Exterior algebra

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Exercise 1 (Tensor products over $\mathbb{R}$). *Reminder: Let V and W be two vector spaces. Then there exists a unique (up to isomorphism) vector space $V \otimes W$ and a bilinear map $\varphi : V \times W \to V \otimes W$ such that any bilinear map $V \times W \to Z$ from $V \times W$ to any vector space Z factors through φ.*

Let E and F be two $\mathbb{R}$-vector spaces of dimension n and m, respectively. We denote by $e = (e_j)$ a basis of E and by $f = (f_i)$ a basis of F.

1. Let $\alpha \in \bigotimes^k E^*$. Identify the coordinates of α in the basis of $\bigotimes^k E^*$ associated with e.
2. Give a natural isomorphism between $E^* \otimes F$ and $\mathcal{L}(E, F)$.
3. Let $L : E \to F$ be a linear map whose matrix is $M = (m^i_j)$ in the bases e and f. We define $L^* : \alpha \mapsto \alpha \circ L$ from F^* to E^*. Give the matrix of L^* in the dual bases e^* and f^*.

Exercise 2 (Pullback). Let E and F be two vector spaces and $L : E \to F$ be a linear map.

1. Show that $L^*(\alpha \wedge \beta) = L^*(\alpha) \wedge L^*(\beta)$, where α and β are exterior forms on E.
2. Let (e_j) be a basis of E and (f_i) a basis of F. We denote by $M = (m^i_j)$ the matrix of L in these bases. Let $J = \{j_1, \dots, j_k\} \subset \{1 \dots, n\}$ be such that $1 \leqslant j_1 < \cdots < j_k \leqslant n$, we denote $e^*_J = e^*_{j_1} \wedge \cdots \wedge e^*_{j_k}$ and use similar notations for F. Let $\omega = \sum \omega_I f^*_I$, where the sum is taken over subsets $I \subset \{1, \dots, n\}$ of cardinal k. Express $L^*(\omega)$ in the basis (e^*_J).

Exercise 3 (Exterior algebra).

1. Is there an exterior form α on a vector space E such that $\alpha \wedge \alpha \neq 0$?
2. Is there a non-zero exterior form commuting with any other?

Exercise 4 (Decomposable forms). Let E be a vector space of dimension n. An exterior form of degree k on E is *decomposable* if it can be written as a exterior product of k linear forms.

1. Prove that linear forms as well as exterior forms of degree n are decomposable.
2. Let $\alpha \in E^* \setminus \{0\}$. Prove that a non zero exterior form ω of degree k is divisible by α (i.e. can be written as $\alpha \wedge \beta$) if and only if $\alpha \wedge \omega = 0$.
3. Let $(\alpha, \beta, \gamma, \delta)$ a linearly independant family of E^*. Is the exterior form $\alpha \wedge \beta + \gamma \wedge \delta$ decomposable?
4. Is an exterior form of degree $(n-1)$ decomposable?
   Hint: consider the map $\phi_\omega : \alpha \mapsto \alpha \wedge \omega$ from E^* to $\bigwedge^n E^*$.

Exercise 5 (Bilinear alternating forms). Let E be a vector space of dimension n, and ω a bilinear alternating form.

1. Prove that there exists a basis (e^i) of E^* such that
$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots + e^{2p-1} \wedge e^{2p},$$
where $n - 2p$ is the dimension of $\ker(\omega) = \{x \in E \mid \omega(x, \cdot) = 0\}$.

2. Prove that p is the smallest integer such that $\omega^{p+1} = 0$.

Exercise 6 (Pfaffian)**.** *Reminder : If A is a skew matrix in $\mathcal{A}_{2n}(\mathbb{R})$, then there exist $a_1, \ldots, a_n \in \mathbb{R}$ and $P \in \mathcal{O}_{2n}(\mathbb{R})$ such that $A = PDiag\begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix} P^{-1}$.*

To any skew matrix $A \in \mathcal{A}_{2n}(\mathbb{R})$ one can associate a 2-linear form ω_A on $\mathbb{R}^{2n}$ defined by

$$\omega_A = \sum_{i<j} a_{i,j} e_i^* \wedge e_j^*,$$

where (e_i) is the canonical basis of $\mathbb{R}^{2n}$.

1. Prove that $\omega_A \in \bigwedge^2(\mathbb{R}^{2n})^*$.
2. Prove that there exists a polynomial map, called the *Pfaffian*, $Pf : \mathcal{A}_{2n}(\mathbb{R}) \to \mathbb{R}$ such that for all $A \in \mathcal{A}_n(\mathbb{R})$, we have $\omega^n/n! = Pf(A)e_1^* \wedge \cdots \wedge e_{2n}^*$.
3. Prove that $Pf(A)^2 = \det(A)$.
4. Prove that $Pf(A) = \dfrac{1}{2^n n!} \displaystyle\sum_{\sigma \in \mathfrak{S}_{2n}} \varepsilon(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}$.