Vector Fields, Flows and Lie Bracket

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Exercise 1 (Straightening theorem for vector fields)**.** Let M be a manifold and X be a smooth vector field on M. Prove that for all p in M such that $X(p) \neq 0$, there exists a chart (U, φ) of M with $p \in U$ such that $X_{|U} = \partial/\partial x_1$.

Exercise 2 (Lie Bracket and commuting flows)**.** Let M be a manifold, X, Y two smooth vector fields on M. Prove that $[X, Y] = 0$ if and only if the flows of X and Y commute: $X^t \circ Y^s = Y^s \circ X^t$ whenever the flows (X^t) and (Y^s) are well defined.

Exercise 3. Let M be a manifold and X a vector field on M such that for all vector fields Y on M, $[X, Y] = 0$. What can you say about X ?

Exercise 4 (Transitivity of Diff(M))**.**

1. Let a and b be two points in the open ball $\mathbb{B} = \{x \in \mathbb{R}^n \mid \|x\|_2 < 1\}$. Prove that there exists a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a) = b$ and $f \equiv id$ outside $\mathbb{B}$.
2. Let M be a connected manifold. Prove that Diff(M) acts transitively on M.
3. Is this action k transitive (for $k \geqslant 1$)?

Exercise 5 (Pseudo-gradient)**.** Let $f : M \to \mathbb{R}$ be a smooth function defined on a smooth manifold M. A *pseudo-gradient* of f is a vector field X of M such that, for all $x \in M \setminus \mathrm{Crit}(f)$, $df(x) \cdot X(x) > 0$.

1. Let $M \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Use the gradient of $\widetilde{f}$ to produce a pseudo-gradient of $\widetilde{f}|_M$.
2. Let M be a submanifold of $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ be the projection on the first coordinate ("height function"). What are the critical points of f? Give an integrable pseudo-gradient.
3. Show the existence of pseudo-gradients in the general case.
4. Given an integrable pseudo-gradient X of $f : M \to \mathbb{R}$, let $(\phi_t)_{t\in\mathbb{R}}$ be its flow. For all $x \in M$, show that $t \mapsto f \circ \phi_t(x)$ is increasing and that
$$\bigcap_{T>0} \overline{\{\phi_t(x) \mid t > T\}} \subset \mathrm{Crit}(f).$$
5. Suppose that f has only isolated critical points, show that, for all $x \in M$, there exist critical points $\alpha, \omega \in M$ such that $\phi_t(x) \to \alpha$ as $t \to -\infty$ and $\phi_t(x) \to \omega$ as $t \to +\infty$.
6. Let M be a closed submanifold of $\mathbb{R}^n$, and $f : M \to \mathbb{R}$ a smooth function. Let $M_t = \{x \in M \mid f(x) \leqslant t\}$. Let $a, b \in \mathbb{R}$ be such that $M \cap f^{-1}([a, b])$ and $\mathrm{Crit}(f)$ are disjoint. Prove that M_a and M_b are diffeomorphic.