ON THE HOFER-ZEHNDER CONJECTURE ON $\mathbb{C}\mathrm{P}^d$ VIA GENERATING FUNCTIONS

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Abstract. We use generating function techniques developed by Givental, Théret and ourselves to deduce a proof in $\mathbb{C}\mathrm{P}^d$ of the homological generalization of Franks theorem due to Shelukhin. This result proves in particular the Hofer-Zehnder conjecture in the non-degenerate case: every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ that has at least $d+2$ non-degenerate periodic points has infinitely many periodic points. Our proof does not appeal to Floer homology or the theory of J-holomorphic curves. An appendix written by Shelukhin contains a new proof of the Smith-type inequality for barcodes of Hamiltonian diffeomorphisms that arise from Floer theory, which lends itself to adaptation to the setting of generating functions.

1. Introduction

Let $\mathbb{C}\mathrm{P}^d$ be the complex d-dimensional projective space endowed with the Fubini-Study symplectic structure ω, that is $\pi^*\omega = i^*\Omega$ where $\pi : \mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$ is the quotient map, $i : \mathbb{S}^{2d+1} \hookrightarrow \mathbb{C}^{d+1}$ is the inclusion map and $\Omega := \sum_j \mathrm{d}q_j \wedge \mathrm{d}p_j$ is the canonical symplectic form of $\mathbb{C}^{d+1} \simeq \mathbb{R}^{2(d+1)}$. We are interested in the study of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$, which are time-one maps of those vector fields X_t satisfying the Hamilton equations $X_t \lrcorner \omega = \mathrm{d}h_t$ for some smooth maps $(h_t) : [0,1] \times \mathbb{C}\mathrm{P}^d \to \mathbb{R}$ called Hamiltonian maps. In 1985, Fortune-Weinstein [12] proved that any Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ has at least $d+1$ fixed points, as was conjectured by Arnol'd. Given a diffeomorphism φ, a k-periodic point of φ is by definition a fixed point of the k-iterated map φ^k. On closed symplectically aspherical manifolds (*e.g.* on tori $\mathbb{T}^{2d}$), every Hamiltonian diffeomorphism has infinitely many periodic points. This result was conjectured by Conley, proven for the tori by Hingston [22] and generalized by Ginzburg [16] after decades of flourishing advances: Conley-Zehnder proved the non-degenerate case for tori [11], Salamon-Zehnder proved the non-degenerate case for aspherical manifolds [27], Franks-Handel and Le Calvez proved the conjecture for surfaces [15, 24]. Contrary to aspherical symplectic manifolds, the Conley conjecture does not hold in $\mathbb{C}\mathrm{P}^d$: there exists Hamiltonian diffeomorphisms with only finitely many periodic points. A simple counter-example is the diffeomorphism

$$[z_1 : z_2 : \cdots : z_{d+1}] \mapsto \left[e^{2i\pi a_1} z_1 : e^{2i\pi a_2} z_2 : \cdots : e^{2i\pi a_{d+1}} z_{d+1}\right],$$

with rationally independent coefficients $a_1, \ldots, a_{d+1} \in \mathbb{R}$. This is indeed a Hamiltonian diffeomorphism whose only periodic points are its fixed points: the projection of the canonical base of $\mathbb{C}^{d+1}$. Notice that this Hamiltonian diffeomorphism has

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the minimal number of periodic points, that is the minimal number of fixed points conjectured by Arnol'd.

In the case $d = 1$, $\mathbb{C}\mathrm{P}^1 \simeq \mathbb{S}^2$, and Hamiltonian diffeomorphisms are the area preserving diffeomorphisms isotopic to identity. Franks [13, 14] proved that such an area preserving homeomorphism has either 2 or infinitely many periodic points. In 1994, Hofer-Zehnder [23, p.263] conjectured a higher-dimensional generalization of this result: every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ has either $d+1$ or infinitely many periodic points (it was stated for more general symplectic manifolds). In this direction, a symplectic proof of Franks result was provided by Collier *et al.* [10] in the smooth setting. Here, we are interested in a version of the Hofer-Zehnder conjecture proved by Shelukhin in 2019: if a Hamiltonian diffeomorphism on a closed monotone symplectic manifold with semisimple quantum homology has a finite number of contractible periodic points then the sum of dimension of the local Floer homologies at its contractible fixed points is equal to the total dimension of the homology of the manifold [28]. As we will see, his proof is based on the theory of barcodes in symplectic topology introduced by Polterovich-Shelukhin in [26].

In [1], we elaborate on the ideas of Givental [18] and Théret [30] to essentially build an analogue of the Floer homology of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$ with classical Morse theory and generating functions. In this article, we complete this construction in order to give an alternative proof of the theorem of Shelukhin on the Hofer-Zehnder conjecture "that could have been given in the 90s". Shelukhin introduced a homology count over a field $\mathbb{F}$ of the number of fixed points of a Hamiltonian diffeomorphism with finitely many fixed points φ of $\mathbb{C}\mathrm{P}^d$. In our setting, it is defined by

$$N(\varphi;\mathbb{F}) := \sum_{x\in \mathrm{Fix}(\varphi)} \dim \mathrm{C}_*(\varphi;x;\mathbb{F}) \in \mathbb{N}, \tag{1}$$

where the precise definition of the local homology $\mathrm{C}_*(\varphi;x;\mathbb{F})$ of the fixed point x over $\mathbb{F}$ is given in Section 3.3. In fact, one can prove that $\mathrm{C}_*(\varphi;x;\mathbb{F})$ is isomorphic to the local Floer homology of x over the field $\mathbb{F}$, so that $N(\varphi;\mathbb{F})$ equals the homological count defined by Shelukhin. One always has $N(\varphi;\mathbb{F}) \geq d+1$, this is an avatar of the Fortune-Weinstein theorem. The theorem of Shelukhin that we prove is the following.

Theorem 1.1 ([28, Theorem A for $M = \mathbb{C}\mathrm{P}^d$])**.** *Every Hamiltonian diffeomorphism φ of $\mathbb{C}\mathrm{P}^d$ with finitely many fixed points such that $N(\varphi;\mathbb{F}) > d+1$ for some field $\mathbb{F}$ has infinitely many periodic points. Moreover, if $\mathbb{F}$ has characteristic 0 in the former assumption, there exists $A \in \mathbb{N}$ such that, for all prime $p \geq A$, φ has a p-periodic point that is not a fixed point; if $\mathbb{F}$ has characteristic $p \neq 0$, φ has infinitely many periodic points with period including in $\{p^k \mid k \in \mathbb{N}\}$.*

In the special case where every fixed point of φ is non-degenerate, one has $\dim \mathrm{C}_*(\varphi;x;\mathbb{F}) = 1$ for every fixed point; hence, $N(\varphi;\mathbb{F})$ equals the number of fixed points of φ. As a special case, every Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ that has at least $d+2$ non-degenerate periodic points has infinitely many periodic points and the number grows at least like the sum of prime numbers (*i.e.* the number of periodic points of period less than k is $\gtrsim \frac{k^2}{\log k}$). One can indeed replace φ with a power k of itself in order to get $N(\varphi^k;\mathbb{F}) \geq d+2$ and apply Theorem 1.1.

Our proof follows the same main steps as the original one and takes advantage of the new proof given by Shelukhin in Appendix B of inequality (2) below. Let us give a short outline of it. Our analogue of the Floer homology of the Hamiltonian diffeomorphism associated with the Hamiltonian map (h_s) defines with its inclusion morphisms a persistence module $(G_*^{(-\infty,t)}(h_s;\mathbb{F}))_t$. Such a persistence module can be represented in a graphical way by a barcode (see Figure 1). Infinite bars of the barcodes always exist in the same cardinality: they are associated with the spectral values of (h_s). On the other hand, finite bars exist if and only if $N(\varphi;\mathbb{F}) > d+1$. The barcodes always have infinitely many bars but there is a natural free $\mathbb{Z}$-action on their collection, preserving the length of bars in such a way that, if φ has finitely many fixed points, there are only finitely many $\mathbb{Z}$-orbits of finite bars. Moreover, if φ has finitely many periodic points, the number of $\mathbb{Z}$-orbits of finite bars associated with its iterations is uniformly bounded. The proof consists in showing that the existence of a finite bar implies an unbounded growth of the number of $\mathbb{Z}$-orbits of finite bars.

Since $\mathbb{Z}$ acts on the collection of finite bars by preserving their length, one can define the sequence of bar length of each $\mathbb{Z}$-orbit $0 < \beta_1((h_s);\mathbb{F}) \leq \beta_2((h_s);\mathbb{F}) \leq \cdots \leq \beta((h_s);\mathbb{F})$. We denote by $\beta_{\mathrm{tot}}((h_s);\mathbb{F})$ the sum of these lengths. The proof relies on two results: the length of a finite bar is bounded above by 1 (with our normalization, see (3) below), the sum of the length satisfies the Smith-type inequality

$$\beta_{\mathrm{tot}}((h_{ps});\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}((h_s);\mathbb{F}_p), \tag{2}$$

for all prime p. These two steps easily imply Theorem 1.1. We refer to Section 2 for a more precise outline of the proof.

Organization of the paper. In Section 2, we give an outline of the proof. In Section 3, we provide the background on persistence modules and generating functions. In Section 4, we define and study the generating function homology associated with Hamiltonian maps of $\mathbb{C}\mathrm{P}^d$. In Section 5, we prove that every finite bar of the barcode associated with a Hamiltonian map of $\mathbb{C}\mathrm{P}^d$ has length less than 1. In Section 6, we show that the sum of the lengths of the representatives of the finite bars of such a barcode satisfies a Smith-type inequality. In Section 7, we prove Theorem 1.1. In Appendix A, we prove fundamental properties of the projective homology join used throughout the paper. In Appendix B, Shelukhin gives a new proof of the aforementioned Smith-type inequality on a closed monotone symplectic manifold in the realm of Floer theory.

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2. Outline of the proof

Here, we introduce the main tools of the proof. We postpone the proof of technical statements and the definition of technical objects to the remaining sections in order to give the proof of Theorem 1.1 at the end of this section.

Let $(h_s) : [0,1] \times \mathbb{C}\mathrm{P}^d \to \mathbb{R}$ be a smooth periodic Hamiltonian map and let (φ_s) be the associated Hamiltonian flow on $\mathbb{C}\mathrm{P}^d$. This Hamiltonian map defines a unique Hamiltonian map $(H_s) : [0,1] \times (\mathbb{C}^{d+1} \setminus 0) \to \mathbb{R}$ that is 2-homogeneous, invariant under the diagonal action of S^1 on $\mathbb{C}^{d+1} \setminus 0$ given by $\lambda \cdot (z_0, \dots, z_d) := (\lambda z_0, \dots, \lambda z_d)$ (we will simply write "S^1-invariant") so that its restriction on the unit sphere $\mathbb{S}^{2d+1} \subset \mathbb{C}^{d+1}$ is a lift of (h_s) under the quotient map $\mathbb{S}^{2d+1} \to \mathbb{C}\mathrm{P}^d$. Let (Φ_s) be the $\mathbb{C}$-equivariant Hamiltonian flow associated with (H_s). We will say that (H_s) and (Φ_s) are the lifted Hamiltonian map and Hamiltonian flow of (h_s). In Section 3.2, we introduce decompositions of (Φ_s) into "small" Hamiltonian diffeomorphisms (see Section 3.2) that we usually write $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ so that $\sigma_k \circ \dots \circ \sigma_1 = \Phi_{k/n}$ for $1 \leq k \leq n$. For such a decomposition $\boldsymbol{\sigma}$, we define homology groups $G_*^{(a,b)}(\boldsymbol{\sigma})$ for almost all $-\infty \leq a < b \leq +\infty$ in Section 4. In the end of Section 4.3, we prove that these homology groups and their natural morphisms do not depend on the choice of the decomposition $\boldsymbol{\sigma}$ of (Φ_s) (up to isomorphism) so that we can write

$$G_*^{(a,b)}(h_s) := G_*^{(a,b)}(\boldsymbol{\sigma}),$$

fixing a decomposition $\boldsymbol{\sigma}$. We call these homology groups "Generating functions homology groups of (h_s)" or simply "GF-homology of (h_s)".

Generating functions homology groups of (h_s) satisfy the same key properties as the Floer homology groups of (h_s) and one can hope that $G_*^{(a,b)}(h_s)$ is isomorphic to $HF_*^{(\pi a, \pi b)}(h_s)$ with commuting inclusion and boundary morphisms (the π factor is due to our normalisation, see (3) below). These groups are homology groups defined over any chosen ring R (in fact over any group G). Given a fixed point $z \in \mathbb{C}\mathrm{P}^d$ of φ_1 and a capping $u : \mathbb{D}^2 \to \mathbb{C}\mathrm{P}^d$, that is a smooth map from the unit 2-disk of $\mathbb{C}$ to $\mathbb{C}\mathrm{P}^d$ so that $u(e^{2i\pi s}) = \varphi_s(z)$, one can define the action $t(\bar{z}) \in \mathbb{R}$ of the capped orbit $\bar{z} := (z, u)$ by

$$t(\bar{z}) = -\frac{1}{\pi}\left(\int_{\mathbb{D}^2} u^*\omega + \int_0^1 h_s \circ \varphi_s(z)\mathrm{d}s\right). \tag{3}$$

Recapping gives a $\mathbb{Z}$-orbit of action values: $t(A\#\bar{z}) = t(\bar{z}) + k$ where $A \in \pi_2(\mathbb{C}\mathrm{P}^d)$ and $\pi k = -\langle [\omega], A\rangle$. On Floer homology, the recapping by the generator $A_0 \in \pi_2(\mathbb{C}\mathrm{P}^d) \simeq \mathbb{Z}$ of symplectic area $\langle [\omega], A_0\rangle = \pi$ induces the quantum operator

$$q : HF_*^{(a,b)}(h_s) \xrightarrow{\simeq} HF_{*-2(d+1)}^{(a-\pi, b-\pi)}(h_s).$$

The analogue isomorphism is defined at (11) (more precisely, its inverse).

Taking R to be a field $\mathbb{F}$, these homology groups are $\mathbb{F}$-vector spaces and the family $(G_*^{(-\infty,t)}(h_s;\mathbb{F}))_t$ together with its inclusion morphisms define a persistent module that we call the persistence module associated with (h_s) over the field $\mathbb{F}$. Assuming that the Hamiltonian diffeomorphism φ_1 has finitely many fixed points, the persistence modules of fixed degree $(G_k^{(-\infty,t)}(h_s;\mathbb{F}))_t$, $k \in \mathbb{Z}$, satisfy suitable finiteness assumptions and one can define a finite barcode for each of them, giving a global countable (graded) barcode for the persistence module associated with (h_s). Let us describe this barcode (see Figure 1 for an example). The isomorphism

$G_*^{(-\infty,t)}(h_s) \simeq G_{*+2(d+1)}^{(-\infty,t+1)}(h_s)$ induces a $\mathbb{Z}$-action on the bars of the barcode sending a bar (a,b) of degree k on a bar $(a+1,b+1)$ of degree $k+2(d+1)$. Therefore, it is enough to describe a set of representatives of bars under this action. In the case where φ_1 is non-degenerate, end-points of representative bars are in one-to-one correspondence with fixed points of $\mathbb{C}\mathrm{P}^d$, the value of an end-point being equal to the action of a capping of the associated fixed point. In general, a fixed point should be counted with multiplicity equal to the dimension of its local homology, which gives $N((h_s);\mathbb{F})$ end-points. Among the representative bars, $d+1$ and only $d+1$ are infinite. This is an avatar of Fortune-Weinstein theorem, in fact the increasing sequence $(c_k(h_s))_{k\in\mathbb{Z}}$ of values of end-points of the infinite bars of the whole barcode corresponds to the sequence of spectral invariants of (h_s) (see Theorem 4.11).

Theorem 1.1 is proved by studying the length of the finite bars of the barcode of (h_s). Let us denote by $I_1,\dots,I_n \subset \mathbb{R}$ representatives of the $\mathbb{Z}$-orbits of finite bars over the field $\mathbb{F}$. Up to a permutation, one can assume that $(\operatorname{length} I_k)_k$ is non-decreasing. Let $\beta_k((h_s);\mathbb{F})$ be the length of I_k, $\beta((h_s);\mathbb{F}) := \beta_n((h_s);\mathbb{F})$ be the length of the longest bar and

$$\beta_{\mathrm{tot}}((h_s);\mathbb{F}) := \sum_k \beta_k((h_s);\mathbb{F}).$$

The number $\beta((h_s);\mathbb{F})$ was first introduced by Usher and called the boundary depth of (h_s) [32]. The first important property is that every finite bar has a length smaller or equal to 1 (see Theorem 5.1). This is the analogue of [28, Theorem B] in the special case $M = \mathbb{C}\mathrm{P}^d$ and the proof follows the same key ideas: we define a product between GF-homologies and use it to find an interleaving between the GF-homology of (h_s) and the one of $(h'_s) \equiv 0$ that does not have any finite bar. The second important property is the Smith inequality (2) stated in Corollary 6.3 which is the analogue of [28, Theorem D] in the special case $M = \mathbb{C}\mathrm{P}^d$. The general strategy follows a new proof of [28, Theorem D] given by Shelukhin in Appendix B. In the realm of generating functions, the proof is rather short and very elementary: it essentially relies on the classical Smith inequality (16); it could seem surprising regarding the extraordinary machinery necessary to prove its Floer theoretical analogue (although the Floer theoretical proof is available for every closed monotone symplectic manifold).

3. Preliminaries

In this section, we fix convention and notation used throughout the article and recall known results that will be used as well.

3.1. **Persistence modules and barcodes.**

Let us fix a field $\mathbb{F}$. A persistence module (V,π) over the field $\mathbb{F}$ is an $\mathbb{R}$-family of $\mathbb{F}$-vector spaces $(V^t)_{t\in\mathbb{R}}$ with a collection of morphisms $\pi_s^t : V^s \to V^t$ for $s \leq t$ such that $\pi_t^t = \mathrm{id}_{V^t}$ and $\pi_s^t \circ \pi_r^s = \pi_r^t$ whenever $r \leq s \leq t$ that one can call persistence morphisms. We will always ask our persistence modules (V,π) to be right continuous: for every $t \in \mathbb{R}$, π_t^s is an isomorphism for every $s > t$ close to t. We extend this definition to families $(V^t)_{t\in\mathbb{R}\setminus S}$ satisfying the same axioms with the exception that $t \in \mathbb{R}\setminus S$ where $S \subset \mathbb{R}$ is discrete by identifying (V^t) with the persistence module $\left(\overline{V}^t\right)_{t\in\mathbb{R}}$ defined by taking the direct limit

$$\overline{V}^t := \varprojlim_{s>t} V^s, \quad \forall t \in \mathbb{R},$$

with the obvious extension of the morphisms π_s^t. Given two persistence modules (V,π) and (V',π'), one defines the direct sum of them in the obvious way $(V \oplus V', \pi \oplus \pi')$ which is a persistence module. A morphism of persistence module $f : (V,\pi) \to (V',\pi')$ is a family of morphisms $f_t : V^t \to V'^t$ commuting with the persistence morphisms.

A persistence module (V^t) is of finite type if $V^t = 0$ for t sufficiently close to $-\infty$, every V^t has a finite dimension and there exists a finite set $S \subset \mathbb{R}$ such that π_s^t is an isomorphism whenever s and t belong to the same connected component of $\mathbb{R} \setminus S$. The fundamental example is given by $V^t := H_*(\{f \leq t\}; \mathbb{F})$, where $f : M \to \mathbb{R}$ is a smooth function on a compact manifold M with finitely many critical points (by taking a general map on any space, we only have a general persistence module). Given an interval $I = [a,b)$, with $b \in (a, +\infty]$, we define the persistence module $\mathbb{F}(I)$ by $V^t = \mathbb{F}$ for $t \in I$ and $V^t = 0$ otherwise, $\pi_s^t = \mathrm{id}$ when t and s belong to the same connected component of $\mathbb{R} \setminus \{a,b\}$ and $\pi_s^t = 0$ otherwise. It is a persistence module of finite type. We think of it as representing a class that appears at $t = a$ and persists until $t = b$ (it persists indefinitely if $b = +\infty$). Graphically, we represent $\mathbb{F}(I)$ by drawing an horizontal bar from $t = a$ to $t = b$ or without right endpoint if $I = [a, +\infty)$. The normal form theorem asserts that for every persistence module V of finite type, there exists a unique finite collection of couples (I_k, m_k), $I_k \subset \mathbb{R}$ being a bar as above and $m_k \in \mathbb{N}^*$, so that there is an isomorphism of persistence modules

$$V \simeq \bigoplus_k \mathbb{F}(I_k)^{\oplus m_k}$$

(see for instance [3, 36]). The collection $\mathcal{B}(V) := \{(I_k, m_k)\}$ is called the barcode of V; it is graphically represented by drawing each horizontal bars of the I_k's with multiplicity in the same figure (see Figure 1 for an example with 6 infinite bars and 5 finite bars; ignore the fact that it is actually part of a larger barcode).

Although we will not need it in its full strength, we recall the isometry theorem between the bottleneck distance between barcodes and the interleaving distance between persistence modules (of finite type). Given $\delta, \delta' \in \mathbb{R}$ with $\delta + \delta' \geq 0$, a (δ,δ')-interleaving between persistence modules (V,π) and (W,κ) is a couple of morphisms of persistence modules $f : (V^t) \to (W^{t+\delta})$ and $g : (W^t) \to (V^{t+\delta'})$ such that $g_{t+\delta} \circ f_t = \pi_t^{t+\delta+\delta'}$ and $f_{t+\delta'} \circ g_t = \kappa_t^{t+\delta+\delta'}$ for all $t \in \mathbb{R}$. When such an interleaving exists, it is said that V and W are (δ,δ')-interleaved. Given $\delta \geq 0$, a δ-interleaving is by definition a (δ,δ)-interleaving. For instance, if (V^t) and (W^t) are (δ,δ')-interleaved, then (V^t) and $(W^{t+\delta_-})$ are δ_+-interleaved, where $\delta_- = \frac{\delta-\delta'}{2}$ and $\delta_+ = \frac{\delta+\delta'}{2}$. The interleaving distance between two persistence modules V and W is defined by

$$d_{\mathrm{int}}(V,W) := \inf\{\delta \geq 0 \mid V \text{ and } W \text{ are } \delta\text{-interleaved}\}.$$

This is a true distance between persistence modules up to isomorphisms taking values in $[0, +\infty]$.

Let $\mathcal{B} := \{(I_k, m_k)\}$ and $\mathcal{B}' := \{(J_l, m_l')\}$ be two barcodes that we see as multisets of intervals. Given an interval $I = (a,b]$ or $(a, +\infty)$, we set $I^\delta := (a-\delta, b+\delta]$ or $(a-\delta, +\infty)$ respectively. Given $\delta \geq 0$, a δ-matching between the barcodes $\mathcal{B}$ and $\mathcal{B}'$ is a bijection of multisets $\mu : \mathcal{B}_0 \to \mathcal{B}_0'$ where $\mathcal{B}_0$ and $\mathcal{B}_0'$ are some submultisets of $\mathcal{B}$ and $\mathcal{B}'$ containing (at least) every interval of length $\geq 2\delta$ and such that $\mu(I) \subset I^\delta$ and $I \subset \mu(I)^\delta$ for every $I \in \mathcal{B}_0$. When such a δ-matching exists, it is said that $\mathcal{B}$

and $\mathcal{B}'$ are δ-matched. The bottleneck distance between two barcodes $\mathcal{B}$ and $\mathcal{B}'$ is defined by

$$d_{\text{bottleneck}}(\mathcal{B},\mathcal{B}') := \inf\{\delta \geq 0 \mid \mathcal{B} \text{ and } \mathcal{B}' \text{ are } \delta\text{-matched}\}.$$

This is a true distance taking values in $[0,+\infty]$.

The isometry theorem between the bottleneck distance and the interleaving distance states that given any persistence modules of finite type V and W,

$$d_{\text{int}}(V,W) = d_{\text{bottleneck}}(\mathcal{B}(V),\mathcal{B}(W)),$$

(see for instance [5]).

3.2. **Generating functions of $\mathbb{C}$-equivariant diffeomorphisms.** In this section, we summarize definitions widely discussed in [1, Section 5]. This use of generating functions was much inspired by the dynamical viewpoint of Chaperon [7] and the work of Givental [18] and Théret [30].

Given a Hamiltonian map $(h_s) : [0,1]\times\mathbb{C}\mathrm{P}^n \to \mathbb{R}$, let (H_s) be the lifted Hamiltonian of $\mathbb{C}^{d+1}$ that is 2-homogeneous and S^1-invariant as defined in the beginning of Section 2 and let (Φ_s) be the associated Hamiltonian flow. The maps Φ_s are smooth diffeomorphisms of $\mathbb{C}^{d+1}\setminus 0$ that extend to homeomorphisms of $\mathbb{C}^{d+1}$ by setting $\Phi_s(0) = 0$. We refer to these maps as $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms because

$$\Phi_s(\lambda z) = \lambda\Phi_s(z), \quad \forall\lambda\in\mathbb{C}, \forall z\in\mathbb{C}^{d+1}.$$

In general, a generating function will denote a map $F : \mathbb{C}^{d+1}\times\mathbb{C}^k \to \mathbb{R}$, $(x;\xi)\mapsto F(x;\xi)$, such that $\partial_\xi F$ is a submersion. We say that $x\in\mathbb{C}^{d+1}$ is the main variable of F and $\xi\in\mathbb{C}^k$ is the auxiliary variable. Let $\Sigma_F\subset\mathbb{C}^{d+1}\times\mathbb{C}^k$ be the submanifold

$$\Sigma_F := \left\{(x;\xi)\in\mathbb{C}^{d+1}\times\mathbb{C}^k \mid \partial_\xi F(x;\xi) = 0\right\}.$$

We say that the generating function F is generating the Hamiltonian diffeomorphism Φ of $\mathbb{C}^{d+1}$ if

$$\forall z\in\mathbb{C}^{d+1}, \exists!(x;\xi)\in\Sigma_F, \quad x = \frac{z+\Phi(z)}{2} \quad \text{and} \quad \partial_x F(x;\xi) = i(z-\Phi(z)).$$

The critical points of a generating function of Φ are in one-to-one correspondence with the fixed points of Φ, the bijection being $(x;\xi)\mapsto x$. Given generating functions $F : \mathbb{C}^{d+1}\times\mathbb{C}^k\to\mathbb{R}$ and $G : \mathbb{C}^{d+1}\times\mathbb{C}^l\to\mathbb{R}$ of Φ and Ψ respectively, the fiberwise sum of F and G denotes the map

$$(F+G)(x;\xi,\eta) := F(x;\xi) + G(x;\eta). \tag{4}$$

Although this is not a generating function of $\Phi\circ\Psi$, the critical points of $F+G$ are also in bijection with the fixed points of $\Phi\circ\Psi$ *via* $(x;\xi,\eta)\mapsto x - i\partial_x G(x;\eta)/2$ (these statements are easy consequences of the definitions).

Let Φ be a $\mathbb{C}$-equivariant Hamiltonian diffeomorphism of $\mathbb{C}^{d+1}$ which can be decomposed in $\Phi = \sigma_n\circ\cdots\circ\sigma_1$ where every Hamiltonian diffeomorphism σ_k is sufficiently C^1-close to id such that they admit generating functions without auxiliary variable $f_k : \mathbb{C}^{d+1}\to\mathbb{R}$. We call such generating functions elementary generating functions. They are uniquely defined up to an additive constant, we will set $f_k(0) = 0$. For all $n\in\mathbb{N}^*$, we will say that the n-tuple $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ is associated to the Hamiltonian flow (Φ_s) if there exists real numbers $0 = t_0\leq t_1\leq\cdots\leq t_n = 1$ such that $\sigma_k = \Phi_{t_k}\circ\Phi_{t_{k-1}}^{-1}$. For all $k\in\mathbb{N}$, we denote $\boldsymbol{\varepsilon}^k$ the k-tuple

$$\boldsymbol{\varepsilon}^k := (\mathrm{id},\dots,\mathrm{id})$$

and we define the k-th power $\boldsymbol{\sigma}^k$ of an n-tuple $\boldsymbol{\sigma}$ as the kn-tuple

$$\boldsymbol{\sigma}^k := (\boldsymbol{\sigma}, \dots, \boldsymbol{\sigma}).$$

A continuous family of such tuples $(\boldsymbol{\sigma}_s)$ will denote a family of tuples of the same size $n \geq 1$, $\boldsymbol{\sigma}_s =: (\sigma_{1,s}, \dots, \sigma_{n,s})$ such that the maps $s \mapsto \sigma_{k,s}$ are C^1-continuous. Every $\mathbb{C}$-equivariant Hamiltonian flow $(\Phi_s)_{s\in[0,1]}$ admit a continuous family of associated tuple $(\boldsymbol{\sigma}_s)$ that is $\boldsymbol{\sigma}_s$ is associated to Φ_s for all $s \in [0,1]$ (and the size can be taken as large as wanted). We denote by $F_{\boldsymbol{\sigma}}$ the following function $(\mathbb{C}^{d+1})^n \to \mathbb{R}$:

$$F_{\boldsymbol{\sigma}}(v_1, \dots, v_n) := \sum_{k=1}^{n} f_k\left(\frac{v_k + v_{k+1}}{2}\right) + \frac{1}{2}\langle v_k, iv_{k+1}\rangle,$$

with convention $v_{n+1} = v_1$, each $f_k : \mathbb{C}^{d+1} \to \mathbb{R}$ being the elementary generating function associated with σ_k. If n is odd, this function is a generating function of Φ with main variable v_1 and auxiliary variable $(v_2, \dots, v_n)$ that will be referred as the generating function associated with $\boldsymbol{\sigma}$. If the opposite is not explicitly supposed, $\boldsymbol{\sigma}$ will have an odd size.

Let (Φ_s) be the lift of the Hamiltonian flow (φ_s) of $\mathbb{C}\mathrm{P}^d$ under the Hamiltonian map (h_s). In this case, we take 2-homogeneous and S^1-equivariant f_k's, that is

$$f_k(\lambda w) = |\lambda|^2 f_k(w), \quad \forall \lambda \in \mathbb{C}, \forall w \in \mathbb{C}^{d+1},$$

so that the resulting $F_{\boldsymbol{\sigma}}$ is also 2-homogeneous and S^1-invariant. Hence, a $\mathbb{C}$-line of critical points $\mathbb{C}\mathbf{v}$ of $F_{\boldsymbol{\sigma}}$ corresponds to a $\mathbb{C}$-line of fixed points $\mathbb{C}v_1$ of Φ_1. The Euler identity for homogeneous maps implies that these $\mathbb{C}$-lines of critical points have value 0. Given a 2-homogeneous and S^1-invariant map $F : \mathbb{C}^{N+1} \to \mathbb{R}$, we denote by $\widehat{F} : \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ the projectivization of F, that is the unique map for which the restriction of F to the unit sphere $\mathbb{S}^{2N+1} \subset \mathbb{C}^{N+1}$ is a lift under the quotient map $\mathbb{S}^{2N+1} \to \mathbb{C}\mathrm{P}^N$. For instance, $h_s = \widehat{H_s}$. Critical $\mathbb{C}$-lines of $F_{\boldsymbol{\sigma}}$ are in one-to-one correspondence with critical points of $\widehat{F_{\boldsymbol{\sigma}}}$ *with value* 0. Generically, $\widehat{F_{\boldsymbol{\sigma}}}$ has no critical point of value 0 at all. Indeed, for all $[z] \in \mathbb{C}\mathrm{P}^d$ with lift $z \in \mathbb{C}^{d+1} \setminus 0$,

$$\varphi_1([z]) = [z] \text{ with action } t \in \mathbb{R}/\mathbb{Z} \quad \Leftrightarrow \quad e^{-2i\pi t}\Phi_1(z) = z,$$

where t is defined by (3) (it is well-defined up to an integer if one does not choose a preferred capping); this is due to Théret [30, Proposition 5.8]. Therefore, for a fixed $m \in \mathbb{N}^*$, we need to define a family $(\boldsymbol{\sigma}_{m,t})$ of tuples generating $(e^{-2i\pi t}\Phi_1)$, $-m \leq t \leq m$.

Let (δ_t) be the family of small $\mathbb{C}$-equivariant diffeomorphisms $\delta_t(z) := e^{-2i\pi t}z$, $t \in (-1/2, 1/2)$. The associated elementary generating function is $w \mapsto -\tan(\pi t)\|w\|^2$. Let us fix once and for all an even number $n_0 \geq 4$ and let $(\boldsymbol{\delta}_t^{(1)})$ be the family of n_0-tuples $(\delta_{t/n_0}, \dots, \delta_{t/n_0})$ generating $z \mapsto e^{-2i\pi t}z$ for $t \in (-2, 2)$. For all $m \in \mathbb{N}^*$, let $(\boldsymbol{\delta}_t^{(m)})$ be a family of mn_0-tuples generating $z \mapsto e^{-2i\pi t}z$ for $t \in (-m-1, m+1)$ and satisfying

$$\boldsymbol{\delta}_t^{(m+1)} = \left(\boldsymbol{\delta}_t^{(m)}, \boldsymbol{\varepsilon}^{n_0}\right), \quad \forall t \in [-m, m]. \tag{5}$$

More precisely, let $\chi : \mathbb{R} \to \mathbb{R}$ be an odd smooth non-decreasing map such that $\chi_m \equiv \mathrm{id}$ on $[-m-1/4, m+1/4]$ and $\chi_m \equiv m + 1/2$ on $[m+3/4, +\infty)$. We set

$$\boldsymbol{\delta}_t^{(m+1)} = \left(\boldsymbol{\delta}_{\chi_m(t)}^{(m)}, \boldsymbol{\delta}_{t-\chi_m(t)}^{(1)}\right), \quad \forall t \in (-m-2, m+2).$$

Finally, we can set

$$\boldsymbol{\sigma}_{m,t} := \left(\boldsymbol{\sigma}, \boldsymbol{\delta}_t^{(m)}\right), \quad \forall t \in [-m, m].$$

Since tan is increasing on $[-\pi/2, \pi/2]$, we deduce that $\partial_t F_{\boldsymbol{\sigma}_{m,t}} \leq 0$ by a straightforward computation.

3.3. **Homology of sublevel sets and local homology of a fixed point.** The study of the homology of sublevel sets of generating functions was introduced by Viterbo [33] who introduced spectral invariants of Hamiltonian diffeomorphisms of $\mathbb{R}^{2d}$ with compact support. This work led to the definition of homology groups of these diffeomorphisms by Traynor [31] (which are in fact isomorphic to their Floer theoretic analogue [34]). Here, we show how to define similar homology groups for Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$. We refer to [1, Section 5] for comprehensive proofs and references of the results stated in this section.

Throughout this paper, $H_*(X)$ and $H^*(X)$ denote respectively the singular homology and the singular cohomology of a topological space or pair X over an indeterminate ring R. If one needs to specify the ring R, one writes $H_*(X; R)$ and $H^*(X; R)$ instead. Let $\boldsymbol{\sigma}$ be an n-tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms. We denote by $Z(\boldsymbol{\sigma}) \subset \mathbb{C}\mathrm{P}^{n(d+1)-1}$ the sublevel set

$$Z(\boldsymbol{\sigma}) := \left\{\widehat{F}_{\boldsymbol{\sigma}} \leq 0\right\}.$$

We denote by $HZ_*(\boldsymbol{\sigma})$ the shifted homology group

$$HZ_*(\boldsymbol{\sigma}) := H_{*+(n-1)(d+1)}(Z(\boldsymbol{\sigma})),$$

and if $Z(\boldsymbol{\sigma}') \subset Z(\boldsymbol{\sigma})$, with $\boldsymbol{\sigma}'$ an n-tuple, we set

$$HZ_*(\boldsymbol{\sigma}, \boldsymbol{\sigma}') := H_{*+(n-1)(d+1)}(Z(\boldsymbol{\sigma}), Z(\boldsymbol{\sigma}')).$$

For $m \in \mathbb{N}^*$ and $a \leq b$ in $[-m, m]$, one has $F_{\boldsymbol{\sigma}_{m,b}} \leq F_{\boldsymbol{\sigma}_{m,a}}$ so $Z(\boldsymbol{\sigma}_{m,a}) \subset Z(\boldsymbol{\sigma}_{m,b})$ and we can set

$$G_*^{(a,b)}(\boldsymbol{\sigma}, m) := HZ_*(\boldsymbol{\sigma}_{m,b}, \boldsymbol{\sigma}_{m,a}),$$

when a and b are not action values of $\boldsymbol{\sigma}$. In [1, Section 5.4], we remarked that this homology group can be naturally identified to the homology of sublevel sets of a map, that is

$$G_*^{(a,b)}(\boldsymbol{\sigma}, m) \simeq H_{*+(n-1)(d+1)}\left(\{\mathcal{T} \leq b\}, \{\mathcal{T} \leq a\}\right),$$

for some C^1-map $\mathcal{T} : M \to \mathbb{R}$ defined on some manifold M that is smooth in the neighborhood of its critical points. The function $\mathcal{T}$ is some kind of finite-dimensional action: critical points of $\mathcal{T}$ are in one-to-one correspondence with capped fixed points of φ_1 with action value inside $[-m, m]$. This correspondence sends critical value to action value and Morse index up to a $(n-1)(d+1)$ shift in degree to the Conley-Zehnder index. More generally, the local homology of $\mathcal{T}$ (up to the same shift in degree) is isomorphic to the local Floer homology of the corresponding capped orbit. Let us denote by $\mathrm{C}_*(f; x)$ the local homology of the critical point x of a map f. According to the above discussion, we can define up to isomorphism

$$\mathrm{C}_*(\boldsymbol{\sigma}; z, t) \simeq \mathrm{C}_*\left(\widehat{F}_{\boldsymbol{\sigma}_{m,t}}; \zeta\right) \simeq \mathrm{C}_{*+(n-1)(d+1)}\left(\mathcal{T}; (\zeta, t)\right),$$

where $\zeta \in \mathbb{C}\mathrm{P}^{n(d+1)-1}$ is the critical point of $\widehat{F}_{\boldsymbol{\sigma}_{m,t}}$ associated with the fixed point $z \in \mathbb{C}\mathrm{P}^d$ of action $t \in [-m, m]$ (see [1, Section 5.5 and 5.7] for details). The independence on m of this definition can also easily be deduced from the isomorphism induced by θ_m^{m+1} (defined in Section 4.2) on the local homologies. Local homologies

$\mathrm{C}_*(\boldsymbol{\sigma}; z, t)$ and $\mathrm{C}_*(\boldsymbol{\sigma}; z, t+1)$ are isomorphic up to a $2(d+1)$ shift in degree so we will not specify the choice of representative $t \in \mathbb{R}$ when the grading is irrelevant. One can prove it without using the isomorphism with local Floer homology as was done in [1, Proposition 5.10] or by using the isomorphism induced by (11) in local homology. We can now define precisely (1) for a choice of tuple $\boldsymbol{\sigma}$ by

$$N(\boldsymbol{\sigma}; \mathbb{F}) := \sum_{z \in \mathrm{Fix}(\varphi_1)} \dim \mathrm{C}_*(\boldsymbol{\sigma}; z; \mathbb{F}) \in \mathbb{N}.$$

We recall that an integer $k \in \mathbb{N}^*$ is said to be admissible for φ at a fixed point z if $\lambda^k \neq 1$ for all eigenvalues $\lambda \neq 1$ of $\mathrm{d}\varphi(z)$. Until the end of the section, φ is associated with a tuple $\boldsymbol{\sigma}$ and the periodic points of φ are isolated in order to simplify the statements. The following proposition was proved by Ginzburg-Gürel in [17] for the local Floer homology.

Proposition 3.1. *Let $k \in \mathbb{N}^*$ be an admissible iteration of φ at the fixed point z. Then as graded R-modules,*

$$\mathrm{C}_*(\boldsymbol{\sigma}^k; z) \simeq \mathrm{C}_{*-i_k}(\boldsymbol{\sigma}; z),$$

for some shift in degree $i_k \in \mathbb{Z}$.

In our setting, one can prove this result by directly applying the shifting theorem of Gromoll-Meyer [20, §3]. Let us give the proof when k is odd.

Proof. One can assume that the fixed point z has action 0 so that $\mathrm{C}_*(\boldsymbol{\sigma}^k; z)$ is isomorphic to the local homology of $\widehat{F}_{\boldsymbol{\sigma}^k}$ at $[\mathbf{v}_0 : \cdots : \mathbf{v}_0]$, where $[\mathbf{v}_0] \in \mathbb{C}\mathrm{P}^N$ is the critical point of $\widehat{F}_{\boldsymbol{\sigma}}$ associated with z. Let $M \subset \mathbb{C}\mathrm{P}^N$ be a characteristic submanifold for $\widehat{F}_{\boldsymbol{\sigma}}$ at $[\mathbf{v}_0]$. Then the image of M under the embedding $\iota : [\mathbf{v}] \mapsto [\mathbf{v} : \cdots : \mathbf{v}]$ is a characteristic submanifold for $\widehat{F}_{\boldsymbol{\sigma}^k}$ if and only if k is admissible (see [1, Equation (11) and above]). According to the shifting theorem, the local homology of a function at a given point is isomorphic to the local homology of the restriction of this function to a characteristic submanifold at the given point up to a shift in degree. Since $\widehat{F}_{\boldsymbol{\sigma}^k} \circ \iota = \widehat{F}_{\boldsymbol{\sigma}}$, the conclusion follows. □

This proof needs small changes when k is even (see Section 6.4 for an idea of the case $k = 2$), but it will not be needed to prove the following corollary.

Corollary 3.2. *For every fixed point z of φ, there exists $B > 0$ such that, for all prime p*

$$\dim \mathrm{C}_*(\boldsymbol{\sigma}^p; z; \mathbb{F}_p) < B.$$

Proof. Applying Proposition 3.1 with $R := \mathbb{Z}$, the $\mathrm{C}_*(\boldsymbol{\sigma}^p; z; \mathbb{Z})$'s are isomorphic up to a shift in degree for sufficiently large prime numbers p. Since these $\mathbb{Z}$-modules are finitely generated, the conclusion is a straightforward application of the universal coefficient theorem. □

4. Generating functions homology of Hamiltonian diffeomorphisms of $\mathbb{C}\mathrm{P}^d$

4.1. **Composition of sublevel sets of generating functions.** Given an odd number $n \in \mathbb{N}$, let A_n be the linear automorphism of $(\mathbb{C}^{d+1})^n$ such that $\mathbf{w} = A_n \mathbf{v}$

with $w_k = \frac{v_k+v_{k+1}}{2}$. We will often omit A_n in our changes of variables and talking about w-variables. Let $Q_n : (\mathbb{C}^{d+1})^n \to \mathbb{R}$ be the S^1-invariant quadratic form

$$Q_n(\mathbf{v}) := F_{\varepsilon^n}(\mathbf{v}) = \frac{1}{2}\sum_{k=1}^{n} \langle v_k, iv_{k+1}\rangle = 2\sum_{k=1}^{n}\sum_{l=1}^{k-1}(-1)^{k+l}\langle w_k, iw_l\rangle ,$$

with conventions $v_{n+1} := v_1$ and $w_{n+1} := w_1$.

The following proposition is a direct consequence of the fact that Q_n is both a quadratic form and a generating function of the identity.

Proposition 4.1. *The quadratic form Q_n has nullity $2(d+1)$. Moreover*

$$Q_n(v_1, v_2, \dots, v_n) = -Q_n(v_1, v_n, v_{n-1}, \dots, v_2)$$

so that

$$\operatorname{ind} Q_n = \operatorname{coind} Q_n = (n-1)(d+1).$$

We denote by $\widetilde{B}_{n,m} : (\mathbb{C}^{d+1})^n \times (\mathbb{C}^{d+1})^m \to (\mathbb{C}^{d+1})^{m+n+1}$ the $\mathbb{C}$-linear map

$$\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}') := \left(\mathbf{w}, \sum_{k=1}^{m}(-1)^{k+1}w'_k, \mathbf{w}'\right),$$

and we denote by $B_{n,m} : \mathbb{C}\mathrm{P}^{(d+1)n-1} * \mathbb{C}\mathrm{P}^{(d+1)m-1} \to \mathbb{C}\mathrm{P}^{(d+1)(n+m+1)-1}$ the associated projective map, $B_{n,m}([a:b]) := \pi(\widetilde{B}_{n,m}(a,b))$ where π is the canonical projection; here $*$ denotes the projective join (see Appendix A). A straightforward computation gives the following proposition.

Proposition 4.2. *Given odd integers $n, m \in \mathbb{N}$, for all $\mathbf{w} \in (\mathbb{C}^{d+1})^n$ and $\mathbf{w}' \in (\mathbb{C}^{d+1})^m$, one has in w-variables,*

$$Q_{n+m+1}\left(\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')\right) = Q_n(\mathbf{w}) + Q_m(\mathbf{w}').$$

Corollary 4.3. *Given tuples $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ of odd respective sizes n and m, one has*

$$F_{(\boldsymbol{\sigma},\varepsilon,\boldsymbol{\sigma}')}\left(\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')\right) = F_{\boldsymbol{\sigma}}(\mathbf{w}) + F_{\boldsymbol{\sigma}'}(\mathbf{w}').$$

Therefore, $\widetilde{B}_{n,m}$ induces an injective map

$$\{F_{\boldsymbol{\sigma}} \leq a\} \times \{F_{\boldsymbol{\sigma}'} \leq b\} \to \left\{F_{(\boldsymbol{\sigma},\mathrm{id},\boldsymbol{\sigma}')} \leq a+b\right\}$$

by restriction and $B_{n,m}$ induces a map

$$Z(\boldsymbol{\sigma}) * Z(\boldsymbol{\sigma}') \to Z(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}').$$

Proof. This is a direct consequence of the form that takes $F_{\boldsymbol{\sigma}}$ in w-variables:

$$F_{\boldsymbol{\sigma}}(\mathbf{w}) = \sum_{k=1}^{n} f_k(w_k) + Q_n(\mathbf{w}).$$

Therefore,

$$\begin{aligned} F_{(\boldsymbol{\sigma},\varepsilon,\boldsymbol{\sigma}')}\left(\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')\right) &= \sum_{k=1}^{n} f_k(w_k) + \sum_{l=1}^{m} f'_l(w'_l) + Q_{n+m+1}\left(\widetilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')\right) \\ &= \sum_{k=1}^{n} f_k(w_k) + Q_n(\mathbf{w}) + \sum_{l=1}^{m} f'_l(w'_l) + Q_m(\mathbf{w}') \\ &= F_{\boldsymbol{\sigma}}(\mathbf{w}) + F_{\boldsymbol{\sigma}'}(\mathbf{w}'). \end{aligned}$$

□

By making use of the homology projective join defined in Appendix A, we are now in position to define a special map in homology involving two different Hamiltonian flows and their composition. Let us fix 2 tuples $\boldsymbol{\sigma}$, $\boldsymbol{\sigma'}$ of odd respective sizes n and n'. According to Corollary 4.3, the map $B_{n,n'}$ induces a natural morphism $H_*(Z(\boldsymbol{\sigma}) * Z(\boldsymbol{\sigma'})) \to H_*(Z(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'}))$. In Appendix A, we define a morphism

$$\mathrm{pj}_* : H_*(A \times B) \to H_{*+2}(A * B)$$

called the homology projective join. The composition of this morphism with the homology projective join defines a natural morphism

$$HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma'}) \to HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})).$$

It generalizes to the relative case $Z(\boldsymbol{\sigma''}) \subset Z(\boldsymbol{\sigma'})$:

$$HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma'}, \boldsymbol{\sigma''}) \to HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'}), (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma''}));$$

the relative HZ_* could be in the left hand side of the tensor product as well, as long as one of the two HZ_*'s is an absolute homology group. In symbols, we will write this map as $\alpha \otimes \beta \mapsto \alpha \circledast \beta$. The naturality of these morphisms under inclusion maps and boundary maps follows directly from the naturality of the homology projective join.

In particular, the following diagram commutes

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma'}) & \xrightarrow{\circledast} & HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})) \\ \downarrow & & \downarrow \\ H_{*+r}(\mathbb{C}\mathrm{P}^N) \otimes H_{*+r'}(\mathbb{C}\mathrm{P}^{N'}) & \xrightarrow{\mathrm{pj}_*} & H_{*+r+r'+2}(\mathbb{C}\mathrm{P}^{N''}) \end{array}, \qquad (6)$$

where the vertical arrows are induced by inclusions, $r := (n-1)(d+1)$, $N := n(d+1)-1$, $r' := (n'-1)(d+1)$, $N' := n'(d+1)-1$, $N'' := (n+n'+1)(d+1)-1$ and we see $\mathbb{C}\mathrm{P}^N$ and $\mathbb{C}\mathrm{P}^{N'}$ as the disjoint subspaces included in $\mathbb{C}\mathrm{P}^{N''}$ *via* $[\mathbf{w}] \mapsto [\mathbf{w} : 0]$ and $[\mathbf{w'}] \mapsto [0 : \mathbf{w'}]$. The commutativity of this diagram follows from the naturality of pj_* and the fact that $B_{n,n'}$ is homotopic to $[\mathbf{w} : \mathbf{w'}] \mapsto [\mathbf{w} : 0 : \mathbf{w'}]$.

Proposition 4.4. *Composition morphisms are associative, that is the following diagram commutes:*

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\sigma'}) \otimes HZ_*(\boldsymbol{\sigma''}) & \longrightarrow & HZ_*(\boldsymbol{\sigma}) \otimes HZ_{*-2d}((\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma''})) \\ \downarrow & & \downarrow \\ HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})) \otimes HZ_*(\boldsymbol{\sigma''}) & \longrightarrow & HZ_{*-4d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma''})) \end{array}.$$

In symbols, given $\alpha \in HZ_*(\boldsymbol{\sigma})$, $\beta \in HZ_*(\boldsymbol{\sigma'})$ *and* $\gamma \in HZ_*(\boldsymbol{\sigma''})$,

$$(\alpha \circledast \beta) \circledast \gamma = \alpha \circledast (\beta \circledast \gamma).$$

*This is also true for the relative case where one of the initial groups (*e.g. $HZ_*(\boldsymbol{\sigma''})$*) is replaced by a relative homology group (*e.g. $HZ_*(\boldsymbol{\sigma''}, \boldsymbol{\sigma}^{(3)})$*) while the other groups are still absolute homology groups.*

Proof. We first remark that

$$\widetilde{B}_{n+n'+1,n''}\left(\widetilde{B}_{n,n'}(\mathbf{w},\mathbf{w}'),\mathbf{w}''\right) = \left(\mathbf{w}, \sum_{k=1}^{n'}(-1)^{k+1}w'_k, \mathbf{w}', \sum_{l=1}^{n''}(-1)^{l+1}w''_l, \mathbf{w}''\right)$$
$$= \widetilde{B}_{n,n'+n''+1}\left(\mathbf{w}, \widetilde{B}_{n',n''}(\mathbf{w}',\mathbf{w}'')\right), \quad \forall \mathbf{w},\mathbf{w}',\mathbf{w}'',$$

so that $B_{n+n'+1,n''}\circ(B_{n,n'} * \mathrm{id}) = B_{n,n'+n''+1}\circ(\mathrm{id} * B_{n',n''})$. Here, we denote by $f*g$ the map $(f*g)[a:b] := [f(a):g(b)]$. Therefore, for all $\alpha\in HZ_*(\boldsymbol{\sigma})$, $\beta\in HZ_*(\boldsymbol{\sigma}')$ and $\gamma\in HZ_*(\boldsymbol{\sigma}'')$ (or $\gamma\in HZ_*(\boldsymbol{\sigma}'',\boldsymbol{\sigma}^{(3)})$),

$$\begin{aligned}
&(B_{n+n'+1,n''})_*\mathrm{pj}_*((B_{n,n'})_*(\mathrm{pj}_*(\alpha\times\beta)\times\gamma))\\
&\quad= (B_{n+n'+1,n''})_*(B_{n,n'} * \mathrm{id})_*\mathrm{pj}_*(\mathrm{pj}_*(\alpha\times\beta)\times\gamma)\\
&\quad= (B_{n,n'+n''+1})_*(\mathrm{id} * B_{n',n''})_*\mathrm{pj}_*(\mathrm{pj}_*(\alpha\times\beta)\times\gamma)\\
&\quad= (B_{n,n'+n''+1})_*(\mathrm{id} * B_{n',n''})_*\mathrm{pj}_*(\alpha\times\mathrm{pj}_*(\beta\times\gamma))\\
&\quad= (B_{n,n'+n''+1})_*\mathrm{pj}_*(\alpha\times(B_{n',n''})_*\mathrm{pj}_*(\beta\times\gamma)),
\end{aligned}$$

where we use the naturality of the homology projective join (22) to get the first and last identity, the previous remark to get the second equality and the associativity of the homology projective join (Proposition A.2) to get the third equality. □

4.2. **The direct system of $G_*^{(a,b)}(\boldsymbol{\sigma})$.** For a fixed $m\in\mathbb{N}$, the long exact sequence of the triple induces inclusion and boundary morphisms fitting into a long exact sequence:

$$\cdots \xrightarrow{\partial_{*+1}} G_*^{(a,b)}(\boldsymbol{\sigma},m) \to G_*^{(a,c)}(\boldsymbol{\sigma},m) \to G_*^{(b,c)}(\boldsymbol{\sigma},m) \xrightarrow{\partial_*} G_{*-1}^{(a,b)}(\boldsymbol{\sigma},m) \to \cdots$$

where $-m\le a\le b\le c\le m$. In order to precisely define these maps without reference of m anymore, we will define an isomorphism

$$\theta_m^{m+1} : G_*^{(a,b)}(\boldsymbol{\sigma},m) \to G_*^{(a,b)}(\boldsymbol{\sigma},m+1), \tag{7}$$

for $-m\le a\le b\le m$, that commutes with the above mentioned inclusion and boundary morphisms and define $G_*^{(a,b)}(\boldsymbol{\sigma})$ as the direct limit of the direct system induced by $(\theta_m^{m+1})_m$:

$$G_*^{(a,b)}(\boldsymbol{\sigma}) := \varinjlim G_*^{(a,b)}(\boldsymbol{\sigma},m).$$

We will then have inclusion and boundary morphisms

$$\cdots \xrightarrow{\partial_{*+1}} G_*^{(a,b)}(\boldsymbol{\sigma}) \to G_*^{(a,c)}(\boldsymbol{\sigma}) \to G_*^{(b,c)}(\boldsymbol{\sigma}) \xrightarrow{\partial_*} G_{*-1}^{(a,b)}(\boldsymbol{\sigma}) \to \cdots$$

for all $a\le b\le c$ that are not action values; one can thus set

$$G_*^{(-\infty,b)}(\boldsymbol{\sigma}) := \varprojlim G_*^{(a,b)}(\boldsymbol{\sigma}), \quad a\to-\infty,$$

and one can then define $G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})$ by taking a direct limit in a similar way. The inclusion and boundary maps thus extend to the extended real numbers line $\overline{\mathbb{R}} := \mathbb{R}\cup\{-\infty,+\infty\}$.

In order to define the isomorphism (7), let us remark that for an odd $n\in\mathbb{N}$, the space $Z(\boldsymbol{\varepsilon}^n)$ retracts on the projectivization of the maximal non-positive linear subspace of Q_n which is a $\mathbb{C}\mathrm{P}^{N-1}$ with $N=(d+1)(n+1)/2$ according to Proposition 4.1.

Therefore,

$$HZ_*(\boldsymbol{\varepsilon}^n) = \bigoplus_{k=-(d+1)(n-1)/2}^{d} Ra_k^{(n)} \simeq H_{*+(n-1)(d+1)}\left(\mathbb{C}\mathrm{P}^{(d+1)(n+1)/2-1}\right),$$

where $a_k^{(n)}$ is the generator of degree $2k$ identified with the class $[\mathbb{C}\mathrm{P}^l]$ of appropriate degree $2l = 2k + (n-1)(d+1)$ under the isomorphism induced by the inclusion of a maximal complex projective subspace of $Z(\boldsymbol{\varepsilon}^n)$. With the help of the composition defined in the previous section, we now define (7) by

$$\theta_m^{m+1}(\alpha) := \alpha \circledast a_d^{(n_0-1)} \in G_*^{(a,b)}(\boldsymbol{\sigma}, m+1), \quad \forall \alpha \in G_*^{(a,b)}(\boldsymbol{\sigma}, m),$$

where $n_0 \in \mathbb{N}$ has been define in the paragraph above Equation (5). This is formally well-defined since

$$HZ_*((\boldsymbol{\sigma}_{m,b}, \boldsymbol{\varepsilon}^{n_0}), (\boldsymbol{\sigma}_{m,a}, \boldsymbol{\varepsilon}^{n_0})) = G_*^{(a,b)}(\boldsymbol{\sigma}, m+1),$$

according to (5).

Proposition 4.5. *For an odd $n \in \mathbb{N}$, the morphism*

$$\begin{cases} HZ_*(\boldsymbol{\sigma}_{m,b}, \boldsymbol{\sigma}_{m,a}) & \to & HZ_*((\boldsymbol{\sigma}_{m,b}, \boldsymbol{\varepsilon}^{n+1}), (\boldsymbol{\sigma}_{m,a}, \boldsymbol{\varepsilon}^{n+1})) \\ \alpha & \mapsto & \alpha \circledast a_d^{(n)} \end{cases}$$

is an isomorphism (and the same is true for $\alpha \mapsto a_d^{(n)} \circledast \alpha$).

Corollary 4.6. *The morphism θ_m^{m+1} is an isomorphism.*

Proof. Let $A := Z(\boldsymbol{\sigma}_{m,b}, \boldsymbol{\sigma}_{m,a})$, $A' := Z((\boldsymbol{\sigma}_{m,b}, \boldsymbol{\varepsilon}^{n+1}), (\boldsymbol{\sigma}_{m,a}, \boldsymbol{\varepsilon}^{n+1}))$, $B := Z(\boldsymbol{\varepsilon}^n)$, n' be the size of $\boldsymbol{\sigma}_{m,b}$ and let us denote by θ the morphism in question. Up to a shift in degree, the morphism θ can be written explicitly as the composition

$$H_*(A) \xrightarrow{\mathrm{pj}_*(\cdot \times a_d^{(n)})} H_{*+i_0}(A * B) \xrightarrow{(B_{n',n})_*} H_{*+i_0}(A'),$$

for some $i_0 \in \mathbb{N}$. According to Corollary A.6, the first morphism is an isomorphism. It remains to show that $(B_{n',n})_*$ is also an isomorphism.

Let C be the following automorphism of $(\mathbb{C}^{d+1})^{n'+n+1}$:

$$C(\mathbf{v}, \mathbf{v}') := (\mathbf{v}, \mathbf{v}' + (v_1, v_{n'}, v_1, v_{n'}, \dots, v_1, v_{n'})),$$

where $\mathbf{v} \in (\mathbb{C}^{d+1})^{n'}$ and $\mathbf{v}' \in (\mathbb{C}^{d+1})^{n+1}$. By a direct computation

$$F_{(\boldsymbol{\sigma}_{m,t}, \boldsymbol{\varepsilon}^{n+1})} \circ C(\mathbf{v}, \mathbf{v}') = F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{v}) + Q_{n+2}(0, \mathbf{v}'),$$

(this is an explicit version of [1, Proposition 5.2]). According to Givental [18, Proposition B.1], $\{\widehat{F}_{(\boldsymbol{\sigma}_{m,t}, \boldsymbol{\varepsilon}^{n+1})} \circ C \leq 0\}$ retracts on $\{\widehat{F}_{\boldsymbol{\sigma}_{m,t}} \leq 0\} * \{\widehat{Q}_{n+2}(0, \cdot) \leq 0\}$. The quadratic form $Q_{n+2}(0, \cdot)$ is non-degenerate with index $(n+1)(d+1)$, so that the sublevel set $\{\widehat{Q}_{n+2}(0, \cdot) \leq 0\}$ retracts on a complex projective space of $\mathbb{C}$-dimension $(d+1)(n+1)/2 - 1$.

The injective linear map $J : (\mathbb{C}^{d+1})^n \to (\mathbb{C}^{d+1})^{n+1}$, $J\mathbf{v} := (\mathbf{v}, v_1)$ satisfies

$$Q_{n+2}(0, J\mathbf{v}) = Q_n(\mathbf{v}).$$

Since the sum of the index and the nullity of Q_n is $(n+1)(d+1)$, $\{\widehat{Q}_n \leq 0\}$ retracts on a projective space of $\mathbb{C}$-dimension $(d+1)(n+1)/2 - 1$ in such a way that J induces an isomorphism in homology

$$J_* : H_*\left(\left\{\widehat{Q}_n \leq 0\right\}\right) \xrightarrow{\simeq} H_*\left(\left\{\widehat{Q}_{n+2}(0, \cdot) \leq 0\right\}\right).$$

Let $P : (\mathbb{C}^{d+1})^{n+1} \to (\mathbb{C}^{d+1})^n$ be the surjective linear map

$$P(v_1, \dots, v_{n+1}) := (v_{n+1}, v_2, v_3, \dots, v_n)$$

so that $PJ = \mathrm{id}$. Let $P' : (\mathbb{C}^{d+1})^{n+n'+1} \to (\mathbb{C}^{d+1})^{n+n'}$ be $P'(\mathbf{v}, \mathbf{v}') = (\mathbf{v}, P\mathbf{v}')$ and let $J' : (\mathbb{C}^{d+1})^{n+n'} \to (\mathbb{C}^{d+1})^{n+n'+1}$ be $J'(\mathbf{v}, \mathbf{v}') = (\mathbf{v}, J\mathbf{v}')$ so that $P'J' = \mathrm{id}$. In v-variables, $\widetilde{B}_{n',n}$ takes the form

$$(\mathbf{v}, \mathbf{v}') \mapsto (\mathbf{v}, v_1, \mathbf{v}' + (v_1' - v_1, v_1 - v_1', v_1' - v_1, \dots, v_1 - v_1')).$$

A direct computation then shows that the endomorphism $\tilde{f} := P'C^{-1}\widetilde{B}_{n',n}$ is invertible. More precisely, $\tilde{f}(\mathbf{v}, \mathbf{v}') = (\mathbf{v}, \tilde{g}(\mathbf{v}') + \tilde{h}(\mathbf{v}))$ where $\tilde{g}$ and $\tilde{h}$ are $\mathbb{C}$-linear and $\tilde{g}$ is invertible. Let $f : A * B \to A * B$ be the induced projective map.

We then have the following commutative diagram:

$$\begin{array}{ccc}
H_*(A * B) & \xrightarrow{\quad (B_{n',n})_* \quad} & H_*(A') \\
\simeq \Big\downarrow f_* & & C_* \Big\uparrow \simeq \\
H_*(A * B) & \underset{P'_*}{\overset{J'_*}{\rightleftarrows}} \; \simeq & H_*\left(A * \left\{\widehat{Q}_{n+2}(0, \cdot) \leq 0\right\}\right)
\end{array},$$

By the above discussion, the induced maps f_*, C_*, J'_* and P'_* are isomorphisms. Therefore, $(B_{n',n})_*$ is an isomorphism and so is θ. $\square$

By construction of the map θ_m^{m+1}, the following diagram commutes.

$$\begin{array}{ccc}
G_*^{(a,b)}(\boldsymbol{\sigma}, m) & \xrightarrow[\simeq]{\quad \theta_m^{m+1} \quad} & G_*^{(a,b)}(\boldsymbol{\sigma}, m+1) \\
\Big\uparrow & & \Big\uparrow \\
HZ_*(\boldsymbol{\sigma}_{m,b}) & \xrightarrow{\quad \cdot \circledast a_d^{(n_0-1)} \quad} & HZ_*(\boldsymbol{\sigma}_{m+1,b}) \\
\Big\downarrow & & \Big\downarrow \\
H_{*+r}(\mathbb{C}\mathrm{P}^N) & & \\
\Big\downarrow & & \Big\downarrow \\
H_{*+r}(\mathbb{C}\mathrm{P}^{N+n_0(d+1)}) & \xrightarrow{\mathrm{pj}_*(\cdot \times [\mathbb{C}\mathrm{P}^{(d+1)n_0/2}])} & H_{*+r+n_0(d+1)}(\mathbb{C}\mathrm{P}^{N+n_0(d+1)})
\end{array}, \tag{8}$$

where the vertical arrows are induced by inclusions, $r := (n_1 + mn_0 - 1)(d+1)$, n_1 is the size of $\boldsymbol{\sigma}$, $N := (n_1 + mn_0)(d+1) - 1$ (see the paragraph above Equation (5) for the definition of n_0).

Applying the commutativity of (6) to $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ and $\boldsymbol{\sigma}' = \boldsymbol{\varepsilon}^{n'}$, one has

$$a_k^{(n)} \circledast a_l^{(n')} = a_{k+l-d}^{(n+n'+1)},$$

for $-(d+1)(n-1)/2 \leq k \leq d$ and $-(d+1)(n'-1)/2 \leq l \leq d$. By associativity of $\circledast$ (Proposition 4.4), we deduce that the isomorphism $\theta_m^{m+k} := \theta_{m+k-1}^{m+k} \circ \dots \circ \theta_m^{m+1}$ is $\theta_m^{m+k}(\alpha) = \alpha \circledast a_d^{(kn_0-1)}$.

By using the same construction as for θ_m^{m+k}, one can define an isomorphism

$$\eta_k : G_*^{(a,b)}(\boldsymbol{\sigma}, m) \xrightarrow{\simeq} G_*^{(a,b)}((\boldsymbol{\varepsilon}^k, \boldsymbol{\sigma}), m),$$

sending α to $a_d^{(k-1)} \circledast \alpha$ for each even $k \in \mathbb{N}$. This isomorphism commutes with the direct system induced by the θ_m^{m+1}'s and the inclusion maps and makes a diagram similar to (8) commute where in particular $\boldsymbol{\sigma}_{m+1,b}$ is replaced by $(\boldsymbol{\varepsilon}^k, \boldsymbol{\sigma})_{m,b}$. The commutation with the direct system induces a natural final isomorphism

$$\eta_k : G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_*^{(a,b)}((\boldsymbol{\varepsilon}^k, \boldsymbol{\sigma})).$$

4.3. **Interpolation isomorphisms.** We start this section with a general statement that is easily deduced from Morse theory.

Proposition 4.7. *Let X be a closed manifold and $m > 0$. Let $f_{s,t} : X \to \mathbb{R}$, $s \in [0,1]$, $t \in [-m,m]$, be a C^1-family of maps. We suppose that for all $s \in [0,1]$, $t \in (-m,m)$ and $x \in X$, $\frac{\mathrm{d}}{\mathrm{d}t} f_{s,t}(x) \leq 0$. If $a, b \in (-m,m)$, $a \leq b$, satisfy that 0 is a regular value of $f_{s,a}$ and $f_{s,b}$ for all $s \in [0,1]$, then the inclusion $X \hookrightarrow [0,1] \times X$, $x \mapsto (s,x)$, induces the following isomorphism in homology for all $s \in [0,1]$*

$$H_* \left(\{f_{s,b} \leq 0\}, \{f_{s,a} \leq 0\}\right) \xrightarrow{\simeq} H_* \left(\{(r,x) \mid f_{r,b}(x) \leq 0\}, \{(r,x) \mid f_{r,a}(x) \leq 0\}\right),$$

where (r,x) are describing $[0,1] \times X$ in the right hand side of the arrow. The analogous non-relative statement is also true: let $f_s : X \to \mathbb{R}$, $s \in [0,1]$, be a C^1-family of maps with 0 as a regular value. Then the inclusion $X \hookrightarrow [0,1] \times X$, $x \mapsto (s,x)$, induces the following isomorphism in homology for all $s \in [0,1]$

$$H_* \left(\{f_s \leq 0\}\right) \xrightarrow{\simeq} H_* \left(\{(r,x) \mid f_r(x) \leq 0\}\right).$$

Proof. For any interval $I \subset [0,1]$, let $f_{I,t} : I \times X \to \mathbb{R}$ be the map $f_{I,t}(r,x) := f_{r,t}(x)$. Let $a \leq b$ be real numbers as above. By compacity of $[0,1]$, there exists an $\varepsilon > 0$ such that $[-\varepsilon, 2\varepsilon]$ contains only regular values of $f_{s,a}$ and $f_{s,b}$ for all $s \in [0,1]$. By compacity, there also exists a $\delta > 0$ such that $\|f_{s,c} - f_{r,c}\|_\infty < \varepsilon$ for $|s - r| \leq \delta$ and $c \in \{a,b\}$.

We recall that if topological pairs $A \subset B \subset C \subset D$ satisfy that the maps induced by inclusions $H_*(A) \to H_*(C)$ and $H_*(B) \to H_*(D)$ are isomorphisms, then the map induced by inclusion $H_*(B) \to H_*(C)$ is also an isomorphism. We apply this result to $A := (\{f_{I,b} \leq -\varepsilon\}, \{f_{I,a} \leq -\varepsilon\})$, $B := (I \times \{f_{s,b} \leq 0\}, I \times \{f_{s,a} \leq 0\})$, $C := (\{f_{I,b} \leq \varepsilon\}, \{f_{I,a} \leq \varepsilon\})$ and $D := (I \times \{f_{s,b} \leq 2\varepsilon\}, I \times \{f_{s,a} \leq 2\varepsilon\})$ for $I \subset [0,1]$ an interval of length less than δ and $s \in I$. Indeed, these topological pairs are increasing for $\subset$ by definition of δ and the needed isomorphisms come from the Morse deformation lemma which can be applied by definition of ε. We thus have the following commutative diagram:

$$\begin{array}{ccc} H_*(I \times \{f_{s,b} \leq 0\}, I \times \{f_{s,a} \leq 0\}) & \xrightarrow{\simeq} & H_*(\{f_{I,b} \leq \varepsilon\}, \{f_{I,a} \leq \varepsilon\}) \\ \simeq \uparrow & & \simeq \uparrow \\ H_*(\{s\} \times \{f_{s,b} \leq 0\}, \{s\} \times \{f_{s,a} \leq 0\}) & \xrightarrow{\simeq} & H_*(\{f_{I,b} \leq 0\}, \{f_{I,a} \leq 0\}) \end{array},$$

where every map is induced by inclusion. The top arrow is an isomorphism by the above general fact. The left hand side arrow is an isomorphism because I retracts on $\{s\}$. The right hand side arrow is an isomorphism by the Morse deformation lemma, since $[0,\varepsilon]$ contains only regular values of $f_{I,a}$ and $f_{I,b}$. Therefore, the bottom arrow is an isomorphism.

According to the Mayer-Vietoris long exact sequence, given topological pairs A and B, if the inclusion maps $H_*(A \cap B) \to H_*(A)$ and $H_*(A \cap B) \to H_*(B)$ are isomorphisms, then the inclusion map $H_*(A \cap B) \to H_*(A \cup B)$ is an isomorphism. We apply this result to $A := (\{f_{I,b} \leq 0\}, \{f_{I,a} \leq 0\})$ and $B := (\{f_{J,b} \leq 0\}, \{f_{J,a} \leq 0\})$ for increasing length of I and length$(J) \leq \delta$ to show inductively the result. □

Another way to proceed is to remark that $(\{f_{I,b} \leq 0\} \setminus \{f_{I,a} < 0\}, \{f_{I,a} = 0\})$ retracts on $(\{f_{s,b} \leq 0\} \setminus \{f_{s,a} < 0\}, \{f_{s,a} = 0\})$ relative to boundaries through the gradient flow of the restriction of the projection $I \times X \to I$, which has no critical points under the hypothesis made on a and b.

Let $s \mapsto \boldsymbol{\sigma}^{(s)}$ be a C^1-family (or more generally C^1-piecewise) of tuples associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism Φ without $\mathbb{C}$-lines of fixed points. We apply Proposition 4.7 to the following family of maps:

$$f_s := \widehat{F}_{\boldsymbol{\sigma}^{(s)}} : \mathbb{C}\mathrm{P}^N \to \mathbb{R}.$$

The assumptions of Proposition 4.7 are fulfilled and we define Δ so that the following diagram commutes:

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}^{(0)}) & \xrightarrow{\simeq} & H_{*+i_0}(A) \\ \simeq\big\downarrow \Delta & \nearrow_{\simeq} & \\ HZ_*(\boldsymbol{\sigma}^{(1)}) & & \end{array},$$

where the non-vertical arrows are the inclusions maps and

$$A := \{(r,x) \mid f_r(x) \leq 0\}.$$

Since these isomorphisms are defined with inclusion maps the above way, they clearly commute with inclusion and boundary morphisms. In the same way, let $(\eta_t^{(s)})$ be a C^1-family of tuples so that for each s the family $s \mapsto \eta_t^{(s)}$ is associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism. If the associated map $f_{s,t} : \mathbb{C}\mathrm{P}^N \to \mathbb{R}$ satisfies the assumption of Proposition 4.7, then we can define the associated isomorphism

$$\Delta : HZ_*(\boldsymbol{\eta}_b^{(0)}, \boldsymbol{\eta}_a^{(0)}) \xrightarrow{\simeq} HZ_*(\boldsymbol{\eta}_b^{(1)}, \boldsymbol{\eta}_a^{(1)}).$$

As an important example, let $s \mapsto \boldsymbol{\sigma}^{(s)}$ be a C^1-family (or more generally C^1-piecewise) of tuples associated with the same $\mathbb{C}$-equivariant Hamiltonian diffeomorphism Φ. If $a \leq b$ are not action values of $\boldsymbol{\sigma}$, then $\eta_t^{(s)} := \boldsymbol{\sigma}_{m,t}^{(s)}$ satisfies the above assumption and Δ is an isomorphism

$$\Delta : G_*^{(a,b)}(\boldsymbol{\sigma}^{(0)}, m) \xrightarrow{\simeq} G_*^{(a,b)}(\boldsymbol{\sigma}^{(1)}, m).$$

We will call the isomorphism Δ the interpolation isomorphism associated with $(\boldsymbol{\sigma}^{(s)})$ and write in symbols $\Delta \longleftrightarrow s \mapsto \boldsymbol{\sigma}^{(s)}$.

Proposition 4.8. *Let Δ, Δ' and Δ'' be the interpolation isomorphisms associated with $(\boldsymbol{\sigma}^{(s)})$, $(\boldsymbol{\eta}_t^{(s)})$ and $(\boldsymbol{\sigma}^{(s)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_t^{(s)})$ respectively. The following diagram commutes:*

$$\begin{array}{ccc} HZ_*\left(\boldsymbol{\sigma}^{(0)}\right) \otimes HZ_*\left(\boldsymbol{\eta}_1^{(0)}, \boldsymbol{\eta}_0^{(0)}\right) & \xrightarrow{\circledast} & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}^{(0)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{(0)}\right), \left(\boldsymbol{\sigma}^{(0)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{(0)}\right)\right) \\ \simeq\big\downarrow \Delta\otimes\Delta' & & \simeq\big\downarrow \Delta'' \\ HZ_*\left(\boldsymbol{\sigma}^{(1)}\right) \otimes HZ_*\left(\boldsymbol{\eta}_1^{(1)}, \boldsymbol{\eta}_0^{(1)}\right) & \xrightarrow{\circledast} & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{(1)}\right), \left(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{(1)}\right)\right) \end{array}.$$

Proof. One can assume that either $(\boldsymbol{\sigma}^{(s)})$ or $(\boldsymbol{\eta}_t^{(s)})$ is independent of s. The proof of the proposition is a consequence of the naturality of $(B_{n,n'})_*$ and a slightly generalized version of pj_* to projective bundles. Let $I := [0,1]$ and let us extend pj_* to subsets of $I \times \mathbb{C}\mathrm{P}^N$ the following way. Let $A \subset \mathbb{C}\mathrm{P}^n$ and $B \subset I \times \mathbb{C}\mathrm{P}^m$, we set $B_s := B \cap \{s\} \times \mathbb{C}\mathrm{P}^m$ and define $A * B \subset I \times \mathbb{C}\mathrm{P}^{m+n+1}$ by $A * B := \bigcup_s \{s\} \times (A * B_s)$. Let $E_{A,B}$

be the set of those $(a,(s,b),(t,c))$'s with $a \in A$, $(s,b) \in B$ and $(t,c) \in A * B$ such that $s = t$ and $c \in (ab)$; let $p_1 : E_{A,B} \to A \times B$ and $p_2 : E_{A,B} \to A * B$ be associated projection maps. Now p_1 is a $\mathbb{C}\mathrm{P}^1$-fiber bundle and $\mathrm{pj}_* : H_*(A \times B) \to H_{*+2}(A * B)$ is defined by $(p_2)_* \circ (p_1)^*$. Since E_{A,B_s} is the restriction of the fiber bundle $E_{A,B}$ to $A \times B_s$, by naturality of the morphisms involved, the following diagram commutes for all $s \in I$:

$$\begin{array}{ccc} H_*(A \times B_s) & \xrightarrow{\mathrm{pj}_*} & H_{*+2}(A * B_s) \\ \downarrow & & \downarrow \\ H_*(A \times B) & \xrightarrow{\mathrm{pj}_*} & H_{*+2}(A * B) \end{array},$$

where the vertical arrows are inclusion morphisms induced by $\{s\} \hookrightarrow I$. By giving to $Z(\boldsymbol{\eta}_t^I)$ the meaning of $\bigcup_s \{s\} \times Z(\boldsymbol{\eta}_t^{(s)})$ and then extending the definition of HZ_* accordingly, we deduce that the following diagram commutes for all $s \in I$:

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}) \otimes HZ_*\left(\boldsymbol{\eta}_1^{(s)}, \boldsymbol{\eta}_0^{(s)}\right) & \xrightarrow{\circledast} & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{(s)}\right), \left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{(s)}\right)\right) \\ \downarrow \simeq & & \downarrow \simeq \\ HZ_*(\boldsymbol{\sigma}) \otimes HZ_*\left(\boldsymbol{\eta}_1^{I}, \boldsymbol{\eta}_0^{I}\right) & \xrightarrow{\circledast} & HZ_{*-2d}\left(\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_1^{I}\right), \left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\eta}_0^{I}\right)\right) \end{array},$$

where the vertical arrows are inclusion morphisms. The conclusion follows. ☐

In particular, the interpolation isomorphism $\Delta : G_*^{(a,b)}(\boldsymbol{\sigma}^{(0)}, m) \to G_*^{(a,b)}(\boldsymbol{\sigma}^{(1)}, m)$ commutes with the direct system (θ_m^{m+1}), so it ultimately defines the interpolation isomorphism

$$\Delta : G_*^{(a,b)}(\boldsymbol{\sigma}^{(0)}) \xrightarrow{\simeq} G_*^{(a,b)}(\boldsymbol{\sigma}^{(1)})$$

that commutes with inclusion and boundary morphisms.

We are now in position to prove that $G_*^{(a,b)}(\boldsymbol{\sigma})$ and its inclusion and boundary morphisms are independent, up to isomorphism, of the choice of continuous family of n-tuples of small Hamiltonian flows $(\boldsymbol{\sigma}^s)$ generating the $\mathbb{C}$-equivariant Hamiltonian flow (Φ_s) lifting (φ_s) with $\boldsymbol{\sigma}^0 = \boldsymbol{\varepsilon}^n$ and $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}$. Indeed, let $(\boldsymbol{\sigma}^s)$ and $((\boldsymbol{\sigma}')^s)$ be a n-tuple and a n'-tuple of small Hamiltonian flows generating (Φ_s) with $n \geq n'$. One can define an isomorphism $G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_*^{(a,b)}(\boldsymbol{\sigma}')$ by composition of reduction maps η_k and interpolation maps Δ in the following way:

$$\begin{array}{ccc} G_*^{(a,b)}(\boldsymbol{\sigma}) & \xrightarrow{\qquad\simeq\qquad} & G_*^{(a,b)}(\boldsymbol{\sigma}') \\ \eta_{2n} \uparrow \simeq & & \eta_{3n-n'} \uparrow \simeq \\ G_*^{(a,b)}((\boldsymbol{\varepsilon}^{2n}, \boldsymbol{\sigma})) & & G_*^{(a,b)}((\boldsymbol{\varepsilon}^{3n-n'}, \boldsymbol{\sigma}')) \\ \simeq \downarrow \Delta & & \Delta'' \uparrow \simeq \\ G_*^{(a,b)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{2n})) & \underset{\simeq}{\xrightarrow{\Delta'}} & G_*^{(a,b)}((\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}, \boldsymbol{\varepsilon}^{n-n'}, \boldsymbol{\sigma}')) \end{array},$$

where Δ, Δ' and Δ'' are interpolation isomorphisms associated with isotopies of $3n$-tuples generating the same Hamiltonian diffeomorphism Φ in the following way

$$\begin{aligned}
\Delta &\longleftrightarrow s \mapsto \begin{cases} (\boldsymbol{\sigma}^{2s}, \boldsymbol{\sigma}^{-2s}, \boldsymbol{\sigma}), & 0 \leq s \leq 1/2, \\ (\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-2(1-s)}, \boldsymbol{\sigma}^{2(1-s)}), & 1/2 \leq s \leq 1, \end{cases} \\
\Delta' &\longleftrightarrow s \mapsto (\boldsymbol{\sigma}, \boldsymbol{\sigma}^{-s}, \boldsymbol{\varepsilon}^{n-n'}, (\boldsymbol{\sigma'})^s), \\
\Delta'' &\longleftrightarrow s \mapsto (\boldsymbol{\sigma}^{1-s}, \boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}^{n-n'}, \boldsymbol{\sigma'}).
\end{aligned}$$

4.4. **Composition of generating functions homologies.** Let us fix 2 tuples $\boldsymbol{\sigma}$, $\boldsymbol{\sigma'}$ of odd respective sizes n and n', $a, b, c \in \mathbb{R}$ that are not action values of $\boldsymbol{\sigma}$ or $\boldsymbol{\sigma'}$. Let $m, m' \in \mathbb{N}$ such that $m > 2m' > 4n$. The composition map has the form

$$HZ_*(\boldsymbol{\sigma'}_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m) \xrightarrow{\circledast} HZ_*(\boldsymbol{\eta}_b, \boldsymbol{\eta}_a),$$

where

$$\boldsymbol{\eta}_t := \left(\boldsymbol{\sigma'}, \boldsymbol{\delta}_c^{(m')}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\delta}_t^{(m)}\right).$$

Let $(\boldsymbol{\eta}_t^s)_s$ be a homotopy of tuples of small Hamiltonians from $\boldsymbol{\eta}_t^0 = \boldsymbol{\eta}_t$ to $\boldsymbol{\eta}_t^1 = (\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})_{t+c,m+m'}$, for $2|t| < m$, generating the same diffeomorphism for a fixed value of t. The condition $m' > 2m > 4n'$ makes the construction of such a homotopy possible, we sketch the successive stages of it:

$$\begin{aligned}
\boldsymbol{\eta}_t = \left(\boldsymbol{\sigma'}, \boldsymbol{\delta}_c^{(m')}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\delta}_t^{(m)}\right) &\rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}^{m'n_0+1}, \boldsymbol{\sigma}, \boldsymbol{\delta}_{t+c}^{(m)}\right) \\
&\rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\sigma}^{-1}, \boldsymbol{\varepsilon}^k, \boldsymbol{\sigma}, \boldsymbol{\delta}_{t+c}^{(m)}\right) \rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{m'n_0}, \boldsymbol{\delta}_{t+c}^{(m)}\right) \\
&\rightsquigarrow \left(\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\delta}_{t+c}^{(m+m')}\right) = (\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})_{t+c,m+m'},
\end{aligned}$$

(see the paragraph above Equation (5) for the definition of n_0). According to the previous section, this homotopy induces an interpolation isomorphism

$$\Delta : HZ_*(\boldsymbol{\eta}_b, \boldsymbol{\eta}_a) \xrightarrow{\simeq} G_*^{(a+c,b+c)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'}), m + m').$$

The composition of the above composition morphism with Δ gives this generating functions homology version of the composition morphism:

$$HZ_*(\boldsymbol{\sigma'}_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m) \to G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}), m + m').$$

We define the same way composition morphism of absolute homology groups:

$$HZ_*(\boldsymbol{\sigma}_{m,t}) \otimes HZ_*(\boldsymbol{\sigma'}_{m',t'}) \to HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})_{m+m',t+t'}).$$

We will denote these maps $\alpha \otimes \beta \mapsto \alpha \diamond \beta$ so that in symbols

$$\alpha \diamond \beta := \Delta(\alpha \circledast \beta).$$

Since interpolation isomorphisms commute with inclusion in the total space, the commutativity of (6) implies the commutativity of the analogous square

$$\begin{array}{ccc}
HZ_*(\boldsymbol{\sigma}_{m,t}) \otimes HZ_*(\boldsymbol{\sigma'}_{m',t'}) & \xrightarrow{\diamond} & HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma'})_{m+m',t+t'}) \\
\downarrow & & \downarrow \\
H_{*+r}(\mathbb{C}\mathrm{P}^N) \otimes H_{*+r'}(\mathbb{C}\mathrm{P}^{N'}) & \xrightarrow{\mathrm{pj}_*} & H_{*+r+r'+2}(\mathbb{C}\mathrm{P}^{N''})
\end{array} . \tag{9}$$

This new form of the composition morphism is also associative.

Corollary 4.9. *The following diagram of composition morphisms commutes:*

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}_{m,c}) \otimes HZ_*(\boldsymbol{\sigma}'_{m',c'}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}'', m'') & \longrightarrow & HZ_*(\boldsymbol{\sigma}_{m,c}) \otimes G_{*-2d}^{(a+c',b+c')}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''), m'+m'') \\ \downarrow & & \downarrow \\ HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}'', m'') & \longrightarrow & G_{*-4d}^{(a+c+c',b+c+c')}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''), m+m'+m'') \end{array}.$$

In symbols, given $\alpha \in HZ_*(\boldsymbol{\sigma}_{m,c})$, $\beta \in HZ_*(\boldsymbol{\sigma}'_{m',c'})$ *and* $\gamma \in G_*^{(a,b)}(\boldsymbol{\sigma}'', m'')$,

$$(\alpha \diamond \beta) \diamond \gamma = \alpha \diamond (\beta \diamond \gamma).$$

Proof. According to Proposition 4.8,

$$(\alpha \diamond \beta) \diamond \gamma = \Delta_2(\Delta_1(\alpha \circledast \beta) \circledast \gamma) = \Delta_2 \circ \widetilde{\Delta}_1((\alpha \circledast \beta) \circledast \gamma),$$

where the interpolation isomorphisms are associated with homotopies in the following way:

$$\begin{aligned} \Delta_1 &\longleftrightarrow (\boldsymbol{\sigma}_{m,c}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}'_{m',c'}) \rightsquigarrow (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}, \\ \widetilde{\Delta}_1 &\longleftrightarrow ((\boldsymbol{\sigma}_{m,c}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}'_{m',c'}), \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''_{m'',t}) \rightsquigarrow ((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''_{m'',t}), \\ \Delta_2 &\longleftrightarrow ((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',c+c'}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}''_{m'',t}) \rightsquigarrow (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}'')_{m+m'+m'',c+c'+t}. \end{aligned}$$

In the same way,

$$\alpha \diamond (\beta \diamond \gamma) = \Delta_2'(\alpha \circledast \Delta_1'(\beta \circledast \gamma)) = \Delta_2' \circ \widetilde{\Delta}_1'(\alpha \circledast (\beta \circledast \gamma)),$$

with convenient interpolation isomorphisms Δ_1', $\widetilde{\Delta}_1'$ and Δ_2'. According to the associativity of $\circledast$ (Proposition 4.4), it is enough to prove that $\Delta_2 \circ \widetilde{\Delta}_1 = \Delta_2' \circ \widetilde{\Delta}_1'$. These two interpolation isomorphisms are associated with homotopies that are themselves homotopic through homotopies preserving the associated family of diffeomorphisms. It is then a simple consequence of the definition of the interpolation isomorphisms. ☐

As a consequence, the following diagram commutes

$$\begin{array}{ccc} HZ_*(\boldsymbol{\sigma}'_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m) & \xrightarrow{\diamond} & G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}), m+m') \\ \simeq \Big\downarrow \mathrm{id} \otimes \theta_m^{m+1} & & \simeq \Big\downarrow \theta_{m+m'}^{m+m'+1} \\ HZ_*(\boldsymbol{\sigma}'_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}, m+1) & \xrightarrow{\diamond} & G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma}), m+m'+1) \end{array}.$$

It ultimately defines a morphism

$$HZ_*(\boldsymbol{\sigma}'_{m',c}) \otimes G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\diamond} G_{*-2d}^{(a+c,b+c)}((\boldsymbol{\sigma}', \boldsymbol{\varepsilon}, \boldsymbol{\sigma})),$$

for almost all $a, b \in \overline{\mathbb{R}}$.

By naturality of the composition morphism, given $t \geq 0$ and $m \in \mathbb{N}^*$, the following diagram commutes

$$\begin{array}{ccc} G_*^{(a,b)}(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\varepsilon}_{m,0}) & \xrightarrow{\diamond} & G_{*-2d}^{(a,b)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^2)) \\ \Big\downarrow \mathrm{id} \otimes \mathrm{inc}_* & & \Big\downarrow \mathrm{inc}_* \\ G_*^{(a,b)}(\boldsymbol{\sigma}) \otimes HZ_*(\boldsymbol{\varepsilon}_{m,t}) & \xrightarrow{\diamond} & G_{*-2d}^{(a+t,b+t)}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^2)) \end{array}. \tag{10}$$

4.5. **Properties of the generating functions homology.** Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ be two different tuples of (h_s). We proved that the graded modules $G_*^{(a,b)}(\boldsymbol{\sigma})$ and $G_*^{(a,b)}(\boldsymbol{\sigma}')$ are isomorphic and that there exists a family of isomorphism compatible with the inclusion maps so it makes sense to define $G_*^{(a,b)}(h_s)$ in Section 2. Nevertheless, we will keep track of the specific choice of $\boldsymbol{\sigma}$ in our statements for the sake of being precise.

Let us first focus on the special case $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}$. Let us denote by $T_{m,t}$ the family of generating functions associated with $(\boldsymbol{\varepsilon}_{m,t})_t$. Since the elementary generating function of δ_s is $u \mapsto -\tan(\pi s)\|u\|^2$, the map $T_{m,t}$ is a quadratic form. Since $T_{m,t}$ is a generating function, its kernel as a quadratic form has dimension $2(d+1)$. We had already remarked that $T_{m,0}$ is equivalent to $-T_{m,0}$ (they both generates the identity) which implies that $\operatorname{ind} T_{m,0} = mn_0(d+1)$ (Proposition 4.1). The variation of index is governed by the Maslov index of $(e^{-2i\pi t})$ so that

$$\operatorname{ind} T_{m,t} - \operatorname{ind} T_{m,0} = 2(d+1)\lfloor t \rfloor,$$

(See [1, Section 3 and Lemma 5.5]). Therefore, there exists an increasing sequence of complex projective subspaces $P_{m,-m} \subset P_{m,-m+1} \subset \cdots \subset P_{m,m}$ such that $P_{m,k} \simeq \mathbb{C}\mathrm{P}^{(d+1)(k+mn_0/2)}$ and $Z(\boldsymbol{\varepsilon}_{m,t})$ retracts on $P_{m,\lfloor t \rfloor}$ inducing an equivalence between the persistence modules $(H_*(P_{m,\lfloor t \rfloor}))$ and $(H_*(Z(\boldsymbol{\varepsilon}_{m,t})))$, $-m \leq t \leq m$. Thus, as a graded R-module,

$$HZ_*(\boldsymbol{\varepsilon}_{m,t}) = \bigoplus_{k=-(d+1)mn_0/2}^{d+(d+1)\lfloor t \rfloor} R a_k^{(mn_0+1)}(t),$$

where $a_k^{(mn_0+1)}(t)$ is the generator of degree $2k$ identified with the class $[\mathbb{C}\mathrm{P}^l]$ of appropriate degree $2l = 2k + (d+1)mn_0$ under the isomorphism induced by $P_{m,\lfloor t \rfloor} \hookrightarrow Z(\boldsymbol{\varepsilon}_{m,t})$. The inclusion morphism $HZ_*(\boldsymbol{\varepsilon}_{m,t}) \to HZ_*(\boldsymbol{\varepsilon}_{m,s})$ maps each $a_k^{(mn_0+1)}(t)$ to $a_k^{(mn_0+1)}(s)$ (for $-m \leq t \leq s \leq m$). Hence,

$$G_*^{(a,b)}(\boldsymbol{\varepsilon}, m) = \bigoplus_{k=d+(d+1)\lfloor a \rfloor}^{d+(d+1)\lfloor b \rfloor} R \alpha_k^{(m)}(a,b),$$

for $-m < a \leq b < m$, where $\alpha_k^{(m)}(a,b)$ is the image of $a_k^{(mn_0+1)}(b)$ under the inclusion morphism $HZ_*(\boldsymbol{\varepsilon}_{m,b}) \to G_*^{(a,b)}(\boldsymbol{\varepsilon}, m)$. According to the commutativity of (8), one has $\theta_m^{m+1}\alpha_k^{(m)}(a,b) = \alpha_k^{(m+1)}(a,b)$. We set $\alpha_k(a,b) := \theta_m^\infty \alpha_k^{(m)}(a,b)$. For $a < b < c$, if $\alpha_k(b,c)$ is well-defined, then $\alpha_k(a,c)$ is also well-defined and sent to the former. We deduce that there exists a well-defined $\alpha_k(-\infty,c) \in G_{2k}^{(-\infty,c)}(\boldsymbol{\varepsilon})$ sent to $\alpha_k(a,c)$ for all $a \leq c$. Let α_k be the image of $\alpha_k(-\infty,c)$ under $G_{2k}^{(-\infty,c)}(\boldsymbol{\varepsilon}) \to G_{2k}^{(-\infty,+\infty)}(\boldsymbol{\varepsilon})$. Finally,

$$G_*^{(-\infty,+\infty)}(\boldsymbol{\varepsilon}) = \bigoplus_{k\in\mathbb{Z}} R\alpha_k,$$

we will show in Theorem 4.11 that this is also the case for any $\boldsymbol{\sigma}$.

In order to show the "periodicity" of the persistence module of $\boldsymbol{\sigma}$, let us define a natural isomorphism

$$G_*^{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G_{*+2(d+1)}^{(a+1,b+1)}(\boldsymbol{\sigma}). \tag{11}$$

In order to simplify the exposition, let us set $a_d := a_d^{(mn_0+1)}(0) \in HZ_{2d}(\boldsymbol{\varepsilon}_{m,0})$ and $a_{2d+1} := a_{2d+1}^{(mn_0+1)}(1) \in HZ_{2(2d+1)}(\boldsymbol{\varepsilon}_{m,1})$. According to Proposition 4.5, the morphism

$G_*^{(a+1,b+1)}(\boldsymbol{\sigma}) \to G_*^{(a+1,b+1)}((\boldsymbol{\varepsilon}^2,\boldsymbol{\sigma}))$, $\alpha \mapsto a_d \diamond \alpha$, is an isomorphism; let us write $\alpha \mapsto a_d^{-1} \diamond \alpha$ its inverse morphism. We define the morphism (11) by $\alpha \mapsto a_d^{-1} \diamond a_{2d+1} \diamond \alpha$.

Proposition 4.10. *The morphism (11) is an isomorphism commuting with inclusion and boundary morphisms.*

Proof. The naturality of this morphism comes from the naturality of $\alpha \mapsto a_d \diamond \alpha$ and $\alpha \mapsto a_{2d+1} \diamond \alpha$. It remains to prove that $\alpha \mapsto a_{2d+1} \diamond \alpha$ is an isomorphism. Let us set $a_{-1} := a_{-1}^{(mn_0+1)}(-1) \in HZ_{-2}(\boldsymbol{\varepsilon}_{m,-1})$. According to the commutativity of (9), $a_{2d+1} \diamond a_{-1} = a_{-1} \diamond a_{2d+1} = a_d$ where a_d is identified with $a_d^{(2mn_0+3)}(0)$. Thus, the following diagram commutes

$$\begin{array}{ccccc} G_*^{(a,b)}(\boldsymbol{\sigma}) & \xrightarrow{a_{2d+1}\diamond\cdot} & G_{*+2(d+1)}^{(a+1,b+1)}((\boldsymbol{\varepsilon}^2,\boldsymbol{\sigma})) & & \\ & \underset{a_d\diamond\cdot}{\overset{\simeq}{\searrow}} & \downarrow{\scriptstyle a_{-1}\diamond\cdot} & \underset{a_d\diamond\cdot}{\overset{\simeq}{\searrow}} & \\ & & G_*^{(a,b)}((\boldsymbol{\varepsilon}^4,\boldsymbol{\sigma})) & \xrightarrow[a_{2d+1}\diamond\cdot]{} & G_{*+2(d+1)}^{(a+1,b+1)}((\boldsymbol{\varepsilon}^6,\boldsymbol{\sigma})) \end{array}$$

so every arrow in it is an isomorphism. ☐

Theorem 4.11. *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms associated with the Hamiltonian diffeomorphism φ of $\mathbb{C}\mathrm{P}^d$. As a graded R-module,*

$$G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma}) = \bigoplus_{k\in\mathbb{Z}} R\alpha_k$$

for some non-zero α_k's with $\deg \alpha_k = 2k$. For all $k \in \mathbb{Z}$, let

$$c_k(\boldsymbol{\sigma}) := \inf\left\{t \in \mathbb{R} \mid \alpha_k \in \operatorname{im}\left(G_*^{(-\infty,t)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})\right)\right\}.$$

Then for all $k \in \mathbb{Z}$, $c_k(\boldsymbol{\sigma}) \in \mathbb{R}$ is an action value of $\boldsymbol{\sigma}$ and $c_{k+d+1}(\boldsymbol{\sigma}) = c_k(\boldsymbol{\sigma}) + 1$. Moreover

$$c_k(\boldsymbol{\sigma}) \leq c_{k+1}(\boldsymbol{\sigma})$$

for all $k \in \mathbb{Z}$, and if there exists $k \in \mathbb{Z}$ such that $c_k(\boldsymbol{\sigma}) = c_{k+1}(\boldsymbol{\sigma})$, then φ has infinitely many fixed points of action $c_k(\boldsymbol{\sigma})$. If $d+1$ consecutive $c_k(\boldsymbol{\sigma})$'s are equal then $\varphi = \mathrm{id}$.

The following proposition is contained in the proof of Theorem 4.11. It makes precise the fact that one could informally think of α_k as "the class $[\mathbb{C}\mathrm{P}^{k+\infty}] \in H_{2(k+\infty)}(\mathbb{C}\mathrm{P}^{2\infty})$".

Proposition 4.12. *Let $\boldsymbol{\sigma}$ be an n_1-tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms and let $k \in \mathbb{Z}$. Let $m \in \mathbb{N}$, $K \in \mathbb{R}$ and $t \in \mathbb{R}$ such that $-m < -K < c_k(\boldsymbol{\sigma}) < t < m$. We set $r := (n_1 + mn_0 - 1)(d+1)$ and $N := (n_1 + mn_0)(d+1) - 1$. Let $\alpha_k' \in G_{2k}^{(-\infty,t)}(\boldsymbol{\sigma})$ be a class sent to α_k under the inclusion map $G_*^{(-\infty,t)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})$. Then the image $\alpha_k'' \in G_{2k}^{(-K,t)}(\boldsymbol{\sigma})$ under the inclusion map is non-zero and there exists $b_k' \in G_{2k}^{(-K,t)}(\boldsymbol{\sigma}, m)$ such that $\theta_m^\infty(b_k') = \alpha_k''$. The class b_k' is the image of a class $b_k \in HZ_{2k}(\boldsymbol{\sigma}_{m,t})$ that is sent to $[\mathbb{C}\mathrm{P}^{k+r/2}] \in H_{2k+r}(\mathbb{C}\mathrm{P}^N)$ under the map induced by inclusion $Z(\boldsymbol{\sigma}_{m,t}) \hookrightarrow \mathbb{C}\mathrm{P}^N$.*

Proof of Theorem 4.11. In the beginning of this section, we have proved the theorem for $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}$; hence for $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ for all odd $n \in \mathbb{N}$. Let us show that the persistence module of any n-tuple $\boldsymbol{\sigma}$ and the persistence module of $\boldsymbol{\varepsilon}^n$ are δ-interleaved for some $\delta > 0$.

Let $\boldsymbol{\sigma}$ be an n-tuple and let us denote $f_1, \dots, f_n : \mathbb{C}^{d+1} \to \mathbb{R}$ the elementary generating functions of $\sigma_1, \dots, \sigma_n$ respectively. Let $\tau \in (0, 1/2)$ be such that

$$\max_{z \in B, j \in \{1,\dots,n\}} |f(z)| \leq \tan(\pi\tau),$$

where $B \subset \mathbb{C}^{d+1}$ denotes the closed unit ball. Then

$$F_{(\delta_\tau,\dots,\delta_\tau)_{m,t}} \leq F_{\boldsymbol{\sigma}_{m,t}} \leq F_{(\delta_{-\tau},\dots,\delta_{-\tau})_{m,t}}.$$

Composing interpolation morphisms associated with $(\delta_{\pm\tau}, \dots, \delta_{\pm\tau})_{m,t} \rightsquigarrow (\boldsymbol{\varepsilon}^n)_{m,t\pm n\tau}$,

$$s \mapsto (\delta_{\pm(1-s)\tau}, \dots, \delta_{\pm(1-s)\tau})_{m,t+sn\tau},$$

with inclusion morphisms induced by

$$Z((\delta_{-\tau}, \dots, \delta_{-\tau})_{m,t}) \subset Z(\boldsymbol{\sigma}_{m,t}) \subset Z((\delta_\tau, \dots, \delta_\tau)_{m,t}),$$

one gets a natural (*i.e.* that commutes with inclusions and the direct system) $n\tau$-interleaving between $G_*^{(-K,t)}(\boldsymbol{\sigma}, m)$ and $G_*^{(-K,t)}(\boldsymbol{\varepsilon}^n, m)$ with $K \in (-m, 1-m)$ fixed, almost every $t \in (-K, m)$ and large m. These natural interleavings induced a $n\tau$-interleaving between $G_*^{(-\infty,t)}(\boldsymbol{\sigma})$ and $G_*^{(-\infty,t)}(\boldsymbol{\varepsilon}^n)$. The fact that $G_*^{(-\infty,+\infty)}(\boldsymbol{\sigma})$ is isomorphic as a graded R-module to $G_*^{(-\infty,+\infty)}(\boldsymbol{\varepsilon}^n)$ is now a direct consequence of the existence of an interleaving between their associated persistence modules (by taking the direct limit of the interleaving morphisms, one gets the desired isomorphism). The characterisation of $c_k(\boldsymbol{\sigma})$ and α_k given by Proposition 4.12 is true for $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ by the above discussion. Since the $n\tau$-interleaving is ultimately induced by inclusions, Proposition 4.12 is still true for $\boldsymbol{\sigma}$ that is "α_k corresponds to $[\mathbb{C}\mathrm{P}^{N(m)+k}]$" seen in $G_*^{(a,b)}(\boldsymbol{\sigma}, m)$ for m, $-a$ and b large enough.

The fact that $c_{k+d+1}(\boldsymbol{\sigma}) = c_k(\boldsymbol{\sigma}) + 1$ is a direct consequence of Proposition 4.10 applied to $a = -\infty$ and $b = c_k(\boldsymbol{\sigma}) + \varepsilon$ for $\varepsilon > 0$. The last statements of the Theorem 4.11 are consequences of the Lyusternik-Schnirelmann theory, as one can see that $G_*^{(a,b)}(\boldsymbol{\sigma}, m)$ is naturally isomorphic to the relative homology of sublevel sets of one single map (see [30, Section 5] or [1, Section 5.4] for details). □

The dual statement holds for the generating functions cohomology groups with the additional structure of R-algebra induced by the cup-product: the R-algebra $G^*_{(-\infty,+\infty)}(\boldsymbol{\sigma})$ is generated by a class e of degree 2 that is invertible and of infinite order:

$$G^*_{(-\infty,+\infty)}(\boldsymbol{\sigma}) = \bigoplus_{k \in \mathbb{Z}} R e^k.$$

Informally one could think of e^k as "the class $u^{k+\infty} \in H^{2(k+\infty)}(\mathbb{C}\mathrm{P}^{2\infty})$" where u is the generator of $H^*(\mathbb{C}\mathrm{P}^\infty)$. Therefore, one can define alternatively the spectral values $c_k(\boldsymbol{\sigma})$ by

$$c_k(\boldsymbol{\sigma}) = \inf\left\{ t \in \mathbb{R} \mid e^k \notin \ker\left(G^*_{(-\infty,+\infty)}(\boldsymbol{\sigma}) \to G^*_{(-\infty,t)}(\boldsymbol{\sigma}) \right) \right\}.$$

Given an n-tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, we denote by $\boldsymbol{\sigma}^{-1}$ the n-tuple $(\sigma_n^{-1}, \dots, \sigma_1^{-1})$. We recall that if $f : \mathbb{C}^{d+1} \to \mathbb{R}$ is an elementary generating function of σ then $-f$ is an elementary generating function of σ^{-1}. Therefore, one has

$$F_{\boldsymbol{\sigma}^{-1}}(v_1, v_2, \dots, v_n) = -F_{\boldsymbol{\sigma}}(v_1, v_n, v_{n-1}, \dots, v_2) \tag{12}$$

(this identity has already been stated in the special case $\boldsymbol{\sigma} = \boldsymbol{\varepsilon}^n$ in Proposition 4.1).

Given an n-tuple $\boldsymbol{\sigma}$ and an m-tuple $\boldsymbol{\sigma}'$, one has

$$F_{(\boldsymbol{\sigma},\boldsymbol{\sigma}')}(\mathbf{v},\mathbf{v}') = F_{(\boldsymbol{\sigma}',\boldsymbol{\sigma})}(\mathbf{v}',\mathbf{v}), \quad \forall \mathbf{v} \in (\mathbb{C}^{d+1})^n, \forall \mathbf{v}' \in (\mathbb{C}^{d+1})^m. \tag{13}$$

Proposition 4.13 (Poincaré duality)**.** *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1} \setminus 0$. There exists a duality isomorphism between generating function homology and cohomology*

$$PD : G^*_{(a,b)}(\boldsymbol{\sigma}) \xrightarrow{\simeq} G^{(-b,-a)}_{2d-*}(\boldsymbol{\sigma}^{-1}),$$

with $-\infty \leq a \leq b \leq +\infty$ and a, b not action values. This isomorphism is natural: it commutes with inclusion and boundary maps.

Proof. Let us recall a version of the classical Poincaré duality (see for instance [21, Theorem 3.43]). Let M be a compact R-orientable n-dimensional manifold whose boundary ∂M is the union of two disjoint manifolds A and B. Then the cap-product with the fundamental class $[M] \in H_n(M, \partial M)$ gives a natural isomorphism $H^*(M, A) \to H_{n-*}(M, B)$. This general statement can be applied to sublevel sets of a C^1 map of a closed R-orientable n-manifold $f : W \to \mathbb{R}$ in the following way. Let $a < b$ be regular values of f so that $A := \{f = a\}$ and $B := \{f = b\}$ are the disjoint pieces of the boundary of the compact R-orientable n-manifold $M := \{a \leq f \leq b\}$. Now, by excision (which can be used because the boundary of M admits a collar neighborhood)

$$H^*(M, A) \simeq H^*\Big(\{f \leq b\}, \{f \leq a\}\Big) \text{ and } H_*(M, B) \simeq H_*\Big(\{-f \leq -a\}, \{-f \leq -b\}\Big).$$

Finally, one has the duality isomorphism

$$PD : H^*\Big(\{f \leq b\}, \{f \leq a\}\Big) \xrightarrow{\simeq} H_{n-*}\Big(\{-f \leq -a\}, \{-f \leq -b\}\Big).$$

Let us apply the same idea in $W := \mathbb{C}\mathrm{P}^N$ with $N := (n + mn_0)(d + 1) - 1$:

$$\begin{aligned} G^*_{(a,b)}(\boldsymbol{\sigma}, m) &= H^{(n+mn_0-1)(d+1)+*}\left(\left\{\widehat{F}_{\boldsymbol{\sigma}_{m,b}} \leq 0\right\}, \left\{\widehat{F}_{\boldsymbol{\sigma}_{m,a}} \leq 0\right\}\right) \\ &\simeq H_{2N-((n+mn_0-1)(d+1)+*)}\left(\left\{\widehat{F}_{\boldsymbol{\sigma}_{m,a}} \geq 0\right\}, \left\{\widehat{F}_{\boldsymbol{\sigma}_{m,b}} \geq 0\right\}\right), \end{aligned}$$

where we have used that 0 is not a critical value of either $\widehat{F}_{\boldsymbol{\sigma}_{m,a}}$ or $\widehat{F}_{\boldsymbol{\sigma}_{m,b}}$. This last homology group is isomorphic to

$$H_{(n+mn_0-1)(d+1)+(2d-*)}\left(\left\{\widehat{F}_{(\boldsymbol{\sigma}^{-1})_{m,-a}} \leq 0\right\}, \left\{\widehat{F}_{(\boldsymbol{\sigma}^{-1})_{m,-b}} \leq 0\right\}\right) = G^{(-b,-a)}_{2d-*}(\boldsymbol{\sigma}^{-1}, m).$$

Indeed, according to identity (12), this homology group is naturally isomorphic to the homology group of a pair of sublevel sets of functions $-\widehat{F}_{(\boldsymbol{\sigma}_{m,t})^{-1}}$. The conclusion follows by applying Proposition 4.7 to an interpolation from $(\boldsymbol{\sigma}_{m,t})^{-1}$ to $(\boldsymbol{\sigma}^{-1})_{m,-t}$.

Let us precise the interpolation argument. One has $(\boldsymbol{\sigma}_{m,t})^{-1} = ((\boldsymbol{\delta}^{(m)}_t)^{-1}, \boldsymbol{\sigma}^{-1})$. According to (13), the sublevel set induced by this tuple is homeomorphic to the one induced by $(\boldsymbol{\sigma}^{-1}, (\boldsymbol{\delta}^{(m)}_t)^{-1})$ so it is enough to find an interpolation from $(\boldsymbol{\delta}^{(m)}_t)^{-1}$ to $\boldsymbol{\delta}^{(m)}_{-t}$. By definition,

$$(\boldsymbol{\delta}^{(m)}_t)^{-1} = (\boldsymbol{\varepsilon}^{qn_0}, \delta_s^{-1}, \delta_1^{-1}, \ldots, \delta_1^{-1})$$

with $s = t - \lfloor t \rfloor$ and for some $q \in \mathbb{N}$. We remark that $\delta_t^{-1} = \delta_{-t}$. It is then easy to find the wanted homotopy among tuples of the form $(\delta_{t_1}, \delta_{t_2}, \ldots)$ with $t_1 + t_2 + \cdots = -t$. □

Using both equivalent definitions of the spectral values $c_k(\boldsymbol{\sigma})$, the Poincaré duality implies the following identity.

Corollary 4.14. *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, then*

$$c_k(\boldsymbol{\sigma}^{-1}) = -c_{d-k}(\boldsymbol{\sigma}), \quad \forall k \in \mathbb{Z}.$$

Finally, let us apply the composition morphisms to spectral classes in order to prove the sub-additivity of the spectral values.

Proposition 4.15. *Let $m, m' \in \mathbb{N}$ and $k, l \in \mathbb{Z}$ be such that $c_k(\boldsymbol{\sigma}) \in I_m$ and $c_l(\boldsymbol{\sigma}') \in I_{m'}$ and let $t \in (c_k(\boldsymbol{\sigma}), m)$ and $t' \in (c_l(\boldsymbol{\sigma}'), m')$. Let $b_k \in HZ_{2k}(\boldsymbol{\sigma}_{m,t})$ and $b'_l \in HZ_{2l}(\boldsymbol{\sigma}'_{m',t'})$ be classes associated with the respective spectral classes α_k and α'_l in the way expressed in Proposition 4.12. Then the composition morphism*

$$HZ_*(\boldsymbol{\sigma}_{m,t}) \otimes HZ_*(\boldsymbol{\sigma}'_{m',t'}) \to HZ_{*-2d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')_{m+m',t+t'})$$

maps the class $b_k \otimes b'_l$ to a class b''_{k+l-d} that is sent to the $[\mathbb{C}\mathrm{P}^r] \in H_*(\mathbb{C}\mathrm{P}^N)$ of appropriate degree under the inclusion morphism.

Proof. This is a direct consequence of the commutativity of (9). □

Corollary 4.16. *Given any tuples $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, one has*

$$c_{k+l-d}((\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}')) \leq c_k(\boldsymbol{\sigma}) + c_l(\boldsymbol{\sigma}'), \quad \forall k, l \in \mathbb{Z}.$$

As explained in Section 2, we can now associate to every $\boldsymbol{\sigma}$ the persistence module $(G_*^{(-\infty,t)}(\boldsymbol{\sigma}))_t$ (see also Section 3.1 for the definition of persistence modules and barcodes). Its associated morphisms $G_*^{(-\infty,s)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,t)}(\boldsymbol{\sigma})$, for $s \leq t$, are the inclusion morphisms. This persistence module satisfies the "periodicity" property $G_*^{(-\infty,t+1)}(\boldsymbol{\sigma}) \simeq G_{*+2(d+1)}^{(-\infty,t)}(\boldsymbol{\sigma})$, the isomorphism being an isomorphism of persistence module according to the naturality of (11). While discussing barcodes properties, we assume that the persistence module is over a field and the number of associated fixed points in $\mathbb{C}\mathrm{P}^d$ is finite. Since this periodicity property shifts the degree by a constant positive integer $2(d+1)$, it induces a permutation of the bars of the barcode sending a bar $[a, b)$ on a bar $[a+1, b+1)$ that generates a free $\mathbb{Z}$-action on the bars. A family of representatives of the bars is given by the union of the barcodes of $(G_k^{(-\infty,t)}(\boldsymbol{\sigma}))_t$ for $0 \leq k \leq 2d+1$. For instance, Figure 1 represents a part of the barcode of some $\boldsymbol{\sigma}$ associated with a Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^1$. This barcode has 2 $\mathbb{Z}$-orbits of finite bars and $d+1 = 2$ $\mathbb{Z}$-orbits of infinite bars corresponding to the spectral values $c_1 + \mathbb{Z}$ and $c_2 + \mathbb{Z}$ where $c_k := c_k(\boldsymbol{\sigma})$.

Lemma 4.17. *Let $\boldsymbol{\sigma}$ be a tuple of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms with a finite number of fixed $\mathbb{C}$-lines and let $a < b$ that are not action values of $\boldsymbol{\sigma}$. For every field $\mathbb{F}$, the number of bars of the barcode of $\boldsymbol{\sigma}$ over $\mathbb{F}$ that intersect $t = a$ or $t = b$ but not both is equal to $\dim G_*^{(a,b)}(\boldsymbol{\sigma}; \mathbb{F})$.*

Proof. Let us consider the long exact sequence

$$\cdots \xrightarrow{\partial_{*+1}} G_*^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F}) \to G_*^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F}) \to G_*^{(a,b)}(\boldsymbol{\sigma}; \mathbb{F}) \xrightarrow{\partial_*} G_{*-1}^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F}) \to G_{*-1}^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F}) \to \cdots. \quad (14)$$

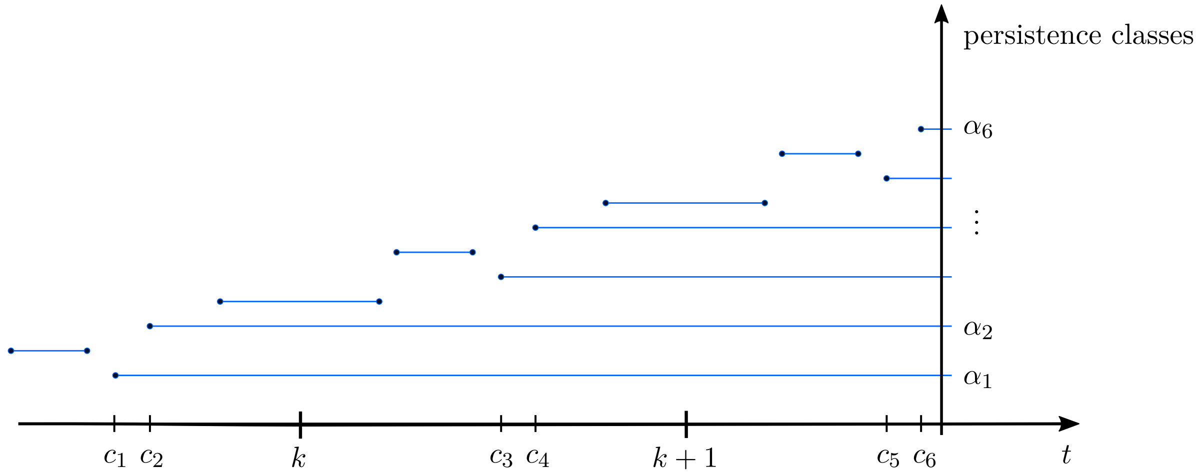


FIGURE 1. Barcode of a Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^1$ in the neighborhood of $[k, k+1]$ for some $k \in \mathbb{Z}$ (bars of degree less than 2 are missing).

Applying the normal form theorem of persistence modules, one can find bases $((\alpha_i), (\delta_j^-))$ and $((\beta_k), (\delta_j^+))$ of the $\mathbb{F}$-vector spaces $G_*^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F})$ and $G_*^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F})$ that are in a canonical bijection with bars of the barcode intersecting $t = a$ and $t = b$ respectively, δ_j^- and δ_j^+ being associated with the same bar for each j while the bars associated with the α_i's do not intersect $t = b$ and the bars associated with the β_k's do not intersect $t = a$ (see Figure 2). In other words, the following diagram commutes

$$\begin{array}{ccc} G_*^{(-\infty,a)}(\boldsymbol{\sigma}; \mathbb{F}) = \bigoplus_i \mathbb{F}\alpha_i \oplus & \bigoplus_j \mathbb{F}\delta_j^- & \\ \downarrow & \downarrow \simeq & , \\ G_*^{(-\infty,b)}(\boldsymbol{\sigma}; \mathbb{F}) = & \bigoplus_j \mathbb{F}\delta_j^+ \oplus \bigoplus_k \mathbb{F}\beta_k & \end{array}$$

where the left arrow is the inclusion morphism and the right arrow sends δ_j^- to δ_j^+ for all j and the α_i's to 0. Let us recall that, according to the finiteness assumption on the number of fixed points, the number of α_i's and β_k's are finite (here a and b are finite). With the above diagram, one can extract a short exact sequence of finite dimensional vector spaces from the long exact sequence (14)

$$0 \to \bigoplus_k \mathbb{F}\beta_k \to G_*^{(a,b)}(\boldsymbol{\sigma}; \mathbb{F}) \to \bigoplus_i \mathbb{F}\alpha_i \to 0.$$

Hence the result. □

Proposition 4.18. *Given a tuple $\boldsymbol{\sigma}$ of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1} \setminus 0$ with a finite number of fixed $\mathbb{C}$-lines, for every field $\mathbb{F}$,*

$$N(\boldsymbol{\sigma}; \mathbb{F}) = d + 1 + 2K(\boldsymbol{\sigma}; \mathbb{F}),$$

where $K(\boldsymbol{\sigma}; \mathbb{F})$ is the number of $\mathbb{Z}$-orbits of finite bars of the persistence module of $\boldsymbol{\sigma}$ over the field $\mathbb{F}$. In other words, $N(\boldsymbol{\sigma}; \mathbb{F})$ is the number of (finite) extremities of a set of representative bars.

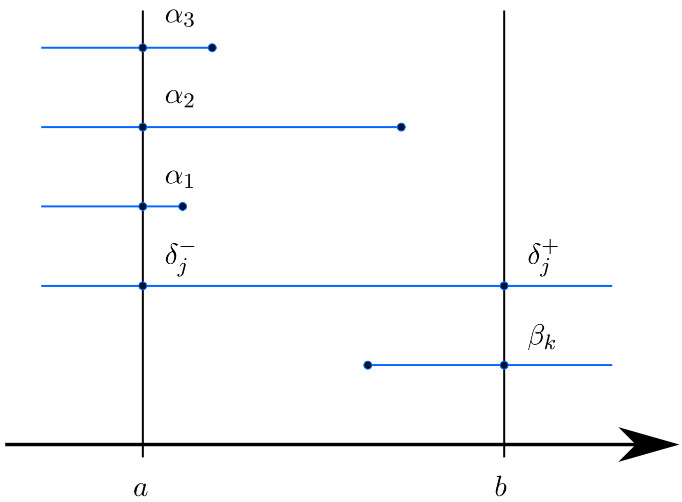


FIGURE 2. Relationship between the barcode of $\boldsymbol{\sigma}$ in the interval (a,b) and its persistence module (there are infinitely many bars that do not appear in the figure). The value $\dim G_*^{(a,b)}(\boldsymbol{\sigma};\mathbb{F})$ gives the number of α_i's and β_k's.

Proof. According to the $\mathbb{Z}$-symmetry of the barcode, it boils down to proving that the number of extremities of the barcode lying inside $[0,1)$ is equal to $N(\boldsymbol{\sigma};\mathbb{F})$. Let $0 \le t_1 < t_2 < \cdots < t_n < 1$ be the action values of $\boldsymbol{\sigma}$ in $[0,1)$, that is the points where extremities of the barcode could appear. Let $t_j^\pm := t_j \pm \varepsilon$ where $\varepsilon > 0$ is strictly less than the minimum distance between two action values so that t_j is the only action value inside $[t_j^-, t_j^+]$. According to Lemma 4.17, $\dim G_*^{(t_j^-,t_j^+)}(\boldsymbol{\sigma};\mathbb{F})$ equals the number of extremities at $t = t_j$. Therefore, we just have to prove that

$$N(\boldsymbol{\sigma};\mathbb{F}) = \sum_{j=1}^{n} \dim G_*^{(t_j^-,t_j^+)}(\boldsymbol{\sigma};\mathbb{F}).$$

Let us denote by φ the Hamiltonian diffeomorphism associated with $\boldsymbol{\sigma}$. Since t_j is the only action value in $[t_j^-, t_j^+]$, an excision argument gives

$$G_*^{(t_j^-,t_j^+)}(\boldsymbol{\sigma}) \simeq \bigoplus_z \mathrm{C}_*(\boldsymbol{\sigma}; z, t_j),$$

where the direct sum is over the fixed points $z \in \mathbb{C}\mathrm{P}^d$ of φ with action value t_j (see [1, Section 5.5]). By taking these isomorphisms over the field $\mathbb{F}$ for all j, the conclusion follows. □

5. Uniform bound on β

Theorem 5.1. *For every tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms $\boldsymbol{\sigma}$ generating a Hamiltonian diffeomorphism of $\mathbb{C}\mathrm{P}^d$ with finitely many fixed points, the longest finite bar of its barcode satisfies*

$$\beta(\boldsymbol{\sigma}) \le c_{d+k}(\boldsymbol{\sigma}) - c_k(\boldsymbol{\sigma}), \quad \forall k \in \mathbb{Z}.$$

In particular, the longest finite bar is less than 1.

As a matter of fact, the proof allows us to give the more precise bound:

$$\beta(\boldsymbol{\sigma}) \leq c_{d+k}(\boldsymbol{\sigma}) - c_k(\boldsymbol{\sigma}),$$

for all $k \in \mathbb{Z}$ (by considering $c_{d+k}(\boldsymbol{\sigma})$ and $c_{d-k}(\boldsymbol{\sigma}^{-1})$ rather than $c_d(\boldsymbol{\sigma})$ and $c_d(\boldsymbol{\sigma}^{-1})$ in the proof). In particular, one can always replace ≤ 1 by < 1 in the inequality.

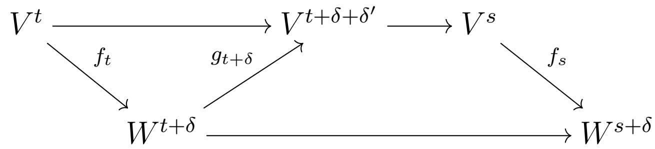
In the proof, we will essentially prove that the persistence modules of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are $(c_d(\boldsymbol{\sigma}), -c_0(\boldsymbol{\sigma}))$-interleaved. The isometry theorem between the interleaving distance and the barcode distance states that the distance between the two associated barcodes is not more than $c_d(\boldsymbol{\sigma}) - c_0(\boldsymbol{\sigma})$. Since the barcode of $\boldsymbol{\varepsilon}$ does not have any finite bar, the conclusion follows.

In order to simplify the proof, we will use a slightly weaker result than the isometry theorem. We recall that the maximal length of a finite bar $\beta(V^t) \geq 0$ in the persistence module (V^t) can alternatively be defined by

$$\beta(V^t) = \sup\left\{\beta \geq 0 \mid \exists t \in \mathbb{R},\ \ker(V^t \to V^{t+\beta}) \neq \ker(V^t \to V^{+\infty})\right\}.$$

Lemma 5.2. *Let $((V^t), \pi)$ be a persistence module and $((W^t), \kappa)$ be a persistence module without any finite bar. If there exist $\delta, \delta' \in \mathbb{R}$ with $\delta + \delta' \geq 0$ and $f : (V^t) \to (W^{t+\delta})$ and $g : (W^t) \to (V^{t+\delta'})$ that are morphisms of persistence modules such that $g_{t+\delta} f_t = \pi_t^{t+\delta+\delta'}$ for all $t \in \mathbb{R}$, then $\beta(V^t) \leq \delta + \delta'$.*

Proof of Lemma 5.2. The proof can be summed up by the following diagram:



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(we do not assume that the right "square" is commutative). By contradiction, let us assume that there exists $t \in \mathbb{R}$ and $v \in V^t$ such that $\pi_t^{t+\delta+\delta'} v \neq 0$ and $\pi_t^{+\infty} v = 0$. Since $\pi_t^{+\infty} v = 0$, there exists $s \geq t + \delta + \delta'$ such that $\pi_t^s v = 0$. By hypothesis, $\pi_t^{t+\delta+\delta'} = g_{t+\delta} f_t$ so $w := f_t v \neq 0$. Since (W^t) does not have any finite bars, $\kappa_{t+\delta}^s w \neq 0$. Since f is a morphism of persistence modules, $f_s \pi_t^s = \kappa_{t+\delta}^{s+\delta} f_t$ so $f_s \pi_t^s v = \kappa_{t+\delta}^{s+\delta} w \neq 0$. A contradiction with $\pi_t^s v = 0$. $\square$

Proof of Theorem 5.1. Let $\varepsilon > 0$, $k \in \mathbb{Z}$ and let $\eta := c_{d+k}(\boldsymbol{\sigma}) + \varepsilon/2$ and $\eta' := c_{d-k}(\boldsymbol{\sigma}^{-1}) + \varepsilon/2$. Let $m > \max(|\eta|, |\eta'|, |c_{d+k}(\boldsymbol{\sigma})|, |c_{d-k}(\boldsymbol{\sigma}^{-1})|)$. One can assume that the size n of $\boldsymbol{\sigma}$ satisfies $2n = mn_0$ for some integer

$$m > \max(|\eta|, |\eta'|, |c_{d+k}(\boldsymbol{\sigma})|, |c_{d-k}(\boldsymbol{\sigma}^{-1})|),$$

by concatenation of $\boldsymbol{\sigma}$ with some $\boldsymbol{\varepsilon}^l$ (see the paragraph above Equation (5) for the definition of n_0). Let $b_{d+k} \in HZ_{2(d+k)}(\boldsymbol{\sigma}_{m,\eta})$ and $b'_{d-k} \in HZ_{2(d-k)}(\boldsymbol{\sigma}^{-1}_{m,\eta'})$ be the respective classes associated with the respective spectral classes α_{d+k} and α_{d-k} in the sense of Corollary 4.14 (well defined by definition of η, η' and m). In order to simplify the exposition, let us set $a_d := a_d^{(2n+1)}(0) \in HZ_{2d}(\boldsymbol{\varepsilon}^{2n+1})$ and $a'_d := a_d^{(2n+1)}(\eta + \eta') \in HZ_{2d}(\boldsymbol{\varepsilon}_{m,\eta+\eta'})$ (we recall that $\boldsymbol{\varepsilon}^{2n+1} = \boldsymbol{\varepsilon}^{mn_0+1} = \boldsymbol{\varepsilon}_{m,0}$ by our assumption on n). Let us denote by Δ_1 the interpolation isomorphism associated with $(\boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{1-s})_{s\in[0,1]}$ and by Δ_2 the interpolation isomorphism associated with $(\boldsymbol{\sigma}^{1-s}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{s-1})_{s\in[0,1]}$. Let us denote by $\widetilde{\Delta}_1$ the interpolation isomorphism associated with $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{1-s})_{s\in[0,1]}$ and by $\widetilde{\Delta}_2$ the one associated with $(\boldsymbol{\sigma}^{1-s}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}^{s-1}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma})_{s\in[0,1]}$.

We define the following morphisms of persistence modules:

$$f_t : \begin{cases} G_*^{(-\infty,t)}(\boldsymbol{\sigma}) & \to & G_*^{(-\infty,t+\eta')}(\boldsymbol{\varepsilon}^{2n+1}), \\ \alpha & \mapsto & \Delta_1(b'_{d-k} \diamond \alpha), \end{cases}$$

$$g_t : \begin{cases} G_*^{(-\infty,t)}(\boldsymbol{\varepsilon}^{2n+1}) & \to & G_*^{(-\infty,t+\eta)}(\boldsymbol{\sigma}), \\ \alpha & \mapsto & a_d^{-1} \diamond \widetilde{\Delta}_2 \circ \widetilde{\Delta}_1^{-1}(b_{d+k} \diamond \alpha). \end{cases}$$

Indeed, these morphisms commute with inclusion morphisms by naturality of the morphisms involved in their definitions. Let us denote by $\pi_t^s : G_*^{(-\infty,t)}(\boldsymbol{\sigma}) \to G_*^{(-\infty,s)}(\boldsymbol{\sigma})$ the inclusion morphism for $t \leq s$.

In order to apply Lemma 5.2, one just needs to show that $g_{t+\eta'} \circ f_t = \pi_t^{t+\eta+\eta'}$ for all $t \in \mathbb{R}$. For all $\alpha \in G_*^{(-\infty,t)}(\boldsymbol{\sigma})$, one has

$$\begin{aligned} g_{t+\eta'} \circ f_t(\alpha) &= a_d^{-1} \diamond \widetilde{\Delta}(b_{d+k} \diamond \Delta_1(b'_{d-k} \diamond \alpha)) \\ &= a_d^{-1} \diamond \widetilde{\Delta}_2(b_{d+k} \diamond b'_{d-k} \diamond \alpha) \\ &= a_d^{-1} \diamond \Delta_2(b_{d+k} \diamond b'_{d-k}) \diamond \alpha \\ &= a_d^{-1} \diamond a'_d \diamond \alpha \\ &= \pi_t^{\eta+\eta'} \alpha, \end{aligned}$$

where the second and third equality comes from Proposition 4.8 and the identity $a_d^{-1} \diamond a'_d = \pi_t^{\eta+\eta'}$ is a direct consequence of the commutativity of (10). The conclusion follows from Lemma 5.2 since the persistence module of $\boldsymbol{\varepsilon}^{2n+1}$ does not have any finite bar (see the beginning of Section 4.5) and

$$\eta + \eta' = c_{d+k}(\boldsymbol{\sigma}) + c_{d-k}(\boldsymbol{\sigma}^{-1}) + \varepsilon = c_{d+k}(\boldsymbol{\sigma}) - c_k(\boldsymbol{\sigma}) + \varepsilon,$$

where the second equality comes from Corollary 4.14.

The fact that $c_{d+k}(\boldsymbol{\sigma}) - c_k(\boldsymbol{\sigma}) \leq 1$ for all $k \in \mathbb{Z}$ and that $c_{d+k_0}(\boldsymbol{\sigma}) - c_{k_0}(\boldsymbol{\sigma}) < 1$ for some k_0 come from the monotonicity of $(c_k(\boldsymbol{\sigma}))_k$ and the periodicity $c_{k+d+1}(\boldsymbol{\sigma}) = c_k(\boldsymbol{\sigma}) + 1$, $\forall k \in \mathbb{Z}$, stated in Theorem 4.11. □

6. Smith inequality

In this section, we show how the classical Smith inequality (16) can be applied to the sublevel sets of generating functions to prove inequality (2). Çineli-Ginzburg used the same kind of argument to prove a Smith inequality between the dimension of the local homology of a Hamiltonian orbit and its p-iterate for p prime [9].

6.1. $\mathbb{Z}/(p)$-action of a p-iterated generating function.

Let us fix a prime number $p \geq 3$. Let us fix $t \in \mathbb{R}$ and study the generating function of $e^{-2i\pi t}\Phi_1$. In order to fix the notation, we recall that

$$F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{v}) := \sum_{k=1}^{n} f_k\left(\frac{v_k + v_{k+1}}{2}\right) + \frac{1}{2}\langle v_k, iv_{k+1}\rangle,$$

where $\mathbf{v} := (v_1, \dots, v_n) \in (\mathbb{C}^{d+1})^n$ and the $f_k : \mathbb{C}^{d+1} \to \mathbb{R}$ are S^1-invariant and 2-homogeneous. Thus $F_{\boldsymbol{\sigma}^p_{m,t}} : (\mathbb{C}^{n(d+1)})^p \to \mathbb{R}$ is invariant under the action of $\mathbb{Z}/(p)$ by cyclic permutation of coordinates generated by

$$(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p) \mapsto (\mathbf{v}_p, \mathbf{v}_1, \dots, \mathbf{v}_{p-1}),$$

(here $\boldsymbol{\sigma}_{m,t}^p$ means $(\boldsymbol{\sigma}_{m,t})^p$). The induced $\widehat{F}_{\boldsymbol{\sigma}_{m,t}^p} : \mathbb{C}\mathrm{P}^{pn(d+1)-1} \to \mathbb{R}$ is then invariant under the $\mathbb{Z}/(p)$-action by permutation of homogeneous coordinates induced by

$$[\mathbf{v}_1 : \mathbf{v}_2 : \cdots : \mathbf{v}_p] \mapsto [\mathbf{v}_p : \mathbf{v}_1 : \cdots : \mathbf{v}_{p-1}].$$

Fixed points $(\mathbb{C}\mathrm{P}^N)^{\mathbb{Z}/(p)}$ of this action are the disjoint union $\bigsqcup_q P_q$ of the p following $(n(d+1)-1)$-complex projective subspaces:

$$P_q := \left\{\left[\mathbf{v} : \zeta^q \mathbf{v} : \zeta^{2q}\mathbf{v} : \cdots : \zeta^{(p-1)q}\mathbf{v}\right] \mid [\mathbf{v}] \in \mathbb{C}\mathrm{P}^{n(d+1)-1}\right\}, \quad \zeta := e^{\frac{2i\pi}{p}}.$$

Using the fact that the f_k's are S^1-invariant and 2-homogeneous,

$$\frac{1}{p} F_{\boldsymbol{\sigma}_{m,t}^p}(\mathbf{v}, \zeta^q \mathbf{v}, \ldots, \zeta^{q(p-1)}\mathbf{v}) = \sum_{k=1}^{n-1} \left[f_k\left(\frac{v_k + v_{k+1}}{2}\right) + \frac{1}{2}\langle v_k, i v_{k+1}\rangle\right] + f_n\left(\frac{v_n + \zeta^q v_1}{2}\right) + \frac{1}{2}\langle v_n, i\zeta^q v_1\rangle .$$

We apply the linear change of variables $\mathbf{v} \mapsto \mathbf{u}$ given by $u_k := v_k + (-1)^k \frac{1-\zeta^q}{2} v_1$ so that

$$\begin{cases} u_1 + u_2 &= v_1 + v_2, \\ u_2 + u_3 &= v_2 + v_3, \\ &\vdots \\ u_{n-1} + u_n &= v_{n-1} + v_n, \\ u_n + u_1 &= v_n + \zeta^q v_1. \end{cases}$$

A direct computation gives

$$\sum_{k=1}^{n-1} \langle v_k, i v_{k+1}\rangle + \langle v_n, i\zeta^q v_1\rangle = \sum_{k=1}^{n} \langle u_k, i u_{k+1}\rangle - 2\tan\left(\frac{q\pi}{p}\right)\|u_1\|^2,$$

for all integer $q \in \left\{\frac{1-p}{2}, \ldots, \frac{p-1}{2}\right\}$, so that

$$F_{\boldsymbol{\sigma}_{m,t}^p}(\mathbf{v}, \zeta^q \mathbf{v}, \ldots, \zeta^{q(p-1)}\mathbf{v}) = p\left[F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{u}) - \tan\left(\frac{q\pi}{p}\right)\|u_1\|^2\right] =: pG_{t,q}(\mathbf{u}),$$

for $q \in \left\{\frac{1-p}{2}, \ldots, \frac{p-1}{2}\right\}$. This last function $G_{t,q}$ is the fiberwise sum of a generating function of $e^{-2i\pi t}\Phi_1$ and a generating function of $e^{-2i\pi q/p}$. We recall that in this case, $\mathbb{C}$-lines of critical points of this function are in one-to-one correspondence with $\mathbb{C}$-lines of fixed points of the composed diffeomorphism $e^{-2i\pi(t+q/p)}\Phi_1$ (see the paragraph surrounding Equation (4)). Let $(f_{s,t})$ be the family of function

$$f_{s,t}(\mathbf{u}) := F_{\boldsymbol{\sigma}_{m,t+(1-s)q/p}}(\mathbf{u}) - \tan\left(s\frac{q\pi}{p}\right)\|u_1\|^2, \quad s \in [0,1].$$

The function $f_{s,t}$ is the fiberwise sum of a generating function of $e^{-2i\pi(t+(1-s)q/p)}\Phi_1$ and a generating function of $e^{-2i\pi sq/p}$ so 0 is a regular value of $f_{s,t}$ if and only if $e^{-2i\pi(t+q/p)}\Phi_1$ does not have any $\mathbb{C}$-line of fixed points, that is if and only if $t + q/p$ is not an action value of $\boldsymbol{\sigma}$. According to Proposition 4.7 applied to the family of maps $\hat{f}_{s,t}$,

$$H_*\left(\left\{\widehat{G}_{b,q} \le 0\right\}, \left\{\widehat{G}_{a,q} \le 0\right\}\right) \simeq G_{*-i_0}^{(a+q/p, b+q/p)}(\boldsymbol{\sigma}, m) \simeq G_{*-i_0}^{(a+q/p, b+q/p)}(\boldsymbol{\sigma}), \tag{15}$$

where $-m \le a + q/p \le b + q/p \le m$, i_0 is some integer and $a + q/p$ and $b + q/p$ are not action values of $\boldsymbol{\sigma}$.

6.2. **Application of Smith inequality.** Let $p \in \mathbb{N}$ be prime and let X be a locally compact space or pair on which the cyclic group $\mathbb{Z}/(p)$ is acting. According to Smith inequality, if $H_*(X;\mathbb{F}_p)$ is finitely generated,

$$\dim H_*(X;\mathbb{F}_p) \geq \dim H_*(X^{\mathbb{Z}/(p)};\mathbb{F}_p), \tag{16}$$

where $\dim H_*$ means the total dimension $\sum_k \dim H_k$ and $X^{\mathbb{Z}/(p)} \subset X$ is the set fixed by the group action (see for instance [6, Chapter IV, §4.1]).

Proposition 6.1. *Given any tuple $\boldsymbol{\sigma}$ of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms, for every prime number p and every $a \leq b$ such that $a+q/p$ and $b+q/p$ are not action values of $\boldsymbol{\sigma}$ for $q \in \mathbb{Z}$ and pa and pb are not action values of $\boldsymbol{\sigma}^p$,*

$$\dim G_*^{(pa,pb)}(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq \sum_{(1-p)/2 \leq q \leq (p-1)/2} \dim G_*^{(a+q/p,b+q/p)}(\boldsymbol{\sigma};\mathbb{F}_p).$$

Proof. Let us assume that $p \geq 3$ and refer the reader to Section 6.4 for the modifications specific to $p = 2$. By concatenating $\boldsymbol{\sigma}$ with some $\boldsymbol{\varepsilon}^k$ if needed, it is not difficult to find a homotopy interpolating $\boldsymbol{\sigma}^p_{m,t} := (\boldsymbol{\sigma}_{m,t})^p$ and $(\boldsymbol{\sigma}^p)_{pm,pt}$. Therefore, the associated interpolation isomorphism gives us

$$HZ_*(\boldsymbol{\sigma}^p_{m,b},\boldsymbol{\sigma}^p_{m,a}) \simeq G_*^{(pa,pb)}(\boldsymbol{\sigma}^p,pm) \simeq G_*^{(pa,pb)}(\boldsymbol{\sigma}^p). \tag{17}$$

Now, we apply the Smith inequality (16) to the couple

$$X := \left(\left\{\widehat{F}_{\boldsymbol{\sigma}^p_b} \leq 0\right\}, \left\{\widehat{F}_{\boldsymbol{\sigma}^p_a} \leq 0\right\}\right).$$

According to the last section,

$$X^{\mathbb{Z}/(p)} \simeq \bigsqcup_{(1-p)/2 \leq q \leq (p-1)/2} \left(\left\{\widehat{G}_{b,q} \leq 0\right\}, \left\{\widehat{G}_{a,q} \leq 0\right\}\right).$$

Therefore, Smith inequality (16), (17) and (15) bring the conclusion. □

6.3. **Computation of β_{tot}.** The arguments of this section follow the proof of Theorem B.1 given by Shelukhin in Appendix B in the realm of Floer theory that applies to every closed monotone symplectic manifold.

Proposition 6.2. *Let $\boldsymbol{\sigma}$ be a tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1} \setminus 0$ with a finite number of associated fixed points in $\mathbb{C}\mathrm{P}^d$. For all $a \in \mathbb{R}$, all integers $n \in \mathbb{N}^*$ and all fields $\mathbb{F}$,*

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}) = \frac{1}{2}\left(\int_0^1 \dim G_*^{(a+t,a+t+n)}(\boldsymbol{\sigma};\mathbb{F})\,\mathrm{d}t - n(d+1)\right).$$

Proof. Let $I_1,\dots,I_n \subset \mathbb{R}$ be representatives of each $\mathbb{Z}$-orbit of finite bars of the persistence module $(G_*^{(-\infty,t)}(\boldsymbol{\sigma};\mathbb{F}))_t$ and let $J_1,\dots,J_{d+1} \subset \mathbb{R}$ be representatives of each $\mathbb{Z}$-orbit of infinite bars of this persistence module. Given an interval $I \subset \mathbb{R}$, let $\chi_I : \mathbb{R} \to \{0,1\}$ denote its characteristic map. According to Lemma 4.17, for $a < b$ which are neither infinite nor action values of $\boldsymbol{\sigma}$, one has

$$\dim G_*^{(a,b)}(\boldsymbol{\sigma};\mathbb{F}) = \sum_{k\in\mathbb{Z}} \left[\sum_{r=1}^{n} \left|\chi_{I_r}(b+k) - \chi_{I_r}(a+k)\right| + \sum_{s=1}^{d+1} \left|\chi_{J_s}(b+k) - \chi_{J_s}(a+k)\right|\right].$$

Therefore, in order to prove the statement, it is enough to prove that for all $I \in \{I_1, \dots, I_n\}$,

$$\sum_{k\in\mathbb{Z}} \int_0^1 \Big|\chi_I(a+t+k+n) - \chi_I(a+t+k)\Big| \mathrm{d}t = 2 \operatorname{length} I, \tag{18}$$

while for all $J \in \{J_1, \dots, J_{d+1}\}$,

$$\sum_{k\in\mathbb{Z}} \int_0^1 \Big|\chi_J(a+t+k+n) - \chi_J(a+t+k)\Big| \mathrm{d}t = n. \tag{19}$$

In order to prove (18), let us write $I = (u, v)$. According to Theorem 5.1, $\operatorname{length} I = v - u \leq 1$. If the integer parts $\lfloor u-a \rfloor$ and $\lfloor v-a \rfloor$ are equal, then only the terms $k = \lfloor u-a \rfloor$ and $k = \lfloor u-a \rfloor + n$ are non-zero, both equal to $\operatorname{length} I$. Otherwise, one must compute the four non-zero terms and gets

$$(1 - \{u-a\}) + \{v-a\} + (1 - \{u-a\}) + \{v-a\} = 2(v-u) = 2 \operatorname{length} I,$$

where $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$ for $x \in \mathbb{R}$ (at the first equality, we have used $\lfloor v-a \rfloor = \lfloor u-a \rfloor + 1$).

Identity (19) is proven by a similar straightforward computation. □

The following corollary is a generating functions analogue of [28, Theorem D] in the special case $M = \mathbb{C}\mathrm{P}^d$.

Corollary 6.3. *For all tuples of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms of $\mathbb{C}^{d+1} \setminus 0$ with a finite number of associated fixed points in $\mathbb{C}\mathrm{P}^d$, for all prime numbers p,*

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p; \mathbb{F}_p) \geq p \beta_{\mathrm{tot}}(\boldsymbol{\sigma}; \mathbb{F}_p).$$

Proof. Let us take the integral over almost all $t \in [0, 1]$ of the inequality stated in Proposition 6.1 for $a = t$ and $b = 1 + t$. On the left hand side,

$$\begin{aligned} \int_0^1 \dim G_*^{(pt, p+pt)}(\boldsymbol{\sigma}^p; \mathbb{F}_p) \mathrm{d}t &= \frac{1}{p} \int_0^p \dim G_*^{(s, p+s)}(\boldsymbol{\sigma}^p; \mathbb{F}_p) \mathrm{d}s \\ &= \frac{1}{p} \sum_{k=0}^{p-1} \int_0^1 \dim G_*^{(k+s, k+s+p)}(\boldsymbol{\sigma}^p; \mathbb{F}_p) \mathrm{d}s \\ &= \frac{1}{p} \sum_{k=0}^{p-1} \Big[2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p; \mathbb{F}_p) + p(d+1)\Big] \\ &= 2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p; \mathbb{F}_p) + p(d+1), \end{aligned}$$

where we have applied Proposition 6.2 at the third line. On the right hand side, by applying Proposition 6.2 once again,

$$\begin{aligned} &\sum_{(1-p)/2 \leq q \leq (p-1)/2} \int_0^1 \dim G_*^{(q/p, q/p+t)}(\boldsymbol{\sigma}; \mathbb{F}_p) \mathrm{d}t \\ &\qquad = \sum_{(1-p)/2 \leq q \leq (p-1)/2} \Big[2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}; \mathbb{F}_p) + d + 1\Big] = 2p\beta_{\mathrm{tot}}(\boldsymbol{\sigma}; \mathbb{F}_p) + p(d+1). \end{aligned}$$

Therefore,

$$2\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p; \mathbb{F}_p) + p(d+1) \geq 2p\beta_{\mathrm{tot}}(\boldsymbol{\sigma}; \mathbb{F}_p) + p(d+1)$$

and the conclusion follows. □

Proposition 6.4. *For every tuple of small $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms $\boldsymbol{\sigma}$ of $\mathbb{C}^{d+1}\setminus 0$ with a finite number of associated fixed points in $\mathbb{C}\mathrm{P}^d$, there exists an integer $N\in\mathbb{N}$ such that for all prime numbers $p\geq N$,*

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p)=\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{Q}).$$

Proof. According to Proposition 6.2, it is enough to prove that for some $N\in\mathbb{N}$ every prime number such that $p\geq N$ satisfies

$$\dim G_*^{(t,t+1)}(\boldsymbol{\sigma};\mathbb{F}_p)=\dim G_*^{(t,t+1)}(\boldsymbol{\sigma};\mathbb{Q}),\quad \forall t\in[0,1]. \tag{20}$$

If there is no action value of $\boldsymbol{\sigma}$ in $[a,b]$ then $\dim G_*^{(a,a+1)}(\boldsymbol{\sigma};\mathbb{F})=\dim G_*^{(b,b+1)}(\boldsymbol{\sigma};\mathbb{F})$ for all field $\mathbb{F}$. Since there is a finite number of action values in $[0,1]$, it is enough to prove (20) for a finite number of value t (one in between each critical value of $[0,1]$). For each t (that is not an action value), $G_*^{(t,t+1)}(\boldsymbol{\sigma};\mathbb{F})\simeq H_{*+i_0}(A_t,B_t;\mathbb{F})$ for a topological pair (A_t,B_t) of some complex projective space independent of $\mathbb{F}$ with a finitely generated homology group with integral coefficients. According to the universal coefficient theorem, there exists $N_t\in\mathbb{N}$ such that $\dim H_*(A_t,B_t;\mathbb{Q})=\dim H_*(A_t,B_t;\mathbb{F}_p)$ for all prime number $p\geq N_t$. The conclusion follows by taking the maximum among the N_t's for our finite set of t's. □

6.4. **The special case $p=2$.** Here, we briefly explain how to modify the above arguments in the special case $p=2$ – that is only useful to prove Theorem 1.1 when $\mathbb{F}$ has characteristic 2.

In order to study the $\mathbb{Z}/(2)$-symmetry of a generating function associated with $(e^{-2i\pi t}\Phi_1)^2$, one cannot take the generating function of $\boldsymbol{\sigma}_{m,t}^2$ since it is an even tuple. Instead, we take the generating function of $(\boldsymbol{\sigma}_{m,t},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t})$ which is invariant under the following action of $\mathbb{Z}/(2)$ written in w-variables:

$$(\mathbf{w}^1,w^2,\mathbf{w}^3)\mapsto\left(\mathbf{w}^3,-w^2+\sum_{k=1}^n(-1)^k(w_k^1+w_k^3),\mathbf{w}^1\right).$$

Indeed, Q_{2n+1} is invariant under this action and, in w-variables,

$$F_{(\boldsymbol{\sigma}_{m,t},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t})}(\mathbf{w^1},w^2,\mathbf{w}^3)=F'(\mathbf{w^1})+F'(\mathbf{w^3})+Q_{2n+1}(\mathbf{w}^1,w^2,\mathbf{w}^3),$$

where F' is the direct sum of the elementary generating functions of $\boldsymbol{\sigma}_{m,t}$. The set of fixed points of the induced action on $\mathbb{C}\mathrm{P}^{(2n+1)(d+1)-1}$ is the disjoint union of the complex projective spaces P_0 and P_1 defined by

$$P_0:=\left\{\left[\mathbf{w}:\sum_{k=1}^n(-1)^k w_k:\mathbf{w}\right]\;|\;\mathbf{w}\in(\mathbb{C}^{d+1})^n\right\},$$
$$P_1:=\left\{[\mathbf{w}:w':-\mathbf{w}]\;|\;(\mathbf{w},w')\in(\mathbb{C}^{d+1})^n\times\mathbb{C}^{d+1}\right\}.$$

The restriction of the generating function to P_0 gives us back the generating function $\widehat{F}_{\boldsymbol{\sigma}_{m,t}}$ whereas, still in w-variables,

$$\frac{1}{2}F_{(\boldsymbol{\sigma}_{m,t},\boldsymbol{\varepsilon},\boldsymbol{\sigma}_{m,t})}(\mathbf{w},w',-\mathbf{w})=F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{w})+2\left\langle\sum_{k=1}^n(-1)^{k+1}w_k,iw'\right\rangle,$$

By the change of variables $A_n\mathbf{v}=\mathbf{w}$ and $\xi=2w'$, one gets, in v-variables, the function

$$(\mathbf{v},\xi)\mapsto F_{\boldsymbol{\sigma}_{m,t}}(\mathbf{v})+\langle v_1,i\xi\rangle$$

which is the fiberwise sum of a generating function of $e^{-i\pi t}\Phi_1$ with the generating function $(x;\xi)\mapsto\langle x, i\xi\rangle$ that generates $-\mathrm{id}$. This time, we can take $(f_{s,t})$ to be the family of function

$$f_{s,t}(\mathbf{v}, v_{n+1}) := F_{\boldsymbol{\sigma}_{m,t+(1-s)/2}}(\mathbf{v}) + \sin\left(s\frac{\pi}{2}\right)\left\langle v_1 - i\cos\left(s\frac{\pi}{2}\right)\frac{\xi}{2}, i\xi\right\rangle,\ s\in[0,1],$$

that interpolates the latter with $(\mathbf{v},\xi)\mapsto F_{\boldsymbol{\sigma}_{m,t+1/2}}(\mathbf{v})$. That being said, it is not difficult to conclude.

7. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\boldsymbol{\sigma}$ be any tuple of $\mathbb{C}$-equivariant Hamiltonian diffeomorphisms associated with φ, so that $N(\boldsymbol{\sigma};\mathbb{F}) = N(\varphi;\mathbb{F})$. Let us denote by $K(\boldsymbol{\sigma};\mathbb{F})$ the number of $\mathbb{Z}$-orbits of finite bars of the barcode associated with $\boldsymbol{\sigma}$ over the field $\mathbb{F}$. According to the universal coefficient theorem, one can assume that $\mathbb{F}=\mathbb{Q}$ if $\mathbb{F}$ has characteristic 0 and $\mathbb{F}=\mathbb{F}_p$ if it has characteristic $p\neq 0$.

Let us assume that $\mathbb{F}=\mathbb{Q}$. According to Proposition 4.18, $N(\boldsymbol{\sigma};\mathbb{Q}) > d+1$ implies that $K(\boldsymbol{\sigma};\mathbb{Q})>0$ so $\beta(\boldsymbol{\sigma};\mathbb{Q})>0$. According to Corollary 6.3, for all prime number $p\geq 3$,

$$K(\boldsymbol{\sigma}^p;\mathbb{F}_p)\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq \beta_{\mathrm{tot}}(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p).$$

Thus, by Proposition 6.4, for all sufficiently large prime p,

$$K(\boldsymbol{\sigma}^p;\mathbb{F}_p)\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p) \geq p\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{Q}) \geq p\beta(\boldsymbol{\sigma};\mathbb{Q}),$$

(the sum of length β_{tot} is obviously larger than the largest length β). That is to say that $K(\boldsymbol{\sigma}^p;\mathbb{F}_p)\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p)$ grows at least linearly with prime numbers p. According to Theorem 5.1, $\beta(\boldsymbol{\sigma}^p;\mathbb{F}_p)\leq 1$ so $K(\boldsymbol{\sigma}^p;\mathbb{F}_p)$ must diverge to $+\infty$ with prime numbers p and so must $N(\boldsymbol{\sigma}^p;\mathbb{F}_p)$ by Proposition 4.18. Let $z_1,\dots,z_n\in\mathbb{C}\mathrm{P}^d$ be the fixed points of φ. According to Corollary 3.2, there exists $B>0$ such that $\dim \mathrm{C}_*(\boldsymbol{\sigma}^p;z_k;\mathbb{F}_p) < B$ for all k and all prime p. Let $A\in\mathbb{N}$ be such that for all prime $p\geq A$, $N(\boldsymbol{\sigma}^p;\mathbb{F}_p) > nB$. Then, for all prime $p\geq A$, there must be at least one fixed point of φ^p that is not one of the z_k's, that is there must be at least one p-periodic point that is not a fixed point. Hence, the conclusion for the case $\mathbb{F}$ of characteristic 0.

Let us assume that $\mathbb{F}=\mathbb{F}_p$ for some prime number p. By contradiction, let us assume that φ has only finitely many periodic point of period p^k for some $k\in\mathbb{N}$. According to Corollary 6.3,

$$\beta_{\mathrm{tot}}(\boldsymbol{\sigma}^{p^k};\mathbb{F}_p) \geq p^k\beta_{\mathrm{tot}}(\boldsymbol{\sigma};\mathbb{F}_p),\quad \forall k\in\mathbb{N},$$

in particular, $N(\boldsymbol{\sigma}^{p^k};\mathbb{F}_p) > d+1$ for all $k\in\mathbb{N}$. Thus, by taking a sufficiently large p^k-iterate of φ, one can assume that every periodic point of φ of period p^k for some k is an admissible fixed point of φ (see the end of Section 3.3 for the definition of an admissible fixed point). According to Proposition 3.1, it implies that $N(\boldsymbol{\sigma}^{p^k};\mathbb{F}_p) = N(\boldsymbol{\sigma};\mathbb{F}_p)$ for all $k\in\mathbb{N}$. But Corollary 6.3 together with Proposition 4.18 imply that the left-hand side of this equation must diverge to $+\infty$ as k grows, a contradiction. $\square$

Appendix A. Projective join

We describe an operation on the homology of subsets of a projective space relating the homology of two subsets A and B to the homology of their projective join $A*B$. The analogous operation for the topological join was already defined by Whitehead in [35]. Granja-Karshon-Pabiniak-Sandon already defined a homology projective join in the real case in [19] for a purpose similar to ours. However, their direct construction seems difficult to extend in the complex case.

Since the proof of some fundamental properties of this operation are only technical and does not shed much light on their applications, we have put these proofs in a specific section. More precisely, proofs of Proposition A.3 and A.5 are postponed to Section A.2

A.1. **Definition and properties.** Let $m, n \in \mathbb{N}$ and let $\pi : \mathbb{C}^{m+n+2}\setminus 0 \to \mathbb{C}\mathrm{P}^{m+n+1}$ be the quotient map. We projectively endow $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^{m+n+1}$ by identifying $\mathbb{C}\mathrm{P}^m$ with $\pi(\mathbb{C}^{m+1} \times 0 \setminus 0)$ and $\mathbb{C}\mathrm{P}^n$ with $\pi(0 \times \mathbb{C}^{n+1} \setminus 0)$ so that $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$ do not intersect. This is equivalent to considering two projective subspaces of respective $\mathbb{C}$-dimension m and n in general position. Let $A \subset \mathbb{C}\mathrm{P}^m$ and $B \subset \mathbb{C}\mathrm{P}^n$ be non-empty sets. Then the projective join $A * B \subset \mathbb{C}\mathrm{P}^{m+n+1}$ is the union of every projective line intersecting A and B. In other words, $A*B = A \cup B \cup \pi(\widetilde{A}\times\widetilde{B})$ where $\widetilde{A}$ and $\widetilde{B}$ are the lifts of A and B to $\mathbb{C}^{n+1} \setminus 0$ and $\mathbb{C}^{m+1} \setminus 0$ respectively. One can remark that $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}^{m+n+1}$ and that if $[a : b] \in \mathbb{C}\mathrm{P}^{m+n+1}$, with $a \in \mathbb{C}^{m+1}$ and $b \in \mathbb{C}^{n+1}$, does not belong to $\mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^n$, then only one projective line intersecting these two subspaces contains $[a : b]$, namely the line joining $\alpha := [a : 0]$ to $\beta := [0 : b]$ denoted by $(\alpha\beta)$.

We need to define a projective join in the level of homology which would be a map $\mathrm{pj}_* : H_*(A \times B) \to H_{*+2}(A * B)$, or dually in the level of cohomology $\mathrm{pj}^* : H^*(A * B) \to H^{*+2}(A \times B)$. One can perhaps proceed directly by defining a projective join at the level of chains, sending two chains $\alpha \in C_i(A)$ and $\beta \in C_j(B)$ to a chain $\alpha*\beta \in C_{i+j+2}(A*B)$ that triangulates the projective join of their images. We will proceed in an indirect way by remaining in the level of homology.

Let $E_{A,B} \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^{m+n+1}$ be the set

$$E_{A,B} := \{(a, b, c) \in A \times B \times (A * B) \mid c \in (ab)\}.$$

Let p_1 and p_2 be the canonical projection of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^{m+n+1}$ on the factor $\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^{m+n+1}$ respectively. Then $p_1|_{E_{A,B}}$ defines a $\mathbb{C}\mathrm{P}^1$-fiber bundle on $A \times B$, the fiber of any $(a, b) \in A \times B$ being $a \times b \times (ab) \simeq (ab)$. As $\mathbb{C}\mathrm{P}^1$ can be identified with the 2-sphere $\mathbb{S}^2$, the Gysin long exact sequence holds:

$$\cdots \xrightarrow{\smile e} H^*(A \times B) \xrightarrow{(p_1)^*} H^*(E_{A,B}) \xrightarrow{(p_1)_*} H^{*-2}(A \times B) \xrightarrow{\smile e} \cdots, \tag{21}$$

where $e \in H^3(A \times B)$ denotes the Euler class of the $\mathbb{S}^2$-bundle $E_{A,B}$.

Definition A.1. The cohomology projective join $\mathrm{pj}^* : H^*(A * B) \to H^{*-2}(A \times B)$ denotes the map $\mathrm{pj}^* := (p_1)_* \circ (p_2)^*$, where $(p_1)_*$ is defined by (21) and $(p_2)^*$ is induced by $p_2 : E_{A,B} \to A * B$. The homology projective join $\mathrm{pj}_* = (p_2)_* \circ (p_1)^* : H_*(A \times B) \to H_{*+2}(A * B)$ is defined dually.

We extend this definition to topological couples $(A, B) \subset \mathbb{C}\mathrm{P}^m$, $(C, D) \subset \mathbb{C}\mathrm{P}^n$ the following way. Let $(A, B) * (C, D) := (A * C, A * D \cup B * C)$, the map p_1 defines a relative $\mathbb{C}\mathrm{P}^1$-fiber bundle $(E_{A,C}, E_{A,D} \cup E_{B,C})$ on $(A, B) \times (C, D)$ while

p_2 maps this bundle on $(A,B)*(C,D)$. Hence, one can set $\mathrm{pj}^* := (p_1)_* \circ (p_2)^*$ and $\mathrm{pj}_* := (p_2)_* \circ (p_1)^*$ as before. By naturality of the maps induced by p_1 and p_2, this extension is natural: projective join commutes with long exact sequences of topological pairs or triples.

These maps are also natural in the following way: let $A, C \subset \mathbb{C}\mathrm{P}^m$ and $B, D \subset \mathbb{C}\mathrm{P}^n$ and assume that $f : A * B \to C * D$ is the restriction of a projective map satisfying $f|_A : A \to C$ and $f|_B : B \to D$, then the $\mathbb{C}\mathrm{P}^1$-fiber bundle $E_{A,B}$ is the pull-back of $E_{C,D}$ by $f|_A \times f|_B$, so that the following diagram commutes:

$$\begin{array}{ccc} H_*(A \times B) & \xrightarrow{\mathrm{pj}_*} & H_{*+2}(A * B) \\ \Big\downarrow{\scriptstyle (f|_A \times f|_B)_*} & & \Big\downarrow{\scriptstyle f_*} \\ H_*(C \times D) & \xrightarrow{\mathrm{pj}_*} & H_{*+2}(C * D) \end{array} . \tag{22}$$

This statement extends to topological pairs in the obvious way.

Proposition A.2. *The homology projective join is associative: given A, B and C included in $\mathbb{C}\mathrm{P}^n$,*

$$\forall(\alpha,\beta,\gamma) \in H_*(A) \times H_*(B) \times H_*(C), \quad \mathrm{pj}_*(\mathrm{pj}_*(\alpha \times \beta) \times \gamma) = \mathrm{pj}_*(\alpha \times \mathrm{pj}_*(\beta \times \gamma)).$$

As R-algebras, one has

$$H^*(\mathbb{C}\mathrm{P}^{m+n+1}) = R[u]/u^{m+n+2} \quad \text{and} \quad H^*(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) = R[u_1,u_2]/(u_1^{m+1}, u_2^{n+1})$$

where u, u_1 and u_2 restrict to orientation classes of $\mathbb{C}\mathrm{P}^1$ (with $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^m$ for u_1 and $\mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^n$ for u_2).

Proposition A.3. *Let pj^* be the cohomology projective join on $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$, with the above notation one has*

$$\mathrm{pj}^* u^k = \sum_{i+j=k-1} u_1^i u_2^j, \quad \forall k \in \mathbb{N}^*.$$

Dually, the homology projective join pj_* on $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$ satisfies

$$\mathrm{pj}_*\left([\mathbb{C}\mathrm{P}^i] \times [\mathbb{C}\mathrm{P}^j]\right) = [\mathbb{C}\mathrm{P}^{i+j+1}], \quad \forall i \in \{0,\dots,m\}, \forall j \in \{0,\dots,n\}.$$

We recall that the cohomological length $\ell(A)$ of a subspace $A \subset \mathbb{C}\mathrm{P}^N$ is the rank of the morphism $H^*(\mathbb{C}\mathrm{P}^N;\mathbb{Z}) \to H^*(A;\mathbb{Z})$ induced by the inclusion (*e.g.* $\ell(\mathbb{C}\mathrm{P}^n) = n+1$). This is also the rank of the morphism $H_*(A;\mathbb{Z}) \to H_*(\mathbb{C}\mathrm{P}^N;\mathbb{Z})$.

Given two subsets $A \neq \emptyset$ and B as above, let $\ell := \ell(B)$. The restriction of $u^\ell \in H^*(\mathbb{C}\mathrm{P}^{m+n+1})$ to $H^*(B)$ is zero and is non-zero in $H^*(A*B)$. Let $v_B \in H^*(A*B, B)$ be one of its inverse image. We recall that there is a well-defined cap-product

$$H_k(A*B,B) \times H^l(A*B,B) \xrightarrow{\frown} H_{k-l}(A*B \setminus B)$$

which is defined by the following commutative diagram:

$$\begin{array}{ccc} H_*(A*B,B) \times H^*(A*B,B) & \searrow^{\frown} & \\ \Big\downarrow{\scriptstyle \simeq} \qquad\qquad {\scriptstyle \simeq}\Big\uparrow & & \\ H_*(A*B,T) \times H^*(A*B,T) & & H_*(A*B \setminus B) , \\ {\scriptstyle \simeq}\Big\uparrow \qquad\qquad \Big\downarrow{\scriptstyle \simeq} & & \\ H_*(A*B \setminus B, T \setminus B) \times H^*(A*B \setminus B, T \setminus B) & \nearrow_{\frown} & \end{array}$$

where $T \subset A * B$ is the restriction of a tubular neighborhood of $\mathbb{C}\mathrm{P}^n$ to $A * B$, the bottom diagonal arrow is the usual cap-product and vertical maps are isomorphisms induced by inclusion maps (the isomorphisms come from retractions at the top and from excision at the bottom). Let $p_A : A * B \setminus B \to A$ be the map $p_A[a:b] := [a:0]$. Let $f : H_*(A*B) \to H_{*-2\ell}(A)$ be the map $f(\alpha) := (p_A)_*(\alpha \frown v_B)$. These definitions extend to the case where A is a topological pair (A_1, A_0) with $A_1 \neq A_0$ by taking $v_B \in H^{\ell(B)}(A_1 * B, B)$ and by using the cap-product

$$H_k(A_1 * B, A_0 * B) \times H^l(A_1 * B, B) \xrightarrow{\frown} H_{k-l}(A_1 * B \setminus B, A_0 * B \setminus B),$$

defined the same way as above.

Corollary A.4 ([18, Corollary A.2]). *For all non-empty subsets $A \subset \mathbb{C}\mathrm{P}^m$ and $B \subset \mathbb{C}\mathrm{P}^n$, one has*

$$\ell(A * B) = \ell(A) + \ell(B).$$

Proof. Let $\alpha \in H_{2\ell(A)-2}(A)$ and $\beta \in H_{2\ell(B)-2}(B)$ be classes that are sent to the class $[\mathbb{C}\mathrm{P}^{\ell(A)-1}] \in H_*(\mathbb{C}\mathrm{P}^m)$ and $[\mathbb{C}\mathrm{P}^{\ell(B)-1}] \in H_*(\mathbb{C}\mathrm{P}^n)$ respectively. According to Proposition A.3 and naturality (22), $\mathrm{pj}_*(\alpha \times \beta)$ is sent to $[\mathbb{C}\mathrm{P}^{\ell(A)+\ell(B)-1}]$ in $H_*(\mathbb{C}\mathrm{P}^{m+n+1})$. Hence $\ell(A * B) \geq \ell(A) + \ell(B)$. The converse inequality comes from the commutativity of the following diagram:

$$\begin{array}{ccc} H_*(A * B) & \xrightarrow{\quad f \quad} & H_{*-2\ell(B)}(A) \\ \downarrow & & \downarrow \\ H_*(\mathbb{C}\mathrm{P}^{m+n+1}) & \xrightarrow{\cdot \frown u^{\ell(B)}} & H_{*-2\ell(B)}(\mathbb{C}\mathrm{P}^{m+n+1}) \end{array}.$$

□

Proposition A.5. *Let $A \subset \mathbb{C}\mathrm{P}^m$, $B \subset \mathbb{C}\mathrm{P}^n$ be non-empty sets and $\ell := \ell(B)$. Let $\beta \in H_*(B)$ be a class that is sent to $[\mathbb{C}\mathrm{P}^{\ell-1}] \in H_*(\mathbb{C}\mathrm{P}^n)$. The following diagram commutes:*

$$\begin{array}{ccc} H_{*+2\ell-2}(A \times B) & \xrightarrow{\mathrm{pj}_*} & H_{*+2\ell}(A * B) \\ {\scriptstyle \cdot\times\beta}\uparrow & & \downarrow{\scriptstyle f} \\ H_*(A) & \xrightarrow[=]{\quad \mathrm{id} \quad} & H_*(A) \end{array},$$

where $f(\alpha) := (p_A)_*(\alpha \frown v_B)$ as defined above. This result also holds when A is a topological pair (A_1, A_0) with $A_1 \neq A_0$.

Corollary A.6. *Let (A_1, A_0) be a topological pair included in $\mathbb{C}\mathrm{P}^m$, the map $\alpha \mapsto$ $\mathrm{pj}_*(\alpha \times [\mathbb{C}\mathrm{P}^n])$ gives an isomorphism*

$$H_*(A_1, A_0) \to H_{*+2(n+1)}(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n)$$

which is the inverse of the isomorphism $f : \alpha \mapsto (p_{A_1})_*\alpha \frown v_{\mathbb{C}\mathrm{P}^n}$.

Proof. According to [1, Proposition 4.1], the inclusion morphism $H_*(\mathbb{C}\mathrm{P}^n) \to H_*(A_i * \mathbb{C}\mathrm{P}^n)$ is an isomorphism in degree $* \leq 2n+1$ and $f_i : H_*(A_i * \mathbb{C}\mathrm{P}^n) \to H_{*-2(n+1)}(A_i)$, $f_i(\alpha) := (p_{A_i})_*\alpha \frown v_{\mathbb{C}\mathrm{P}^n}$ is an isomorphism in degree $* \geq 2(n+1)$, for $i \in \{0,1\}$ (it is stated in cohomology but the proof also holds in this dual setting). Using the long exact sequences of the pairs (A_1, A_0) and $(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n)$, we deduce that $f : H_*(A_1 * \mathbb{C}\mathrm{P}^n, A_0 * \mathbb{C}\mathrm{P}^n) \to H_{*-2(n+1)}(A_1, A_0)$ is an isomorphism. The result is now a direct consequence of Proposition A.5. □

A.2. **Technical proofs.** We will denote $E_{\mathbb{CP}^m,\mathbb{CP}^n}$ by $E_{m,n}$. The bundle $E_{A,B}$ is the restriction of the bundle $E_{m,n}$ to $A\times B$, hence e is the pullback of the Euler class of $E_{m,n}$ which lies in $H^3(\mathbb{CP}^m\times\mathbb{CP}^n)=0$. Therefore $e=0$ and (21) reduces to the short exact sequence

$$0\to H^*(A\times B)\xrightarrow{(p_1)^*}H^*(E_{A,B})\xrightarrow{(p_1)_*}H^{*-2}(A\times B)\to 0. \tag{23}$$

Proof of Proposition A.2. Let us first define a projective join with 3 entries pj^3_* : $H_*(A\times B\times C)\to H_{*+4}(A*B*C)$ then prove that

$$\mathrm{pj}_*(\mathrm{pj}_*(\alpha\times\beta)\times\gamma)=\mathrm{pj}^3_*(\alpha\times\beta\times\gamma)=\mathrm{pj}_*(\alpha\times\mathrm{pj}_*(\beta\times\gamma)). \tag{24}$$

Given three points a, b and c of some projective space $\mathbb{CP}^N$ that are projectively independent, we use the classical notation $(abc)\subset\mathbb{CP}^N$ to denote the complex projective plane Let $E_{A,B,C}\subset(\mathbb{CP}^n)^3\times\mathbb{CP}^{3n+2}$ be the set

$$E_{A,B,C}:=\{(a,b,c,z)\in A\times B\times C\times(A*B*C)\mid z\in(abc)\}$$

and let $P_1:E_{A,B,C}\to A\times B\times C$ and $P_2:E_{A,B,C}\to A*B*C$ be the associated projection maps. The map P_1 defines a $\mathbb{CP}^2$-fiber bundle which is the restriction of the fiber bundle $E_{\mathbb{CP}^n,\mathbb{CP}^n,\mathbb{CP}^n}\to(\mathbb{CP}^n)^3$, so the action of $\pi_1(A\times B\times C)$ on the homology group $H_*(\mathbb{CP}^2)$ of a fiber is the restriction of the action of $\pi_1((\mathbb{CP}^n)^3)=0$ *i.e.* trivial. We define pj^3_* by $\mathrm{pj}^3_*:=(P_2)_*\circ(P_1)^*$ where $(P_1)^*:H_*(A\times B\times C)\to H_{*+4}(E_{A,B,C})$ denotes the morphism dual to the integration along the fiber of the fibration P_1 (the complex structure of $\mathbb{CP}^2$ gives a natural identification $H_4(\mathbb{CP}^2)\simeq R$). We refer to Section A.3 for the definition and properties of this morphism.

In order to prove (24), let us introduce the set

$$E_{(A,B),C}:=\{(a,b,c,x,z)\in A\times B\times C\times(A*B)\times(A*B*C)\mid x\in(ab)\text{ and }z\in(xc)\}$$

with the projection maps $P'_2:E_{(A,B),C}\to A*B*C$, $P'_1:E_{(A,B),C}\to E_{A,B}\times C$ sending (a,b,c,x,z) to $(a,b,x;c)$, $\tilde f:E_{(A,B),C}\to E_{A*B,C}$ sending (a,b,c,x,z) to (x,c,z) and $g:E_{(A,B),C}\to E_{A,B,C}$ sending (a,b,c,x,z) to (a,b,c,z). The map P'_1 is a $\mathbb{CP}^1$-fiber bundle, in fact $\tilde f$ is a morphism of fiber bundle with base-space morphism $f:=p_2\times\mathrm{id}_C$. In order to summarize the situation, we have the following commutative diagram:

$$\begin{array}{ccccc}
 & & A*B*C & & \\
 & \overset{P_2}{\nearrow} & \uparrow{\scriptstyle P'_2} & \overset{p'_2}{\nwarrow} & \\
E_{A,B,C} & \xleftarrow[g]{} & E_{(A,B),C} & \xrightarrow{\tilde f} & E_{A*B,C}\\
 & \searrow & \downarrow{\scriptstyle P'_1} & & \downarrow{\scriptstyle p'_1} \\
 & {\scriptstyle P_1}\searrow & E_{A,B}\times C & \xrightarrow{f} & (A*B)\times C\\
 & & \downarrow{\scriptstyle p_1\times\mathrm{id}_C} & & \\
 & & A\times B\times C & &
\end{array}\quad . \tag{25}$$

According to the naturality of the integration along the fiber (35), it follows that $(p'_2)_*(p'_1)^*f_*(p_1\times\mathrm{id}_C)^*=(P'_2)_*(P'_1)^*(p_1\times\mathrm{id}_C)^*$, which means that

$$\mathrm{pj}_*(\mathrm{pj}_*(\alpha\times\beta)\times\gamma)=(P'_2)_*(P'_1)^*(p_1\times\mathrm{id}_C)^*(\alpha\times\beta\times\gamma). \tag{26}$$

The map g commutes with the Serre fibration $q:=(p_1\times\mathrm{id}_C)\circ P'_1$ and the fiber bundle P_1. Let us fix a base-point $(a,b,c)\in A\times B\times C$, of fiber $F=a\times b\times c\times(abc)\simeq(abc)$

for P_1 and of fiber $F' \simeq \{(x,z) \mid x \in (ab) \text{ and } z \in (xc)\}$ for q. According to Proposition A.7 and the remark after it, in order to show that the following diagram commutes:

$$\begin{array}{ccc} H_{*+4}(E_{A,B,C}) & \xleftarrow{\;g_*\;} & H_{*+4}(E_{(A,B),C}) \\ & \nwarrow^{P_1^*} & \uparrow{q^*} \\ & & H_*(A \times B \times C) \end{array} \tag{27}$$

(with coefficients of every H_* in the same ring R), one must prove that $g_* : H_4(F') \to H_4(F)$ commutes with the identity of R under the isomorphisms $H_4(F) \simeq R$ and $H_4(F') \simeq R$ given by the local complex orientation. This comes from the fact that the quotient space $F'/((ab) \times c)$ is canonically homeomorphic to $(abc) \simeq F$ (in particular, preserving the orientation), the homeomorphism being induced by $g|_{F'}$. The long exact sequence of the couple $(F', (ab) \times c)$ concludes. Therefore, diagram (27) commutes. Thus, according to the left hand side of the diagram (25) together with the composition property $q^* = (P_1')^*(p_1 \times \mathrm{id}_C)^*$, one has $(P_2')_*(P_1')^*(p_1 \times \mathrm{id}_C)^* = (P_2)_*(P_1)^*$ and (26) gives the first equality of (24).

The second equality is proven in a symmetric way. □

Proof of Proposition A.3. For now, let us work on $E_{m,n}$. First, let us see that $\mathrm{pj}^* u = 1$. By naturality (22), it boils down to showing that $\mathrm{pj}^* : H^2((ab)) \to H^0(a \times b)$ maps the restriction of u to (ab) to $1 \in H^0(a \times b)$ for all $(a,b) \in \mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n$. Now $E_{a,b} = a \times b \times (ab)$ so that $(p_2)^*$ is an isomorphism sending the orientation class of (ab) to the orientation class of $E_{a,b}$. According to (23), $(p_1)_*$ is also an isomorphism (preserving the orientation), hence the result.

Let $u_0 := (p_2)^* u \in H^2(E_{m,n})$. We must now study $(p_1)_* u_0^k$ for $k \in \mathbb{N}^*$.

Let $T \subset \mathbb{C}\mathrm{P}^{m+n+1}$ be a tubular neighborhood of $\mathbb{C}\mathrm{P}^n$ that is a deformation retract. Its pullback $p_2^{-1}(T)$ is a deformation retract of $p_2^{-1}(\mathbb{C}\mathrm{P}^n)$, hence

$$\begin{aligned} H^*\left(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n)\right) &\simeq H^*\left(E_{m,n}, p_2^{-1}(T)\right) \\ &\simeq H^*\left(E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n), p_2^{-1}(T \setminus \mathbb{C}\mathrm{P}^n)\right), \end{aligned} \tag{28}$$

where the isomorphisms are induced by inclusion, the second one coming from excision. The space $E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ is the $\mathbb{C}\mathrm{P}^1$-fiber bundle $E_{m,n}$ with one global section taken away, so it is a $\mathbb{C}$-fiber bundle. Let $t \in H^2(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n))$ be its Thom class (under the natural identification given by (28)). Therefore, according to Thom isomorphism theorem, the map

$$H^*(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) \to H^{*+2}(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n)), \quad \alpha \mapsto t \smile (p_1)^*\alpha \tag{29}$$

is an isomorphism. By looking at restrictions to $E_{a,b}$'s, we see that t is non-zero on $H^*(E_{m,n})$ and is sent to $1 \in H^0(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n)$ by $(p_1)_*$. According to (23), on $H^*(E_{m,n})$ we have $u_0 - t = (p_1)^* v$ for some $v \in H^2(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n)$. In order to find

v, we consider the following commutative diagram:

$$\begin{array}{ccccc} & & & & H^*(\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n) \\ & & & \swarrow^{(p_1)^*} & \downarrow^{(p_1)^*} \\ H^*\left(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n)\right) & \longrightarrow & H^*(E_{m,n}) & \longrightarrow & H^*\left(p_2^{-1}(\mathbb{C}\mathrm{P}^n)\right) \\ {\scriptstyle (p_2)^*}\uparrow & & {\scriptstyle (p_2)^*}\uparrow & & {\scriptstyle (p_2)^*}\uparrow \\ H^*(\mathbb{C}\mathrm{P}^{m+n+1}, \mathbb{C}\mathrm{P}^n) & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^{m+n+1}) & \longrightarrow & H^*(\mathbb{C}\mathrm{P}^n) \end{array}, \qquad (30)$$

where horizontal arrows are induced by inclusion and form exact sequences. The restriction of p_1 to $p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ induces a homeomorphism $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^n \simeq p_2^{-1}(\mathbb{C}\mathrm{P}^n)$. Under this identification, the restriction of p_2 to $p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ is the projection on the second factor $\mathbb{C}\mathrm{P}^n$. Hence, the right-hand side vertical arrow $(p_2)^*$ of (30) sends u^k to $(p_1)^*u_2^k$. In particular, by commutativity of (30), $u_0 \in H^2(E_{m,n})$ is sent to $(p_1)^*u_2$. Since $t \in H^2(E_{m,n})$ is in the image of the top left arrow, it is sent to 0 in $H^2(p_2^{-1}(\mathbb{C}\mathrm{P}^n))$ by exactness. Thus $u_0 - t \in H^2(E_{m,n})$ is sent to $(p_1)^*u_2$ whereas $(p_1)^*v \in H^2(E_{m,n})$ is sent to $(p_1)^*v \in H^2(p_2^{-1}(\mathbb{C}\mathrm{P}^n))$ by commutativity of the up right triangle (with a slight abuse of notation). Therefore $v = u_2$.

In order to study the powers of u_0, we now study the powers of $t \in H^2(E_{m,n})$. Seen in $H^4(E_{m,n}, p_2^{-1}(\mathbb{C}\mathrm{P}^n))$, $t^2 = t \smile (p_1)^*(\lambda u_1 + \mu u_2)$ for some $\lambda, \mu \in \mathbb{Z}$ according to Thom isomorphism (29). Let us first find the value of λ by restricting the complex line bundle $E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n)$ to the base space $\mathbb{C}\mathrm{P}^m \times \mathbb{C}\mathrm{P}^0$. This complex line bundle is $E_{m,0} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^0)$ and its Thom class t' is the restriction of t to

$$H^2\left(E_{m,0}, p_2^{-1}(\mathbb{C}\mathrm{P}^0)\right) \simeq H^2\left(E_{m,0}/p_2^{-1}(\mathbb{C}\mathrm{P}^0)\right),$$

so that $t'^2 = \lambda t' \smile (p_1)^*u_1$. Since $\mathbb{C}\mathrm{P}^0$ is just a point, p_2 factors in a homeomorphism between $E_{m,0}/p_2^{-1}(\mathbb{C}\mathrm{P}^0)$ and $\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0$. Thus p_2 induces an isomorphism of $\mathbb{Z}$-algebras

$$H^*(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0) \xrightarrow{\simeq} H^*(E_{m,0}, p_2^{-1}(\mathbb{C}\mathrm{P}^0)). \qquad (31)$$

According to the long exact sequence of the couple $(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0)$, $H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0) \simeq H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0)$ so that the generator $u \in H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0)$ can naturally be seen in $H^2(\mathbb{C}\mathrm{P}^m * \mathbb{C}\mathrm{P}^0, \mathbb{C}\mathrm{P}^0)$. Under the isomorphism (31), u is mapped to t' so that u^2 is mapped to t'^2. Thus t'^2 must be a generator of $H^4(E_{m,0}, p_2^{-1}(\mathbb{C}\mathrm{P}^0))$, hence $\lambda = \pm 1$. By applying the orientation preserving morphism $(p_1)_*$, we see that $\lambda = 1$.

Now, since $u_0 = t + (p_1)^*u_2$, one has

$$u_0^2 = t \smile (p_1)^*(u_1 + (\mu + 2)u_2) + (p_1)^*u_2^2$$

hence $(p_1)_*u_0^2 = u_1 + (\mu + 2)u_2$. By symmetry of $(p_1)_*u_0^2$ in u_1 and u_2, μ must be -1. Indeed, the above identity is still true by restricting ourselves to $E_{1,1}$ and must be invariant under the map induced by $(a, b, c) \mapsto (b, a, c)$ that swaps u_1 and u_2. Since $u_0 = t + (p_1)^*u_2$, one has that

$$u_0^k = \sum_{i+j=k} \binom{k}{i} t^i \smile (p_1)^*u_2^j.$$

Using $t^i = t \smile (p_1)^*(u_1 - u_2)^{i-1}$ for $i \in \mathbb{N}$ and $(p_1)_*(t \smile (p_1)^* w) = w$, one finally gets

$$(p_1)_* u_0^k = \sum_{i+j=k} \binom{k}{i} (u_1 - u_2)^{i-1} u_2^j = \sum_{i+j=k-1} u_1^i u_2^j, \tag{32}$$

the last equality can be obtained by identification of coefficients of the polynomial expression in u_1 and u_2. □

Proof of Proposition A.5. We first remark that there is a well-defined cap-product $H_*(E_{A,B}) \times H^*(E_{A,B}, p_2^{-1}(B)) \to H_*(E_{A,B} \setminus p_2^{-1}(B))$ compatible with the one defines in $A * B$ through the map p_2 and defined the same way. This is summed up by saying that the left hand side of the following diagram is "commutative":

$$\begin{array}{ccccc} H_*(E_{A,B}) \times H^*(E_{A,B}, p_2^{-1}(B)) & \xrightarrow{\ \frown\ } & H_*(E_{A,B} \setminus p_2^{-1}(B)) & \xrightarrow[\simeq]{(p_1)_*} & H_*(A \times B) \\ \big\downarrow (p_2)_* \qquad (p_2)^* \big\uparrow & & \big\downarrow (p_2)_* & & \big\downarrow (\mathrm{pr}_1)_* \\ H_*(A * B) \times H^*(A * B, B) & \xrightarrow{\ \frown\ } & H_*(A * B \setminus B) & \xrightarrow[\simeq]{(p_A)_*} & H_*(A) \end{array} . \tag{33}$$

In this diagram, $\mathrm{pr}_1 : A \times B \to A$ is the projection on the first factor. The commutativity of the right hand side of the diagram comes from the obvious commutativity of the associated continuous maps: $p_A \circ p_2 = \mathrm{pr}_1 \circ p_1$ on $E_{A,B} \setminus p_2^{-1}(B)$. Let $v'_B := (p_2)^* v_B \in H^{2\ell}(E_{A,B}, p_2^{-1}(B))$. The space $E_{A,B} \setminus p_2^{-1}(B)$ is the restriction to $A \times B$ of the $\mathbb{C}$-fiber bundle $E_{m,n} \setminus p_2^{-1}(\mathbb{C}\mathrm{P}^n)$, so that the following map is an isomorphism for the same reason the map (29) was:

$$H^*(A \times B) \to H^{*+2}(E_{A,B}, p_2^{-1}(B)), \quad w \mapsto t' \smile (p_1)^* w,$$

where $t' \in H^2(E_{A,B}, p_2^{-1}(B))$ is the restriction of the class t in (29). Therefore, $v'_B = t' \smile (p_1)^* w$ for some $w \in H^{2\ell-2}(A \times B)$ satisfying $w = (p_1)_*(v'_B)$ (we recall that $(p_1)_* t' = 1$). Seen in $H^{2\ell}(A * B)$, the class v_B is the restriction of u^ℓ, so that, seen in $H^{2\ell}(E_{A,B})$, v'_B is the restriction of u_0^ℓ. According to the identity (32), one has

$$(p_1)_* v'_B = w = \sum_{i+j=\ell-1} u_1^i u_2^j, \tag{34}$$

identifying u_1 and u_2 with their restrictions to $H^*(A \times B)$ by a slight abuse of notation. We can now compute, for all $\alpha \in H_*(A)$,

$$\begin{aligned} (p_A)_*(\mathrm{pj}_*(\alpha \times \beta) \frown v_B) &= (p_A)_* \circ (p_2)_*((p_1)^*(\alpha \times \beta) \frown v'_B) \\ &= (\mathrm{pr}_1)_* \circ (p_1)_*((p_1)^*(\alpha \times \beta) \frown v'_B) \\ &= (\mathrm{pr}_1)_* \left((\alpha \times \beta) \frown \sum_{i+j=\ell-1} u_1^i u_2^j \right) \\ &= \sum_{i+j=\ell-1} (\mathrm{pr}_1)_* \left((\alpha \frown u^i) \times (\beta \frown u^j) \right) \\ &= \left\langle u^{\ell-1}, \beta \right\rangle \alpha \frown u^0 = \alpha. \end{aligned}$$

The second equality follows from commutativity of the diagram (33), the third uses (34) together with the projection formula $p_*(p^*\gamma \frown w) = \gamma \frown p_* w$ where p is a sphere bundle. By grading issues, only the indices $(i, j) = (0, \ell - 1)$ contribute to the sum and, by definition of β, $\left\langle u^{\ell-1}, \beta \right\rangle = 1$. The result of this computation is the statement we wanted to prove. □

A.3. **Integration along the fiber.** In this section, we recall some well known properties of the morphism of integration along fiber (see [8, Section A.2] and references therein). Since we cannot find any reference concerning the proof of the composition property, we give a proof of this key property used to prove Proposition A.2.

Let G be a group. Throughout this section, $H_*(X)$ and $H^*(X)$ will denote respectively the singular homology and the singular cohomology of the topological space or pair X with coefficients in G. If we want to put another group of coefficients G', we will write down explicitly $H_*(X;G')$ or $H^*(X;G')$ so that for instance $H_*(X;H_d(Y)) = H_*(X;H_d(Y;G))$ where Y is a topological space or pair.

Let us assume that $\pi : X \to B$ is a Serre fibration with fiber F which has the type of a CW complex of dimension d (throughout this section, we will simply write that the fibers of π have dimension d). In order to simplify the statements, we will always assume that $\pi_1(B)$ acts trivially on $H_*(F)$. Then the Serre spectral sequence in homology $(E^r_{p,q})$ of π satisfies $E^2_{p,q} \simeq H_p(B;H_q(F)) = 0$ for $q > d$. Hence, it gives natural morphisms $E^2_{p,d} \to E^\infty_{p,d} \to H_{p+d}(X)$ for all p. Let $\pi^* : H_*(B;H_q(F)) \to H_{*+d}(X)$ denote the composition of the Serre isomorphism $H_*(B;H_d(F)) \simeq E^2_{*,d}$ with this map $E^2_{*,d} \to H_{*+d}(X)$. Dually, one can define a morphism $\pi_* : H^*(X) \to H^{*-d}(B;H^d(F))$. In de Rham cohomology, the map π_* can be easily defined on compact smooth fiber bundles as induced by the integration of differential forms along the fibers.

In the special case $F \simeq \mathbb{S}^n$, the map π_* corresponds to the Gysin morphism. This definition extends directly to relative fibrations $\pi : (X,X') \to (B,B')$ and the induced maps commute with the long exact sequences of pair and triple by naturality of the Serre spectral sequence. Given a commuting square of (possibly relative) fibrations

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & Y \\ {\scriptstyle F}\big\downarrow{\scriptstyle p} & & {\scriptstyle F'}\big\downarrow{\scriptstyle q} \\ B & \xrightarrow{f} & C \end{array},$$

with fibers F and F' of the same dimension d, the naturality of the Serre spectral sequence induces the commutative square

$$\begin{array}{ccc} H_{*+d}(X) & \xrightarrow{\tilde{f}_*} & H_{*+d}(Y) \\ {\scriptstyle p^*}\big\uparrow & & {\scriptstyle q^*}\big\uparrow \\ H_*(B;H_d(F)) & \xrightarrow{f_*} & H_*(B;H_d(F')) \end{array}, \qquad (35)$$

where f_* sends a class $[\sigma \otimes h]$, $\sigma \in C_*(B;\mathbb{Z})$ and $h \in H_d(F)$, on $[f_*\sigma \otimes \tilde{f}_* h]$.

Finally, it satisfies the following composition property. Let $\pi_1 : Y \to X$ and $\pi_2 : X \to B$ be (possibly relative) Serre fibrations of respective fibers F_1 and F_2 of dimension d_1 and d_2. Then $\pi := \pi_2 \circ \pi_1$ is a fibration whose fiber F is a fibration over F_2 with fibers F_1. According to the Serre spectral sequence, it has dimension $d_1 + d_2$ and $H_{d_1+d_2}(F)$ is naturally isomorphic to $H_{d_2}(F_2;H_{d_1}(F_1))$.

Proposition A.7. *The following diagram commutes*

$$\begin{array}{ccc} H_{*+d_2}(X;H_{d_1}(F_1)) & \xrightarrow{\pi_1^*} & H_{*+d_1+d_2}(Y) \\ {\scriptstyle \pi_2^*}\uparrow & & {\scriptstyle \pi^*}\uparrow \\ H_*(B;H_{d_2}(F_2;H_{d_1}(F_1))) & \xrightarrow{\simeq} & H_*(B;H_{d_1+d_2}(F)) \end{array}$$

where the bottom isomorphism is induced by the isomorphism $H_{d_2}(F_2;H_{d_1}(F_1)) \simeq H_{d_1+d_2}(F)$ between the groups of coefficients.

In our article, the fibers F_1, F_2 and F are naturally oriented so that there are canonical isomorphisms between their top degree homology groups and the coefficient group and one does not have to bother with changes of coefficient group. However, if one wants to avoid the change of coefficients in the naturality statement (35) when $H_d(F) \simeq G$ and $H_d(F') \simeq G$, the map $\tilde{f}_* : H_d(F) \to H_d(F')$ must send preferred generator to preferred generator.

Proof of Proposition A.7. Without loss of generality, one can assume that B and X are actual CW complexes and that π_2 is a locally trivial cellular fibration [4]. We denote by E, E_1 and E_2 Serre spectral sequences of π, π_1 and π_2 respectively, E_2 having $H_{d_1}(F_1)$ coefficients. Let B^p and X^p denote respectively the p-skeleton of B and the p-skeleton of X. Let $X_p := \pi_2^{-1}(B^p)$, $Y_p := \pi^{-1}(B^p)$ and $Y_{1;p} := \pi_1^{-1}(X^p)$ denote the filtration of the spaces X and Y associated with E_2, E and E_1 respectively. Therefore, for instance the first page of E_2 is given by $E^1_{2;p,q} := H_{p+q}(X_p, X_{p-1})$.

Since π_2 is a cellular fibration with fibers of dimension d_2, $X_p = \pi_2^{-1}(B^p)$ is included in X^{p+d_2}. Hence, $Y_p \subset Y_{1;p+d_2}$ and this inclusion between filtrations induces a morphism of spectral sequences (with a shift in degree) $E^r_{p,q} \to E^r_{1;p+d_2,q-d_2}$. Therefore, one gets the following commutative diagram

$$\begin{array}{ccccccc} & & H_{*+d_1+d_2}(Y) & = & H_{*+d_1+d_2}(Y) & & \\ & {\scriptstyle \pi^*}\nearrow & \uparrow & & \uparrow & \nwarrow{\scriptstyle \pi_1^*} & \\ & & E^\infty_{*,d_1+d_2} & \longrightarrow & E^\infty_{1;*+d_2,d_1} & & \\ & & \uparrow & & \uparrow & & \\ H_*(B;H_{d_1+d_2}(F)) & \xleftarrow{\simeq} & E^2_{*,d_1+d_2} & \longrightarrow & E^2_{1;*+d_2,d_1} & \xrightarrow{\simeq} & H_{*+d_2}(X;H_{d_1}(F_1)) \end{array},$$

where both the left and the right "squares" are the ones defining morphisms π^* and π_1^*. The bottom row allows us to define a morphism $f : H_*(B;H_{d_1+d_2}(F)) \to H_{*+d_2}(X;H_{d_1}(F_1))$. According to the last diagram, it is enough to prove that $f = \pi_2^*$ under the identification $H_{d_1+d_2}(F) \simeq H_{d_2}(F_2;H_{d_1}(F_1))$ to conclude.

We recall that the Serre isomorphism $E^2_{*,d_1+d_2} \simeq H_*(B;H_{d_1+d_2}(F))$ is induced by a chain isomorphism between the chain complex $E^1_{*,d_1+d_2} = H_{*+d_1+d_2}(Y_*, Y_{*-1})$ and the chain complex of the cellular filtration (B^p) of B that is $H_*(B^*, B^{*-1}; H_{d_1+d_2}(F))$. We denote by $\Psi : H_{*+d_1+d_2}(Y_*, Y_{*-1}) \to H_*(B^*, B^{*-1}; H_{d_1+d_2}(F))$ the chain isomorphism associated with π and by Ψ_1 and Ψ_2 the chain isomorphisms associated with π_1 and π_2 respectively. Since these chain isomorphisms are natural (this is included in the proof of the naturality of the Serre spectral sequence), one gets the following commutative diagram of chain complexes:

$$\begin{array}{ccc}
H_{*+d_1+d_2}(Y_*, Y_{*-1}) & \longrightarrow & H_{*+d_1+d_2}(Y_{1;*+d_2}, Y_{1;*+d_2-1}) \\
\simeq \big\downarrow \Psi & & \simeq \big\downarrow \Psi_1 \\
H_*(B^*, B^{*-1}; \mathbb{Z}) \otimes H_{d_1+d_2}(F) & & H_{*+d_2}(X^{*+d_2}, X^{*+d_2-1}; \mathbb{Z}) \otimes H_{d_1}(F_1) \\
\simeq \big\downarrow \mathrm{id}\otimes u & & \big\uparrow \\
H_*(B^*, B^{*-1}; \mathbb{Z}) \otimes H_{d_2}(F_2; H_{d_1}(F_1)) & \underset{\Psi_2}{\overset{\simeq}{\longleftarrow}} & H_{*+d_2}(X_*, X_{*-1}) \otimes H_{d_1}(F_1)
\end{array},$$

where u denotes the natural isomorphism $H_{d_1+d_2}(F) \to H_{d_2}(F_2; H_{d_1}(F_1))$. By passing to homology, one gets the following commutative diagram:

$$\begin{array}{ccccc}
E^2_{*,d_1+d_2} & \longrightarrow & E^2_{1;*+d_2,d_1} & & \\
\simeq \big\downarrow \Psi_* & & \simeq \big\downarrow (\Psi_1)_* & & \\
H_*(B; H_{d_1+d_2}(F)) & \xrightarrow{f} & H_{*+d_2}(X; H_{d_1}(F_1)) & \nwarrow & \\
\simeq \big\downarrow (\mathrm{id}\otimes u)_* & & \big\uparrow & & \\
H_*(B; H_{d_2}(F_2; H_{d_1}(F_1))) & \underset{(\Psi_2)_*}{\overset{\simeq}{\longleftarrow}} & E^2_{2;*,d_2} & \longrightarrow & E^\infty_{2;*,d_2}
\end{array},$$

where only the commutativity of the triangle is not a direct consequence of the previous diagram. The commutativity of the triangle is a consequence of the naturality of the morphism induced by the inclusion of filtrations $X_p \subset X^{p+d_2}$ between the associated spectral sequences. Indeed, since (X^{p+d_2}) is a cellular filtration, the associated spectral sequence whose first page is $(H_{p+q}(X^{p+d_2}, X^{p+d_2-1}))_{p,q}$ degenerates at the second page. The bottom part of the diagram shows that f is indeed π_2^* under the identification induced by u. ☐

Appendix B. Smith-type inequality for barcodes of Hamiltonian diffeomorphisms via the minimal Novikov field by Egor Shelukhin

We present an alternative argument proving [28, Theorem D], which is the key Smith-type inequality for barcodes of Hamiltonian diffeomorphisms for the argument proving the Hofer-Zehnder conjecture for closed monotone symplectic manifolds with semi-simple even quantum homology algebra. While the approach described in [28] is more algebraically conceptual, it relies on the use of Floer complexes with coefficients in the universal Novikov ring, which are quite difficult to make sense of geometrically. The current approach, at the cost of certain more ad hoc arguments, reduces the consideration to filtered Floer homology with the standard minimal Novikov field (with quantum variable of degree twice the minimal Chern number). This choice of coefficients is more geometric, and it is this new proof that is most amenable to interpretation in the setting of generating functions.

To set up the result and its proof we recall a few basic notions on barcodes of Hamiltonian diffeomorphisms. We follow the exposition in [28] and [2], whereto we refer for further detail and discussion of relevant notions. Suppose that (M, ω) is a closed monotone symplectic manifold, that is there exists $\kappa \in \mathbb{R}_{>0}$ such that for each $A \in H_2^S(M; \mathbb{Z})$, $\langle [\omega], A \rangle = \kappa \langle c_1(TM), A \rangle$, where $H_2^S(M; \mathbb{Z})$ is the image of the Hurewicz map $\pi_2(M) \to H_2(M; \mathbb{Z})$. The minimal Chern number of (M, ω) is the positive generator $N > 0$ of the image of the map $c_1(TM) : H_2^S(M;\mathbb{Z}) \to \mathbb{Z}$, given by $A \mapsto \langle c_1(TM), A\rangle$. Set the rationality constant $\rho \in \mathbb{R}_{>0}$ on (M,ω) to be $\rho = \kappa \cdot N$.

Fix a ground field $\mathbb{K}$. For a time-dependent Hamiltonian H on a closed monotone symplectic manifold (M,ω), denote by $\mathrm{Spec}(H) \subset \mathbb{R}$ the set of all actions

$$\mathcal{A}_H(\overline{x}) = \int_0^1 H(t,x(t)) - \int_{\overline{x}} \omega,$$

where $\overline{x}$ runs over contracting disks (considered up to the equivalence relation of having the same ω-areas) of 1-periodic orbit x of the Hamiltonian flow of H. If H is non-degenerate, that is its time-one map ϕ has its graph transverse to the diagonal in $M \times M$, we denote by $CF(H,J)$ the Floer complex of H with respect to a generic ω-compatible time-dependent almost complex structure $J = (J_t)_{t\in S^1}$. This complex is filtered by a natural extension of $\mathcal{A}_H$ to $CF(H,J)$, in the sense that $\mathcal{A}_H(d_{H,J}x) < \mathcal{A}_H(x)$ for all non-zero $x \in CF(H,J)$. For $a,b \notin \mathrm{Spec}(H)$, $-\infty \le a < b \le +\infty$ we call the interval $I = (a,b)$ admissible, and denote by $CF(H,J)^{<a}$, $CF(H,J)^{<b}$ the subcomplexes given by elements of filtration level smaller than a and b respectively, and set $CF(H,J)^I = CF(H,J)^{<b}/CF(H,J)^{<a}$ for the quotient complex. We denote by $HF(H)^I$ the homology of this complex, called the Floer homology of H in action window I, and the collection of $HF(H)^I$ for all admissible intervals I, together with natural maps $HF(H)^I \to HF(H)^{I'}$ for $I = (a,b), I' = (a',b')$ with $a \le a', b \le b'$ induced by inclusions of complexes, is called the filtered Floer homology of H. For later use, we call an admissible interval $I = (a,b)$ p-admissible for $p \in \mathbb{Z}_{>0}$ if $pa, pb \notin \mathrm{Spec}(H)^{+p}$, where the p-fold sum A^{+p} of a subset $A \subset \mathbb{R}$ is the set of all sums $a_1 + \ldots + a_p$, where $a_j \in A$ for all $1 \le j \le p$.

Note that for $I = (-\infty,\infty)$, (a suitable completion of) $CF(H,J) = CF(H,J)^I$, as well as $HF(H) = HF(H)^I$ can be considered to be modules over the minimal Novikov field

$$\Lambda_{\min,\mathbb{K}} = \mathbb{K}[[q^{-1},q]$$

where q is a formal variable of degree $2N$. By the PSS-isomorphism [25] $HF(H) \cong QH(M;\Lambda_{\min,\mathbb{K}})$, the right-hand side denoting the quantum homology algebra of (M,ω). As a $\Lambda_{\min,\mathbb{K}}$-module, $QH(M;\Lambda_{\min,\mathbb{K}}) = H_*(M;\mathbb{K}) \otimes_{\mathbb{K}} \Lambda_{\min,\mathbb{K}}$, and the PSS-isomorphism is an isomorphism of $\Lambda_{\min,\mathbb{K}}$-modules. We note that ignoring the grading, for each admissible interval I, we have $HF(H)^I \cong HF(H)^{I+\rho}$, the isomorphism being given by multiplication by q. We express this fact by saying that the area, or the valuation, of q is ρ.

To a time-dependent Hamiltonian H on a closed monotone symplectic manifold (M,ω), the set $\mathrm{Fix}_c(\phi)$ of whose contractible fixed points is finite, we can associate a ρ-periodic barcode, $\mathcal{B}(H;\mathbb{K}) = \{(I_j,m_j)\}$, where I_j, called *bars*, are intervals of the form $(a,b]$, or (a,∞), with endpoints in $\mathrm{Spec}(H)$ and $m_j \in \mathbb{Z}_{>0}$ are their multiplicities. The ρ-periodicity means that if $(I,m) \in \mathcal{B}(H;\mathbb{K})$ then $(I+\rho,m) \in \mathcal{B}(H;\mathbb{K})$, or alternatively that the group $\rho\cdot\mathbb{Z}$ acts on $\mathcal{B}(H;\mathbb{K})$. This barcode has a number of properties summarized in [2, Proposition 22]:

(i) For each window $J = (a,b)$ in $\mathbb{R}$, with $a,b \notin \mathrm{Spec}(H)$, only a finite number of intervals I with $(I,m) \in \mathcal{B}$ have endpoints in J. Furthermore,

$$\dim_{\mathbb{K}} HF(H)^J = \sum_{(I,m)\in\mathcal{B}(H),\ \#\partial I\cap J = 1} m,$$

where for an interval $I = (a,b]$, $\partial I = \{a,b\}$, and for $I = (a,\infty)$, $\partial I = \{a\}$.

(ii) In particular for $a \in \mathrm{Spec}(H)$, and $\epsilon > 0$ sufficiently small, so that we have $(a-\epsilon, a+\epsilon) \cap \mathrm{Spec}(H) = \{a\}$,

$$\dim_{\mathbb{K}} HF(H)^{(a-\epsilon,a+\epsilon)} = \sum_{(I,m)\in\mathcal{B}(H),\, a\in\partial I} m,$$

$$\dim_{\mathbb{K}} HF(H)^{(a-\epsilon,a+\epsilon)} = \sum_{\mathcal{A}(\overline{x})=a} \dim_{\mathbb{K}} HF^{\mathrm{loc}}(H,\overline{x}).$$

(iii) There are $K(\phi,\mathbb{K})$ orbits of finite bars counted with multiplicity, and $B(\mathbb{K})$ orbits of infinite bars counted with multiplicity, under the $\rho\cdot\mathbb{Z}$ action on $\mathcal{B}(H)$. These numbers satisfy:

$$B(\mathbb{K}) = \dim_{\mathbb{K}} H_*(M;\mathbb{K})$$

and

$$N(\phi,\mathbb{K}) = 2K(\phi,\mathbb{K}) + B(\mathbb{K}),$$

where

$$N(\phi,\mathbb{K}) = \sum \dim_{\mathbb{K}} HF^{\mathrm{loc}}(\phi,x)$$

is the *homological count of the fixed points of* ϕ, the sum running over all the set $\mathrm{Fix}_c(\phi)$ of its fixed points.

(iv) There are $K(\phi,\mathbb{K})$ *bar-lengths* corresponding to the finite orbits,

$$0 < \beta_1(\phi,\mathbb{K}) \leq \ldots \leq \beta_{K(\phi,\mathbb{K})}(\phi,\mathbb{K}),$$

which depend only on ϕ. We call

$$\beta(\phi,\mathbb{K}) = \beta_{K(\phi,\mathbb{K})}(\phi,\mathbb{K})$$

the *boundary-depth* of ϕ, and

$$\beta_{\mathrm{tot}}(\phi,\mathbb{K}) = \sum_{1\leq j\leq K(\phi,\mathbb{K})} \beta_j(\phi,\mathbb{K})$$

its *total bar-length*.

(v) Each spectral invariant $c(\alpha,H) \in \mathrm{Spec}(H)$ for $\alpha \in QH_*(M)\backslash\{0\}$ is a starting point of an infinite bar in $\mathcal{B}(H)$, and each such starting point is given by a spectral invariant.

(vi) If H' is another Hamiltonian generating ϕ, then $\mathcal{B}(H') = \mathcal{B}(H)[c]$, for a certain constant $c \in \mathbb{R}$, where $\mathcal{B}(H)[c] = \{(I_i - c, m_i)\}_{i\in\mathcal{I}}$.

(vii) If $\mathbb{K}$ is a field extension of $\mathbb{F}$, and H is a Hamiltonian, then $\mathcal{B}(H;\mathbb{K}) = \mathcal{B}(H;\mathbb{F})$. In particular $\mathcal{B}(H;\mathbb{K}) = \mathcal{B}(H;\mathbb{F}_p)$ if $\mathrm{char}(\mathbb{K}) = p$, and $\mathcal{B}(H;\mathbb{K}) = \mathcal{B}(H;\mathbb{Q})$ if $\mathrm{char}(\mathbb{K}) = 0$.

We are now in a situation to formulate the key Smith-type inequality.

Theorem B.1 ([28, Theorem D])**.** *Let* $\phi \in \mathrm{Ham}(M,\omega)$ *be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold* (M,ω)*. Suppose that* $\mathrm{Fix}_c(\phi^p)$ *is finite. Then*

$$p\cdot\beta_{\mathrm{tot}}(\phi,\mathbb{F}_p) \leq \beta_{\mathrm{tot}}(\phi^p,\mathbb{F}_p).$$

We now give a new proof of this result, that stays within the realm of filtered Floer homology with coefficients in $\Lambda_{\min,\mathbb{K}}$, temporarily passing through a slightly larger field.

Proof of Theorem B.1. Suppose for this proof that p is odd, as the case of $p = 2$ is simpler and is left to the reader.

We start by observing that for a p-admissible interval $I = (a, b)$, the $\mathbb{Z}/(p)$-equivariant pants product

$$\mathcal{P}^{p \cdot I} : \widehat{H}(\mathbb{Z}/(p), (CF(H)^{\otimes p})^{p \cdot I}) \to \widehat{HF}_{eq}(H^{(p)})^{p \cdot I}$$

is well-defined by arguments from [28, Section 6], and is an isomorphism by the same spectral sequence argument as in [29]. It is crucial to note that here, the coefficients are taken in the Novikov field $\Lambda_{\mathbb{K}} = \mathbb{K}[[q^{-1}, q]$, where our base field is the field $\mathbb{K} = \mathbb{F}_p[u^{-1}, u]]$ of Laurent polynomials in a formal variable u, and the tensor product of Floer complexes $CF(H)^{\otimes p}$ is taken over $\Lambda_{\mathbb{K}}$.

Disregarding grading, we define a new Novikov field

$$\Lambda_{\mathbb{K}}^{1/p} = \Lambda_{\mathbb{K}}(q^{1/p}) = \mathbb{K}[[t^{-1}, t],$$

into which $\Lambda_{\mathbb{K}}$ embeds by sending q to t^p. Observe that t has valuation ρ/p. The theory of quasi-Frobenius maps as in [29] now yields an isomorphism

$$F^I : HF(H; \Lambda_{\mathbb{K}}^{1/p})^I \otimes \mathbb{K}\langle\theta\rangle \to \widehat{H}(\mathbb{Z}/(p), (CF(H)^{\otimes p})^{p \cdot I}),$$

where $HF(H; \Lambda_{\mathbb{K}}^{1/p})^I$ denotes the Floer homology of the Hamiltonian H with coefficients in $\Lambda_{\mathbb{K}}^{1/p}$ taken in the action window I, and $\langle\theta\rangle$ is the exterior algebra over $\mathbb{F}_p$ on the formal variable θ of degree 1. Note that $\dim_{\mathbb{F}_p} \langle\theta\rangle = 2$.

Now rewrite $HF(H; \Lambda_{\mathbb{K}}^{1/p})^I$ in terms of usual filtered Floer homology, considered as ungraded:

$$HF(H; \Lambda_{\mathbb{K}}^{1/p})^I \cong \bigoplus_{0 \leq j < p} HF(H)^{I + j\rho/p}.$$

Note that as $HF(H)^{I+\rho} \cong HF(H)^I$, we can replace the interval $0 \leq j < p$, by any other interval with p integers. For example we can pick the symmetric interval $-(p-1)/2 \leq j \leq (p-1)/2$ when p is odd. Taking dimensions over $\mathbb{K}$, and using the inequality

$$\dim_{\mathbb{K}} \widehat{HF}_{eq}(H^{(p)})^{p \cdot I} \leq 2 \dim_{\mathbb{K}} HF(H^{(p)})^{p \cdot I},$$

whose proof follows that in [29] verbatim, we obtain after canceling a multiple of 2 that

$$\sum_{j=0}^{p-1} \dim_{\mathbb{K}} HF(H)^{I + j\rho/p} \leq \dim_{\mathbb{K}} HF(H^{(p)})^{p \cdot I}. \tag{36}$$

In fact, we could replace the right-hand side by the generally smaller quantity[1] $\dim_{\mathbb{K}}(HF(H^{(p)})^{p \cdot I})^{\mathbb{Z}/(p)}$, the invariant subspace being considered with respect to a natural $\mathbb{Z}/(p)$-action.

We proceed by considering an interval I_0 of length $l = |I_0| = k\rho$, with $k \in \mathbb{Z}_{>0}$ sufficiently large, so that $l > \beta(\phi, \mathbb{K})$, and $pl > \beta(\phi^p, \mathbb{K})$. Set $I_t = I_0 + t$ for the interval I_0 shifted by $t \in [0, l)$. We shall consider Equation (36) for $I = I_t$ and integrate it over t in $[0, l)$, keeping in mind that for an interval J, $\dim HF(H)^J$ is given as the number of bars $(a, b]$ in the barcode of H such that $\#(J \cap \{a, b\}) = 1$. Note that any admissible interval of length $l = k\rho$ contains precisely $k \cdot B(\mathbb{K})$ infinite

[1]This quantity can again be reduced. We refer to [29] for this argument.

bars. Furthermore, each finite bar of length β contributes $2k\beta$ to the integral. Therefore we obtain that

$$p \cdot l(kB(\mathbb{K})) + p \cdot 2k\beta_{\mathrm{tot}}(\phi) \leq \frac{1}{p}\Big((pl)(pkB(\mathbb{K})) + 2pk\beta_{\mathrm{tot}}(\phi^p)\Big),$$

since $p \cdot I$ is of length $pl = pk\rho$. This simplifies to

$$p \cdot \beta_{\mathrm{tot}}(\phi, \mathbb{K}) \leq \beta_{\mathrm{tot}}(\phi^p, \mathbb{K}),$$

which yields the desired inequality, since by property (vii) of barcodes of Hamiltonian diffeomorphisms $\beta_{\mathrm{tot}}(\phi, \mathbb{K}) = \beta_{\mathrm{tot}}(\phi, \mathbb{F}_p)$, $\beta_{\mathrm{tot}}(\phi^p, \mathbb{K}) = \beta_{\mathrm{tot}}(\phi^p, \mathbb{F}_p)$. □

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