Manifolds, Tangent spaces and Differentials

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Exercise 1 (The projective space)**.** Let $\mathbb{RP}^n$ be the space defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation "belonging to the same vector line". For $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we denote by $[x_0 : \dots : x_n]$ its class in $\mathbb{RP}^n$.

1. Show that $\mathbb{RP}^n$ is compact Hausdorff and that the canonical projection $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is open.
2. Let $i \in \{0, \dots, n\}$. Show that $U_i = \{[x_0 : \dots : x_n] \in \mathbb{RP}^n \mid x_i \neq 0\}$ is open in $\mathbb{RP}^n$, and construct a homeomorphism $\varphi_i : U_i \to \mathbb{R}^n$.
3. Show that $\mathbb{RP}^n$ equipped with the atlas $(\varphi_i)_{i \in \{0,\dots,n\}}$ is a smooth manifold.
4. Show that p is smooth and that a map $f : \mathbb{RP}^n \to M$ is smooth if and only if $f \circ p$ is smooth, given any smooth manifold M.
5. Show that the restriction of p to $\mathbb{S}^n$ is a local diffeomorphism.
6. Show that $\mathbb{RP}^1$ is diffeomorphic to $\mathbb{S}^1$.
7. Let $A \in GL_{n+1}(\mathbb{R})$, we denote by $h_A : \mathbb{RP}^n \to \mathbb{RP}^n$ its induced map (why is it well defined?). Show that h_A is a diffeomorphism and give its differential.
8. Given $A \in GL_{n+1}(\mathbb{R})$, what are the fixed points of h_A?

Exercise 2 (Tangent space of a submanifold)**.** Let $M \subset \mathbb{R}^m$ and $N \in \mathbb{R}^n$ be a submanifolds of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively

1. Describe the tangent space $T_pM \subset \mathbb{R}^m$ of the submanifold M of $\mathbb{R}^m$ at a point p, for each of the four characterizations of a submanifold. Why is there no ambiguity in identifying the tangent space $\subset \mathbb{R}^m$ at a point p of M seen as a submanifold with its tangent space at p where M is seen as a manifold (endowed with the differentiable structure naturally induced)?
2. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open neighborhoods. Let $\widetilde{f} : U \to V$ be a smooth map such that $f := \widetilde{f}|_{M \cap U} : M \cap U \to N \cap V$. Show that f is a smooth map between manifolds and that
$$\mathrm{d}f_p = \left(\mathrm{d}\widetilde{f}_p\right)\Big|_{T_pM} : T_pM \to T_{f(p)}N.$$

Exercise 3 (Tangent space of the torus)**.** Let $p : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map and let M be a smooth manifold.

1. Using charts, show that $\mathbb{T}^n$ is parallelizable.
2. Show that p is smooth and that a map $f : \mathbb{T}^n \to M$ is smooth if and only if $f \circ p$ is smooth. Express the differential of $f : \mathbb{T}^n \to M$ by means of $f \circ p$.
3. We identify matrices $A \in \mathcal{M}_n(\mathbb{Z})$ with their induced maps $A : \mathbb{T}^n \to \mathbb{T}^n$ (why is it well defined?). In which condition is $A \in \mathcal{M}_n(\mathbb{Z})$ a diffeomorphism of $\mathbb{T}^n$?

Exercise 4 (Tangent space of a product)**.**

1. Let M and N be two smooth manifolds, show that $T(M \times N)$ is diffeomorphic to $TM \times TN$.
2. Show that $T(\mathbb{S}^n) \times \mathbb{R}$ is diffeomorphic to $\mathbb{S}^n \times \mathbb{R}^{n+1}$. Deduce that $\mathbb{S}^n \times \mathbb{S}^1$ is parallelizable.

Exercise 5 ($\mathbb{S}^3$ is parallelizable)**.** A Lie group is smooth manifold G endow with a group structure such that the multiplication $\mu : G \times G \longrightarrow G$ and the inverse $\eta : G \longrightarrow G$ are smooth maps.

1. Show that if G is a Lie group, then G parallelizable.
2. Show that $\mathrm{SU}(2)$ is a Lie group diffeomorphic to $\mathbb{S}^3$
3. Deduce that $\mathbb{S}^3$ is parallelizable

Exercise 6 (Computation of a differential)**.** Compute the differential of $\overline{F} : \mathbb{T}^2 \to \mathbb{S}^2$ defined as the quotient of the map from $\mathbb{R}^2$ to $\mathbb{S}^2$:

$$F : (x, y) \mapsto (\cos(2\pi x)\cos(2\pi y), \cos(2\pi x)\sin(2\pi y), \sin(2\pi x)).$$

On which set is $\overline{F}$ a local diffeomorphism? Is $\overline{F}$ restricted to this domain a global diffeomorphism?

Exercise 7.

1. Show that every immersed n-manifold (that is a manifold of dimension $n > 0$) of $\mathbb{R}^n$ is parallelizable.
2. Is it possible to immerse a compact n-manifold in $\mathbb{R}^n$?
3. What is the minimal number of charts an atlas of $\mathbb{S}^n$ can have?