Calculus and submanifold

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Exercise 1 (Calculus)**.** Show that the following maps are differentiable, and compute their differentials.

1. $(f,g) \longmapsto g \circ f$ from $\mathcal{L}(E,F) \times \mathcal{L}(F,G)$ to $\mathcal{L}(E,G)$.
2. $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$.
3. $f \longmapsto f^{-1}$ from $GL(E)$ to itself, where E is a real vector space of finite dimension.
4. Let $\Omega \subset \mathbb{R}^n$ be an open set of compact closure and V a vector space of finite dimension of functions $\mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^1$. We consider $\mathrm{ev} : (f,x) \mapsto f(x)$ from $V \times \Omega$ to $\mathbb{R}$.

Exercise 2 (Diffeomorphisms)**.**

1. Let $f : U \to V$ be a homeomorphism between open sets of $\mathbb{R}^n$. Show that f is a $\mathcal{C}^1$-diffeomorphism if and only if f is $\mathcal{C}^1$ and for all x in U, df_x is invertible.
2. Show that $B^n := \{x \in \mathbb{R}^n \mid \|x\|_2 < 1\}$ is diffeomorphic to $]-1;1[^n$, and to $\mathbb{R}^n$.
3. Generalize the last result to any open convex nonempty set.

Exercise 3 (Study of an implicitly defined curve)**.** Let us define the following curve:

$$C := \left\{(x,y) \in \mathbb{R}^2 \mid x^4 + y^3 - y^2 + x - y = 0\right\}.$$

Give the tangent line of C and the relative position of C with reference to it at the points $(0,0)$ and $(0,1)$.

Exercise 4 (Roots of separable polynomials)**.** Let $U \subset \mathbb{C}_d[X] \simeq \mathbb{C}^{d+1}$ be the open subset of separable polynomials of degree d (*i.e.* polynomials of degree d with distinct roots). Show that, for all $P \in U$, there exists a neighborhood $V \subset U$ of P and a smooth map $f : V \to \mathbb{C}^d$ such that,

$$\forall Q \in V, \quad \{z_1, \dots, z_d\} \text{ is the set of roots of } Q \text{ where } (z_1, \dots, z_d) := f(Q).$$

Exercise 5 (Submanifolds)**.**

1. Recall the four equivalent definitions of a d-dimensional submanifold of $\mathbb{R}^n$ (without proving that they are equivalent).
2. Among the following sets, which ones are submanifolds of $\mathbb{R}^n$? Give their dimensions.
   (a) The sphere of radius 1 in $\mathbb{R}^n$ for the euclidean norm.
   (b) The set $M_r := \{(x,y,z) \mid x^2 + y^2 - z^2 + r = 0\}$ of $\mathbb{R}^3$ for a given parameter $r \in \mathbb{R}$.
   (c) A disjoint union of a straight line and a plane in $\mathbb{R}^3$.
   (d) The subset $\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\}$.
3. Let Ω be an open subset of $\mathbb{R}^d$ and $h : \Omega \to \mathbb{R}^n$ an injective immersion, is $h(\Omega)$ always a submanifold of $\mathbb{R}^n$?

Exercise 6 (Classical groups of matrices)**.** Show that the following subgroups of $GL_n(\mathbb{R}) \subset \mathcal{M}_n(\mathbb{R})$ are submanifolds of $\mathcal{M}_n(\mathbb{R})$, give their respective dimensions and their connected components.

1. $GL_n(\mathbb{R})$,
2. $SL_n(\mathbb{R})$ (subgroup of matrices of determinant 1),
3. $O_n(\mathbb{R})$ (subgroup of orthogonal matrices).

Exercise 7 (Quotient topology)**.** Let X be a topological space and $\sim$ be an equivalence relation on X. We denote by $p : X \to X/\sim$ the canonical projection.

1. Recall the definition of the quotient topology on $X/\sim$.
2. Let $f : X/\sim \to Y$. Show that f is continuous if and only if $f \circ p$ is.
3. Show that if G is a discrete group acting properly discontinuously on a locally compact Hausdorff space X, *i.e.*, for all $g \in G$, $x \mapsto g \cdot x$ is a continuous map and
$$\{g \in G \mid g \cdot K \cap K \neq \emptyset\} \text{ is finite}, \quad \forall K \subset X \text{ compact subset},$$
then the quotient X/G is Hausdorff.
4. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Show that $\mathbb{T}^n$ is compact and Hausdorff, and that p is an open map.
5. Let $f : K \to Y$ be continuous and bijective with K compact Hausdorff and Y Hausdorff. Show that f is a homeomorphism. Give a counter-example if Y is not Hausdorff.
6. We define $\mathbb{S}^1$ as $\{z \in \mathbb{C} \mid |z| = 1\}$. Show that $\mathbb{T}^1$ is homeomorphic to $\mathbb{S}^1$. More generally, show that $\mathbb{T}^n$ is homeomorphic to $(\mathbb{S}^1)^n \subset \mathbb{C}^n$.
7. Let $\mathbb{RP}^n$ be the space defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation "belonging to the same vector line". Show that $\mathbb{RP}^n$ is compact Hausdorff and that p is open.