HOMOLOGICALLY VISIBLE CLOSED GEODESICS ON COMPLETE SURFACES

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Abstract. In this article, we give multiple situations when having one or two geometrically distinct closed geodesics on a complete Riemannian cylinder $M \simeq S^1 \times \mathbb{R}$ or a complete Riemannian plane $M \simeq \mathbb{R}^2$ leads to having infinitely many geometrically distinct closed geodesics. In particular, we prove that any complete cylinder with isolated closed geodesics has zero, one or infinitely many homologically visible closed geodesics; this answers a question of Alberto Abbondandolo.

1. Introduction

The problem of the existence and multiplicity of closed geodesics plays an important role in both Riemannian geometry and dynamics. Going back to Hadamard and Poincaré [17, 20], it is still open for a lot of Riemannian manifolds. Given a complete Riemannian manifold (M, g), a famous question is whether it possesses a closed geodesic for every Riemannian metric g. This is always true if M is closed [9, 19, 13]. We can then ask whether the number of closed geodesics is infinite or not. It is known that every closed surface has infinitely many geometrically distinct closed geodesics [14, 5, 18]. However, this question is still open for spheres of higher dimension. In this article, we are interested in non-compact complete Riemannian surfaces for which even the existence of one closed geodesic fails in general: planes and cylinders. For instance, the Euclidean plane does not possess any closed geodesic. Nevertheless, under specific geometric conditions, interesting results can be stated. In 1980, Bangert proved that any complete Riemannian cylinder, plane or Möbius band of finite area has infinitely many closed geodesics [4]. For the plane and the cylinder he proved the same result even under the weaker assumption of just the existence of a convex neighborhood of infinity. We will discuss this result in greater depth as it is used extensively in our proofs. The purpose of this article is to give simple conditions under which the existence of one or two distinct closed geodesics implies that a complete Riemannian cylinder or plane contains infinitely many geometrically distinct closed geodesics.

Let $S^1 := \mathbb{R}/\mathbb{Z}$ and let $M \simeq S^1 \times \mathbb{R}$ be a complete Riemannian cylinder. Let ΛM be its loop space. Two loops $\alpha, \beta \in \Lambda M$ are said to be geometrically distinct if their images are distinct: $\alpha(S^1) \neq \beta(S^1)$. Throughout the article, by writing that two closed geodesics are distinct we will always mean that they are geometrically distinct. A closed geodesic $\gamma \in \Lambda M$ is said to be homologically visible if the local homology of the critical circle $S^1 \cdot \gamma \subset \Lambda M$ of the energy functional is non-zero (see Section 2 for precise definitions).

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Theorem 1.1. *Let M be a complete Riemannian cylinder with isolated closed geodesics and assume one of the following hypothesis:*

1. *there exists a contractible closed geodesic,*
2. *there exists a self-intersecting closed geodesic,*
3. *there exist two distinct closed geodesics that intersect,*
4. *there exists a closed geodesic of non-zero average index,*
5. *there exist two homologically visible closed geodesics.*

Then M contains infinitely many geometrically distinct homologically visible closed geodesics.

Notice that according to Bott iteration theory, a closed geodesic c has a non-zero average index if and only if some iterate c^m has a non-zero index. The fact that hypothesis 5 implies that there exists infinitely many closed geodesics proves a conjecture of Abbondandolo:

Corollary 1.2. *Any complete Riemannian cylinder with isolated closed geodesics has zero, one or infinitely many homologically visible closed geodesics.*

Similar results can also be obtained when $M \simeq \mathbb{R}^2$ is a complete plane:

Theorem 1.3. *Let M be a complete Riemannian plane with isolated closed geodesics and assume one of the following hypothesis:*

1. *there exists a self-intersecting closed geodesic,*
2. *there exist two distinct closed geodesics that intersect,*
3. *there exists a closed geodesic of non-zero average index,*
4. *there exists a homologically visible closed geodesic.*

Then M contains infinitely many geometrically distinct homologically visible closed geodesics.

Corollary 1.4. *Any complete Riemannian plane with isolated closed geodesics has zero or infinitely many geometrically distinct homologically visible closed geodesics.*

It is easy to give counter-examples to Theorem 1.1 when none of the assumptions 1–5 hold by considering embedded cylinders of revolution

$$(\theta, z) \mapsto (r(z)\cos\theta, r(z)\sin\theta, z), \tag{1}$$

for well-chosen smooth maps $r : \mathbb{R} \to (0, +\infty)$. A complete cylinder may have no closed geodesic at all: take $r' > 0$. It can have an arbitrary large finite number $k \in \mathbb{N}$ of homologically invisible closed geodesics: take $r'(z) > 0$ for all $z \in \mathbb{R} \setminus \{z_1, \dots, z_k\}$ and $r'(z_i) = 0$. It can also have a unique visible closed geodesic: take $r' < 0$ on $(-\infty, 0)$, $r'(0) = 0$ and $r' > 0$ on $(0, +\infty)$ (one can as well add to this cylinder an arbitrary large finite number of homologically invisible closed geodesic the same way as before). Remark that in our examples closed geodesics are without self-intersections and not contractible as implied by the theorem. Counter-examples where the theorem fails by lack of completeness can be found as well by choosing embedded cylinders of revolution restricting the domain of the embedding (1) to $(\theta, z) \in S^1 \times (a, b)$ for $a, b \in \mathbb{R}$. We could proceed as follows: take an even $r : [-1, 1] \to (0, +\infty)$ with $r' > 0$ on $[-1, 0)$ such that $z = 0$ is the only closed geodesic of the associated compact embedded cylinder. One can find such an r by slightly modifying a Tannery surface: a sufficient condition is that the metric g on the interior of the cylinder can be written as

$$g = [\alpha + h(\cos\rho)]^2 \,\mathrm{d}\rho^2 + \sin^2\rho \,\mathrm{d}\theta^2,$$

for a good choice of coordinates $(\rho, \theta) \in (0, \pi) \times S^1$, where α is irrational and $h : (-1, 1) \to (-\alpha, \alpha)$ is a smooth odd function (see for instance [8, Theorem 4.13]). Then extend r to a smooth map $(-3, 1] \to (0, +\infty)$ with $r|_{(-3,-1)} < r(-1)$, $r' < 0$ on $(-3, -2)$ and $r' > 0$ on $(-2, -1)$. Then $z = -2$ and $z = 0$ are the only closed geodesic of the cylinder embedded by $r|_{(-3,1)}$ and are both visible.

In a similar way, we can give examples of complete planes with nothing but an arbitrary finite number of homologically invisible closed geodesics by considering surfaces of revolution (1) parametrized by $\mathbb{R}/2\pi\mathbb{Z} \times [0, +\infty)$ with $r : [0, +\infty) \to [0, +\infty)$ being increasing and smooth on $(0, +\infty)$ with $r(0) = 0$ and $r'(z) \to +\infty$ when $z \to 0$ in a suitable way (*i.e.* so that the surface is smooth at the origin). Then, as above, we get homologically invisible closed geodesics on the inflexion points of r, and nowhere else.

We say that $C_- \subset M$ (resp. C_+) is a neighborhood of $-\infty$ (resp. of $+\infty$) if C_- contains $S^1 \times (-\infty, a)$ for some $a \in \mathbb{R}$ (resp. $S^1 \times (b, +\infty)$ for some $b \in \mathbb{R}$) for an arbitrarily fixed identification of M with $S^1 \times \mathbb{R}$. In order to prove Theorem 1.1, we will extensively use the following theorem due to Bangert:

Theorem 1.5 ([4, Theorem 3, Remark 2])**.** *Let M be a complete Riemannian cylinder with isolated closed geodesics and suppose there exists disjoint locally convex open neighborhoods C_- and C_+ of $-\infty$ and $+\infty$ respectively such that the boundaries $\partial C_\pm$ are not totally geodesic. Then M contains infinitely many homologically visible closed geodesics intersecting $M \setminus (C_- \cup C_+)$ and at least one without self-intersections.*

Since Bangert did not give the precise proof of that statement, for the sake of completeness we give a comprehensive proof in the paper. The proof of Theorem 1.3 is quite similar and relies extensively on the analogous theorem of Bangert when M is a plane with isolated closed geodesics: if there exists an open neighborhood of infinity $C \neq M$ with a boundary ∂C which is not totally geodesic, M contains infinitely many homologically visible closed geodesics [4, Theorem 3]. These two theorems were originally used by Bangert to prove that any complete Riemannian plane of finite area has infinitely many closed geodesics.

In fact, Theorem 1.1 extends *verbatim* to the case where M is a complete reversible Finsler manifold as we will essentially use variational properties of geodesics in our proof with no concern for geometric notion specific to Riemannian manifold. However, nothing can be said concerning the more general case of a complete (asymmetrical) Finsler manifold. The major issue is that, in the asymmetrical case, a closed subset of M which is bounded by a geodesic is not locally convex. In this direction, we point out that the related question of whether or not infinitely many closed geodesics exist on every irreversible Finsler cylinder of finite area is still open [11, Question 2.3.2].

In order to put these results in perspective, we recall some known results concerning existence of closed geodesics on complete non-compact Riemannian manifolds. In 1978, Thorbergsson proved the existence of closed geodesics on a complete Riemannian manifold M if it contains a convex compact set which is not homotopically trivial or if M has a non-negative sectional curvature outside some compact set [22]. In the 1990s, Benci and Giannoni proved that any complete d-dimensional Riemannian manifold such that the limit superior of its sectional curvature at infinity is non-positive and the homology of its free loop space is non-trivial in some degree larger than $2d$ possesses a closed geodesic [6, 7]. In 2017, Asselle and Mazzucchelli showed the existence of infinitely many closed geodesics for complete d-dimensional Riemannian manifolds which have no close conjugate points at infinity and a free loop space with unbounded Betti numbers in degrees larger than d [1]. They also improved the result of Benci and Giannoni by replacing the asymptotic curvature assumption by an assumption on the conjugated points at infinity and by improving the bound on the homology of the free loop space. However, the existence of one closed geodesic in any complete Riemannian manifold of finite volume is still an open problem (see for instance the following recent review of the subject [11]).

Organization of the paper. In Section 2, we fix notation and recall results of the variational theory of geodesics that we will need. In Section 3, we give a comprehensive proof of Theorem 1.5 of Bangert. In Section 4, we prove Theorem 1.1 when hypothesis 1, 2 or 3 is assumed. In Section 5, we prove Theorem 1.1 when hypothesis 4 is assumed. In Section 6, we prove the last case of Theorem 1.1. In Section 7, we prove Theorem 1.3.

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2. Preliminaries

In this section, we recall some results of Riemannian geometry that we will use in our proofs and fix some notation. For the extension of these notions to the Finsler case, the reader may for instance look at [12, Section 2].

2.1. The energy functional.

Given a complete Riemannian manifold with boundary M, we denote by ΛM the space of H^1-maps $S^1 \to M$. In fact, if one wants to avoid analytic questions, we can always reduce our space to a finite-dimensional manifold of broken geodesics. For $\gamma \in \Lambda M$ and $m \in \mathbb{N}^*$, the iterated loop $\gamma^m \in \Lambda M$ is defined by $t \mapsto \gamma(mt)$. A geodesic is an immersed path $\gamma : \mathbb{R} \to M$ such that

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

where ∇ denotes the Levi-Civita connection of the metric and $\dot\gamma$ stands for the derivative of γ. Therefore, in our convention, geodesics have constant speed. A closed geodesic is a geodesic γ which is periodic: $\gamma(t+1) = \gamma(t)$ so that $\gamma \in \Lambda M$. Closed geodesics of M are the critical points with non-zero critical value of the energy functional $E : \Lambda M \to [0, +\infty)$,

$$E(\gamma) := \int_{S^1} g_\gamma(\dot\gamma, \dot\gamma)\mathrm{d}t, \quad \forall \gamma \in \Lambda M.$$

The energy functional E is C^2. If M is a locally convex compact manifold (possibly with boundary), E also satisfies the Palais-Smale condition and the $(-\nabla E)$-flow is defined for all time $t \geq 0$. We notice that every closed geodesic lies on a critical circle $S^1 \cdot \gamma$, where S^1 acts on ΛM by $t \cdot \gamma := \gamma(t + \cdot)$. In our study we assume that E has only isolated critical circles (except for the constant loops which have zero value). Two closed geodesics c_1 and c_2 are said to be geometrically distinct if they do not have the same image in M.

2.2. **Finite-dimensional approximation of the loop space.** Morse's finite-dimensional approximation of the curve space over M, as presented by Bangert in [4] consist of the following data: an open set $\mathcal{O} \subset M$, an energy bound $\kappa > 0$ and a parameter $j \in \mathbb{N}$ satisfying $\frac{1}{j} < \frac{\varepsilon^2}{\kappa}$ where $\varepsilon > 0$ is smaller than the injectivity radius on $\mathcal{O}$. The positivity of ε will be fulfilled if for instance $\mathcal{O}$ has compact closure, as will be the case in our considerations. The finite-dimensional approximation $\Omega = \Omega(\mathcal{O}, \kappa, j)$ is constructed as follows: it is the set of all curves $\gamma \in \Lambda M$ such that $E(\gamma) < \kappa$, $\gamma(i/j) \in \mathcal{O}$ and such that $\gamma|_{[i/j,(i+1)/j]}$ is a geodesic of length less than ε for $0 \leq i \leq j-1$.

Let Ω be a finite-dimensional approximation of ΛM and $C \subset M$ a locally convex set with compact boundary such that $C \subset \mathcal{O}$. By the local convexity of C, there exists an $\varepsilon > 0$ such that for two points $p, q \in C$ with Riemannian distance $d(p,q) < \varepsilon$, there exists a unique geodesic of length $= d(p,q)$ joining p and q, and contained entirely in C. The negative gradient of the restriction of the energy functional to Ω is given by

$$-\nabla E|_{\Omega}(\gamma) = -2\Big(\dot\gamma_1(1/j) - \dot\gamma_2(1/j), \ldots, \dot\gamma_{j-1}((j-1)/j) - \dot\gamma_j((j-1)/j)\Big)$$

for $\gamma \in \Omega$, where $\gamma_i = \gamma|_{[(i-1)/j,i/j]}$ for $1 \leq i \leq j$ (see [15, p. 252]). Now from our choice of j and Cauchy-Schwarz inequality, we get

$$d(\gamma((i-1)/j), \gamma(i/j))^2 \leq \frac{1}{j} E(\gamma|_{((i-1)/j,i/j)}) \leq \frac{\varepsilon^2}{\kappa}\kappa = \varepsilon^2$$

and consequently by local convexity of C, the negative gradient flow of the finitedimensional approximation of the energy functional respects Ω.

2.3. **Index of a closed geodesic.** The index of a closed geodesic γ is the Morse index of E:

$$\mathrm{ind}(\gamma) := \mathrm{ind}(E, \gamma).$$

It is always finite. The behavior of this index under iteration $k \mapsto \mathrm{ind}(\gamma^k)$ was extensively studied by Bott in [10]. We simply recall that

$$\mathrm{ind}(\gamma^k) \geq k\, \overline{\mathrm{ind}}(\gamma) - \dim(M) + 1, \quad k \in \mathbb{N}, \tag{2}$$

where $\overline{\mathrm{ind}}(\gamma) \geq 0$ is the average index of γ defined by

$$\overline{\mathrm{ind}}(\gamma) := \lim_{k\to\infty} \frac{\mathrm{ind}(\gamma^k)}{k}.$$

Let $p \in M$ and $\Omega_p M \subset \Lambda M$ be the set of loops based at p, that is H^1-paths $\gamma : [0,1] \to M$ such that $\gamma(0) = \gamma(1) = p$. Given a closed geodesic $\gamma \in \Lambda M$, we denote by $\mathrm{ind}_{\Omega}(\gamma) \in \mathbb{N}$ the Morse index

$$\mathrm{ind}_{\Omega}(\gamma) := \mathrm{ind}\left(E|_{\Omega_{\gamma(0)}M}, \gamma\right).$$

By inclusion, $\mathrm{ind}_\Omega(\gamma) \leq \mathrm{ind}(\gamma)$. In fact, we have the concavity inequality [2, Eq. (1.5)]:

$$\mathrm{ind}(\gamma) - \dim(M) + 1 \leq \mathrm{ind}_\Omega(\gamma) \leq \mathrm{ind}(\gamma). \tag{3}$$

A Jacobi field of the geodesic path γ is a smooth map $J : \mathbb{R} \to \gamma^* TM$, satisfying

$$J(t) \in T_{\gamma(t)}M, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \nabla^2_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma,$$

where R denotes the Riemann tensor. Let $\mu(t)$ be the number of linearly independent Jacobi fields of γ such that $J(0) = J(t) = 0$; the Morse index theorem states that

$$\mathrm{ind}_\Omega(\gamma) = \sum_{0<t<1} \mu(t). \tag{4}$$

The local homology of a critical circle $S^1 \cdot \gamma$ is by definition

$$\mathrm{C}_*(S^1 \cdot \gamma) := H_*(\{E < E(\gamma)\} \cup S^1 \cdot \gamma, \{E < E(\gamma)\}),$$

where the set $\{E < E(\gamma)\}$ is $\{\delta \in \Lambda M \mid E(\delta) < E(\gamma)\}$, and H_* denotes the singular homology with integral coefficients. A closed geodesic is said to be homologically visible if $\mathrm{C}_*(S^1 \cdot \gamma) \neq 0$ and is said to be homologically invisible otherwise. We will be interested in the properties of the local homology $\mathrm{C}_*(S^1 \cdot \gamma)$ only in the case where γ is a closed geodesic of average index $\overline{\mathrm{ind}}(\gamma) = 0$ and whose image $\gamma(S^1)$ lies in the interior of M ($\overline{\mathrm{ind}}(\gamma) = 0$ is equivalent to the fact that $\mathrm{ind}(\gamma^m)$ vanishes for all $m \geq 1$). Let $\gamma \in \Lambda M$ be such a closed geodesic. Given $m \in \mathbb{N}$, we denote by $\psi_m : \Lambda M \to \Lambda M$ the iteration map $\psi_m(\delta) := \delta^m$. According to a theorem of Gromoll-Meyer [16, Theorem 3], the local homology of $\mathrm{C}_d(S^1 \cdot \gamma)$ is zero in degrees $d \geq 2 \dim M$ and there exists infinitely many positive integers m such that the induced map in homology

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot \gamma) \to \mathrm{C}_*(S^1 \cdot \gamma^m) \tag{5}$$

is an isomorphism. On the other hand, a theorem of Bangert-Klingenberg [3, Corollary 1] states that there exists $m_0 \in \mathbb{N}$ above which for all $m \geq m_0$, there exists $e_m > m^2 E(\gamma)$ such that the composition

$$\mathrm{C}_*(S^1 \cdot \gamma) \xrightarrow{(\psi_m)_*} \mathrm{C}_*(S^1 \cdot \gamma^m) \xrightarrow{\mathrm{inc}_*} H_* \left(\left\{E < e_m\right\}, \left\{E < m^2 E(\gamma)\right\}\right) \tag{6}$$

is zero.

3. Proof of Bangert theorem

A closed geodesic γ is a mountain pass if, for all neighborhoods $U \subset \Lambda M$ of $S^1 \cdot \gamma$, $U \cap E^{-1}([0, E(\gamma)))$ is not connected. For the proof of Theorem 1.5, we need the following statement, which tells us that isolated closed geodesics cannot remain mountain pass critical points of the energy functional when sufficiently iterated. A geometric proof is given by Bangert [4].

Theorem 3.1 ([4, Theorem 2])**.** *Let γ be an isolated closed geodesic on M, where $\dim M = 2$. Then there exists an integer $m_\gamma \in \mathbb{N}$ such that the following holds: For all integer $m \in \mathbb{N}$ with $m \geq m_\gamma$, there is a neighborhood U of $S^1 \cdot \gamma$ in ΛM such that $U \cap E^{-1}([0, E(\gamma^m)))$ is connected.*

According to Gromoll-Meyer [15], given an isolated closed geodesic γ, there exists a connected neighborhood $U \subset \Lambda M$ of the critical circle $S^1 \cdot \gamma$ such that

$$\mathrm{C}_*(S^1 \cdot \gamma) \simeq H_* \left(U, U \cap E^{-1}([0, E(\gamma)))\right).$$

If γ and all its iterates are homologically invisible, Theorem 3.1 is thus true for $m_\gamma = 1$.

Proof of Theorem 1.5. Assume there are only finitely many prime closed geodesics $\gamma_1, \dots, \gamma_k$ which have homologically visible iterates and which intersect $M \setminus (C_- \cup C_+)$. We will now derive a contradiction from this assumption. We will define a suitable finite-dimensional approximation $\Omega = \Omega(\mathcal{O}, \kappa, j)$. Now as the statement of Theorem 3.1 remains true in a finite-dimensional approximation, we get that there exists $m_0 \in \mathbb{N}$ such that for all integers $m \geq m_0$ and for all $i \in \{1, \dots, k\}$ the following holds:

i) There exists a neighborhood U of $S^1 \cdot \gamma_i^m$ in Ω such that $U \cap E^{-1}([0, E(\gamma_i^m)))$ is connected.

Set $A := \max\{E(\gamma_i^{m_0}) \,|\, i \in \{1, \dots, k\}\}$, and notice that A is larger than the energy of a closed geodesic of mountain pass type. We fix an identification of $\pi_1(M)$ with $\mathbb{Z}$ and denote by $[\gamma] \in \mathbb{Z}$ the class of a loop $\gamma \in \Lambda M$. We define the following sets of curves:

$$P_j^{\pm} := \{\gamma \in \Omega \,|\, \gamma(S^1) \subset \mathrm{int}(C_\pm),\ [\gamma] = j\}.$$

In the following for each $U, V \subset M$, we will denote

$$\mathrm{dist}(U, V) := \inf_{x \in U,\, y \in V} d(x, y).$$

Choose $\delta > 0$. Then there exists an $n \in \mathbb{N}$ such that for any curve $\gamma \in P_n^{\pm}$ and $\mathrm{dist}(\gamma(S^1), M \setminus (C_- \cup C_+)) < \delta$ it holds that $E(\gamma) \geq A$. We can now say how exactly the finite-dimensional approximation has to be chosen:

- Choose a $\kappa > 0$ large enough such that there exists a homotopy $h : [0,1] \to E^{-1}([0, \kappa))$ in Ω from $h_0 \in P_n^-$ to $h_1 \in P_n^+$ with
$$\mathrm{dist}\Big(h_t(S^1), M \setminus (C_- \cup C_+)\Big) < \delta, \quad \forall t \in [0,1].$$
- Set $\mathcal{O} := \{p \in M \,|\, \mathrm{dist}(p, M \setminus (C_- \cup C_+)) < R\}$, where $R > 2\kappa^{\frac{1}{2}} + \delta$ such that $\mathcal{O}$ contains $\gamma_1, \dots, \gamma_k$.
- Choose k such that the $(-\nabla E)$-flow of the finite-dimensional approximation respects $C_\pm$, as described above.

A technical issue is given by the fact that the gradient flow of $-\nabla E$ may not be defined for all times as the sublevel sets of $E|_\Omega$ are not compact. Ultimately we are only going to be interested in curves intersecting the compact set $M \setminus (C_- \cup C_+)$, i.e. the subset

$$K := \{\gamma \in \Omega \,|\, \gamma(S^1) \cap (M \setminus (C_- \cup C_+)) \neq \varnothing\}$$

of Ω. We introduce a smooth function $g : \Omega \to [0,1]$ with the property that

$$\begin{cases} g(\gamma) = 1 & \text{if } \mathrm{dist}(\gamma(S^1), K) \leq \frac{1}{2}\kappa^{\frac{1}{2}}, \\ g(\gamma) = 0 & \text{if } \mathrm{dist}(\gamma(S^1), K) > \frac{3}{2}\kappa^{\frac{1}{2}}. \end{cases}$$

Then the flow ϕ_t of $-g\nabla E$ is defined for all times $t \geq 0$ and coincides with the negative gradient flow for curves in K. Two crucial observations about the set K are the following: firstly, for all $\bar{\kappa} < \kappa$ the set $K \cap E^{-1}([0, \bar{\kappa}])$ is compact. Secondly, if $\phi_t(\gamma) \in K$ for some $\gamma \in \Omega$ and some time $t \geq 0$, we already have $\gamma \in K$ as the flow ϕ_t respects the convex sets $C_\pm$. From this it follows:

ii) Let $0 < \kappa_0 < \kappa_0 + \varepsilon < \kappa$. Let V denote a neighborhood of the closed geodesics in K of energy κ_0. Suppose there is no closed geodesic in $K \cap E^{-1}((\kappa_0, \kappa_0 + \varepsilon])$. Then there exists a time $\tau > 0$, such that

$$\phi_\tau \left(E^{-1}\left([0, \kappa_0 + \varepsilon]\right)\right) \cap K \subset E^{-1}([0, \kappa_0)) \cup V.$$

This is just the deformation lemma; for a proof see for instance [21, Lemma 3.4]. We are now set to complete the proof of the theorem. Define the set of homotopies

$$\Pi := \left\{ \beta \,|\, \beta : [0,1] \to \Omega,\ \beta_0 \in P_n^-,\ \beta_1 \in P_n^+ \right\}.$$

Note that Π is not empty, as $h \in \Pi$. Furthermore, $\phi_t \circ \beta \in \Pi$ for all $\beta \in \Pi$ and all $t \geq 0$ as the flow respects the convex sets $C_\pm$ and therefore $\phi_t(\beta_0) \in P_n^-$ and $\phi_t(\beta_1) \in P_n^+$ for all $t \geq 0$. Define now

$$\kappa_0 := \inf_{\beta \in \Pi} \max_{\substack{t \in [0,1] \\ \beta_t \in K}} E(\beta_t)\,.$$

By definition of κ, one has $\kappa_0 < \kappa$. For every $\beta \in \Pi$ for time $t_0 := \min\{t \in [0,1] \,|\, \beta_t \notin P_n^-\}$ it holds that $\beta_{t_0} \in K$ and $E(\beta_{t_0}) \geq A$ (as E and β are continuous and there exists a sequence $(t_k) \nearrow t_0$ such that $\beta_{t_k} \in P_n^-$ and $\mathrm{dist}(\beta_{t_k}, M \setminus (C_- \cup C_+)) < \delta$). Consequently, we get $\kappa_0 \geq A$. Since $\kappa_0 < \kappa$, for $\varepsilon > 0$ small enough, the subset $K \cap E^{-1}([0, \kappa_0 + \varepsilon])$ is compact and there are only finitely many S^1-orbits of closed geodesics inside (we assumed every orbit to be isolated). Let $\{S^1 \cdot d_j\}_{1 \leq j \leq l}$ denote the critical circles of energy κ_0 in K. By definition of A and by using i) (when d_j is homologically visible), there exist disjoint neighborhoods U_j of the $S^1 \cdot d_j$'s such that $U_j \cap E^{-1}([0, \kappa_0))$ is connected for all j. Since $\partial C_\pm$ are not totally geodesic, we know that the d_j's are not contained in ∂K and we can assume that $U_j \subset K$. Now because there are only finitely many closed geodesics in $K \cap E^{-1}([0, \kappa_0 + \varepsilon])$ for $\varepsilon > 0$ small enough, one can take $\varepsilon > 0$ such that there is no closed geodesic in $K \cap E^{-1}((\kappa_0, \kappa_0 + \varepsilon])$. By the definition of κ_0 there exists a homotopy $\beta \in \Pi$ satisfying $E(\beta_t) \leq \kappa_0 + \varepsilon$ for all $t \in [0,1]$ such that $\beta_t \in K$. Choose neighborhoods V_j of $S^1 \cdot d_j$ such that $\overline{V_j} \subset \mathrm{int}(U_j)$ and use property ii) on the neighborhood $V := \bigcup_{k=1}^{l} V_j$ of closed geodesics of energy κ_0 in K to obtain a $\tau > 0$ with the property that for the homotopy $\phi_\tau \circ \beta$ we have that $(\phi_\tau \circ \beta)_t \in K$ implies $E((\phi_\tau \circ \beta)_t) < \kappa_0$ or $(\phi_\tau \circ \beta)_t \in V$. Now $(\phi_\tau \circ \beta)^{-1}(V) = \bigcup_{r=1}^{m} (t_r, t_r')$ and by our choice of the V_j we have $(\phi_\tau \circ \beta)([t_r, t_r']) \subset U_j$ and for the endpoints $(\phi_\tau \circ \beta)_{t_r}, (\phi_\tau \circ \beta)_{t_r'} \in U_j \cap E^{-1}([0, \kappa_0))$ for some $j \in \{1, \ldots, l\}$ (which is why we applied ii) only to V and not to $\bigcup_{j=1}^{l} U_j$ directly). Now, by using i) if d_j is homologically visible, we know that $U_j \cap E^{-1}([0, \kappa_0))$ is connected and consequently we can replace $(\phi_\tau \circ \beta)|_{[t_r, t_r']}$ by a path in $E^{-1}([0, \kappa_0))$ with the same endpoints. After m steps we obtain a homotopy $\hat{\beta} : [0,1] \to \Omega$ such that $E(\hat{\beta}_t) < \kappa_0$ when $\hat{\beta}_t \in K$. Since $(\phi_\tau \circ \beta)_0, (\phi_\tau \circ \beta)_1 \notin K$ it follows that $(\phi_\tau \circ \beta)_0, (\phi_\tau \circ \beta)_1 \notin \bigcup_{j=1}^{l} U_j$ and therefore $\hat{\beta}_0 \in P_n^-$,$\hat{\beta}_1 \in P_n^+$, hence $\hat{\beta} \in \Pi$. This contradicts the minimality of κ_0. $\square$

4. Contractible and intersecting closed geodesics

Here M still denotes a complete Riemannian cylinder. We assume that there exists a contractible closed geodesic $c \in \Lambda M$. Let us consider the unbounded components of $M \setminus c(S^1)$. Since $c(S^1)$ is bounded, there are at most two distinct unbounded components. If there are two distinct unbounded components C_- and C_+, one can

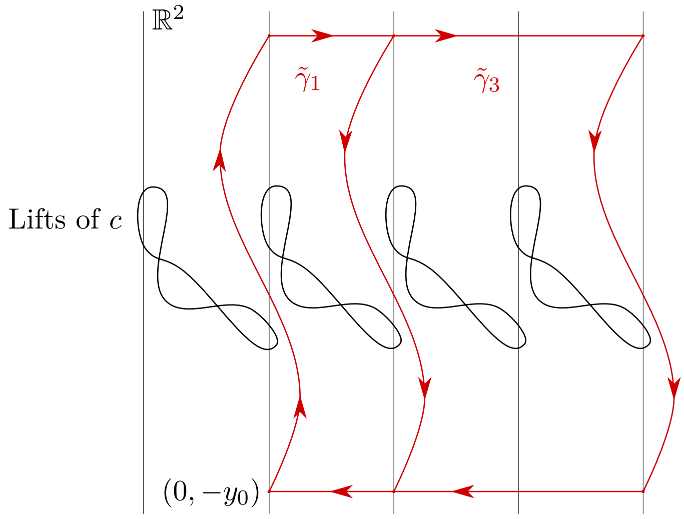

FIGURE 1. The family of loops $(\tilde\gamma_n)$

assume that C_- is a neighborhood of $-\infty$ and C_+ is a neighborhood of $+\infty$. By $C_\pm$ we will mean any of these two neighborhoods. Then $\partial C_\pm$ is a broken geodesic with angles strictly less than π inside $C_\pm$ since c is a closed geodesic (see Figure 2 for an instance of ∂C_+). Hence $C_\pm$ is locally convex. Moreover if the boundary were totally geodesic, then $\partial C_\pm$ would be parametrised by c which is impossible for c is contractible. We can thus apply Theorem 1.5 in this case.

We now assume that $M \setminus c(S^1)$ has only one unbounded component C. Let us identify M with $S^1 \times \mathbb{R}$ in the remaining of this proof in order to fix notations. Let $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ be the universal cover of $S^1 \times \mathbb{R}$. By compactness of $c(S^1)$, there exists $A > 0$ such that $c(S^1) \subset S^1 \times (-A, A)$. Let $y_0 > A$, since $S^1 \times (-\infty, -A)$ and $S^1 \times (A, +\infty)$ belong to the same component of $M \setminus c(S^1)$, there exists a smooth path $\alpha : [0,1] \to M \setminus c(S^1)$ such that $\alpha(0) = (0, -y_0)$ and $\alpha(1) = (0, y_0)$. Let β_0 be the smooth lift of α in $\mathbb{R}^2$ such that $\beta_0(0) = (0, -y_0)$ and $\beta_0(1) = (n_0, y_0)$ for some $n_0 \in \mathbb{Z}$ that we can take equal to $n_0 = 0$ by chaining α with $t \mapsto (tn_0 \bmod 1, y_0)$. Let $\delta_{n,\pm} : [0,1] \to \mathbb{R}^2$ be the path $t \mapsto (nt, \pm y_0)$ and $\beta_n : [0,1] \to \mathbb{R}^2$ be the family of lifts $\beta_n := (n, 0) + \beta_0$, $n \in \mathbb{N}$. We define the family of loops $\tilde\gamma_n \in \Lambda\mathbb{R}^2$ by

$$\tilde\gamma_n := \beta_0 \cdot \delta_{n,+} \cdot \beta_n^{-1} \cdot \delta_{n,-}^{-1}.$$

They project to $\gamma_n := \pi \circ \tilde\gamma_n$ in $M \setminus c(S^1)$. Let $q_0 \in \mathbb{R}^2$ be a lift of some point of $c(S^1)$ and define $q_n := q_0 + (n, 0)$. Then the first homology group $H_1(\mathbb{R}^2 \setminus \{q_n\}_{n\in\mathbb{Z}})$ is the free abelian group with generators $(g_n)_{n\in\mathbb{Z}}$, and by construction the class of $\tilde\gamma_n$ is $g_1 + g_2 + \cdots + g_n$. Since the covering transformations of $\mathbb{R}^2 \setminus \{q_n\}_{n\in\mathbb{Z}} \to S^1 \times \mathbb{R} \setminus \pi(q_0)$, which form a group isomorphic to $\mathbb{Z}$, act on the first homology group by $k \cdot g_i = g_{i+k}$, we see that, if $n \neq m$ and $k, l \in \mathbb{Z}^*$, the iterated loops γ_n^k and γ_m^l are not freely homotopic in $M \setminus \pi(q_0)$ and hence in the unbounded component C of $M \setminus c(S^1)$. We want to apply the $(-\nabla E)$-flow to γ_n, $n \in \mathbb{N}$. Since c is a closed geodesic, the unbounded component C of $M \setminus c(S^1)$ is a locally convex neighborhood. Hence ΛC is preserved by the $(-\nabla E)$-flow. Since M is complete, the set of points at distance less than $\ell > 0$ from $c(S^1)$ is compact. Moreover, the image of γ_n by the flow (when defined) is kept inside this compact set for $\ell = \frac{1}{2}\sqrt{E(\gamma_n)}$. Thus one can apply the $(-\nabla E)$-flow to γ_n at all time $t > 0$ and ultimately get a closed geodesic of C

homotopic to γ_n. We thus get a family of closed geodesics which are not homotopic and not iterations of each other.

Now that Theorem 1.1 is proved under hypothesis 1, in order to prove it when there is one self-intersecting closed geodesic c or two intersecting ones c_1 and c_2, one can assume that these geodesics are not contractible. Therefore, in both respective cases, $M \setminus c(S^1)$ or $M \setminus (c_1(S^1) \cup c_2(S^1))$ has exactly two unbounded connected components C_- and C_+, which are locally convex by construction. The intersection hypothesis then implies that none of the boundaries $\partial C_\pm$ is totally geodesic. Hence the conclusion follows by applying Theorem 1.5.

5. Geodesic of non-zero average index

We assume that there exists a closed geodesic $c \in \Lambda M$ of average index $\overline{\mathrm{ind}}(c) > 0$. If c is contractible or self-intersecting, we already know that there are infinitely many closed geodesics. Let us assume that c is an embedded curve generating $\pi_1(M) \simeq \mathbb{Z}$. By a slight abuse of notation, we identify the loop $c : S^1 \to M$ with its lift $\mathbb{R} \to M$.

Lemma 5.1. *There exist $k \in \mathbb{N}^*$ and $\delta \in (0,1]$ such that for all $s \in \mathbb{R}$, there exists a Jacobi field $J : \mathbb{R} \to c^*TM$ of c with*

(1) $J(s) = 0$,
(2) $J|_{(s,s+\delta)}$ non-vanishing,
(3) $J(s+t) = 0$ for some $t \in [\delta, k]$.

Proof. Given a closed geodesic $\gamma \in \Lambda M$ and $s \in S^1$, let us denote by $\gamma_s : [0,1] \to M$ the geodesic path $\gamma_s(t) := \gamma(s+t)$. Since $\overline{\mathrm{ind}}(c) > 0$, Bott iteration inequality (2) and the concavity bound (3) imply that there exists $k \in \mathbb{N}^*$ such that

$$\mathrm{ind}_\Omega((c^k)_s) \geq 1, \quad \forall s \in S^1.$$

Let us fix such a $k \geq 1$. Then according to the Morse index theorem (4) for all $s \in \mathbb{R}$ we can find a non-zero Jacobi field J of c satisfying conditions (1) and (3). Let $r > 0$ be the injectivity radius along the curve c. We define $\delta > 0$ by

$$\delta = \frac{r}{|\dot{c}(0)|},$$

(we recall that c has constant speed $|\dot{c}(0)|$). Then by definition of r, $\delta < 1$. Given any Jacobi field $J \in \mathbb{J}(c) \setminus 0$, there exists a smooth family of geodesics (γ_s) with $\gamma_0 = c$ such that $J = \partial_s \gamma_s|_{s=0}$. Suppose that $J(0) = J(t) = 0$ with $t < \delta$, then for some s close to 0, γ_s must intersect c at $c(t_0)$ and $c(t_1)$ with $2|t_0|$ and $2|t_1 - t|$ strictly less than $\delta - t$ so that $|t_1 - t_0| < \delta$ which contradicts the definition of the injectivity radius r. Hence δ fulfills the condition of the lemma for any non-zero Jacobi fields vanishing at 0. ☐

In order to fix notation, let us identify the image of the loop c to $S^1 \times \{0\}$, with $c(s) = (s, 0)$ for $s \in S^1$, so that $M \setminus c(S^1)$ is the disjoint union of the neighborhood $S^1 \times (-\infty, 0)$ of $-\infty$ and the neighborhood $S^1 \times (0, +\infty)$ of $+\infty$ (we only need this identification to be a homeomorphism). We now use Lemma 5.1 to find $n \leq \lceil 1/k(\delta - \varepsilon) \rceil + 1$ geodesic chords $\alpha_1, \dots, \alpha_n$ lying inside $S^1 \times [0, +\infty)$ and intersecting c only at endpoints so that the unbounded component C_+ of $S^1 \times (0, +\infty) \setminus \bigcup_i \alpha_i([0,1])$ is locally convex and not a closed geodesic. We can do the same on the other side and eventually find two locally convex neighborhoods

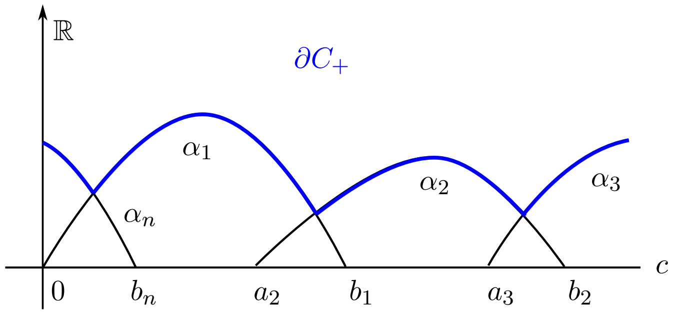

FIGURE 2. Construction of the locally convex neighborhood C_+

$C_\pm$ of $\pm\infty$ whose boundaries are not totally geodesic and with non-intersecting closure. To make the statement precise, let $p : S^1 \to [0,1)$ be the natural bijection of $S^1 = \mathbb{R}/\mathbb{Z}$ to the fundamental domain $[0,1)$ and $\pi : M \simeq S^1 \times \mathbb{R} \to [0,1)$ be the induced projection so that $\pi \circ c(t) = t$ for $t \in [0,1)$.

Proof. Let $k \in \mathbb{N}^*$ and $\delta > 0$ be given by Lemma 5.1. Let $\varepsilon < \frac{\delta}{4}$ be a small non-zero real number. Up to a translation of the parametrisation of c, we construct inductively a family $\alpha_1, \dots, \alpha_n$, $n \leq \lceil 1/k(\delta - \varepsilon) \rceil$, of geodesic chords satisfying:

(1) $\alpha_1(0) = c(0)$.
(2) $\alpha_i : [0,1] \to S^1 \times [0,+\infty)$ is a geodesic path intersecting c only at $\alpha_i(0)$ and $\alpha_i(1)$.
(3) If $a_i := \pi \circ \alpha_i(0)$ and $b_i := \pi \circ \alpha_i(1)$ denote the positions of endpoints, then
$$b_i - a_i > \delta - 2\varepsilon \quad \text{and} \quad [a_i, b_i] \subset \pi \circ \alpha_i([0,1])$$
for $1 \leq i \leq n$. And for $i = n$,
$$[0, b_n] \cup [a_n, 1) \subset \pi \circ \alpha_n([0,1]).$$
(4) The position of endpoints satisfy
$$\begin{cases} a_i < a_{i+1} < b_i < b_{i+1}, & 1 \leq i \leq n-2, \\ a_{n-1} < a_n < b_{n-1}, \\ a_1 = 0 \leq b_n. \end{cases}$$

Suppose $\alpha_1, \dots, \alpha_q$ satisfies properties (2) and (3) and relation of property (4) for $i \in \{1, \dots, q-1\}$. To construct α_1, we take $b_0 := 0$ in the following then we translate the parametrisation of c once for all so that property (1) is fulfilled. By definition, the position $b_q \in [0,1)$ satisfies $c(b_q) = \alpha_q(1)$. According to Lemma 5.1, there exists a Jacobi field along c such that $J(b_q - \varepsilon) = 0$, J does not vanish on $I := (b_q - \varepsilon, b_q + \delta - \varepsilon)$ and $J(b_q - \varepsilon + t) = 0$ for some $t \in [\delta, 1]$. Up to a change of sign, one can assume that $J|_I$ is pointing inside $S^1 \times (0,+\infty)$. Since there exists a smooth family $(\beta_s)_{s \in (-1,1)}$ of geodesic paths such that $J|_I = \frac{\partial \beta_s}{\partial s}|_{s=0}$, it implies that there exists a geodesic path $\alpha_{q+1} : [0,1] \to S^1 \times [0,+\infty)$ intersecting c (transversally) only at its endpoints with $a_{q+1} < b_q < b_{q+1} - \delta - 2\varepsilon$ unless $\pi \circ \alpha_{q+1}([0,1]) = [0, b_{q+1}] \cup [a_{q+1}, 1)$, in this case we set $n := q+1$. In the exceptional case where $b_n = 0$, we construct a last geodesic chord α_{n+1} satisfying property (2), with at least one endpoints different from $c(0)$ and whose image projects on a neighborhood of $c(0) \in S^1$.

Thus we get a family of geodesic chords such that consecutive chords intersect in $S^1 \times (0,+\infty)$ and the union of their images $\bigcup_i \alpha_i([0,1])$ projects onto S^1. By construction, the unbounded component C_+ of $S^1 \times (0, +\infty) \setminus \bigcup_i \alpha_i([0, 1])$ has a boundary which is broken geodesic with angles strictly less than π. By symmetry, we get two disjoint neighborhoods of $+\infty$ and $-\infty$ respectively which are locally convex and whose boundaries are not totally geodesic, we can thus apply Theorem 1.5. □

6. Two homologically visible geodesics

Here M denotes a complete Riemannian cylinder. We fix an identification of $\pi_1(M)$ with $\mathbb{Z}$ and denote by $[\gamma] \in \mathbb{Z}$ the class of a loop $\gamma \in \Lambda M$. We assume that there exists two geometrically distinct and homologically visible closed geodesics. We suppose by contradiction that there are only finitely many (geometrically distinct) closed geodesics in M. By the previous cases of Theorem 1.1, every prime closed geodesic of M must be embedded, non-contractible, without intersections with another closed geodesic, and of zero average index. Thus the closed geodesics of $M \simeq S^1 \times \mathbb{R}$ are naturally ordered by their intersection with $* \times \mathbb{R}$ where $*$ denotes any point of S^1. The order is independent of the choice of $* \in S^1$. We will say that two closed geodesics are consecutive if they are so with respect to this order. Since we assume that the S^1-orbits of closed geodesics are isolated, given a closed geodesic, one can talk about the next and the previous one with respect to this order.

Lemma 6.1. *There exists two closed embedded geodesics c_1 and c_2 of M with degree $[c_1] = [c_2] = 1$ bounding a compact locally convex cylinder $C \simeq S^1 \times [0, 1]$ such that*

(1) c_1 is a local minimum of $E|_{\Lambda C}$,
(2) c_2 is not a local minimum of $E|_{\Lambda C}$,
(3) c_1 and c_2 are the only closed geodesics of M inside C that have homologically visible iterates.

Proof. We first show that two consecutive closed geodesics among closed geodesics that possess homologically visible iterates cannot be both local minima of $E|_{\Lambda C'}$ if C' is the compact cylinder that they bound. By contradiction, let us assume so and let us call γ_0 and γ_1 these two geodesics. Up to a change of parametrization, one can assume that $[\gamma_0] = [\gamma_1]$ and thus that these two geodesics are homotopic in $\Lambda C'$. Let

$$\Pi := \{h : [0, 1] \to \Lambda C' \text{ continuous} \mid h(0) := \gamma_0 \text{ and } h(1) := \gamma_1\}$$

denote the set of homotopy of loops in C' starting at γ_0 and ending at γ_1. We consider the following min-max:

$$\tau = \inf_{h \in \Pi} \max E \circ h.$$

Let $e := \max(E(\gamma_0), E(\gamma_1))$. Since γ_0 and γ_1 are local minima of $E|_{\Lambda C'}$, $\tau > e$. By compactness of C', $E|_{\Lambda C'}$ satisfies Palais-Smale (alternatively, one can work in the compact finite-dimensional manifold of k-broken-geodesics of energy $\leq c + \varepsilon$ for a large $k \in \mathbb{N}$ and $\varepsilon > 0$). By local convexity of C', the $(-\nabla E)$-flow preserves $\Lambda C'$. By the minimax principle, τ is thus a critical value of $E|_{\Lambda C'}$ and there exists a homologically visible closed geodesic $\gamma \in \Lambda C'$ of energy τ. Hence γ_0 and γ_1 are not consecutive, a contradiction.

By a similar argument, we show that one out of two consecutive closed geodesics among those that possess homologically visible iterates is a local minimum of $E|_{\Lambda C'}$.

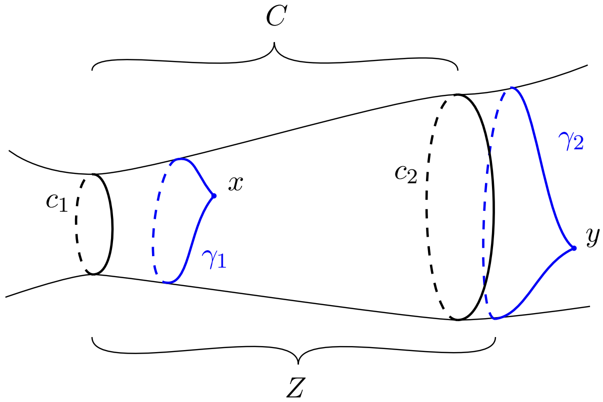

FIGURE 3. Construction of cylinder Z

Indeed, otherwise one has that

$$\inf_{\substack{\gamma\in\Lambda C' \\ [\gamma]=1}} E(\gamma) < \min(E(\gamma_0), E(\gamma_1)),$$

and this infimum is reached for some closed geodesic in C' by compactness and local convexity of C' (and this is not a point since $E(\gamma) \geq (2r)^2$ for all $\gamma \in \Lambda C'$ of degree $[\gamma] = 1$ where $r > 0$ denotes the injectivity radius of the compact Riemannian manifold with boundary C'). This new closed geodesic is a local minimum of E by definition and thus homologically visible.

The requirements of the lemma are thus fulfilled by taking any consecutive homologically visible closed geodesics. □

Proof of Theorem 1.1. Let c_1 and c_2 be closed geodesics of M satisfying Lemma 6.1. We will reach a contradiction by finding a homologically visible geodesic which is not c_1 or c_2 and arbitrarily close to C.

Let $x \in \mathrm{Int}(C)$ and let $\gamma_1 \in \Lambda C$ be the loop of degree $[\gamma_1] = 1$ based at x of minimal length. It exists by local convexity and compactness of C. The loop γ_1 is not a periodic geodesic (this is a geodesic as a path $[0,1] \to C$ but not as a loop $S^1 \to C$) since there is no local minimum of $E|_{\Lambda C}$ but c_1. This loop lies inside $\mathrm{Int}(C)$ so that either the connected component of $C \setminus \gamma_1(S^1)$ containing c_1 or the connected component containing c_2 is locally convex – depending on the angle of γ_1 at $\gamma_1(0) = \gamma_1(1) = x$. If the connected component containing c_2 were locally convex, then the infimum of E among loops of degree one lying inside the locally convex compact cylinder bounded by γ_1 and c_2 would give a closed geodesic loop $\neq c_1$ which would be a local minimum. Thus the connected component of $C \setminus \gamma_1(S^1)$ containing c_1 is a locally convex compact cylinder. Hence the unbounded component of $M \setminus \gamma_1(S^1)$ containing c_1 is a locally convex neighborhood of $-\infty$ which is not totally geodesic since γ_1 is not a closed geodesic.

Let c_3 be the closed geodesic succeeding c_2 among closed geodesic (possibly non homologically visible), if it exists. Let C' be either the compact cylinder that c_2 and c_3 bound or the infinite cylinder $\simeq S^1 \times [0, +\infty)$ with boundary c_2 and ending at $+\infty$, depending on the existence of c_3 (so that $C \cap C' = c_2(S^1)$ in both cases). Let $y \in \mathrm{Int}(C')$ and let $\gamma_2 \in \Lambda C'$ be a loop of degree $[\gamma_2] = 1$ based at y of minimal length. Since C' is complete and locally convex, it exists. It cannot be a closed geodesic for c_3 succeeds c_2. One of the two unbounded components of $M \setminus \gamma_2(S^1)$ is thus locally convex, depending on the angle of γ_2 at $\gamma_2(0) = \gamma_2(1) = y$. If the neighborhood of $+\infty$ was the locally convex one, by Theorem 1.5 applied to the locally convex neighborhood of $-\infty$ defined above with γ_1 and this neighborhood of $+\infty$, there would be infinitely many closed geodesics. Thus the neighborhood of $-\infty$ is the locally convex unbounded component of $M \setminus \gamma_2(S^1)$. Restricting this neighborhood to the compact cylinder $C \cup C'$, one gets a compact locally convex cylinder Z intersecting only two geodesics c_1 and c_2 that possess homologically visible iterates, moreover $c_1(S^1) \subset \partial Z$ and $c_2(S^1) \subset \mathrm{Int}(Z)$.

Let $k \in \mathbb{Z}^*$ be such that $\mathrm{C}_*(S^1 \cdot c_2^k) \neq 0$. Let $\Lambda_m \subset \Lambda Z$ be the connected component of loops $\gamma \in \Lambda Z$ of degree $[\gamma] = m$. Let $\psi_m : \Lambda_k \to \Lambda_{km}$ be the iteration map $\psi_m(\gamma) := \gamma^m$. According to Bangert-Klingenberg theorem (6), there exists $m_0 \in \mathbb{N}$ above which for all $m \geq m_0$ there exists $e_m > m^2 E(c_2^k)$ such that the composition

$$\mathrm{C}_*(S^1 \cdot c_2^k) \xrightarrow{(\psi_m)_*} \mathrm{C}_*(S^1 \cdot c_2^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E|_{\Lambda_{km}} < e_m\right\}, \left\{E|_{\Lambda_{km}} < m^2 E(c_2^k)\right\}\right)$$

is zero. According to Gromoll-Meyer theorem (5), since $\overline{\mathrm{ind}}(c_2^k) = k\,\overline{\mathrm{ind}}(c_2) = 0$, there exists infinitely many m such that

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot c_2^k) \to \mathrm{C}_*(S^1 \cdot c_2^{km})$$

is an isomorphism. Let $m \geq m_0$ be such an integer, then the inclusion induces a zero map

$$\mathrm{C}_*(S^1 \cdot c_2^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E|_{\Lambda_{km}} < e_m\right\}, \left\{E|_{\Lambda_{km}} < m^2 E(c_2^k)\right\}\right),$$

which contradicts the fact that c_2^{km} is the homologically visible critical points of $E|_{\Lambda_{km}}$ of maximal value. Indeed, since Z is locally convex, critical points of $E|_{\Lambda_{km}}$ are closed geodesics of Z of degree km. Thus $S^1 \cdot c_1^{km}$ and $S^1 \cdot c_2^{km}$ are the only homologically visible critical circle of $E|_{\Lambda_{km}}$ (and $E(c_2^{km}) > E(c_1^{km})$ since c_1 is the only local minimum in C). Moreover Z is compact and has only isolated closed geodesics, we can thus apply Morse theoretical arguments since $E|_{\Lambda_{km}}$ has isolated critical circles and satisfies Palais-Smale or, alternatively, one can restrict E to the finite-dimensional subspace of k-broken-geodesics of Λ_{km} of energy less than $e_m + \varepsilon$ for some large $k \in \mathbb{N}$ and some $\varepsilon > 0$. Thus, if $S^1 \cdot c_2^{km}$ were the only homologically visible critical circle of energy $\geq m^2 E(c_2^k)$, Morse deformation lemma would imply inc_* to be an isomorphism. □

7. The case of the plane

Let $M \simeq \mathbb{R}^2$ be a complete Riemannian plane with isolated closed geodesics. Using what we have seen in the previous sections, we now give the proof of Theorem 1.3.

Proof of Theorem 1.3. When hypothesis 1, 2 or 3 is assumed, the conclusion follows from the same argument as in the case of the cylinder: by construction of an open neighborhood $C \neq M$ of infinity. More precisely, this neighborhood C is the unbounded component of $M \setminus c(S^1)$ or $M \setminus (c_1(S^1) \cup c_2(S^1))$ if c is self-intersecting or c_1 and c_2 are intersecting closed geodesics. In the case when there exists a closed geodesic c of non-zero average index, C is constructed by "integrating Jacobi fields" along c as was done in Section 5.

Now, let us assume that all the closed geodesics of M are without self-intersection and with zero average index. In order to complete the proof, we must prove that this last case implies the existence of infinitely many closed geodesics. Let c be a homologically visible closed geodesic such that there is not any homologically visible closed geodesic inside the disk D bounded by c. Let $G = \bigcup_\gamma \gamma(S^1) \subset M$ be the union of the images of the closed geodesics γ of M. Let U be the connected component of $M \setminus (D \cup G)$ that contains $c(S^1)$ in its boundary. Since U contains loops that are not contractible in $\mathbb{R}^2 \setminus D$ (by taking loops close to the boundary $c(S^1)$), U is not simply connected. Let $y \in M$ and let $\gamma \in \Lambda\overline{U}$ be a loop minimizing the length among the non-contractible loops of $\overline{U}$ based at y (it exists since $\overline{U}$ is complete). Since ∂U is a disjoint union of closed geodesics, γ lies in the interior of U and is a geodesic path. Depending on the angle that γ makes at y, either the unbounded component of $M \setminus \gamma(S^1)$ is locally convex and not totally geodesic or the bounded component containing c is locally convex. In the first case, one can apply Bangert's theorem to complete the proof.

We can thus assume that c lies in the interior of a compact and locally convex subset $K \subset M$ and that c is the only homologically visible closed geodesic of K. Since $\overline{\mathrm{ind}}(c) = 0$, the local homology groups $C_d(S^1 \cdot c^m)$ are trivial in degrees $d \geq 4$ for all $m \in \mathbb{N}$. Let $d \in \{0, 1, 2, 3\}$ be the maximal degree such that $C_d(S^1 \cdot c^m) \neq 0$ for some $m \in \mathbb{N}^*$. Let $k \in \mathbb{N}^*$ be such that $C_d(S^1 \cdot c^k) \neq 0$. According to Gromoll-Meyer theory, there exists infinitely many $m \in \mathbb{N}^*$ such that the map induced by the iteration map

$$(\psi_m)_* : \mathrm{C}_*(S^1 \cdot c^k) \to \mathrm{C}_*(S^1 \cdot c^{km})$$

is an isomorphism. As above, according to Bangert-Klingenberg theorem (6), there exists $m_0 \in \mathbb{N}^*$ such that, for all such $m \in \mathbb{N}^*$ greater than m_0, the inclusion of sublevel sets of $E|_{\Lambda K}$ induces the zero map

$$\mathrm{C}_*(S^1 \cdot c^{km}) \xrightarrow{\mathrm{inc}_*} H_*\left(\left\{E|_{\Lambda K} < e_m\right\}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\}\right),$$

for some $e_m > m^2 E(c^k)$. Thus, for such an m, the long exact sequence of the triple

$$\left(\left\{E|_{\Lambda K} < e_m\right\}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\} \cup S^1 \cdot c^{km}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\}\right)$$

implies that

$$H_{d+1}\left(\left\{E|_{\Lambda K} < e_m\right\}, \left\{E|_{\Lambda K} < m^2 E(c^k)\right\} \cup S^1 \cdot c^{km}\right) \neq 0.$$

Therefore, by Morse deformation lemma applied to the smooth map $E|_{\Lambda K}$ which satisfies the Palais-Smale condition and whose anti-gradient flow preserves ΛK (by compactness and local convexity of K), there must be a closed geodesic $\gamma \in \Lambda K$ such that $\mathrm{C}_{d+1}(S^1 \cdot \gamma) \neq 0$. By maximality of d, γ and c are geometrically distinct. But c is the only homologically visible closed geodesic of K, a contradiction. $\square$

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