 2.2).
2. In our situation it should also be possible to define geodesics as critical points of the length functional. Indeed, even if it is not clear that the length functional is smooth in general, it should be the case at paths satisfying the previous definition (see [23, Chapter 12] and [17, Section 1.2]).

The main result of this section consists of giving a complete characterization of smooth geodesics of the spectral and SCH distances. To do so we introduce the following notion extending the terminology introduced by Bialy-Polterovich [6, Definition 1.3.C] (see also Section 2.1 for the definition of Hamiltonian maps).

Definition 1.7. A Legendrian isotopy $(\Lambda_t)_{t\in I}$ is *α-quasi-autonomous* if there exist $\epsilon \in \{\pm 1\}$ and a continuous path (x_t) of points belonging to a same α-Reeb orbit such that $x_t \in \Lambda_t$ and $\epsilon H_t(x_t) = \max |H_t|$, $\forall t \in I$ where (H_t) denotes the α-Hamiltonian map of (Λ_t).
A Legendrian isotopy $(\Lambda_t)_{t\in I}$ is *locally* α-quasi-autonomous if for all $t \in I$ there exists a neighborhood $J \subset I$ of t such that $(\Lambda_t)_{t\in J}$ is α-quasi-autonomous.

Theorem 1.8. *A Legendrian isotopy (Λ_t) in an orderable $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is a geodesic for d^α if and only if it is α-quasi-autonomous, where d^α denotes either $\mathrm{d}^\alpha_{\mathrm{spec}}$ or $\mathrm{d}^\alpha_{\mathrm{SCH}}$.*

The above Theorem 1.8 and Definition 1.7 have their direct analogue for contact isotopies (see Section 5).

1.3. **Discussion.** In the symplectic context similar statements have been proven for the Hofer norm and for the Viterbo's type spectral norms [21, 26, 29] on the group of Hamiltonian symplectomorphisms [6, 16, 17, 22]. Bialy-Polterovich [6] characterized the geodesics of the Hofer norm of compactly supported Hamiltonian symplectomorphisms of the standard symplectic Euclidean space after proving the local flatness. Lalonde-McDuff [16] went the other way around by first characterizing geodesics of symplectic manifolds that do not admit short loops, i.e. 0 is an isolated point of the Hofer length spectrum, and derive from it local flatness for this family of symplectic manifolds. Finally McDuff [18] was able to get rid of the assumption about the non-existence of short loops and thus characterized in full generality the geodesics and proved the local flatness of the Hofer norm for all closed symplectic manifolds. To do so, she generalized the non-squeezing theorem of Gromov [13] to non-trivial symplectic fibrations over the 2-sphere. Her proof involves technical tools such as Seidel morphism and Gromov-Witten invariants.

Surprisingly enough our proofs of local flatness and characterization of geodesics in the contact context involve only elementary arguments relying essentially on the axiomatic properties of the Legendrian spectral distance that we list below (see Theorem 2.2). As in [1], the explanation of this surprising ease comes from the orderability assumption. Indeed, non trivial machineries, such as Floer Homology or generating functions techniques, are hidden behind this assumption. Nevertheless once this assumption has been made, the contact spectral distance is well defined and seems easier and more intuitive to handle than its symplectic cousins.

As illustrated by the present article, it is very common that one adapts statements from symplectic geometry to statements in contact geometry. However according to the previous paragraph it should also be interesting in the future to go the other way around and see whether one can use the contact spectral distance to shed some light on the Hofer or Viterbo's type distances of symplectic geometry.

Organization of the article. In Section 2, we fix the notations and the conventions on the objects that we will use throughout the paper. In Section 3, we prove Theorem 1.1 and Corollary 1.3. In Section 4, we prove Theorem 1.8 characterizing geodesics of the spectral distance. Finally, in Section 5, the extension to universal covers of some contactomorphism groups is discussed.

Acknowledgment. The second author wants to thank Sheila Sandon for introducing him to symplectic geometry with the paper of Bialy-Polterovich [6]. We both want to thank Alberto Abbondandolo, Lukas Nakamura, Stefan Nemirovski and Sheila Sandon for their questions and remarks that have motivated and enlightened

this work.
The first author was funded by the postdoctoral fellowships of the *Interdisciplinary Thematic Institute IRMIA++*, as part of the *ITI* 2021-2028 program of the University of Strasbourg, CNRS and Inserm, supported by *IdEx Unistra (ANR-10-IDEX-0002)* and by *SFRI-STRAT'US project (ANR-20-SFRI-0012)* under the framework of the *French Investments for the Future Program.* The second author is partially supported by the *Deutsche Forschungsgemeinschaft* under the *Collaborative Research Center SFB/TRR 191 - 281071066 (Symplectic Structures in Geometry, Algebra and Dynamics).*

2. Preliminaries

2.1. **Conventions.** Let (M,ξ) be a cooriented contact manifold and fix a contact form α supporting ξ and its coorientation whose Reeb vector field is complete and denote by $\phi_t^\alpha \in \mathrm{Cont}_0(M,\xi)$ its flow at time $t\in\mathbb{R}$. Here $\mathrm{Cont}_0(M,\xi)$ denotes the group of contactomorphisms isotopic to the identity. We denote by $\mathcal{G}$ the group of contactomorphisms isotopic to the identity through compactly supported contactomorphisms and we endow it with the C^1-topology. We denote by $\widetilde{\mathcal{G}}$ its universal cover and $\Pi:\widetilde{\mathcal{G}}\to\mathcal{G}$ the covering map. By a slight abuse of notation, we still call the identity and denote $\mathrm{id}\in\widetilde{\mathcal{G}}$ the class of the constant isotopy $s\mapsto \mathrm{id}$.

Fix a closed Legendrian $\Lambda\subset M$ and denote by $\mathcal{L}(\Lambda)=\{\phi(\Lambda)\mid \phi\in\mathrm{Cont}_0(M,\xi)\}$ (or simply $\mathcal{L}$) its isotopy class that we endow with the C^1-topology. We denote by $\widetilde{\mathcal{L}}$ its universal cover and $\Pi:\widetilde{\mathcal{L}}\to\mathcal{L}$ the covering map.

Everywhere in the paper, $I\subset\mathbb{R}$ will denote an interval. Let us recall that a Legendrian isotopy $(\Lambda_t)_{t\in I}$ in $\mathcal{L}$ is a path of Legendrian submanifolds such that there exists a smooth map $j: I\times\Lambda\to M$ whose restriction j_t to $\{t\}\times\Lambda$ is an embedding onto Λ_t for all $t\in I$. The α-Hamiltonian map of (Λ_t) is the family of maps $(h_t:\Lambda_t\to\mathbb{R})$ defined by $h_t\circ j_t=\alpha(\partial_t j_t)$ for any smooth parametrization j. Given a (smooth) contact isotopy $(\phi_t)\subset\mathrm{Cont}_0(M,\xi)$, we recall that its α-Hamiltonian map $h: I\times M\to\mathbb{R}$ is defined by $h_t\circ\phi_t=\alpha(\partial_t\phi_t)$. We say that a path $(\Lambda_t)\subset\widetilde{\mathcal{L}}$ (resp. $(\phi_t)\subset\widetilde{\mathcal{G}}$) is an isotopy if its projection to $\mathcal{L}$ (resp. $\mathcal{G}$) is an isotopy and we define its α-Hamiltonian to be the α-Hamiltonian of its projection.

On O being either $\mathcal{G},\widetilde{\mathcal{G}},\mathcal{L}$ or $\widetilde{\mathcal{L}}$ we write $x\preceq y$, or equivalently $y\succeq x$, if there exists a non-negative isotopy from x to y, *i.e.* an isotopy whose Hamiltonian is non-negative. O is called orderable if and only if $\preceq$ defines a partial order. This relation has been introduced by Eliashberg-Polterovich [12] and has been widely studied [1, 5, 8, 10, 11, 25].

2.2. **Hofer type distances.** The Shelukhin-Chekanov-Hofer (resp. Shelukhin-Hofer) length functional is a functional defined on the space of Legendrian isotopies $(\Lambda_t)\subset\mathcal{L}$ (resp. contact isotopies $(\phi_t)\subset\mathcal{G}$) as follows: the length of any isotopy of α-Hamiltonian $(H_t)_{t\in I}$ is given by

$$\int_I \max|H_t|dt,$$

(see also [24, Section 7] and [27]). We denote this length functional by $\mathrm{L}^\alpha_{\mathrm{SCH}}$ (resp. $\mathrm{L}^\alpha_{\mathrm{SH}}$). The Shelukhin-Chekanov-Hofer (SCH) pseudo-distance $\mathrm{d}^\alpha_{\mathrm{SCH}}$ on $\mathcal{L}$ or $\widetilde{\mathcal{L}}$ is defined for any $\Lambda_0,\Lambda_1\in\mathcal{L}$ or $\widetilde{\mathcal{L}}$ as

$$\mathrm{d}^\alpha_{\mathrm{SCH}}(\Lambda_1,\Lambda_0):=\inf\{\mathrm{L}^\alpha_{\mathrm{SCH}}(\Lambda_t)\mid(\Lambda_t)\subset\mathcal{L}\text{ or }\widetilde{\mathcal{L}}\text{ joining }\Lambda_0\text{ to }\Lambda_1\}.$$

The Shelukhin-Hofer pseudo-norm $|\cdot|_{\mathrm{SH}}^{\alpha}$ on $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ is defined for any $\phi \in \mathcal{G}$ or $\widetilde{\mathcal{G}}$ as

$$|\phi|_{\mathrm{SH}}^{\alpha} = \inf\{\mathrm{L}_{\mathrm{SH}}^{\alpha}(\phi_t) \mid (\phi_t)_{t\in[0,1]} \subset \mathcal{G} \text{ or } \widetilde{\mathcal{G}} \text{ such that } \phi_0 = \mathrm{id} \text{ and } \phi_1 = \phi\}.$$

When M is closed Shelukhin [27] showed that $|\cdot|_{\mathrm{SH}}^{\alpha}$ is a genuine norm on $\mathcal{G}$. Hedicke [15] showed that $\mathrm{d}_{\mathrm{SCH}}^{\alpha}$ is a genuine distance when $\mathcal{L}$ is orderable (even for non-closed M).

Remark 2.1. Note that Theorem 1.1 together with Corollary 1.3 imply that $\mathrm{Length}_{\mathrm{d}_{\mathrm{SCH}}^{\alpha}}$ and $\mathrm{Length}_{\mathrm{d}_{\mathrm{SH}}^{\alpha}}$ (as defined in (1)) restricted to isotopies correspond respectively to $\mathrm{L}_{\mathrm{SCH}}^{\alpha}$ and $\mathrm{L}_{\mathrm{SH}}^{\alpha}$, where $\mathrm{d}_{\mathrm{SH}}^{\alpha}$ denotes the right-invariant distance associated with the norm $|\cdot|_{\mathrm{SH}}^{\alpha}$. See also [14, Proposition 1.6] and the remark following it.

2.3. **Order spectral selectors and induced distances.** Following [1] let us define two functions $\ell_{\pm}^{\alpha} : \mathcal{L} \times \mathcal{L} \to \mathbb{R} \cup \{\mp\infty\}$ (resp. $\widetilde{\mathcal{L}} \times \widetilde{\mathcal{L}} \to \mathbb{R} \cup \{\mp\infty\}$) by

$$\ell_{+}^{\alpha}(\Lambda_1, \Lambda_0) := \inf\{t \in \mathbb{R} \mid \Lambda_1 \preceq \phi_t^{\alpha} \cdot \Lambda_0\} \text{ and } \ell_{-}^{\alpha}(\Lambda_1, \Lambda_0) := \sup\{t \in \mathbb{R} \mid \phi_t^{\alpha} \cdot \Lambda_0 \preceq \Lambda_1\},$$

for $\Lambda_1, \Lambda_0 \in \mathcal{L}$ (resp. $\Lambda_1, \Lambda_0 \in \widetilde{\mathcal{L}}$), where $\phi_t^{\alpha} \cdot \Lambda_0$ denotes the natural action of the Reeb flow at time $t \in \mathbb{R}$ on $\Lambda_0 \in \mathcal{L}$ (resp. $\Lambda_0 \in \widetilde{\mathcal{L}}$). Recall that the α-spectrum of $(\Lambda_1, \Lambda_0) \in \mathcal{L}^2$ (resp. $\widetilde{\mathcal{L}}^2$) is the set of lengths of α-Reeb chords joining Λ_0 to Λ_1 (resp. $\Pi(\Lambda_0)$ to $\Pi(\Lambda_1)$) that is

$$\mathrm{Spec}^{\alpha}(\Lambda_1, \Lambda_0) := \{t \in \mathbb{R} \mid \Lambda_1 \cap \phi_t^{\alpha}\Lambda_0 \neq \emptyset\} \text{ (resp. } \mathrm{Spec}^{\alpha}(\Lambda_1, \Lambda_0) := \mathrm{Spec}^{\alpha}(\Pi\Lambda_1, \Pi\Lambda_0)).$$

Theorem 2.2 ([1]). *The maps $\ell_{\pm}^{\alpha}$ are real-valued if and only if $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable. Moreover when real valued they satisfy the following properties for every $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$),*

1. *(normalization) $\ell_{\pm}^{\alpha}(\Lambda_0, \Lambda_0) = 0$ and $\ell_{\pm}^{\alpha}(\phi_t^{\alpha}\Lambda_1, \Lambda_0) = t + \ell_{\pm}^{\alpha}(\Lambda_1, \Lambda_0)$, $\forall t \in \mathbb{R}$,*
2. *(monotonicity) $\Lambda_2 \preceq \Lambda_1$ implies $\ell_{\pm}^{\alpha}(\Lambda_2, \Lambda_0) \leq \ell_{\pm}^{\alpha}(\Lambda_1, \Lambda_0)$,*
3. *(triangle inequalities) $\ell_{+}^{\alpha}(\Lambda_2, \Lambda_0) \leq \ell_{+}^{\alpha}(\Lambda_2, \Lambda_1) + \ell_{+}^{\alpha}(\Lambda_1, \Lambda_0)$ and $\ell_{-}^{\alpha}(\Lambda_2, \Lambda_0) \geq \ell_{-}^{\alpha}(\Lambda_2, \Lambda_1) + \ell_{-}^{\alpha}(\Lambda_1, \Lambda_0)$,*
4. *(Poincaré duality) $\ell_{+}^{\alpha}(\Lambda_1, \Lambda_0) = -\ell_{-}^{\alpha}(\Lambda_0, \Lambda_1)$,*
5. *(compatibility) $\ell_{\pm}^{\alpha}(\varphi(\Lambda_1), \varphi(\Lambda_0)) = \ell_{\pm}^{\varphi^*\alpha}(\Lambda_1, \Lambda_0)$, for every φ in $\mathrm{Cont}_0(M, \xi)$ (resp. in its universal cover),*
6. *(non-degeneracy) $\ell_{+}^{\alpha}(\Lambda_1, \Lambda_0) = \ell_{-}^{\alpha}(\Lambda_1, \Lambda_0) = t$ for some $t \in \mathbb{R}$ implies $\Lambda_1 = \phi_t^{\alpha}\Lambda_0$ (resp. it only implies the equality $\Pi\Lambda_1 = \phi_t^{\alpha}\Pi\Lambda_0$ in $\mathcal{L}$).*
7. *(spectrality) $\ell_{\pm}^{\alpha}(\Lambda_1, \Lambda_0) \in \mathrm{Spec}^{\alpha}(\Lambda_1, \Lambda_0)$.*

As a consequence the map $\mathrm{d}_{\mathrm{spec}}^{\alpha} := \max\{\ell_{+}^{\alpha}, -\ell_{-}^{\alpha}\}$ is a distance (resp. a pseudo-distance) on $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) whenever it is orderable. Thanks to the last property of Theorem 2.2 we call this (pseudo-)distance the Legendrian spectral distance. A consequence of normalization and monotonicity properties are the following inequalities. If $(\Lambda_t)_{t\in[0,1]}$ is an isotopy of $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), then

$$\int_0^1 \min H_t \mathrm{d}t \leq \ell_{-}^{\alpha}(\Lambda_1, \Lambda_0) \leq \ell_{+}^{\alpha}(\Lambda_1, \Lambda_0) \leq \int_0^1 \max H_t \mathrm{d}t, \tag{2}$$

where (H_t) is the associated α-Hamiltonian map [1, Lemma 3.3]. In particular, the spectral distance is dominated by the SCH distance:

$$\mathrm{d}_{\mathrm{spec}}^{\alpha} \leq \mathrm{d}_{\mathrm{SCH}}^{\alpha}, \tag{3}$$

which subsequently implies the C^1-continuity of $\ell_{\pm}^{\alpha}$ [1, Corollary 3.4].

Assuming that M is closed, similarly we defined in [1, 3] two functions $c_\pm^\alpha : \mathcal{G} \to \mathbb{R} \cup \{\mp\infty\}$ (resp. $\widetilde{\mathcal{G}} \to \mathbb{R} \cup \{\mp\infty\}$)

$$c_+^\alpha(\phi) = \inf\{t \in \mathbb{R} \mid \phi \preceq \phi_t^\alpha\} \quad \text{and} \quad c_-^\alpha(\phi) = \sup\{t \in \mathbb{R} \mid \phi_t^\alpha \preceq \phi\}.$$

We showed that the maps $c_\pm^\alpha$ take values in $\mathbb{R}$ if and only if $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) is orderable and moreover $c_\pm^\alpha$ satisfy properties analogous to the ones of $\ell_\pm^\alpha$ except for the spectrality. Let us recall that the α-spectrum of $\phi \in \mathcal{G}$ (resp. $\phi \in \widetilde{\mathcal{G}}$) is the set of translations of its α-translated points that is

$$\operatorname{Spec}^\alpha(\phi) := \{t \in \mathbb{R} \mid \exists x \in M,\ \phi(x) = \phi_t^\alpha(x),\ (\phi^*\alpha)_x = \alpha_x\} \tag{4}$$

$$\text{(resp. } \operatorname{Spec}^\alpha(\phi) := \operatorname{Spec}^\alpha(\Pi(\phi))).$$

Nevertheless this allows us to define a norm (resp. a pseudo-norm)

$$|\cdot|_{\text{spec}}^\alpha := \max\{c_+^\alpha, -c_-^\alpha\}$$

that we still call the spectral norm on $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$). Since normalization and monotonicity are still satisfied by $c_\pm^\alpha$, one has analogues of (2) and (3). Given $\phi \in \mathcal{G}$ (resp. $\in \widetilde{\mathcal{G}}$),

$$\int_0^1 \min H_t \mathrm{d}t \leq c_-^\alpha(\phi) \leq c_+^\alpha(\phi) \leq \int_0^1 \max H_t \mathrm{d}t, \tag{5}$$

for any α-Hamiltonian map (H_t) generating ϕ. As a consequence $|\cdot|_{\text{spec}}^\alpha \leq |\cdot|_{\text{SH}}^\alpha$.

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 let us assume $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) to be orderable and let us fix once for all Λ in it. Let us be more explicit on the construction of the parametrization Φ centered at Λ in order to fix notation. By the Weinstein neighborhood theorem, there exists Ψ a diffeomorphism between a neighborhood $\mathcal{V} \subset M$ of Λ and a neighborhood $V \subset J^1\Lambda$ of the 0-section j^10 that moreover satisfies $\Psi(\Lambda) = j^10$, and more precisely $\Psi(x)$ is the image of the 0-section at x for all $x \in \Lambda$, and $\Psi^*\alpha_0 = \alpha$ where $\alpha_0 = \mathrm{d}z - p \cdot \mathrm{d}q$ denotes the canonical 1-form of $J^1\Lambda$. Let U be a sufficiently C^2-small open neighborhood of the 0-function in $C^\infty(\Lambda, \mathbb{R})$ such that $j^1f \subset V$ for any $f \in U$. Then one can show that the map

$$\Phi : U \to \mathcal{L} \qquad f \mapsto \Psi^{-1}(j^1f)$$

is injective and open (see for example the third paragraph of [28] for more details).

One can have the same discussion for the universal cover $\widetilde{\mathcal{L}}$ of $\mathcal{L}$. Indeed in this situation, ensuring that $\Phi(U)$ is small enough, one can use the projection $\Pi : \widetilde{\mathcal{L}} \to \mathcal{L}$, which is a local homeomorphism, to construct the local homeomorphism $\Pi|_{\Phi(U)}^{-1} \circ \Phi : U \to \widetilde{\mathcal{L}}$. By a slight abuse of notation we also denote this latter local homeomorphism by Φ.

Before proving Theorem 1.1 let us prove the following lemmata.

Lemma 3.1. *Let $f \in U$. If an α-Reeb chord between Λ and $\Phi(f)$ of length $\ell \in \mathbb{R}$ is contained in $\mathcal{V}$, i.e. $\{\phi_{t\ell}^\alpha(x)\}_{t\in[0,1]} \subset \mathcal{V}$ for some $x \in \Lambda$ and $\phi_\ell^\alpha(x) \in \Phi(f)$, then ℓ is a critical value of f.*

Proof. Since $\Psi : \mathcal{V} \to V$ is a strict contactomorphism $\Psi(\phi_{t\ell}^\alpha(x)) = (x, 0, t\ell) \in V \subset T^*X \times \mathbb{R}$. Therefore $(x, 0, \ell) \in j^1f = \{(x, \mathrm{d}f(x), f(x)) \mid x \in \Lambda\}$ which brings the conclusion. □

Let $\varepsilon_0 > 0$ be sufficiently small so that:

(i) $\phi_t^\alpha(\Lambda) \subset \mathcal{V}$ for any $t \in (-\varepsilon_0, \varepsilon_0)$

(ii) there exists an open set of the form $V_{\varepsilon_0} := \underline{V} \times (-\varepsilon_0, \varepsilon_0) \subset J^1\Lambda = T^*\Lambda \times \mathbb{R}$ that is contained in V.

From now on, for any positive $\varepsilon \leq \varepsilon_0$ we will denote by U_ε a convex C^2-neighborhood of the 0-function such that $j^1 f \in V_\varepsilon := \underline{V} \times (-\varepsilon, \varepsilon)$ for all $f \in U_\varepsilon$.

Corollary 3.2. *Let $f \in U_\varepsilon$ then $\ell_\pm^\alpha(\Phi(f), \Lambda)$ are critical values of f.*

Proof. Since $f \in U_\varepsilon$ by (2) it implies that $-\varepsilon < \min f \leq \ell_-^\alpha(\Phi(f), \Lambda) \leq \ell_+^\alpha(\Phi(f), \Lambda) \leq \max f < \varepsilon$. Since $\ell_\pm := \ell_\pm^\alpha(\Phi(f), \Lambda)$ are spectral values it implies that there exists $x_\pm \in \Lambda$ such that $\{\phi_\alpha^{t\ell_\pm}(x_\pm)\}$ are Reeb chords between Λ and $\Phi(f)$. By (i) these Reeb chords are included in $\mathcal{V}$ and therefore we conclude by Lemma 3.1. ☐

Lemma 3.3. *There exists a positive $\delta_0 \leq \varepsilon_0$ such that for any positive $\delta \leq \delta_0$*

$$\ell_\pm^\alpha(\Phi(f), \Phi(g)) = \ell_\pm^\alpha(\Phi(f - g), \Lambda) \quad \textit{for any } f, g \in U_\delta.$$

Proof. Note that if there exists a contactomorphism $\phi \in \mathrm{Cont}_0(M, \ker\alpha)$ that commutes with the Reeb flow, or equivalently $\phi^*\alpha = \alpha$, such that

$$\phi(\Phi(h)) = \Phi(h - g) \text{ for any } h \in U_\delta \tag{6}$$

then Lemma 3.3 follows from the compatibility property of Theorem 2.2. This is the case when $(M, \ker\alpha) = (J^1\Lambda, \ker\alpha_0)$ with the trivial α_0-Weinstein parametrization centered at the zero section, i.e. $\Phi(h) = j^1 h$, since the contact isotopy (ϕ_t), $\phi_t : (q,p,z) \mapsto (q, p - t\mathrm{d}g(q), z - tg(q))$, is an isotopy of strict contactomorphisms whose time 1 satisfies (6). The α_0-Hamiltonian of this isotopy is given by the autonomous function $H : (q,p,z) \mapsto -g(q)$.

When $(M, \ker\alpha)$ is a general contact manifold and for δ small enough, we get the desired result by cutting off the Hamiltonian function H properly. More precisely let δ be small enough so that $(f_t := f - tg)$, $(g_t := (1-t)g)$ and $(g_{t,s}^\alpha := g_s + (2t-1)2\delta) \subset U$ for all $f, g \in U_\delta$ and $t, s \in [0,1]$. Consider $\rho : J^1\Lambda \to \mathbb{R}$ a cutoff function supported in V such that ρ is equal to 1 on a neighborhood containing $(j^1 f_t)$, $(j^1 g_t)$ and $(j^1 g_{t,s}^\alpha)$. Let $K : M \to \mathbb{R}$ be the compactly supported α-Hamiltonian function defined by $x \mapsto \rho(\Psi(x))H(\Psi(x))$ if $x \in \mathcal{V}$ and $x \mapsto 0$ otherwise. It is easy to check that its time 1-flow ϕ commutes with ϕ_α^t on $\Phi(g)$ for $t \in [-2\delta, 2\delta]$ and satisfies (6) when h is either f or g. Moreover by the previous Corollary 3.2 and triangle inequality we get that $\ell_\pm^\alpha(\Phi(f), \Phi(g)) \in (-2\delta, 2\delta)$. The conclusion follows from the definition of $\ell_\pm^\alpha$. ☐

Proof of Theorem 1.1. We will show Theorem 1.1 for $U' \to \mathcal{L}$ (resp. $U' \to \widetilde{\mathcal{L}}$) where $U' := U_{\delta_0/2}$ and δ_0 is the positive constant of Lemma 3.3.

Remark that for any $f \in U_{\delta_0}$ convexity ensures that $tf \in U_{\delta_0}$ for all $t \in [0,1]$. Since the set of critical values $\mathrm{CV}(tf) = t\mathrm{CV}(f)$ of tf is nowhere dense, we deduce by continuity of $\ell_+^\alpha(\cdot, \Lambda)$ (see (3) and below) and Corollary 3.2 that there exists $x_0 \in \Lambda$ a critical point of f such that $\ell_+^\alpha(\Phi(tf), \Lambda) = tf(x_0)$. It thus remains to show that $f(x_0) = \max f$ for any $f \in U_{\delta_0/2}$ to get the desired equality for ℓ_+^α.

Let us first assume $f \geq 0$ and $f \in U_{\delta_0}$ is Morse. In particular $f \neq 0$. Let $\mathcal{B}$ be a Morse neighborhood of a maximum x_1 of f: there exist $\varepsilon > 0$ and a diffeomorphism $Q := (q_1, \ldots, q_n)$ from $\mathcal{B}$ to the Euclidean ball of radius ε centered at 0 such that $Q(x_1) = 0$ and $f(x) := \max f - \sum q_i(x)^2$ for any $x \in \mathcal{B}$. It is then easy to construct a cut-off function $\rho : M \to [0,1]$ supported in $\mathcal{B}$ such that $\rho(x_1) = 1$ and $\mathrm{CV}(\rho f) := \{\max f, 0\}$. Note that $\rho f \leq f$ since $f \geq 0$ but ρf may not be contained in U. However there exists $\mu > 0$ small enough so that $t\rho f \in U_\delta$ for all $t \in [0,\mu]$. Therefore thanks to Corollary 3.2 $\ell_+^\alpha(\Phi(t\rho f), \Lambda) \in \{t \max f, 0\}$. Moreover $\Phi(t\rho f) \neq \Lambda$ (resp. $\Pi(\Phi(t\rho f)) \neq \Pi(\Lambda)$) for $t \in (0,\mu)$ since $f \neq 0$ therefore by non-degeneracy of the selectors $\ell_+^\alpha(\Phi(t\rho f), \Lambda) = t \max f$. Thus by monotonicity we deduce that $\ell_+^\alpha(\Phi(tf), \Lambda) = t \max f$.

Let us now assume $f \in U_{\delta_0/2}$ is Morse but $-\delta_0/2 < m := \min f < 0$. Consider then $g := f - m \geq 0$. Remark that $\max g = \max f - m < \delta_0/2 + \delta_0/2 = \delta_0$. Therefore $g \in U_{\delta_0}$ is Morse and non-negative. So we deduce by the previous case that $\ell_+^\alpha(\Phi(g), \Lambda) = \max f - m$. Moreover $\phi_m^\alpha(\Phi(g)) = \Phi(f)$ so by normalization we deduce that $\ell_+^\alpha(\Phi(f), \Lambda) = \max f$.

Finally, let f be any function in $U_{\delta_0/2}$. Since Morse functions are C^2-dense, one can find a sequence (f_n) of Morse functions in $U_{\delta_0/2}$ that converges to f in the C^2-topology. Therefore $(\Phi(f_n))$ C^1-converges to $\Phi(f)$. By C^1-continuity of ℓ_+^α (see (3) and below) and the previous cases, we deduce that $\ell_+^\alpha(\Phi(f), \Lambda) = \max f$.

Thanks to Lemma 3.3 $\ell_+^\alpha(\Phi(f), \Phi(g)) = \ell_+(\Phi(f-g), \Lambda) = \max(f-g)$.

We deduce the analogous result for ℓ_-^α by a similar reasoning or simply by using the Poincaré duality property, and this concludes the proof. $\square$

Let us conclude this section by deducing the C^1-local coincidence of $\mathrm{d}_{\mathrm{spec}}^\alpha$ and $\mathrm{d}_{\mathrm{SCH}}^\alpha$.

Proof of Corollary 1.3. As recalled at (3), one always has $\mathrm{d}_{\mathrm{spec}}^\alpha \leq \mathrm{d}_{\mathrm{SCH}}^\alpha$. In the neighborhood of $\Lambda = \Phi(0)$, one has $\mathrm{d}_{\mathrm{spec}}^\alpha(\Phi(f), \Phi(g)) = \max|f-g|$. The α_0-Hamiltonian map associated with $(j^1 f_t)$ for $f_t := (1-t)f + tg$, $t \in [0,1]$, is $(q,p,z) \mapsto g(q) - f(q)$, therefore $\mathrm{L}_{\mathrm{SCH}}^{\alpha_0}(j^1 f_t) = \max|f-g|$. Since the SCH-length of an isotopy is invariant under strict contactomorphism $\mathrm{L}_{\mathrm{SCH}}^\alpha(\Phi(f_t)) = \max|f-g|$. As a consequence, one gets the reverse inequality $\mathrm{d}_{\mathrm{SCH}}^\alpha(\Phi(f), \Phi(g)) \leq \max|f-g|$. $\square$

4. Characterization of the geodesics

To prove Theorem 1.8 let us first state and prove the two following lemmata.

Lemma 4.1. *Given a continuous map $g : [a,b]_t \times N \to \mathbb{R}$ on a compact set, the following two conditions are equivalent:*

(1) $\int_a^b \max|g_t|\mathrm{d}t = \max_x \left|\int_a^b g_t(x)\mathrm{d}t\right|$,

(2) there exist $\epsilon \in \{\pm 1\}$ and $x_0 \in \Lambda$ such that $\forall t \in [a,b]$, $\epsilon g_t(x_0) = \max|g_t|$.

Proof. The implication (2) $\Rightarrow$ (1) is clear. Conversely, let $x_0 \in N$ and $\epsilon \in \{\pm 1\}$ be such that $\epsilon \int_a^b g_t(x_0)\mathrm{d}t = \max_x|\int_a^b g_t(x)\mathrm{d}t|$. Then $t \mapsto \max|g_t| - \epsilon g_t(x_0)$ is a non-negative continuous map the integral of which vanishes over $[a,b]$ by assumption. The conclusion follows. $\square$

Lemma 4.2. *Let $f : [0,1]_t \times N \to \mathbb{R}$ be a smooth map on a closed manifold N. Then $(j^1 f_t)$ is a minimizing geodesic for d^{α_0} if and only if it is α_0-quasi-autonomous, where α_0 denotes the canonical 1-form of $J^1 N$ and d^{α_0} either $\mathrm{d}_{\mathrm{spec}}^{\alpha_0}$ or $\mathrm{d}_{\mathrm{SCH}}^{\alpha_0}$.*

Proof. It is enough to prove the result for $\mathrm{d}_{\mathrm{SCH}}^{\alpha_0}$ thanks to Corollary 1.3 and the first point in Remark 1.6. Moreover thanks to Remark 2.1 $(j^1 f_t)$ is a geodesic if and only if $\mathrm{d}_{\mathrm{SCH}}^{\alpha_0}(j^1 f_0, j^1 f_1) = \int_0^1 \max|H_t|\mathrm{d}t$ where (H_t) is the α_0-Hamiltonian map of $(j^1 f_t)$. But $H_t = \partial_t f_t \circ \pi|_{j^1 f_t}$ where $\pi : J^1 N \to N$ is the bundle map and $\mathrm{d}^{\alpha_0}_{\mathrm{SCH}}(j^1 f_0, j^1 f_1) = \max |f_1 - f_0|$ thanks to Remark 1.2, so it boils down to

$$\int_0^1 \max |\partial_t f_t| \mathrm{d}t = \max_q \left| \int_0^1 \partial_t f_t(q) \mathrm{d}t \right|.$$

By Lemma 4.1, this is equivalent to the existence of $q_0 \in N$ and $\epsilon \in \{\pm 1\}$ such that $\forall t \in [0,1]$, $\epsilon \partial_t f_t(q_0) = \max |\partial_t f_t|$.

Let assume that $(j^1 f_t)$ is a geodesic. In particular, q_0 is a critical point of $\partial_t f_t$ where $t \in [0,1]$ is fixed. As the time-derivative commutes with the differential operator on N, $\partial_t(\mathrm{d}f_t)$ vanishes at q_0 so $t \mapsto \mathrm{d}f_t(q_0)$ is constant. Therefore $(j^1 f_t)$ is quasi-autonomous, by taking $x_t = (q_0, \mathrm{d}f_0(q_0), f_t(q_0))$.

Conversely, if $(j^1 f_t)$ is quasi-autonomous, the associated path of $x_t \in j^1 f_t$ belonging to a same Reeb orbit such that $\epsilon H_t(x_t) = \max |H_t|$, for some $\epsilon \in \{\pm 1\}$, is necessarily of the form $(q_0, \mathrm{d}f_0(q_0), f_t(q_0))$, $t \in [0,1]$. As $H_t = \partial_t f_t \circ \pi|_{j^1 f_t}$, the conclusion follows from the beginning of the proof. □

Proof of Theorem 1.8. It is enough to prove the result for $\mathrm{d}^\alpha_{\mathrm{spec}}$ thanks to Corollary 1.3. Let $(\Lambda_t)_{t \in I}$ be an isotopy of $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) and let us fix $t_0 \in I$. Let us consider an α-Weinstein parametrization $\Phi : U \to \mathcal{L}$ (resp. $U \to \widetilde{\mathcal{L}}$) centered at Λ_{t_0} and $U' \subset U$ given by Theorem 1.1. Let $J \subset I$ be a neighborhood of t_0 such that $(\Lambda_t)_{t \in J} \subset \Phi(U')$ and denote $f_t := \Phi^{-1}(\Lambda_t)$ for $t \in J$. According to Theorem 1.1,

$$\mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda_t, \Lambda_s) = \mathrm{d}^{\alpha_0}_{\mathrm{spec}}(j^1 f_t, j^1 f_s), \quad \forall t, s \in J,$$

(see also Remark 1.2). Therefore, $(\Lambda_t)_{t \in J}$ is a minimizing geodesic for $\mathrm{d}^\alpha_{\mathrm{spec}}$ if and only if $(j^1 f_t)$ is a minimizing geodesic for $\mathrm{d}^{\alpha_0}_{\mathrm{spec}}$ in $J^1 \Lambda_{t_0}$. The conclusion now follows from Lemma 4.2 as the notion of quasi-autonomy is stable under strict contactomorphisms. □

5. The case of contactomorphisms

Let us deduce from the Legendrian case the analogous statements for contactomorphisms when a stronger orderability condition is assumed.

5.1. **C^1-local flatness.** Let us first recall that for any *closed* cooriented contact manifold $(M, \xi := \ker \alpha)$, the 1-form $\beta := \pi_2^* \alpha - e^\theta \pi_1^* \alpha$ is a contact form on $M \times M \times \mathbb{R}$ where $\pi_i : M \times M \times \mathbb{R} \to M$, $(x_1, x_2, \theta) \mapsto x_i$. Its cooriented kernel $\Xi := \ker \beta$ does not depend (up to isomorphism) on the choice of the contact form α supporting ξ. Note that the diagonal $\Delta := \{(x, x, 0) \mid x \in M\}$ is a closed Legendrian of $(M \times M \times \mathbb{R}, \Xi)$. For any contactomorphism ϕ of $(M, \ker \alpha)$ isotopic to the identity, the graph of ϕ, $gr_\alpha(\phi) := \{(x, \phi(x), g(x)) \mid x \in M\}$ where $\phi^* \alpha = e^g \alpha$, lies in $\mathcal{L}(\Delta)$. Moreover the map $\mathcal{G} \to \mathcal{L}(\Delta)$ (resp. $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{L}}(\Delta)$) we have just described is a local homeomorphism at the identity.

In particular this allows again to construct a continuous embedding $U \to \mathcal{G}$ (resp. $U \to \widetilde{\mathcal{G}}$) sending the zero map to the identity where U denotes a C^2-neighborhood of the zero map in $C^\infty(M, \mathbb{R})$. Such a map is called an α-Weinstein parametrization centered at the identity.

Corollary 5.1. *If $(M, \ker \alpha)$ is closed and $\widetilde{\mathcal{L}}(\Delta)$ is orderable then endowed with the spectral pseudo-norm $\widetilde{\mathcal{G}}$ is C^1-locally flat. More precisely, for every α-Weinstein* *parametrization* $\Phi : U \to \widetilde{\mathcal{G}}$ *centered at the identity, there exists* $U' \subset U$ *a* C^2-*neighborhood of the zero map such that for all* $f \in U'$

$$c_+^\alpha(\Phi(f)) = \max f \quad \text{and} \quad c_-^\alpha(\Phi(f)) = \min f$$

in particular $|\Phi(f)|^\alpha_{\text{spec}} = \max |f|$.

Proof. Let us define the map $\mathcal{C}^\alpha_\pm(\phi) := \ell_\pm(gr_\alpha(\phi), \Delta)$ where by a slight abuse of notation $\Delta \in \widetilde{\mathcal{L}}(\Delta)$ denotes the class of the constant path and $gr(\phi) \in \widetilde{\mathcal{L}}(\Delta)$ denotes the class of the path $(gr_\alpha(\phi_t))$ for a path $(\phi_t) \subset \mathcal{G}$ representing $\phi \in \widetilde{\mathcal{G}}$. By Theorem 1.1 $\mathcal{C}^\alpha_+(\Phi(f)) = \max f$, $\mathcal{C}^\alpha_-(\Phi(f)) = \min f$. By maximality of $c^\alpha_\pm$ (see the discussion at the end of [1, Section 1.3]) we deduce that $c^\alpha_+(\Phi(f)) \geq \max f$ and $c^\alpha_-(\Phi(f)) \leq \min f$. The reverse inequalities come from (5). □

A proof similar to that of Corollary 1.3 then brings the corresponding statement for $\widetilde{\mathcal{G}}$.

Corollary 5.2. *Suppose that* $(M, \ker\alpha)$ *is closed and* $\widetilde{\mathcal{L}}(\Delta)$ *is orderable. Then for every* $\phi \in \widetilde{\mathcal{G}}$ *that is* C^1-*close to the identity,*

$$|\phi|^\alpha_{\text{SH}} = |\phi|^\alpha_{\text{spec}}.$$

Therefore endowed with the Shelukhin-Hofer pseudo norm $\widetilde{\mathcal{G}}$ *is* C^1-*locally flat.*

By mimicking the proof of Theorem 1.1 one can see that Corollary 5.1 and Corollary 5.2 actually hold whenever there exist maps $c_- \leq c_+$ that are spectral, compatible with the partial order, non-degenerate and normalized in the sense of [1]. In particular it holds for $\widetilde{\mathcal{G}}$ of lens spaces [2]. When $(M, \ker\alpha)$ is a closed contact manifold such that $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) is orderable, it is conjectured in [1] that $c^\alpha_\pm$ on $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) are spectral. It is interesting to note that for elements that can be joined to the identity by a minimizing geodesic this conjecture holds. More precisely denote by

$$\mathcal{E}_- := \left\{\phi \in \mathcal{G} \text{ (resp. } \in \widetilde{\mathcal{G}}) \mid \phi^\alpha_{c^\alpha_-(\phi)} \preceq \phi\right\} \qquad \mathcal{E}_+ := \left\{\phi \in \mathcal{G} \text{ (resp. } \in \widetilde{\mathcal{G}}) \mid \phi \preceq \phi^\alpha_{c^\alpha_+(\phi)}\right\}.$$

Proposition 5.3. *Let* $\phi \in \mathcal{E}_\pm$ *then* $c^\alpha_\pm(\phi) \in \text{Spec}^\alpha(\phi)$.

It seems however unlikely that $\mathcal{E}_+ = \mathcal{E}_- = \mathcal{G}$ – which is equivalent to saying that $\{\phi \succeq \text{id}\} \subset \mathcal{G}$ is closed for the C^1-topology. See Section 5.2 for discussions.

Given a pair of points x, y in $\mathcal{G}$ or $\widetilde{\mathcal{G}}$, let us write $x \lll y$ if there exists a positive isotopy joining x to y. The statement of Proposition 5.3 follows directly from the following lemma.

Lemma 5.4. *Let* $\phi \in \mathcal{G}$ *(resp.* $\widetilde{\mathcal{G}}$*) such that* $\text{id} \preceq \phi$. *If* ϕ *(resp.* $\Pi(\phi)$*) does not have any discriminant point then* $\text{id} \lll \phi$.

Note that if $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) is not orderable the proposition is trivial.

Proof of Lemma 5.4. Consider $(\phi_t) \subset \mathcal{G}$ a non-negative path starting at the identity such that $\phi_1 = \phi \in \mathcal{G}$ (resp. $[(\phi_t)] = \phi$). Since ϕ_1 does not have discriminant point there exists $\varepsilon_0 > 0$ such that the path of closed Legendrian submanifolds $(\text{gr}(\phi^\alpha_{-t\varepsilon} \circ \phi_1))_{t\in[0,1]}$ does not intersect Δ for any $\varepsilon \in (0, \varepsilon_0)$. Therefore there exists a compactly supported contactomorphism (ψ_ε) of $M \times M \times \mathbb{R}$ isotopic to the identity that sends $(\text{gr}(\phi_1))$ on $\text{gr}(\phi^\alpha_{-\varepsilon} \circ \phi_1)$ and that fixes Δ. Moreover, for ε sufficiently small, one can construct ψ_ε sufficiently C^1-small such that the non-negative path of Legendrians $(\psi_\varepsilon(\text{gr}(\phi_t))$ starting at Δ is graphical, *i.e.* there exists an isotopy

$(\varphi_t) \subset \mathcal{G}$ starting at the identity such that $\psi_\varepsilon(\mathrm{gr}(\phi_t)) = \mathrm{gr}(\varphi_t)$. This implies that $\mathrm{id} \preceq \varphi_1 = \phi^\alpha_{-\varepsilon} \circ \phi_1 \lll \phi_1$ and concludes the proof. □

Proof of Proposition 5.3. Let $t := c^\alpha_-(\phi)$. Since $\phi^\alpha_t \preceq \phi$ it implies that $\mathrm{id} \preceq \phi^\alpha_{-t}\phi$. Suppose by contradiction that t is not in the spectrum. Therefore $\mathrm{id} \lll \phi^\alpha_{-t}\phi$ by Lemma 5.4 which contradicts the definition of t. To deduce the result for c^α_+ one can use Poincaré duality. □

5.2. **Geodesics.** Identifying contactomorphisms with their graphs in the contact product of M, the definition of quasi-autonomous contact isotopies is straightforward.

Definition 5.5. A contact isotopy $(\phi_t) \subset \mathcal{G}$ is α-quasi-autonomous if the corresponding Legendrian isotopy $(gr_\alpha(\phi_t)) \subset (M \times M \times \mathbb{R}, \ker\beta)$ is β-quasi-autonomous.

Note that a contact isotopy (ϕ_t) starting at the identity is α-quasi-autonomous if and only if there exist a point $x \in M$ and $\epsilon \in \{\pm 1\}$ such that x is an α-translated point of ϕ_t (*cf.* (4) and above) and $\epsilon H_t(\phi_t(x)) = \max |H_t|$ for all $t \in [0,1]$ where H denotes the α-Hamiltonian function of (ϕ_t).

The characterization of geodesics in this context is now a direct consequence of Theorem 1.8 and the C^1-local isometry between $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{L}}(\Delta)$. Let us recall that a geodesic for the group pseudo-norm $|\cdot|$ is by definition a geodesic for the right-invariant pseudo-distance $(g,h) \mapsto |gh^{-1}|$ (in fact it will also be a geodesic for the associated left-invariant pseudo-distance in our case).

Corollary 5.6. *Let $(M, \ker\alpha)$ be a closed contact manifold such that $\widetilde{\mathcal{L}}(\Delta)$ is orderable. A contact isotopy (ϕ_t) is a geodesic for $|\cdot|^\alpha$ if and only if it is α-quasi-autonomous, where $|\cdot|^\alpha$ denotes either $|\cdot|^\alpha_{\mathrm{spec}}$ or $|\cdot|^\alpha_{\mathrm{SH}}$.*

The second author in [4] characterized some minimizing geodesics of the Shelukhin-Hofer norm on the identity component of the group of compactly supported contactomorphisms of $\mathbb{R}^{2n} \times \mathbb{S}^1$ endowed with its standard contact form α_{st} . Since $\mathbb{R}^{2n} \times \mathbb{S}^1$ is non compact, the previous Corollary 5.6 does not cover this case. However the geodesics characterized in [4] are indeed special cases of α_{st}-quasi-autonomous isotopies. It would be interesting to extend the selectors $\ell^\alpha_\pm$ and $c^\alpha_\pm$ to compactly supported isotopies and extend the results of this paper to the non compact settings (see also [1, Remark 1.4]).

Questions 5.7.

(1) *Does it exist $\Lambda_1 \in \mathcal{L}(\Lambda_0)$ or $\widetilde{\mathcal{L}}$ (resp. $\phi_1 \in \mathcal{G}$ or $\widetilde{\mathcal{G}}$) that cannot be attained by minimizing smooth geodesics,* i.e. *for any isotopy $(\Lambda_t) \subset \mathcal{L}$ or $\widetilde{\mathcal{L}}$ (resp. $(\phi_t) \subset \mathcal{G}$ or $\widetilde{\mathcal{G}}$)*

$$\mathrm{L}^\alpha_{\mathrm{SCH}}(\Lambda_t) = \mathrm{Length}_{\mathrm{d}^\alpha_{\mathrm{spec}}}(\Lambda_t) > \mathrm{d}^\alpha_{\mathrm{SCH}}(\Lambda_1, \Lambda_0) \geq \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda_1, \Lambda_0)$$

$$\textit{(resp. } \mathrm{L}^\alpha_{\mathrm{SH}}(\phi_t) = \mathrm{Length}_{|\cdot|^\alpha_{\mathrm{spec}}}(\phi_t) > |\phi_1|^\alpha_{\mathrm{SH}} \geq |\phi_1|^\alpha_{\mathrm{spec}})?$$

(2) *Does Shelukhin-Chekanov-Hofer type distance (resp. norm) agree with the spectral distance (resp. spectral norm)?*

(3) *Does orderable $\mathcal{L}$ (resp. $\mathcal{G}$) endowed with the spectral distance (resp. spectral norm) is an intrinsic metric space (when taking the infimum of length over continuous paths for the topology induced by the distance,* i.e. *the interval topology* [9, 19]*)?*

Note that a negative answer to question (3) implies a negative answer to question (2). Moreover a negative answer to question (2) implies a positive answer to question (1) for the spectral type distances.

To answer positively to question (1) one can try to adapt some construction of Lalonde-McDuff. Indeed in the Hamiltonian case Lalonde-McDuff [16, Prop 5.1 Part I] constructed examples of $\widetilde{\psi} \in \widetilde{\mathrm{Ham}}(\mathbb{C}\mathrm{P}^1)$ that cannot be joined to the identity by any minimizing geodesics $\{\psi_t\} \subset \mathrm{Ham}(\mathbb{C}\mathrm{P}^1)$ for the Hofer length. It would be interesting to investigate whether certain lifts $\widetilde{\varphi} \in \widetilde{\mathcal{G}}(\mathbb{R}\mathrm{P}^3)$ of $\widetilde{\psi}$ satisfy a similar property : they cannot be joined to the identity by minimizing geodesics of the Hofer-Shelukhin length. Moreover since Corollary 5.2 and the discussion following it imply that the Hofer-Shelukhin length and the spectral length agree on smooth paths in this context, such $\widetilde{\varphi}$ would be examples of elements lying inside $\widetilde{\mathcal{G}} \setminus \mathcal{E}_\pm$.

To answer negatively to question (2) it would be enough to show that the Shelukhin-Hofer type distance (resp. norm) is not compatible with the partial order.

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S. Allais, Université de Strasbourg, IRMA UMR 7501, F-67000 Strasbourg, France
Email address: simon.allais@math.unistra.fr
URL: https://irma.math.unistra.fr/~allais/

P.-A. Arlove, Ruhr-Universität Bochum, Fakultät für Mathematik, 44780 Bochum, Germany
Email address: pierre-alexandre.arlove@ruhr-uni-bochum.de