THE DYNAMICS OF CONFORMAL HAMILTONIAN FLOWS: DISSIPATIVITY AND CONSERVATIVITY

SIMON ALLAIS$^{\circ\dagger}$ AND MARIE-CLAUDE ARNAUD$^{\circ\dagger\ddagger}$

Abstract. We study in detail the dynamics of conformal Hamiltonian flows that are defined on a conformal symplectic manifold (this notion was popularized by Vaisman in 1976). We show that they exhibit some conservative and dissipative behaviours. We also build many examples of various dynamics that show simultaneously their difference and resemblance with the contact and symplectic case.

Contents

1. INTRODUCTION

Symplectic dynamics models many conservative movements. Yet, other phe-
nomena are dissipative and require another setting. This is the case of the damped
mechanical systems: they are modelled by conformal Hamiltonian dynamics, which
alter the symplectic form up to a scaling factor.

This notion of conformal symplectic dynamics can be placed in a broader context.
To define such a dynamics, we only need to know in charts an equivalence class of
2-forms for the relation $\omega_1 \sim \omega_2$,

  where $\omega_1 \sim \omega_2$ if $\omega_1 = f\omega_2$ for some non-vanishing function f.

A manifold endowed with such an equivalence class of local 2-forms, one of
them being closed, is called a conformal symplectic manifold, a notion popularized
by Vaisman in [12]. An equivalent notion is the notion of conformal structure
(M, η, ω), a manifold M endowed with a 1-form called the Lee form and a 2-form
called the conformal form, whose precise definition is given in section 2.1. A proof
of the equivalence of the two notions is given in [2].

We will study autonoumous conformal Hamiltonian flows (CHF in short) $(\varphi_s)_{s\in\mathbb{R}}$
of compact manifolds, see definition in Section 2.2. They alter the conformal form
up to a non-constant scaling factor. As the volume ω^n can increase or decrease at
different points of the manifold under the action of the dynamics, we can expect dif-
ferent behaviours, some of them being *conservative* e.g. completely elliptic periodic
orbits, invariant foliations with compact leaves and some other being *dissipative*,
e.g. attractors or repulsors.

A precise definition of what we call conservative or dissipative requires the in-
troduction of a notion related to the shape of the orbits. The *winding* of a point
$x \in M$ through time is defined as the map $t \mapsto r_t(x)$ (r stands for "rotation"),

$$r_t(x) := \int_0^t \eta(\partial_s\varphi_s(x))\mathrm{d}s, \quad \forall t \in \mathbb{R}.$$

Then $\varphi_t^*\omega = e^{r_t}\omega$, see Lemma 1, and a point $x \in M$ is

- either (positively) dissipative when $\lim_{t\to+\infty} |r_t(x)| = +\infty$;
- or (positively) conservative.

Our main result, Proposition 7, asserts that for every CHF (φ_t^H), if $\mathcal{D}_+$ is the set of
positively dissipative points and if $\mathcal{C}_+$ the set of positively conservative points, then
up to a set of zero volume, $\mathcal{C}_+$ coincides with the set of positively recurrent points,
and then $\mathcal{D}_+$ with the set of positively non-recurrent points. Also, the ω-limit set
$\omega(x)$ of every $x \in \mathcal{D}_+$ is contained in $\{H = 0\}$.
Some examples of conservative and dissipative points are

- every attractor intersects $\{H = 0\}$, has non-trivial homology and almost
  every point in its basin of attraction that doesn't belong to the attractor is
  in $\mathcal{D}_+$, Corollary 9;
- if x is a periodic points that is not a critical point of H, then

- when $x \in \mathcal{C}_+$, the first return map to a Poincaré section preserves a closed 2-form and a foliation into (local) hypersurfaces;
- when $x \in \mathcal{D}_+$, then $H(x) = 0$ and the first return map to a Poincaré section alters a certain closed 2-form up to a constant factor that is different from 1;

• every fixed point of the flow is conservative.

We will provide also an example of wild conservative points: points that are recurrent, in $\{H \neq 0\}$ but whose ω-limit set intersects $\{H = 0\}$, see Section 3.5. Hence these points satisfy $\liminf_{t \to +\infty} |r_t(x)| = +\infty$. The origin of most of our examples is contact geometry. In particular, in Section 2.4, we introduce a notion of twisted conformal symplectization that is crucial to the elaboration of examples and counter-examples.

In a similar way, switching H to $-H$, the set $\mathcal{C}_-$ of negatively conservative points is the set of $x \in M$ such that $\lim_{t \to -\infty} |r_t(x)| = +\infty$ and $\mathcal{D}_- = M \backslash \mathcal{C}_-$ is the set of negatively dissipative points. We prove in Proposition 10 that $\mathcal{C}_-$ and $\mathcal{C}_+$ are always equal up to a set of zero volume. It is not true that for a general flow, the set of positively recurrent points is equal to the set of negatively recurrent points up to a set of volume zero.

A CHF (φ_t) is (positively) conservative when $\mathcal{C}_+ = M$ and dissipative when $\mathcal{C}_+$ has zero volume. We highlight a strong relation between the topology of $\{H = 0\}$ and the property of being convervative: when $\{H = 0\}$ has a neighbourhood V such that for every loop $\gamma : \mathbb{T} \to V$, $\int_\gamma \eta = 0$, then (φ_t^H) is conservative, Section 4.2. This contains the case when H doesn't vanish, Section 4.1. But there exist some examples of conservative CHF that are not in this case, Section 4.5. As the non-vanishing property is open in C^0-topology, we obtain C^0-open sets Hamiltonians H such that the associated CHF flows are conservative.

Among the conservative CHF, the Lee flows are those that correspond to the Hamiltonian $H = 1$ for some choice of representative (η, ω) of the conformally symplectic structure. They are an extension of the Reeb flows in the contact setting. We will provide in every dimension examples of Lee flows

• that are transitive, Section 4.3; this is different from the Hamiltonian symplectic case, where the level sets of H are preserved;
• that have no periodic orbits, Section 4.4; Weinstein conjecture in the contact setting and Arnol'd conjecture in the symplectic setting assert the existence of periodic orbits. This example emphasizes one difference between the CHF and the Reeb flows as well as the symplectic Hamiltonian flows.

We will give a 2-dimensional example of Lee flow that is minimal (Section 1.1.2), but we don't know if there is such an example in higher dimension.

We will give in Part 5.2.1 of Section 5.2 an example of dissipative CHF, with one normally hyperbolic attractor that is a Lagrangian submanifold, one normally hyperbolic repulsor that is also a Lagrangian submanifold and the remaining part of the manifold that is filled with heteroclinic connections. This gives a C^1-open set of CHF that are dissipative. See also Section 1.1.1.

There also exist C^1-open sets of CHF such that both $\mathcal{C}_+$ and $\mathcal{D}_+$ have positive volume. This happens when there is a normally hyperbolic periodic attractor and one non degenerate local minimum of $e^\theta H$ where θ is a local primitive of η.

Another feature of the conformally symplectic dynamics is that they preserve isotropy (this is even a characterization of these dynamics). This is a common point with symplectic dynamics and contact dynamics. Therefore, we extend or amend some classical results for the invariant submanifolds of Hamiltonian flows.

In Section 2.5, we prove that the CHF have a codimension 1 invariant foliation and explain in Section 6.2 the relation for a submanifold between being tangent to this foliation, being invariant and being isotropic (or coisotropic). This is reminiscent of Hamilton-Jacobi equation in the usual Hamiltonian setting.

We deduce that on a conformal cotangent bundle (see Section 2.3), a Lagrangian invariant graph is necessarily contained in the zero level set, which is a major difference with the usual Hamiltonian setting.

Motivated by the result of Herman in the exact symplectic setting, [6], which asserts that every invariant torus on which the dynamics is C^1-conjugate to a minimal rotation is isotropic, we consider tori $\mathcal{T}$ that are invariant by a CHF and such that the restricted dynamics is topologically conjugate to a rotation. In Section 6.2, we recall the definition of the asymptotic cycle of an invariant measure and introduce in a similar way the asymptotic cycle for flows on tori that are C^0-conjugate to a non necessarily minimal rotation. We prove that if the product of the cohomology class of the Lee form by the asymptotic cycle of $\mathcal{T}$ is non-zero, then $\mathcal{T}$ is isotropic. In particular, when the cohomology class of the Lee form is rational and when the rotation is minimal, the invariant torus is isotropic.

1.1. **2-dimensional examples.**

1.1.1. *A dissipative example.* Let us discuss a simple two-dimensional dissipative example that illustrates some of our results. Let $(M, \eta, \omega) = (\mathbb{T}^2, \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)$ where $\mathbb{T}^2$ denotes the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ and let $H : \mathbb{T}^2 \to \mathbb{R}$ be the Hamiltonian function $H(x, y) = \sin(2\pi y)$. We have pictured integral curves of the associated dynamics on Figure 1. In this figure, we see that the only level set of H that is
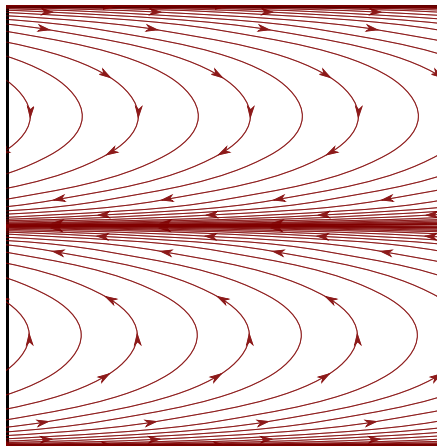
FIGURE 1. Dynamics of $H(x, y) = \sin(2\pi y)$ in the fundamental domain $[0, 1]^2$

preserved is $\{H = 0\}$ and that it has two connected components: an attractive circle and a repelling one. Such a picture can be drawn in any dimension: *if the Lee form is not exact, there exist Hamiltonian flows with attractive or repelling hyperbolic orbits* (*cf.* Proposition 24). Attractors (or repellers) are not necessarily contained in $\{H = 0\}$: lifting this dynamics on the cover $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\mathbb{Z}$, one could see any of the two cylinders bounding the two attracting circles as attractors. However, as we will see attractors always intersect $\{H = 0\}$. On Figure 1, we see that the attractor is winding in the x's direction. In general, *the Lee form is not exact in any neighborhood of the intersection of an attractor with $\{H = 0\}$* (*cf.* Corollary 9). In particular, an attractor cannot be finite and must intersect $\{H = 0\}$.

1.1.2. *A conservative example.* In the opposite direction, let us point out the existence of conformal Hamiltonian dynamics that preserve the symplectic form ω but the behavior of which nonetheless differs from the symplectic Hamiltonian case. As a simple 2-dimensional example, let us consider the 2-torus $\mathbb{T}^2$ endowed with its canonical area form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ once again. Let us fix $a, b \in \mathbb{R}$ and choose the Lee form $\eta := a\mathrm{d}x + b\mathrm{d}y$. The Hamiltonian flow of $H \equiv 1$, which is called the Lee flow associated to the representative of the conformally symplectic structure (the *gauge*) (η, ω), is $\varphi_t(x, y) = (x + bt, y - at)$. If a and b are rationally independent, this flow is minimal. This is a striking difference with autonomous Hamiltonian flows of symplectic manifolds, where trajectories are never dense and there usually are plenty of periodic orbits. In general, we prove that *there exist topologically transitive Lee flows in any dimension* and that *there exist Lee flows without periodic orbit in any dimension* (*cf.* Propositions 15 and 16). In both cases, the Lee form η is not completely resonant (*i.e.* the set of its integrals along the loops is a dense subgroup of $\mathbb{R}$), which is necessary in order to have dense trajectories. One could ask whether there always is a periodic orbit when η is completely resonant, but this question is harder than solving the Weinstein conjecture (*i.e.* the existence of a periodic orbit for any Reeb flow of a closed contact manifold).

1.2. **Structure of the article.**

- In section 2, we introduce the notions of conformal manifold and conformal Hamiltonian dynamics and prove some of their properties, Then we give some examples: the conformal cotangent bundle, the twisted conformal symplectization, and describe the invariant foliation.
- In section 3, we characterize the global conservative-dissipative decomposition of the dynamics in term of recurrence. We prove the almost everywhere coincidence of the behaviours in the past and in the future. We also prove that the boundedness of the winding number implies the existence of invariant measures. We also provide an example of orbits that are conservative and have a strange oscillating behaviour.
- In section 4, we give some topological conditions on $\{H = 0\}$ that imply that the dynamics is conservative. We gave some examples of such dynamics that are transitive and some others that have no periodic orbit. We give an example of a conservative dynamics for which the topological condition for $\{H = 0\}$ is not satisfied.
- In section 5, we begin by studying some ergodic measures whose support is dissipative. Then we give examples of dissipative dynamics with Lagrangian attractors and repulsors, and also examples with periodic attractors and repulsors. We also give sufficient conditions implying that some connected component of $\{H = 0\}$ cannot be an atttractor.
- In section 6, we give some condition that implies that a component of $\{H = 0\}$ is in the closure of a non-compact leaf of the invariant distribution. Then we study invariant submanifolds from different points of view: their position relatively to the invariant foliation, and when they are rotational tori, the relations between their asymptotic cycle and their isotropy.
- Finally, there is an appendix dealing with isotropic submanifolds.

2. Preliminaries

2.1. **Conformal symplectic manifolds.** Given a closed 1-form η, the associated Lichnerowicz-De Rham differential d_η is defined on the differential forms α by $\mathrm{d}_\eta\alpha := \mathrm{d}\alpha - \eta \wedge \alpha$. It satisfies $\mathrm{d}_\eta^2 = 0$ and $\mathrm{d}_{\eta+\mathrm{d}f}\alpha = e^f\mathrm{d}_\eta(e^{-f}\alpha)$. If $\mathrm{d}_\eta\alpha = 0$, one says that α is η-closed.

Given an even dimensional manifold M, a conformal symplectic structure is an equivalence class of couples (η,ω) where η is a closed 1-form of M and ω is a non-degenerate 2-form that is η-closed, two such couples (η_i,ω_i), $i \in \{1,2\}$, being equivalent if there exists a map $f : M \to \mathbb{R}$ such that $\eta_2 = \eta_1 + \mathrm{d}f$ and $\omega_2 = e^f\omega_1$. A conformal symplectic manifold is an even dimensional manifold M endowed with a conformal symplectic structure, we will often work with a specific representative (η,ω) and write (M,η,ω) the conformal symplectic manifold. A notion that does not depend on the specific choice of representative (η,ω) is called gauge invariant or well defined up to gauge equivalence. The closed 1-form η is called the Lee form of (M,η,ω), its cohomology class $[\eta] \in H^1(M;\mathbb{R})$ is gauge invariant. A conformal symplectomorphism $\varphi : (M_1,\eta_1,\omega_1) \to (M_2,\eta_2,\omega_2)$ is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*\eta_2 = \eta_1 + \mathrm{d}f$ and $\varphi^*\omega_2 = e^f\omega_1$ for some $f : M_1 \to \mathbb{R}$ (this notion is gauge invariant). When $\dim M \geqslant 4$, the second equality implies the first one.

Similarly to the symplectic case, a submanifold N of a conformal symplectic manifold (M,η,ω) is called isotropic if $TN \subset TN^\omega$, coistropic if $TN^\omega \subset TN$ and lagrangian if $TN = TN^\omega$ (where E^ω denotes the ω-orthogonal bundle of the bundle E), this notion is gauge invariant.

A symplectic manifold (M,ω) has a natural conformal symplectic structure $(0,\omega)$ (which is the same as $(0,\lambda\omega)$ for $\lambda \in \mathbb{R}^*$); conversely, a conformal symplectic structure (η,ω) comes from a symplectic structure if and only if η is exact.

2.2. **Hamiltonian dynamics.** Given a map $H : M \to \mathbb{R}$ defined on a conformal symplectic manifold (M,η,ω), we define its associated Hamiltonian vector field X by $\iota_X\omega = \mathrm{d}_\eta H$, conversely H is the Hamiltonian of X. When the cohomology class of η is not 0, then H is unique. This matching Hamiltonian-vector field does depend on the choice of representative (η,ω) but not the algebra of Hamiltonian vector fields: the previous vector field X is the same as the one induced by $e^f H$ for the Lee form $\eta+\mathrm{d}f$. One can extend this definition to time-dependent Hamiltonian maps but we will focus on autonomous Hamiltonian in this paper. When $H \equiv 1$, the associated vector field L^η is called the Lee vector field of η and its flow is called the Lee flow.

Let us assume that the vector field X associated with H is complete. Let us denote (φ_t) its flow and

$$r_t^H(x) := \int_0^t \eta(X \circ \varphi_s(x))\mathrm{d}s, \quad \forall x \in M, \forall t \in \mathbb{R}.$$

When the choice of H is clear, we set $r_t := r_t^H$.

Lemma 1. *Given a complete Hamiltonian flow (φ_t) on (M,η,ω) associated with a Hamiltonian H, for all $t \in \mathbb{R}$,*

$$\varphi_t^*\omega = e^{r_t}\omega, \ \varphi_t^*\mathrm{d}_\eta H = e^{r_t}\mathrm{d}_\eta H, \ H \circ \varphi_t = e^{r_t}H \ \text{and} \ \varphi_t^*\eta = \eta + \mathrm{d}r_t.$$

In particular, the level set $\{H = 0\}$ is invariant under the flow and $\frac{1}{H}\omega$ is an invariant 2-form on $\{H \neq 0\}$.

Proof. By taking the Lie derivative of ω,

$$\mathcal{L}_X\omega = \mathrm{d}(\mathrm{d}_\eta H) + \iota_X(\eta \wedge \omega) = \eta \wedge \mathrm{d}H + \eta(X)\omega - \eta \wedge (\iota_X\omega)$$

Thus,

$$\mathcal{L}_X \omega = \eta \wedge \mathrm{d}H + \eta(X)\omega - \eta \wedge \mathrm{d}H + \eta \wedge \eta H = \eta(X)\omega,$$

which implies the first equality of the statement.

We deduce that

$$
\begin{aligned}
\varphi_t^*(\mathrm{d}_\eta H) \quad &= \varphi_t^*(\iota_X \omega) \quad = \omega(X \circ \varphi_t, \mathrm{d}\varphi_t \cdot) = \omega(\mathrm{d}\varphi_t X, \mathrm{d}\varphi_t \cdot) \\
&= \iota_X(\varphi_t^* \omega) \quad = \iota_X(e^{r_t}\omega) = e^{r_t}\mathrm{d}_\eta H.
\end{aligned}
$$

Injecting X in $\iota_X \omega = \mathrm{d}_\eta H$, one gets $\mathrm{d}H \cdot X = \eta(X)H$, which implies that $h(t) :=$ $H \circ \varphi_t(x)$, for a fixed $x \in M$, satisfies $h'(t) = \eta(X \circ \varphi_t(x))h(t)$ and the third statement follows. Finally, the last statement is due to $\mathcal{L}_X \eta = \mathrm{d}(\eta(X))$. $\qquad\square$

We remark that the relations $\varphi_t^* \omega = e^{r_t}\omega$ and $\varphi_t^* \eta = \eta + \mathrm{d}r_t$ are also satisfied in the time-dependent setting. This indeed implies that conformal Hamiltonian diffeomorphisms are conformal symplectomorphisms.

2.3. Conformal cotangent bundles.
Given a manifold L endowed with a closed 1-form β, one can define a conformal symplectic structure on T^*L denoted $T_\beta^* L$ in the following way. Let $\pi : T^*L \to L$ be the cotangent bundle map and λ the associated Liouville form: $\lambda_{(q,p)} \cdot \xi := p(\mathrm{d}\pi \cdot \xi)$. The conformal structure defining $T_\beta^* L$ is $(\eta, \omega) := (\pi^*\beta, -\mathrm{d}_\eta \lambda)$. The neighborhood of the 0-section of $T_\beta^* L$ is a model of a neighborhood of a Lagrangian embedding of L pulling back the Lee form to β (see Section A.2).

Let us recall how one can canonically extend diffeomorphisms and flows of M to conformal symplectomorphisms and Hamiltonian flows of $T_\beta^* M$. Let $f : M \to N$ be a diffeomorphism, one can symplectically extend it to $\hat{f} : T^*M \to T^*N$ by the well-known formula:

$$\hat{f}(q,p) = \left(f(q), p \circ \mathrm{d}f_q^{-1}\right), \quad \forall(q,p) \in T^*M.$$

Now if the diffeomorphism $f : M \to N$ satisfies $f^*\beta = \alpha + \mathrm{d}r$, for closed 1-forms α, β and some map $r : M \to \mathbb{R}$, the extension $\hat{f} : T_\alpha^* M \to T_\beta^* N$ defined by

$$\hat{f}(q,p) = (f(q), e^{r(q)}p \circ \mathrm{d}f_q^{-1}), \quad \forall(q,p) \in T^*M$$

is conformally symplectic. Indeed, let us denote by π_M, π_N the associated cotangent bundle maps, λ_M, λ_N the associated Liouville forms. Then $\hat{f}^*\lambda_N = e^{r \circ \pi_M}\lambda_M$:

$$\left(\hat{f}^*\lambda_N\right)_{(q,p)} \cdot \xi = e^{r(q)}p \circ \mathrm{d}f^{-1} \circ \mathrm{d}\pi_N \circ \mathrm{d}\hat{f} \cdot \xi = e^{r(q)}p \circ \mathrm{d}\pi_M \cdot \xi$$

as $\pi_N \circ \hat{f} = f \circ \pi_M$. We deduce $\hat{f}^*(\mathrm{d}_{\pi_N^*\beta}\lambda_N) = e^{r \circ \pi_M}\mathrm{d}_{\pi_M^*\alpha}\lambda_M$:

$$
\begin{aligned}
\hat{f}^*(\mathrm{d}\lambda_N - \pi_N^*\beta \wedge \lambda_N) &= \mathrm{d}(e^{r \circ \pi_M}\lambda_M) - \pi_M^*(\alpha + \mathrm{d}r) \wedge (e^{r \circ \pi_M}\lambda_M) \\
&= e^{r \circ \pi_M}(\mathrm{d}\lambda_M - \pi_M^*\alpha \wedge \lambda_M).
\end{aligned}
$$

Now given a flow $f_t : M \to M$, with $f_0 = \mathrm{id}$, of associated vector field X_t, one has $f_t^*\beta = \beta + \mathrm{d}r_t$ with $r_t(q) := \int_0^t \beta(X_s \circ f_s(q))\mathrm{d}s$ so the associated conformal symplectic flow $(\hat{f}_t)$ is well-defined and one checks that it corresponds to the Hamiltonian flow of $H_t(q,p) = p(X_t(q))$.

2.4. Twisted conformal symplectizations.
A large class of conformal symplectic manifold that are non-symplectic is given by the conformal symplectizations of contact manifolds. Let (Y^{2n+1}, α) be a manifold endowed with a contact form α (*i.e.* a 1-form satisfying $\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$), its conformal symplectization $\mathcal{S}^{\mathrm{conf}}(Y, \alpha)$ is the manifold $Y \times S^1$ endowed with the structure $(\eta = -\mathrm{d}\theta, \omega = -\mathrm{d}_\eta(\pi^*\alpha))$ where

$S^1 = \mathbb{R}/\mathbb{Z}$ whereas $\theta : Y \times S^1 \to S^1$ and $\pi : Y \times S^1 \to Y$ are the canonical projections. The conformal symplectization only depends on the oriented contact distribution $\ker \alpha$. Indeed, when $(\eta', \omega') = (\eta - \mathrm{d}f, -\mathrm{d}_{\eta'}(e^{-f}\alpha))$, $(x, \theta) \mapsto (x, \theta - f(x))$ is a conformally symplectic diffeomorphism between (ω, η) and (ω', η').

Given a closed 1-form β of Y, we also define the β-twisted conformal symplectization of (V, α) by replacing η in the previous definition with $\eta = \pi^*\beta - \mathrm{d}\theta$, we denote it $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$. We check that ω is non-degenerate by showing that ω^{n+1} does not vanish:

$$(-1)^{n+1}\omega^{n+1} = (n+1)(\mathrm{d}\theta - \pi^*\beta) \wedge \pi^*\alpha \wedge (\mathrm{d}(\pi^*\alpha))^n = (n+1)\mathrm{d}\theta \wedge \pi^*(\alpha \wedge (\mathrm{d}\alpha)^n) \neq 0,$$

the second equality comes from the fact that $\beta \wedge \alpha \wedge (\mathrm{d}\alpha)^n = 0$ for a degree reason and the contact hypotheses implies the non-vanishing of the last expression. When the choice of the contact form α is clear, the couple $(\eta, \omega) = (\pi^*\beta - \mathrm{d}\theta, -\mathrm{d}_\eta(\pi^*\alpha))$ as well as the associated Lee vector field and Hamilton equations will be implicitly chosen or referred to as standard.

Let us show how the study of conformal Hamiltonian dynamics will also inform us about contact Hamiltonian dynamics , see also Proposition 23. We recall that the contact Hamiltonian vector field X associated with the contact Hamiltonian map $H : Y \to \mathbb{R}$ is defined by

$$(1) \qquad \begin{cases} \alpha(X) = H, \\ \iota_X \mathrm{d}\alpha = (\mathrm{d}H \cdot R)\alpha - \mathrm{d}H, \end{cases}$$

where R is the Reeb vector field defined by $\alpha(R) = 1$ and $\iota_R \mathrm{d}\alpha = 0$ (the Hamiltonian vector field associated with $H = 1$).

Lemma 2. *Let $H : Y \to \mathbb{R}$ be a contact Hamiltonian map of the contact manifold (Y, α) with fixed contact form α associated with the Reeb vector field R and let X be the associated Hamiltonian vector field. The conformal Hamiltonian vector field on $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$ associated with $\widetilde{H} : (x, \theta) \mapsto H(x)$ is*

$$\widetilde{X} = X \oplus (\beta(X) - \mathrm{d}H \cdot R)\partial_\theta \in TY \oplus TS^1.$$

In particular, the standard Lee vector field is $L := R \oplus \beta(R)\partial_\theta$.

Proof. Let us first derivate the expression of the Lee vector field L. Let $\eta := \pi^*\beta - \mathrm{d}\theta$ be the Lee form and $\omega := -\mathrm{d}_\eta(\pi^*\alpha)$ be associated symplectic form. Let us write $L = V \oplus f\partial_\theta$, V being a vector field of Y and $f : Y \to \mathbb{R}$. Since $\iota_L \omega = -\eta$, one has $\eta(L) = 0$, that is $f = \beta(V)$. Developing the Lee equation, one then gets

$$\iota_L \mathrm{d}(\pi^*\alpha) + \alpha(V)(\pi^*\beta - \mathrm{d}\theta) = \pi^*\beta - \mathrm{d}\theta.$$

By identification, $\alpha(V) = 1$ and $\iota_V \mathrm{d}\alpha = \beta - \alpha(V)\beta = 0$, therefore $V = R$. The general case is also deduced by identification, once we have remarked that

$$\eta(\widetilde{X}) = \omega(\widetilde{X}, L) = \mathrm{d}\widetilde{H} \cdot L - \widetilde{H}\eta(L) = \mathrm{d}H \cdot R.$$

$\square$

Therefore the conformal Hamiltonian flow (Φ_t) of $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$ lifting the contact Hamiltonian flow (φ_t) is $\Phi_t(x, \theta) = (\varphi_t(x), \theta + \rho_t(x) - r_t(x))$ where

$$r_t(x) = r_t^{\widetilde{H}}(x, \theta) = \int_0^t (\mathrm{d}H(\varphi_s(x)) \cdot R)\mathrm{d}s \quad \text{and} \quad \rho_t(x) = \int_0^t \beta(\partial_s \varphi_s(x))\mathrm{d}s.$$

The expression of r_t is consistent with the following general fact for conformal Hamiltonian vector fields X of (M, η, ω):

$$\eta(X) = \omega(X, L) = \mathrm{d}H \cdot L - \eta(L)H = \mathrm{d}H \cdot L,$$

where L is the Lee vector field. An isotropic embedding $i : L \hookrightarrow (Y, \alpha)$ is by definition an embedding such that $i^*\alpha = 0$, it is Legendrian when the dimension of L is maximal: $2 \dim L + 1 = \dim Y$. One can associate to every isotropic submanifold $L \subset Y$ the isotropic lift $L \times S^1 \subset \mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$. Therefore, dynamical properties of contact Hamiltonians can be deduced from properties of conformal Hamiltonians "by projection $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha) \to Y$". See Part 5.2.1 of section 5.2.

2.5. **The invariant distribution $\mathcal{F}$.** In the conformal setting, the Hamiltonian map H is not an integral of motion. But the (singular) distribution $\mathcal{F} := \ker \mathrm{d}_\eta H$ is still invariant since $\varphi_t^* \mathrm{d}_\eta H = e^{r_t} \mathrm{d}_\eta H$ (Lemma 1). Moreover, we have

$$\mathrm{d}(\mathrm{d}_\eta H) = \eta \wedge \mathrm{d}H = \eta \wedge \mathrm{d}_\eta H$$

hence by Frobenius theorem, at every regular point the Pfaffian distribution $\ker \mathrm{d}_\eta H$ is integrable.

However, in dynamical systems with dissipative behaviors, its regular leaves are often non-compact (the important exception being $\{H = 0\}$). Let us describe the major properties of $\mathcal{F}$.

Lemma 3. *If $\gamma : [0, 1] \to M$ is a path tangent to $\mathcal{F}$, then $H(\gamma(1)) = e^{\int_\gamma \eta} H(\gamma(0))$.*

Proof. Similarly to the proof of Lemma 1, $h := H \circ \gamma$ satisfies $h' = \eta(\dot\gamma)h$. $\qquad\square$

Corollary 4. *Every connected submanifold $L \subset M$ tangent to $\mathcal{F}$ is either included in $\{H = 0\}$ or in $\{H \neq 0\}$. In the case where the pull-back of the Lee form to L is not exact, L is included in $\{H = 0\}$.*

In the symplectic case, regular levels of H admit invariant volume forms (see *e.g.* [3, §I.8]), the following proposition generalizes this phenomenon.

Proposition 5. *Let (M, η, ω) be a $2n$-dimensional closed conformal symplectic manifold and $H : M \to \mathbb{R}$ a Hamiltonian, the flow of which is (φ_t). Let $i : \Sigma \hookrightarrow M$ be an embedded leaf tangent to $\mathcal{F}$. Then there exists a volume form μ of Σ such that $\varphi_t^* \mu = e^{(n-1)r_t} \mu$. Moreover, there exists a $(2n-1)$-form μ_0 on M such that $\mu = i^* \mu_0$ and $\mu_0 \wedge \mathrm{d}_\eta H = \omega^n$ in the neighborhood of Σ.*

Proof. Let μ_1 be a $(2n-1)$-form on M such that $i^* \mu_1$ is a volume form of Σ (which is oriented by $\mathrm{d}_\eta H$). By assumption, $(\mathrm{d}_\eta H)_x \neq 0$ for $x \in \Sigma$ whereas $i^* \mathrm{d}_\eta H = 0$ so $(\mu_1 \wedge \mathrm{d}_\eta H)_x \neq 0$ for every $x \in \Sigma$. Since Σ is an embedded leaf, there exists an open neighborhood U of Σ on which $\mu_1 \wedge \mathrm{d}_\eta H$ does not vanish. There exists $f : M \to \mathbb{R}$ that does not vanish on U such that $f\mu_1 \wedge \mathrm{d}_\eta H = \omega^n$ restricted to U. Let us show that $\mu_0 := f\mu_1$ and the volume form $\mu := i^* \mu_0$ are the desired forms.

Let us recall that $\mathcal{L}_X \omega = \eta(X)\omega$ and $\mathcal{L}_X \mathrm{d}_\eta H = \eta(X)\mathrm{d}_\eta H$ (Lemma 1). Let us apply $\mathcal{L}_X$ to the equation $\mu_0 \wedge \mathrm{d}_\eta H = \omega^n$:

$$(\mathcal{L}_X \mu_0) \wedge \mathrm{d}_\eta H + \mu_0 \wedge \eta(X)\mathrm{d}_\eta H = n\eta(X)\omega^n.$$

Therefore,

$$(\mathcal{L}_X \mu_0) \wedge \mathrm{d}_\eta H = (n-1)\eta(X)\mu_0 \wedge \mathrm{d}_\eta H$$

so that

$$i^*(\mathcal{L}_X \mu_0) = (n-1)\eta(X)i^* \mu_0.$$

Since the flow (φ_t) preserves Σ, $i^*(\mathcal{L}_X \mu_0) = \mathcal{L}_X(i^* \mu_0)$ and the conclusion follows. $\qquad\square$

Corollary 6. *Every embedded leaf of $\mathcal{F}$ outside $\{H = 0\}$ admits an invariant volume form*

Proof. Let μ be the volume form associated with Σ by Proposition 5. The volume form $\frac{\mu}{H^{n-1}}$ is invariant. $\qquad\square$

When the embedded leaf of $\mathcal{F}$ is not compact, this invariant volume can be unbounded.

3. A GLOBAL DECOMPOSITION OF THE PHASE SPACE: CONSERVATIVE VERSUS DISSIPATIVE

Let us introduce three notions of attractors that will be used in different parts of this article.

- An invariant compact subset $A \subset M$ is a *weak attractor* if there exists an open subset $U \supset A$, called a basin of attraction of A, such that $\bigcup_{x \in U} \omega(x) \subset A$ where $\omega(x)$ is the omega-limit set of x. The basin of attraction is not necessarily unique.
- A subset $A \subset M$ is a *strong attractor* if there exists an open subset $U \supset A$, such that $\forall t > 0, \varphi_t(\overline{U}) \subset U$ and $A = \bigcap_{t>0} \varphi_t(U)$ (which implies that A is compact and invariant).
- an invariant closed submanifold $N \subset M$ is *normally hyperbolically attractive* if there exists a tubular neighbourhood $V = j(N \times [-\varepsilon_0, \varepsilon_0])$ of N where $j : N \times [-\varepsilon_0, \varepsilon_0] \hookrightarrow M$ is an embedding, $\tau > 0$ and $a \in (0,1)$ such that $\varphi_\tau^H(V) \subset \mathrm{Int}(V)$ and if we denote $V_\epsilon = j(N \times [-\varepsilon, \varepsilon])$, then
$$\forall \varepsilon \in (0, \varepsilon_0], \varphi_\tau^H(V_\varepsilon) \subset V_{a\varepsilon}.$$

Observe that a strong attractor is always a weak attractor.

3.1. The conservative-dissipative decomposition. We defined in the introduction the partition in invariant sets $M = \mathcal{C}_+ \sqcup \mathcal{D}_+$ with

$$(2) \quad \mathcal{C}_+ := \left\{ x \in M \mid \liminf_{t \to +\infty} |r_t(x)| < +\infty \right\} = \left\{ x \in M \mid \liminf_{p \to +\infty} |r_p(x)| < +\infty \right\}$$

and

$$(3) \qquad\qquad \mathcal{D}_+ := \left\{ x \in M \mid \lim_{t \to +\infty} |r_t(x)| = +\infty \right\}.$$

The second definition of $\mathcal{C}_+$ is to be understood with $p \in \mathbb{N}$; the equality between both definitions is due to $|\partial_t r_t(x)| \leqslant \|\eta(X)\|_\infty < +\infty$ (see Lemma 1).

Proposition 7. *Up to a set with zero Lebesgue measure, the set of positively recurrent points coincides with $\mathcal{C}_+$. The ω-limit set of every point in $\mathcal{D}_+$ is in $\{H = 0\}$. Almost every point in $\mathcal{D}_+$ is in $\{H \neq 0\}$ and if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, $r_t(x) \to -\infty$ as $t \to +\infty$ and every neighbourhood of $\omega(x) = A$ contains a closed curve γ such that $\int_\gamma \eta \neq 0$. Hence A is infinite.*
Moreover, for every embedded leaf Σ included in $\{H \neq 0\}$ with a proper inclusion map, up to a set with zero $(n-1)$-dimensional volume, $\mathcal{C}_+ \cap \Sigma$ coincides with the set of positively recurrent points in Σ.

Proof. Let us first remark that ω^n-almost every point of $\{H = 0\}$ is trivially recurrent and in $\mathcal{C}_+$: every point of the subset $\{H = 0\} \cap \{\mathrm{d}H = 0\}$ is fixed by the dynamics whereas $\{H = 0\} \cap \{\mathrm{d}H \neq 0\}$ is negligible.

Since $H \circ \varphi_t = e^{r_t} H$ by Lemma 1, a point $x \in M$ satisfying $r_t(x) \to +\infty$ must be in $\{H = 0\} \cap \{\mathrm{d}H \neq 0\}$ which is a negligible set. Hence if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, $r_t(x) \to -\infty$ as $t \to +\infty$ and then $\lim_{t\to\infty} H(\varphi_t x) = 0$ and x is not positively recurrent.

Let us now show that almost every point of $\mathcal{C}'_+ := \mathcal{C}_+ \cap \{H \neq 0\}$ is recurrent. According to Lemma 1, $H \circ \varphi_t = e^{r_t} H$, so

$$\mathcal{C}'_+ = \left\{ x \in M \mid \limsup_{p \to +\infty} |H(\varphi_p(x))| \neq 0 \right\}.$$

For $k \in \mathbb{N}^*$, let us define the following compact sets

$$\mathcal{H}_k := \left\{ x \in M \mid |H(x)| \geqslant \frac{1}{k} \right\}.$$

Then $\mathcal{C}'_+$ is the increasing union of the $\mathcal{C}'_k$'s defined by

$$\mathcal{C}'_k := \mathcal{H}_k \cap \bigcap_{N \in \mathbb{N}} \bigcup_{p \geqslant N} \varphi_p^{-1}(\mathcal{H}_k), \quad \forall k \in \mathbb{N}^*.$$

For each $k \in \mathbb{N}^*$, there is a well-defined first-return measurable map $f_k : \mathcal{C}'_k \circlearrowleft$, $f_k(x) := \varphi_{n(x)}(x)$ where $n(x) := \min\{p \in \mathbb{N}^* \mid \varphi_p(x) \in \mathcal{H}_k\}$. Since the 2-form $\frac{\omega}{H}$ of $\{H \neq 0\}$ is preserved by φ_p for all $p \in \mathbb{N}$ (Lemma 1), the measurable maps f_k's are preserving the measure $\nu : A \mapsto \int_A \frac{\omega^n}{H^n}$. Since, for $k \in \mathbb{N}^*$, $\mathcal{C}'_k$ has a countable basis of open sets and a measure $\nu(\mathcal{C}'_k) \leqslant \nu(\mathcal{H}_k) \leqslant k^n \omega^n(M)$ which is finite, the Poincaré's recurrence theorem implies that almost every point of $\mathcal{C}'_k$ is recurrent for f_k.

Let us prove that if $x \in \mathcal{D}_+ \cap \{H \neq 0\}$, every neighbourhood V of $\omega(x)$ contains a closed curve γ such that $\int_\gamma \eta \neq 0$. By compacity of $\omega(x)$, one can assume that V is a finite union of path-connected contractible open sets V_j. Let $K > 0$ be such that $|\int_\gamma \eta| < K$ for every $\gamma : [0,1] \to V_j$ and every j (where η denotes the Lee form). Let $T > 0$ be such that for all $t \geqslant T$, $\varphi_t(x) \in V$ and let j_0 be such that there exist arbitrarily large t's satisfying $\varphi_t(x) \in V_{j_0}$. Let $t_1 > t_0 > T$ be such that $r_{t_0}(x) - r_{t_1}(x) > K$ and $\varphi_{t_i}(x) \in V_{j_0}$ for $i \in \{0,1\}$. Then concatenating $t \mapsto \varphi_t(x)$, $t \in [t_0, t_1]$, with a path of V_{j_0} connecting $\varphi_{t_1}(x)$ to $\varphi_{t_0}(x)$, one gets a loop $\gamma : I \to V$ satisfying $\int_\gamma \eta \neq 0$. The conclusion follows.

Finally, let $\Sigma \subset \{H \neq 0\}$ be an embedded leaf of $\mathcal{F}$ with a proper inclusion map. We have seen that no point in $\mathcal{D}_+ \cap \Sigma$ is positively recurrent. Since $\mathcal{C}_+ \cap \Sigma = \mathcal{C}'_+ \cap \Sigma$, it is enough to prove that almost every point of $\mathcal{C}'_k \cap \Sigma$ is recurrent, for all $k \in \mathbb{N}^*$. Let μ be the volume form associated with Σ by Proposition 5, then the first return maps $f_k|_{\Sigma \cap \mathcal{C}'_k}$ are preserving the measure $\nu_\Sigma : A \mapsto \int_A \frac{\mu}{H^{n-1}}$ defined on Σ. Since $\Sigma \hookrightarrow \{H \neq 0\}$ is proper, the $\Sigma \cap \mathcal{C}'_k$'s are compact, so the $\nu_\Sigma(\Sigma \cap \mathcal{C}'_k)$'s are finite. The conclusion follows. □

We recall that $U \subset M$ is a wandering set if $\exists T > 0, \forall t \geqslant T, \varphi_t(U) \cap U = \varnothing$.

Corollary 8. *Let U be a wandering set. Then almost every point of U belongs to $\mathcal{D}_+ \cap \mathcal{D}_-$ and satisfies $\lim_{t \to +\infty} H(\varphi_t^H(x)) = \lim_{t \to -\infty} H(\varphi_t^H(x)) = 0$.*

Corollary 9. *Let A be a weak attractor with basin U, then for almost every point x of $U \backslash A$, $r_t(x) \to -\infty$ as $t \to +\infty$. In particular, the Lee form is not exact in any neighborhood of $A \cap \{H = 0\}$.*

As a consequence, an attractor (or repeller) of (φ_t) is never a finite set.

Proof of Corollary 9. By definition, points of $U \backslash A$ are not recurrent so almost every point of $U \backslash A$ is in $\mathcal{D}_+$ by Proposition 7. The same proposition implies the other results. □

3.2. **Almost everywhere coincidence of past and future.** In the remaining of the article, we will say that a property is almost everywhere satisfied with reference to every volume form. We have of course that $\mathcal{C}_-$ coincides with the set of negatively recurrent points. What is surprising is that the set of negatively recurrent points coincide with the set of positively recurrent points up to a set with zero volume.

Proposition 10. *The sets $\mathcal{C}_+$ and $\mathcal{C}_-$ coincide up to a set with zero volume. Moreover, for every embedded leaf Σ included in $\{H \neq 0\}$ with a proper inclusion map, up to a set with zero $(n-1)$-dimensional volume, $\mathcal{C}_+ \cap \Sigma$ coincides $\mathcal{C}_- \cap \Sigma$.*

Proof. We will prove that up to a set with zero volume $\mathcal{C}_+ \subset \mathcal{C}_-$ and we will deduce the first part of Proposition 10. We keep the notation of the proof of Proposition 7. The first return map $f_k : \mathcal{C}'_k \circlearrowleft$ preserves the finite volume $\frac{1}{H^n}\omega^n$, hence almost every point of $\mathcal{C}'_k$ is negatively recurrent for f_k. This implies that up to a set with zero volume, $\mathcal{C}'_+$ and hence $\mathcal{C}_+$ is a subset of $\mathcal{C}_-$.

The proof of the last part is similar. $\qquad\square$

3.3. **An example where $\mathcal{C}_+ \neq \mathcal{C}_-$.** Adapting the construction made in Section 3.5 and the shadowing lemma, it is not hard to obtain an orbit that is negatively dissipative and positively conservative, i.e. such that $\mathcal{C}_- \neq \mathcal{C}_+$.

3.4. **Boundedness of r_t and invariant measures.** Let us assume that $L \subset M$ is an invariant measurable set of the dynamics on which $(t,x) \mapsto r_t(x)$ is a bounded map $\mathbb{R} \times L \to \mathbb{R}$ (in particular $L \subset \mathcal{C}_+ \cap \mathcal{C}_-$). Inspired by the proof of [7, Theorem 5.1.13], let us define the bounded measurable map $h : L \to \mathbb{R}$,

$$(4) \qquad h(x) := \sup_{t \in \mathbb{R}} r_t(x).$$

Then $h \circ \varphi_t = h - r_t$, so that for instance Lemma 1 implies that, restricted to L,

$$\varphi_t^*(e^h\omega) = e^h\omega, \ \ \varphi_t^*(e^h \mathrm{d}_\eta H) = e^h \mathrm{d}_\eta H \text{ and } (e^h H) \circ \varphi_t = e^h H, \quad \forall t \in \mathbb{R}.$$

Corollary 11. *An invariant measurable set $L \subset M$ of positive measure on which $(t,x) \mapsto r_t(x)$ is bounded admits an invariant Borel measure of positive density.*

Proof. The measure $A \mapsto \int_A e^{nh}\omega^n$ is positive and invariant. $\qquad\square$

Corollary 12. *If L is an embedded leaf of $\mathcal{F}$ on which $(t,x) \mapsto r_t(x)$ is bounded, then it admits an invariant Borel measure of positive density.*

Proof. The desired measure is $A \mapsto \int_A e^{(n-1)h}\mu$, where μ is given by Proposition 5. $\qquad\square$

We will see in Section 4.2 that when η is exact in the neighborhood of $\{H = 0\}$, the flow is conservative and Corollaries 11 and 12 apply.

One of the dynamical importance of these corollaries is signified by Poincaré's recurrence theorem: if the invariant measures in question are also finite, almost every point of the invariant set is recurrent. However, with the exception of regular leafs of $\mathcal{F}$ included in $\{H = 0\}$, the recurrence can also be deduced from Proposition 7.

3.5. **An oscillating behavior.** In this subsection, we give an example of a flow possessing orbits included in $\mathcal{C}_+$ that are in $\{H \neq 0\}$, positively recurrent and whose ω-limit set intersects $\{H = 0\}$. Hence they have an unbounded associated winding $t \in [0, +\infty) \mapsto r_t(x)$.

Proposition 13. *There exists a Hamiltonian map $H : M \to \mathbb{R}$ on some conformal symplectic manifold, the flow of which satisfying*

$$\limsup_{t \to +\infty} r_t(x) > \liminf_{t \to +\infty} r_t(x) = -\infty,$$

for some point $x \in \{H \neq 0\}$.

In order to prove this proposition, let us briefly recall the statement of the Shadowing Lemma for flows (see *e.g.* [7, Theorem 18.1.6]). Let (φ_t) be a smooth flow on a Riemannian manifold M, the infinitesimal generator of which is X_t. A differentiable curve $c : I \to \mathbb{R}$, $I \subset \mathbb{R}$ interval, is called an ε-orbit if $\|\dot{c}(t) - X_t(c(t))\| \leqslant \varepsilon$ for all $t \in I$. A differentiable curve $c : I \to \mathbb{R}$ is said to be δ-shadowed by the orbit $(\varphi_t(x))_{t \in J}$ if there exists $s : J \to I$ with $|s' - 1| < \delta$ such that $d(c(s(t)), \varphi_t(x)) < \delta$ for all $t \in J$ (d denoting the Riemannian distance). The Shadowing Lemma states that, given a hyperbolic set Λ of (φ_t), there is a

neighborhood $U \supset \Lambda$, so that for every $\delta > 0$ there is an $\varepsilon > 0$ such that every ε-orbit included in U is δ-shadowed by an orbit of (φ_t).

Proof. Let Σ be a closed hyperbolic surface, let us denote $\pi : T^1\Sigma \to \Sigma$ the associated unit tangent bundle and let β be a non-exact closed 1-form of Σ. Let us denote (G_t) the geodesic flow on $T^1\Sigma$ and X the associated vector field. Let (M, η, ω) be a conformal symplectic closed manifold associated with $(T^1\Sigma, \pi^*\beta)$ by Lemma 35 in Appendix A: that is one may assume that $L := T^1\Sigma \times S^1$ is a Lagrangian submanifold of M and that the restriction of η to this submanifold is $\alpha := \pi^*\beta - \mathrm{d}\theta$ (we identify $\pi^*\beta$ with its pull-back by the projection by a slight abuse of notation). Let W be a Weinstein neighborhood of L: identifying the 0-section of $T^*_\alpha L$ with L, one can see W as a neighborhood of the 0-section of $T^*_\alpha L$ (see [2, Theorem 2.11] or Section A.2). Let us identify the vector field X of $T^1\Sigma$ with the vector field $X \oplus 0$ of L and let $H : M \to \mathbb{R}$ be a Hamiltonian function satisfying $H(q, p) = p(X(q))$ on $W \subset T^*_\alpha L$ (shrinking W if necessary). Let us prove that H satisfies the statement of the proposition.

Let us first find an orbit $(\gamma, \dot\gamma) : \mathbb{R}_+ \to T^1\Sigma$ of the geodesic flow (G_t) such that

$$(5) \qquad \begin{cases} \exists K > 0, \forall t > 0, \quad \int_{\gamma|[0,t]} \beta \leqslant K, \\ \limsup_{t \to +\infty} \int_{\gamma|[0,t]} \beta > \liminf_{t \to +\infty} \int_{\gamma|[0,t]} \beta = -\infty. \end{cases}$$

Such an orbit can be found applying the Shadowing Lemma to (G_t). Indeed, let us fix $\delta \in (0, 1)$ and take an $\varepsilon > 0$ associated by the Shadowing Lemma. Let $a : \mathbb{R}/T\mathbb{Z} \to \Sigma$ be a closed geodesic of unit speed such that $\int_a \beta > 0$ (up to reparametrization, such an a can be obtained as a minimizer of the energy functional among loops homotopic to a loop b satisfying $\int_b \beta > 0$). By topological transitivity of (G_t), there exists an $\varepsilon/2$-orbit $(c, \dot c) : [0, T'] \to T^1\Sigma$ such that $\dot c(0) = \dot a(0)$ and $\dot c(T') = -\dot a(0)$. By successively concatenated c or c^{-1} with higher and higher iterations of a and a^{-1}, one can thus build an ε-orbit $(\tilde\gamma, \dot{\tilde\gamma}) : \mathbb{R}_+ \to T^1\Sigma$ satisfying conditions (5) where γ is replaced with $\tilde\gamma$. The Shadowing Lemma applied to this ε-orbit gives us the desired γ.

According to Section 2.3, on $W \subset T^*_\alpha L$, the Hamiltonian flow (φ_t) of H takes the form

$$\varphi_t(q, p; z) = (G_t(q), e^{r_t(q)} p \circ (\mathrm{d}G_t(q))^{-1}; z),$$

where $(q, p) \in T^*(T^1\Sigma)$, $z \in T^*S^1$ and

$$r_t(q) := \int_0^t \pi^*\beta(\partial_s G_s(q))\mathrm{d}s,$$

as long as $\varphi_s(q, p; z)$ stays inside W for $s \in [0, t]$. As (G_t) is Anosov, one has the bundle decomposition $T(T^1\Sigma) = E^s \oplus \mathbb{R}X \oplus E^u$ which is preserved by (G_t) with $\mathrm{d}G_t \cdot X = X \circ G_t$. Let $q \mapsto P_q$ be the section of $T^*(T^1\Sigma)$ vanishing on $E^s \oplus E^u$ and such that $P(X) \equiv 1$; it satisfies $P_q \circ (\mathrm{d}G_t(q))^{-1} = P_{G_t(q)}$ for all q. For fixed $z \in T^*S^1$ and $\lambda > 0$, let us consider the $\mathbb{R}_+$-orbit generated by $(\dot\gamma(0), \lambda P_{\dot\gamma(0)}; z)$ (where γ satisfies (5)). By the first condition of (5), $r_t(\dot\gamma(0))$ is bounded from above so this orbit keeps inside W for a sufficiently small λ. The second condition of (5) implies the statement for $x = (\dot\gamma(0), \lambda P_{\dot\gamma(0)}; z)$ (the orbit is in $\{H \neq 0\}$ since $P(X) \equiv 1$). $\qquad\square$

4. GLOBAL CONSERVATIVE BEHAVIORS

As we have seen in Corollary 9, a necessary condition for attractors to appear is the non-exactness of the Lee form in the neighborhood of $\{H = 0\}$. Here, we study the opposite case: Hamiltonian flow (φ_t) of H on a closed conformal symplectic manifold (M^{2n}, η, ω) in the case where η is exact in the neighborhood

of the invariant set $\{H = 0\}$. That is, we assume that there exists an open set U containing $\{H = 0\}$ such that $[\eta|_U] = 0$ in $H^1(U; R)$. This hypotheses is thus gauge invariant.

4.1. **When H does not vanish.** Let us first assume that H does not vanish and denote X_H^η its associated vector field for the Lee form η. Possibly reversing time, we will assume that H is positive. Since $X_H^\eta = X_{e^f H}^{\eta + \mathrm{d}f}$, by setting $f = -\log \circ H$, we see that X_H^η is the Lee vector field of $\eta' = \eta - \mathrm{d}(\ln \circ H)$. Therefore, Hamiltonian flows of non-vanishing H are Lee flows.

We now assume that $H \equiv 1$ for the choice of gauge (η, ω), so that the vector field is L^η. Since $\eta(L^\eta) = 0$, $r_t \equiv 0$ and $\mathcal{C}_+ = \mathcal{C}_- = M$, i.e. the flow is positively and negatively conservative with the terminology given in the introduction. Thus almost every point is positively and negatively recurrent.

Lemma 1 implies that ω is preserved by the flow. Let us point out that this flow is not conjugated to a symplectic flow in general since one can have $H^2(M; \mathbb{R}) = 0$ (*e.g.* the conformal symplectization of the contact sphere $(\mathbb{S}^{2n-1}, \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x))$). The volume form ω^n is preserved so almost every point is recurrent according to Poincaré's recurrence theorem. More precisely, almost every point of a proper embedded leaf of $\mathcal{F}$ is recurrent according to Proposition 7 and Corollary 12. Let us remark that in the case where η is completely resonant, (*i.e.* the subgroup $\{\int_\gamma \eta; \gamma : S^1 \to M\}$ is discrete), there exists $k \in \mathbb{R}^*$ and a map $\theta : M \to \mathbb{R}/k\mathbb{Z}$ such that $\eta = \mathrm{d}\theta$ and the invariant foliation $(\theta^{-1}(s))_{s \in \mathbb{R}/k\mathbb{Z}}$ has compact leafs.

4.2. **When η is exact in the neighborhood of $\{H = 0\}$.** Let us move on to the general case: there exists an open neighborhood U of $\{H = 0\}$ on which $\eta|_U = \mathrm{d}\theta$ for some $\theta : U \to \mathbb{R}$.

Proposition 14. *Under the hypotheses of this section, the map $(t, x) \mapsto r_t(x)$ is bounded on $\mathbb{R} \times M$.*

Proof. Let $\varepsilon_0 > 0$ be such that the neighborhood $V_0 := H^{-1}([-\varepsilon_0, \varepsilon_0])$ is included in U and let $V := H^{-1}([-\varepsilon, \varepsilon])$ for some $\varepsilon \in (0, \varepsilon_0)$. Let $A := \max_{V_0} \theta - \min_{V_0} \theta$, $b := \inf_{M \setminus V} |H|$ and $B := \sup_{M \setminus V} |H|$. We will show that

$$|r_t(x)| \leqslant 2A + \log(B/b), \quad \forall (t, x) \in \mathbb{R} \times M.$$

Let $(t, x) \in \mathbb{R} \times M$, if $\varphi_s(x) \in V_0$ for all $s \in [0, t]$ ($[t, 0]$ if $t < 0$), then $|r_t(x)| \leqslant A$. If $x \in M \setminus V$ and $\varphi_t(x) \in M \setminus V$, then $b/B \leqslant |H(\varphi_t(x))/H(x)| \leqslant B/b$ so $|r_t(x)| \leqslant \log(B/b)$ according to Lemma 1. If $x \in V$ and $\varphi_t(x) \notin V$, we assume $t > 0$, let $t_0 \in [0, t]$ such that $\varphi_s(x) \in V_0$ for all $s \in [0, t_0]$ and $\varphi_{t_0}(x) \in M \setminus V$, then $|r_{t_0}(x)| \leqslant A$ whereas $|r_{t-t_0}(\varphi_{t_0}(x))| \leqslant \log(B/b)$ by the above case, implying $|r_t(x)| \leqslant A + \log(B/b)$. The same is symmetrically true for $t < 0$ and with x and $\varphi_t(x)$ intertwined. The last case is when $x \in V$ and $\varphi_t(x) \in V$, we assume $t > 0$ (the other case is symmetrical), and $\varphi_{s_0}(x) \notin V_0$ for some $s_0 \in [0, t]$. One can find $t_1 < s_0 < t_2$ such that $H(\varphi_{t_1}(x)) = H(\varphi_{t_2}(x)) = \varepsilon$ and $\varphi_s(x) \in V$ for $s \in [0, t_1] \cup [t_2, t]$. By Lemma 1, $r_{t_2}(x) = r_{t_1}(x)$ and the conclusion follows from the first case treated. $\qquad\square$

Therefore, according to the conservative-dissipative decomposition of Section 3.1, $M = \mathcal{C}_- = \mathcal{C}_+$ and almost every points of M or an embedded leaf of $\mathcal{F}$ in $\{H \neq 0\}$ is positively and negatively recurrent. Moreover, according to Corollaries 11 and 12, M and every embedded leaf of $\mathcal{F}$ in $\{H \neq 0\}$ admits an invariant Borel measure of positive density. In particular, almost every point of a closed regular leaf of $\mathcal{F}$ in $\{H = 0\}$ is also positively and negatively recurrent.

4.3. **A topologically transitive Lee flow.** Our goal is to provide examples of topologically transitive Lee flow in every dimension.

We have seen in Section 1.1.2 that in dimension 2, there are very simple examples of minimal Lee flow. We recall it. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the 2-torus with canonical coordinates $x, y \in \mathbb{R}/\mathbb{Z}$. Let us fix $a, b \in \mathbb{R}$ and endow $\mathbb{T}^2$ with the conformal symplectic structure $(\eta, \omega) = (a\mathrm{d}x + b\mathrm{d}y, \mathrm{d}x \wedge \mathrm{d}y)$, the Lee flow of which is $\varphi_t(x, y) = (x + bt, y - at)$. This flow is minimal if and only if a and b are rationally independent.

One way to extend this example is to remark that in the case $a = -1$, it corresponds to the β-twisted conformal symplectization of the contact manifold $(S^1, \mathrm{d}y)$ with $\beta = b\mathrm{d}y$.

Proposition 15. *Let (Y, α) be a closed connected contact manifold with a fixed contact form, the Reeb flow of which is Anosov and possess a periodic orbit γ such that $\int_\gamma \beta$ is irrational for some closed 1-form β. Then, the standard Lee flow of $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha)$ is topologically transitive.*

Such a (Y, α) can be found in every dimension. Indeed, let N be a closed Riemannian manifold with negative sectional curvature and a non-trivial real homology group of degree 1: $H^1(N; \mathbb{R}) \neq 0$. Let thus β' be a non-exact closed 1-form such that $\int_c \beta'$ is irrational for some loop c. Let $\pi : T^1N \to N$ be the unit sphere bundle of N endowed with its usual contact structure (the Reeb flow of which is the geodesic flow), the β-twisted conformal symplectization of $Y := T^1N$ with $\beta := \pi^*\beta'$ follows the hypothesis of the statement. Indeed, by taking a minimum of the energy functional among loops homotopic to c, one gets a closed geodesic homotopic to c, so a periodic orbit γ of the geodesic flow such that $\int_\gamma \beta = \int_c \beta'$.

However, Lee flows induced by these choices of (Y, α) have a lot of periodic orbits in dimension $2n \geqslant 4$ so are not minimal.

Proof of Proposition 15. Let (φ_t) be the Reeb flow of Y. According to Lemma 2, the Lee flow (Φ_t) of $\mathcal{S}_\beta^{\mathrm{conf}}(Y, \alpha) = Y \times S^1$ takes the following form: for all $(x, \theta) \in Y \times S^1$, $t \in \mathbb{R}$,

$$\Phi_t(x, \theta) = (\varphi_t(x), \theta + \rho_t(x)), \text{ where } \rho_t(x) := \int_0^t \beta(\partial_s\varphi_s(x))\mathrm{d}s.$$

In order to show topological transitivity, it is enough to prove that for every pair of product non-empty open sets $U_i \times V_i \subset Y \times S^1$, $i \in \{1, 2\}$, there is some $(x, \theta) \in U_1 \times V_1$ and some $t \in \mathbb{R}$ such that $\Phi_t(x, \theta) \in U_2 \times V_2$ (see *e.g.* [7, Lemma 1.4.2]). We can assume that the V_i are arcs of length $\ell > 0$.

By hypothesis, there exists a point $y \in Y$ such that $\varphi_{t_2}(y) = y$ for some $t_2 > 0$ and $\rho_{t_2}(y)$ is irrational.

We choose $x_i \in U_i$ for $i = 1, 2$. Let $\delta > 0$ be so small that if a curve $c : [a, b] \to Y$ with $c(a) = x_1$ and $c(b) = x_2$ is δ-shadowed by an orbit $\nu : [a', b'] \to Y$, then $\nu(a') \in U_1$, $\nu(b') \in U_2$ and $|\int_c \beta - \int_\nu \beta| < \ell/3$.

By assumption, (φ_t) is a topologically transitive Anosov flow (contact Anosov flows on connected manifolds are topologically mixing [7, Theorem 18.3.6]).

Let $\varepsilon > 0$ be associated with δ by the Shadowing Lemma. By transitivity, there exist $\varepsilon/2$-orbits $c_1 : [0, t_1] \to Y$, $c_3 : [0, t_3] \to Y$, with $c_1(0) = x_1$, $c_1(t_1) = y$ and $c_3(0) = y$, $c_3(t_3) = x_2$. Then (up to a small deformation at the connecting points) the concatenated paths $\nu_k := c_1 \cdot c_2^k \cdot c_3$, $k \in \mathbb{N}$, are ε-orbits for (φ_t). Let $R_k := \int_{\nu_k} \beta \mod 1 \in S^1$. Since $\int_{c_2} \beta$ is irrational, (R_k) is dense in S^1. Let us fix $k \in \mathbb{N}$ such that $\theta + [R_k - \ell/3, R_k + \ell/3] \in V_2$ for some $\theta \in V_1$ (the length of the arcs V_i's being $> \ell$). Applying the Shadowing Lemma to ν_k, we find an orbit $\nu : [0, T] \to Y$ such that $\nu(0) \in U_1$, $\nu(T) \in U_2$ and $\int_\nu \beta \mod 1$ is $\ell/3$-close to R_k so that $\theta + \int_\nu \beta \in V_2$. $\square$

4.4. **A Lee flow with no periodic orbit.** Relaxing the transitivity hypothesis, one can easily produce a Lee flow without periodic orbit.

Proposition 16. *Let $\mathbb{T}^n$ be the flat n-torus with canonical coordinates $x_i \in \mathbb{R}/\mathbb{Z}$, and let $\beta := a_1 \mathrm{d}x_1 + \cdots + a_n \mathrm{d}x_n$ for some fixed $a_i \in \mathbb{R}$. The standard Lee flow of $\mathcal{S}_\beta^{\mathrm{conf}}(T^1\mathbb{T}^n)$ does not have any periodic orbit if and only if the family $(1, a_1, \ldots, a_n)$ is rationally independent.*

This flow is clearly not minimal and, in general, there is not much hope for the standard Lee flow of a closed twisted conformal symplectization to be minimal in dimension $2n \geqslant 4$. Indeed, the Weinstein conjecture states that every Reeb flow of a closed contact manifold (Y, α) should possess a closed orbit γ, so $\gamma \times S^1$ would be a closed invariant set of the standard Lee flow of the twisted conformal symplectizations of (Y, α).

Proof of Proposition 16. The Reeb flow of $T^1\mathbb{T}^n \simeq \mathbb{T}^n \times \mathbb{S}^{n-1}$ is $\varphi_t(x, v) := (x + tv, v)$. The associated $\rho_t(x, v) := \int_0^t \beta(\partial_s \varphi_s(x, v)) \mathrm{d}s \bmod 1$ satisfies $\rho_t(x, v) = \sum_i a_i t v_i \bmod 1$. Therefore, a point $(x, v, \theta) \in T^1\mathbb{T}^n \times S^1$ is a τ-periodic point of the Lee flow if and only if $\tau v \in \mathbb{Z}^n$ and $\sum_i a_i \tau v_i \in \mathbb{Z}$. $\square$

4.5. **A conservative behavior with $\eta_{|\{H=0\}}$ non exact.**

Proposition 17. *Let $\mathbb{T}^n$ be the flat n-torus with canonical coordinates $q_i \in \mathbb{R}/\mathbb{Z}$, and let us consider on $\mathcal{S}^{\mathrm{conf}}(T^1\mathbb{T}^n)$ the Hamiltonian $H(q_1, \ldots, q_n, p_1, \ldots, p_n, \theta) = p_1$. The Hamiltonian flow is*

$$\varphi_t^H(q_1, \ldots, q_n, p_1, \ldots, p_n, \theta) = (q_1 + t, q_2, \ldots, q_n, p_1, \ldots, p_n, \theta).$$

This flow preserves the conformal 2-form and the zero level $\{H = 0\}$ contains a loop γ such that

$$\int_\gamma \eta \neq 0.$$

Proof. The contact form is the restriction of the Liouville 1-form λ to $T^1\mathbb{T}^n$ and the Reeb vector field R at (q, p) is $(p, 0)$. Hence $dH \cdot R = 0$ and the contact Hamiltonian flow X_H satisfies $\iota_{X_H} d\lambda = -dp_1$ and $X_H = (1, 0, \ldots, 0)$. As $dH \cdot R = 0$, we deduce from Lemma 2 that the conformal Hamiltonian vector field is $(1, 0, \ldots, 0)$. $\square$

5. DISSIPATIVE BEHAVIORS

5.1. **Dissipative ergodic measures.** Let ν be an ergodic measure and let us denote

$$\bar{r}(\nu) := \int_M \eta(X) \mathrm{d}\nu.$$

The following proposition is the ergodic counterpart of Corollary 4.

Proposition 18. *Given an ergodic measure ν, $\nu(\{H = 0\}) \in \{0, 1\}$ and in the case where $\bar{r}(\nu) \neq 0$, the support of ν is included in $\{H = 0\}$.*

Proof. The first statement is obvious since $\{H = 0\}$ is an invariant set. If $\bar{r}(\nu) \neq 0$ and $\mathrm{supp}\,\nu \not\subseteq \{H = 0\}$, there exists $x \in \{H \neq 0\}$ such that $r_t(x) \sim t\bar{r}(\nu)$ as $t \to \pm\infty$. However H is bounded and $H \circ \varphi_t = e^{r_t} H$ (Lemma 1), a contradiction. $\square$

Let us recall the result of Liverani-Wojtkowski about the Lyapunov spectrum of conformally symplectic cocycles [8]. We state the results in the invertible case. Let (M, ν) be a probability space with an inversible ergodic map $T : M \to M$ and let $A : M \to GL(\mathbb{R}^{2n})$ be a measurable map such that both measurable maps $\log_+ \|A^{\pm 1}\|$ are integrable defining the cocycle $A^m(x) := A(T^{m-1}x) \cdots A(x)$ for $m \in \mathbb{Z}$. According to Oseledets multiplicative ergodic theorem, there exists real

numbers $\lambda_1 < \cdots < \lambda_s$ called the Lyapunov exponents of A and an associated decomposition of $\mathbb{R}^{2n}$ (that we will call the Lyapunov decomposition of $\mathbb{R}^{2n}$) in linear subspaces $F_1(x) \oplus \cdots \oplus F_s(x)$ defined for ν-almost every $x \in M$, such that

$$\lim_{m \to \pm \infty} \frac{1}{m} \log \|A^m(x)v\| = \lambda_k, \quad \forall v \in F_k(x), \forall k \in \{1, \ldots, s\}.$$

The positive integer $d_k := \dim F_k$ is well-defined and called the multiplicity of λ_k, these multiplicities satisfy

$$\sum_{k=1}^{s} d_k \lambda_k = \int_M \log |\det A| \mathrm{d}\nu.$$

Liverani-Wojtkowski showed a symmetry of the Lyapunov spectrum in the case where A takes its values in the conformally symplectic linear group $CSp(2n) := CSp(\mathbb{R}^{2n}, \omega_0)$. A conformally symplectic linear map $S \in CSp(E, \omega)$ is a linear map of a symplectic linear space (E, ω) satisfying $S^*\omega = \beta\omega$ for some $\beta \in \mathbb{R}^*$ called the conformal factor of S.

Theorem 19 ([8, Theorem 1.4]). *Let (M, ν) be a probability space with an invertible ergodic map $T : M \to M$ and let $A : M \to CSp(2n)$ be a measurable cocycle such that $\log_+ \|A^{\pm 1}\|$ are integrable. Let $\beta : M \to \mathbb{R}^*$ be such that $A(x)^*\omega_0 = \beta(x)\omega_0$ for all $x \in M$. Then we have the following symmetry of the Lyapunov spectrum $\lambda_1 < \cdots < \lambda_s$ of A:*

$$\lambda_k + \lambda_{s-k+1} = b, \; \text{where } b := \int_M \log |\beta| \mathrm{d}\nu,$$

for every $k \in \{1, \ldots, s\}$. Moreover, the subspace $F_1 \oplus \cdots \oplus F_{s-k}$ is the ω_0-orthogonal complement of $F_1 \oplus \cdots \oplus F_k$.

Let us come back to our setting and consider an ergodic measure ν of M for the Hamiltonian flow (φ_t). Let us fix a Riemannian metric g on M. By taking a measurable symplectic trivialization of TM, one naturally extends the Oseledets multiplicative ergodic theorem for measurable maps $A : M \to GL(\mathbb{R}^{2n})$ such that $\log_+ \|A^{\pm 1}\|$ are integrable for ν to measurable section $A : M \to GL(TM)$ of the fiber bundle $GL(TM)$ such that $\log_+ \|A^{\pm 1}\|$ are integrable where $\| \cdot \|$ is the Riemannian operator norm associated with g. The section $A : x \mapsto \mathrm{d}\varphi_1(x)$ satisfies the integrability condition and the cocycle A^m corresponds to $\mathrm{d}\varphi_m$ for $m \in \mathbb{Z}$. The associated Lyapunov exponents $\lambda_1 < \cdots < \lambda_s$ define the Lyapunov exponents of the flow (φ_t) for the ergodic measure ν. By compactness of M, $t \mapsto \partial_t(\log \|\mathrm{d}\varphi_t(x)v\|)$ is bounded, $x \in M$ and $v \in T_xM \backslash \{0\}$ being fixed, so

$$\lim_{m \to \pm \infty} \frac{1}{m} \log \|\mathrm{d}\varphi_m(x)v\| = \lim_{t \to \pm \infty} \frac{1}{t} \log \|\mathrm{d}\varphi_t(x)v\|,$$

for every $(x, v) \in TM$ for which one of the limit is defined, where $m \in \mathbb{Z}^*$ and $t \in \mathbb{R}^*$.

Corollary 20. *Let ν be an ergodic measure of the Hamiltonian flow (φ_t) of the closed conformal symplectic manifold M. Let $\lambda_1 < \cdots < \lambda_s$ be the associated Lyapunov spectrum and $F_1, \ldots, F_s$ be the associated Lyapunov decomposition of TM. For every $k \in \{1, \ldots, s\}$,*

$$\lambda_k + \lambda_{s-k+1} = \bar{r}(\nu)$$

and the subbundle $F_1 \oplus \cdots \oplus F_{s-k}$ is the ω-orthogonal complement of $F_1 \oplus \cdots \oplus F_k$.

Proof. We apply Theorem 19 to the section $x \mapsto \mathrm{d}\varphi_1(x)$ of the subbundle of conformally symplectic linear maps of TM, with associated conformal factor $\beta : x \mapsto$

$e^{r_1(x)}$. We only need to prove that $b := \int_M \log|\beta| \mathrm{d}\nu$ equals $\bar{r}(\nu)$. By Fubini's theorem and invariance of ν,

$$b = \int_M r_1 \mathrm{d}\nu = \int_M \int_0^1 \eta(X \circ \varphi_t(x)) \mathrm{d}t \mathrm{d}\nu(x) = \int_0^1 \int_M \eta(X) \mathrm{d}\nu \mathrm{d}t = \bar{r}(\nu).$$

$\square$

Let us remark that the fact that $F_1 \oplus \cdots \oplus F_{s-k}$ is the ω-orthogonal complement of $F_1 \oplus \cdots \oplus F_k$ for every k implies that

(6) $$F_k^\omega \cap F_{s-k+1} = 0, \quad \forall k \in \{1, \dots, s\},$$

ν-almost everywhere.

Corollary 21. *Let ν be an ergodic measure of a Hamiltonian flow (φ_t) of the closed conformal symplectic manifold M. There is a measurable sub-bundle F of the Lyapunov decomposition of TM which is transverse to $\mathcal{F}$ on which, for ν-almost every $x \in M$,*

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|\mathrm{d}\varphi_t(x)v\| = \bar{r}(\nu), \quad \forall v \in F(x) \backslash \{0\}.$$

Proof. Let $\lambda_1 < \cdots < \lambda_s$ be the associated Lyapunov spectrum and $F_1, \dots, F_s$ be the associated decomposition of TM. Let X be the vector field of (φ_t). Since $\mathbb{R}X$ is invariant with $\mathrm{d}\varphi \cdot X = X \circ \varphi$, there is $k \in \{1, \dots, s\}$ such that $\lambda_k = 0$ and $\mathbb{R}X \subset F_k$ ν-almost everywhere. Let us show that $F := F_{s-k+1}$ is the desired sub-bundle. According to Corollary 20, $\lambda_{s-k+1} = \bar{r}(\nu)$. According to (6), $\omega(X, v) \neq 0$ for some $v \in F \backslash \{0\}$ when $X \neq 0$. Since $\mathrm{d}_\eta H = \iota_X \omega$, the conclusion follows. $\square$

Let $r \in \{1, \dots, s\}$ be the maximal integer such that $\lambda_r < 0$. According to the non-linear ergodic theorem of Ruelle [10, Theorem 6.3], for every $k \in \{1, \dots, r\}$ and for ν-almost every $x \in M$, the set

$$V_k(x) := \left\{ y \in M \mid \limsup_{t \to +\infty} \frac{1}{t} \log d(\varphi_t(x), \varphi_t(y)) \leqslant \lambda_k \right\},$$

where d denotes the Riemannian distance, is the image of $F_1(x) \oplus \cdots \oplus F_k(x)$ by a smooth injective immersion tangent to identity at x. Therefore the last corollary implies the following proposition.

Corollary 22. *Let ν be an ergodic measure of a Hamiltonian flow (φ_t) of the closed conformal symplectic manifold M such that $\bar{r}(\nu) < 0$ and such that $\operatorname{supp} \nu$ is included in a connected component Σ of $\{H = 0\}$ without critical point of H. For ν-almost every point x of Σ, there exists an immersed submanifold $V \subset M$ transverse to Σ and containing x such that*

$$\limsup_{t \to +\infty} d(\varphi_t(x), \varphi_t(y)) \leqslant \bar{r}(\nu), \quad \forall y \in V.$$

5.2. Examples of isotropic attractors.

5.2.1. *Legendrian attractors.* Contact Hamiltonian dynamical systems can provide examples of conformal dynamical systems by taking their lift to the conformal symplectization (which is closed if the contact manifold is closed).

Proposition 23. *Every Legendrian submanifold is a hyperbolic attractor for some autonomous contact Hamiltonian flow.*

Proof. Let L be a Legendrian submanifold of a contact manifold. According to the contact Weinstein neighborhood theorem, one can assume that L is the 0-section of $(T^*L \times \mathbb{R}, \mathrm{d}z - y\mathrm{d}x)$, with local coordinates $(x, y) \in T^*L$ and $z \in \mathbb{R}$ (see *e.g.*

[4, Corollary 2.5.9 and Example 2.5.11]). Given $H : T^*L \times \mathbb{R} \to \mathbb{R}$, the contact Hamilton equations (1) takes the form

$$\begin{cases} \dot{x} = -\partial_y H, \\ \dot{y} = \partial_x H + y\partial_z H, \\ \dot{z} - y\dot{x} = H. \end{cases}$$

Choosing $H(x,y,z) = -z$, the flow is $\varphi_t(x,y,z) = (x, e^{-t}y, e^{-t}z)$. $\qquad\square$

Let us give some explicit global examples. Let us first consider the standard contact sphere $(\mathbb{S}^{2n-1}, \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x))$. Since $(\mathbb{C}^n\backslash 0, \mathrm{d}x \wedge \mathrm{d}y)$ is the symplectization of the standard sphere, every contact Hamiltonian flow can be obtained in the following way: let $H : \mathbb{C}^n\backslash\{0\} \to \mathbb{R}$ be a positively 2-homogeneous Hamiltonian, the flow of which is (Φ_t), then

$$\varphi_t(z) := \frac{\Phi_t(z)}{\|\Phi_t(z)\|}, \quad \forall z \in \mathbb{S}^{2n-1}, \forall t \in \mathbb{R},$$

defines a contact Hamiltonian flow of $\mathbb{S}^{2n-1}$. Let $H(x,y) := \frac{1}{2}(\|x\|^2 - \|y\|^2)$, so that $\Phi_t(x,y) = (\cosh(t)x + \sinh(t)y, \sinh(t)x + \cosh(t)y)$. The associated contact flow has one Legendrian attractor $L_+ := \{x = y\}$ and one Legendrian repeller $L_- := \{x = -y\}$, every point outside of them having its α-limit set inside L_- and its ω-limit set inside L_+.

Let us now consider a vector field X on some closed manifold M generating a flow (f_t). According to Section 2.3, this flow extends to a Hamiltonian flow $(\hat{f}_t)$ (identifying the 0-section with M) on T^*M which is fiberwise homogeneous: $\hat{f}_t(q, ap) = a\hat{f}_t(q,p)$, $\forall (q,p) \in T^*M$, $\forall a \in \mathbb{R}$. Let us endow M with a Riemannian metric, the flow $(\hat{f}_t)$ induces a Contact Hamiltonian flow (φ_t) on the unit cotangent bundle $(S^*M, i^*\lambda)$ (λ being the Liouville form and $i : S^*M \hookrightarrow T^*M$ the inclusion) by

$$\varphi_t(q,p) := \frac{\hat{f}_t(q,p)}{\|\hat{f}_t(q,p)\|}, \quad \forall (q,p) \in S^*M.$$

A hyperbolically attracting (resp. repelling) fixed point $x \in M$ of (f_t) corresponds to a normally hyperbolically attracting (resp. repelling) Legendrian fiber S_x^*M of $(\hat{f}_t)$.

Let us remark that in both examples, one can directly work in the conformal symplectization by taking the flow induced by (Φ_t) (resp. $(\hat{f}_t)$) on the quotient space $(\mathbb{C}^n\backslash\{0\})/(z \sim ez)$ (resp. $(T^*M\backslash\{0\})/((q,p) \sim (q,ep))$), where $e := \exp(1)$.

5.2.2. *Hyperbolic attractive and repulsive closed orbit in every non-symplectic manifold.* Here, by a non-symplectic manifold, we mean a conformally symplectic manifold, the conformal structure of which is not $\sim (0, \omega)$.

Proposition 24. *Let (M, η, ω) be a conformally symplectic manifold and let $\gamma : S^1 \hookrightarrow M$ be an embedded loop such that $\int_\gamma \eta < 0$. There exists a Hamiltonian $H : M \to \mathbb{R}$ admitting γ as a hyperbolic attracting periodic orbit.*

In particular, every non-symplectic manifold admits a conformal Hamiltonian flow that has a hyperbolic attractive periodic orbit and a hyperbolic repulsive periodic orbit Hamiltonian.

Proof. Let us first remark that γ is included in an open Lagrangian submanifold. According to Theorem 36 in Appendix A, using a cut-off function to define H globally, one can replace M with the normal bundle $N \to S^1$ of $\gamma \simeq S^1$, identified with the 0-section. As a vector bundle N equals $W \oplus T^*S^1$, where W is a symplectic vector bundle. Since the Lagrangian Grassmannian is connected, one can find a

lagrangian subbundle $L \subset W$. Then L is a Lagrangian submanifold of N containing γ.

One can thus assume that $M = T^*_\beta L$ with γ included in the 0-section identified with L. The closed form β of L is the pull-back of the Lee form η to L, in particular $r := \int_\gamma \beta < 0$. Let X be a complete vector field of L inducing a flow (f_t) for which γ is a 1-periodic hyperbolic orbit such that the eigenvalues μ's of $\mathrm{d}f_1(\gamma(0))$ satisfy $e^r < \mu < 1$. Let $(\hat{f}_t)$ be the lifted Hamiltonian flow of $T^*_\beta L$ properly cut-off outside a neighborhood of γ (see the end of Section 2.3). The differential of $\hat{f}_1$ at $\gamma(0)$ is equivalent to $\mathrm{d}f_1(\gamma(0)) \oplus e^r(\mathrm{d}f_1(\gamma(0)))^{-1}$ so its eigenvalues are in $(0,1)$. $\qquad\square$

5.3. **Connected components of $\{H = 0\}$ and attraction.** Here we wonder if a connected component of $\{H = 0\}$ can be an attractor. In Section 1.1.1, we gave a 2-dimensional example where a connected component of $\{H = 0\}$ is attractive. This is the only example that we know, and here we give conditions that ensure that such a component cannot be attractive. We will say that a subset Σ of M separates locally M in two connected components if in every neighbourhood V of Σ, there exists an open neighbourhood $U \subset V$ of Σ such that $U \backslash \Sigma$ has exactly two connected components. The manifold M being connected, we say that Σ separates globally M if $M \backslash \Sigma$ is not connected.

Proposition 25. *Assume that Σ is an isolated connected component of $\{H = 0\}$ that separates globally and locally M in two connected components. Then Σ cannot be a strong attractor.*

Proof. Let us assume that Σ is a strong attractor. Then there exists an open neighbourhood U_0 of Σ such that $U_0 \cap \{H = 0\} = \Sigma$. As Σ separates locally M in two connected components, there exists a neighbourhood U of Σ such that $U \subset U_0$ has two connected components, U_- and U_+. We denote by $\varepsilon_\pm \in \{-1, 1\}$ the sign of $H_{|U_\pm}$.

We know that $M \backslash \Sigma$ is not connected, and the boundary of each of its connected components intersects Σ and thus contains U_- or U_+. This implies that $M \backslash \Sigma$ has exactly two connected components, M_- that contains U_- and M_+ that contains U_+. We denote by $\varepsilon : M \backslash \Sigma \to \{-1, 1\}$ the function such that $\varepsilon_{|M_\pm} = \varepsilon_\pm$.

We choose a smooth bump function $\chi : M \to [0, 1]$ such that χ is equal to 1 in a neighbourhood of Σ and the support of χ is contained in U. Then the Hamiltonian $K : M \to \mathbb{R}$ is defined by $K = \chi H + (1 - \chi)\varepsilon$. We have $\Sigma = \{K = 0\}$. The Hamiltonian flow of K coincides with the flow of H in a neighborhood of Σ and then Σ is also a strong attractor for (φ^K_t). If x is a generic point in the basin of attraction of Σ for K but not in Σ, x is wandering. Observe that a wandering point is wandering for (φ^K_t) and (φ^{-K}_t). We deduce from Corollary 8 that $\lim_{t \to -\infty} K(\varphi^K_t(x)) = 0$ since x was taken generically. But as $\Sigma = \{K = 0\}$ is a strong attractor, this is not possible. $\qquad\square$

We do not know whether a similar statement is true without the separation assumption. We obtain the following result when we assume normal hyperbolic attraction.

Theorem 26. *Let us assume (M, η, ω) has dimension $2n \geqslant 4$ and let $H : M \to \mathbb{R}$ be Hamiltonian. Let Σ be a closed connected component of $\{H = 0\}$ without critical point of H. Then Σ cannot be hyperbolically normally attracting.*

Proof. Let us assume that such a Σ is hyperbolically normally attracting and reach a contradiction. Let us restrict ourself to a neighborhood of Σ. One can assume that $\Sigma = \{H = 0\}$ and that $M = \Sigma \times (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ with $H(x, y) = y$ for all $(x, y) \in \Sigma \times (-\varepsilon_0, \varepsilon_0)$ by a change of variables in a tubular neighborhood of

Σ. Let $V_\varepsilon := \Sigma \times (-\varepsilon, \varepsilon)$, then Σ being normally hyperbolic means that one can assume that there exists $a \in (0, 1)$ and $\tau > 0$ such that

$$(7) \qquad \varphi_\tau(V_\varepsilon) \subset V_{a\varepsilon}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

According to Proposition 5 applied to the leaf Σ, there exists a volume form μ of Σ such that $\varphi_t^* \mu = e^{(n-1)r_t} \mu$ and which is the pull-back of a form μ_0 of M such that $\mu_0 \wedge dy = \omega^n$ in the neighborhood of Σ. Let $\pi : M \to \Sigma$ be the projection on the first factor. By decreasing ε_0, one can assume that $\pi^* \mu \wedge dy$ does not vanish so that there exists a non-vanishing map $f : M \to \mathbb{R}_+$

such that $f\pi^* \mu \wedge dy = \omega^n$. Since $\mu_0 \wedge dy = \omega^n$ and μ_0 and $\pi^* \mu$ agree on $T\Sigma$, $f|_\Sigma \equiv 1$. By (7),

$$(8) \qquad \omega^n(\varphi_\tau(V_\varepsilon)) \leqslant \omega^n(V_{a\varepsilon}), \quad \forall \varepsilon \in (0, \varepsilon_0).$$

On the one hand,

$$(9) \qquad \omega^n(V_\varepsilon) = \int_{x \in \Sigma} \left(\int_{-\varepsilon}^{\varepsilon} f(x,y) dy \right) \mu_x \overset{\varepsilon \to 0}{\sim} 2\varepsilon \cdot \mu(\Sigma).$$

On the other hand,

$$(10) \qquad \omega^n(\varphi_\tau(V_\varepsilon)) = \int_{V_\varepsilon} e^{nr_\tau} \omega^n \overset{\varepsilon \to 0}{\sim} 2\varepsilon \int_\Sigma e^{nr_\tau} \mu.$$

Therefore, (7),(9), (10) imply

$$(11) \qquad \int_\Sigma e^{nr_\tau} \mu \leqslant a\mu(\Sigma).$$

By Hölder's inequality,

$$\int_\Sigma e^{(n-1)r_\tau} \mu \leqslant \left(\int_\Sigma \mathbf{1}^n \mu \right)^{\frac{1}{n}} \left(\int_\Sigma e^{nr_\tau} \mu \right)^{\frac{n-1}{n}},$$

since $\varphi_\tau^* \mu = e^{(n-1)r_\tau} \mu$, it follows from (11) that

$$\mu(\Sigma) = \mu(\varphi_\tau(\Sigma)) \leqslant \mu(\Sigma)^{\frac{1}{n}} a^{\frac{n-1}{n}} \mu(\Sigma)^{\frac{n-1}{n}},$$

so $a \geqslant 1$, a contradiction. $\qquad\square$

6. INVARIANT DISTRIBUTION AND SUBMANIFOLDS

We introduced in section 2.5 the invariant distribution $\mathcal{F}$.

6.1. **Holonomy of embedded leafs of $\mathcal{F}$.** Let us study the holonomy of a regular leaf F of $\mathcal{F}$. By definition, one can find an open neighborhood U of F on which $\mathcal{F}$ defines a non-singular foliation. The holonomy of F is well-defined as the holonomy of F in U for this foliation.

Let us recall the definition of the holonomy $\pi_1(F) \to G$ of a leaf F of a foliation $\mathcal{G}$ of codimension p on a manifold N^n. We refer to [5]. A distinguished map $f : V \to \mathbb{R}^p$ of $\mathcal{G}$ is a map on a trivialization neighborhood $V \simeq \mathbb{R}^p \times \mathbb{R}^{n-p}$ that factors by the projection $\mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}^p$. Given a point $z \in F$, let G be the group of germs of local homeomorphisms of $\mathbb{R}^p$ fixing 0 defined up to internal automorphisms (i.e. up to conjugacy by such germs). Given a loop $\gamma : S^1 \to F$ based at z and a germ of distinguished map f sending z to 0, there is a unique continuous lift (f_t) of γ in the space of germs of distinguished maps such that $f_0 = f$ and f_t sends $\gamma(t)$ to 0. There exists a unique germ $g : \mathbb{R}^p \to \mathbb{R}^p$ fixing the origin such that $f_1 = g \circ f_0$. This germ only depends on f and the homotopy class of γ. If one takes another germ f' of distinguished map at z, the same procedure will give a germ $g' : \mathbb{R}^p \to \mathbb{R}^p$ fixing the origin that is conjugated to g. Therefore, one defines the holonomy of F (based at z) as the morphism $\pi_1(F, z) \to G$ sending the

class of the loop γ to the class of the germ g. The holonomy group of F is the image of the holonomy. Up to isomorphisms, these notions do not depend on the base point z (a leaf being path-connected).

Proposition 27. *The holonomy of an embedded leaf outside $\{H = 0\}$ is trivial. Let $\Sigma \subset \{H = 0\}$ be a connected component of $\{H = 0\}$ without critical point of H, the holonomy of Σ is*

$$[\gamma] \mapsto \left[y \mapsto e^{\int_\gamma \eta} y \right].$$

In particular, the holonomy group of Σ is isomorphic to the subgroup $\langle [\eta], \pi_1(\Sigma) \rangle$ of $\mathbb{R}$.

Proof. One can prove this proposition by considering the global distinguished map $e^{-\theta} H \circ p$ defined on the universal cover $p : \widetilde{M} \to M$ with $\mathrm{d}\theta = p^*\eta$. Let us give a more intrinsec proof.

If F is an embedded leaf outside $\{H = 0\}$, the pull-back of the Lee form η to F is exact according to Corollary 4, so $\eta = \mathrm{d}\theta$ on a tubular neighborhood U of F. Therefore $\mathcal{F}$ is trivially fibered by $e^{-\theta} H$ in U and the holonomy is thus trivial.

Let Σ be a connected component of $\{H = 0\}$ without critical point. Let $i : \Sigma \hookrightarrow M$ be the inclusion map. If $i^*\eta$ is exact, the holonomy is trivial, as above. Otherwise, let us fix $z \in \Sigma$ such that $(i^*\eta)_z \neq 0$, which implies that $\ker \eta_z$ is transversed to $T_z \Sigma$. Let $T \subset M$ be an open connected 1-dimensional manifold containing z and tangent to $\ker \eta$. By shrinking T, one can assume that H induces an isomorphism $H|_T : T \to (-\varepsilon, \varepsilon)$ sending z to 0, for some $\varepsilon > 0$. Let us remark that there exist a distinguished map f in the neighborhood of z such that $f|_T = H|_T$. Indeed, in the neighborhood of z, let θ be such that $\theta(z) = 0$ and $\mathrm{d}\theta = \eta$, then $f := e^{-\theta} H$ is a distinguished map. Since T is tangent to $\ker \eta$, $\theta|_T \simeq 0$ so that $f|_T = H|_T$.

Let $\gamma : [0, 1] \to \Sigma$ be a smooth loop based at z. According to [5, §2.5] for every $x \in W$ a connected neighborhood of z in T, there are smooth paths $\gamma_x : [0, 1] \to M$ tangent to $\mathcal{F}$ and C^0-close to γ such that, $\gamma_x(0) = x$, $\gamma_x(1) \in T$ and the image of the holonomy $\pi_1(\Sigma, z) \to G$ at $[\gamma]$ is the class of the germ

$$y \mapsto H\left(\gamma_{H|_T^{-1}(y)}(1) \right),$$

(here, we used that $H|_T = f|_T$ where f is a distinguished map). According to Lemma 3,

$$H\left(\gamma_{H|_T^{-1}(y)}(1) \right) = e^{\int_{\gamma_x} \eta} y, \quad \text{with} \quad x := H|_T^{-1}(y).$$

Since T is tangent to $\ker \eta$, by concatenating γ_x with the image of the segment $[H(\gamma_x(0)), H(\gamma_x(1))]$ under $H|_T^{-1}$, one gets a loop $\tilde{\gamma}_x$ such that $\int_{\tilde{\gamma}_x} \eta = \int_{\gamma_x} \eta$. Since γ_x is C^0-close to γ, one can reparametrize $\tilde{\gamma}_x$ such that this loop is C^0-close to γ, so $\tilde{\gamma}_x$ is homotopic to γ and the conclusion follows.   □

A consequence is the following (see [5, §2.5]).

Corollary 28. *Let $\Sigma \subset \{H = 0\}$ be a connected component of $\{H = 0\}$ without critical point of H. If the pull-back of the Lee form to Σ is not trivial, there exist leafs of $\mathcal{F}$ different from Σ, the closure of which contains Σ.*

Examples 1.1.2 and 4.3 show that a non-compact leaf can go far away from $\{H = 0\}$.

6.2. **Invariance and isotropy.**

Proposition 29. *Let L be a submanifold of M.*

   (1) If L is isotropic and invariant, then it is tangent to $\mathcal{F}$.

(2) *If L is coisotropic and tangent to $\mathcal{F}$, then it is invariant.*

(3) *If L is invariant and tangent to $\mathcal{F}$, then the pull-back of ω to L is degenerate. In particular, if L is of even dimension $2k$, the pull-back of ω^k to L is zero. An invariant surface tangent to $\mathcal{F}$ is thus isotropic.*

Proof. If L is an invariant isotropic submanifold, then X is tangent to L so $\forall v \in TL$, $\mathrm{d}_\eta H \cdot v = \omega(X, v) = 0$. Conversely, if L is coisotropic and tangent to $\mathcal{F}$, for $x \in L$, $T_x L \supset (T_x L)^\omega \supset (\mathcal{F}_x)^\omega = \mathbb{R} X(x)$ so L is invariant. If $i : L \hookrightarrow M$ is invariant and tangent to $\mathcal{F}$, then $X|_L$ is in the kernel of $i^* \omega$. $\qquad \square$

Combining Proposition 29 and Corollary 4, one gets the following result.

Corollary 30. *An isotropic invariant submanifold on which the pull-back of the Lee form is not exact is included in $\{H = 0\}$.*

Corollary 30 can be applied to Lagrangian graphs of $T_\beta^* Q$ for a non-exact closed 1-form β of Q. Indeed, for every β-closed 1-form α of Q, $q \mapsto \alpha_q$ defines a Lagrangian section $Q \hookrightarrow T_\beta^* Q$ pulling back the Lee form to the non-exact form β.

We now are interested in how the dynamics can force the isotropy.
Following [11], we recall that a point $x \in M$ is *quasi-regular* if for every continuous map $f : M \to \mathbb{R}$, the following limit exists

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t f(\varphi_s x) ds.$$

Then we can associate to every quasi-regular point its *asymptotic cycle* $A(x) \in H_1(M, \mathbb{R})$ that satisfies for every continuous closed 1-form ν on M

$$\langle [\nu], A(x) \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t \nu(X_H \circ \varphi_s(x)) ds.$$

Moreover, if μ is an invariant Borel probability by (φ_t), μ almost every point is quasi-regular and the asymptotic cycle $A(\mu) \in H_1(M, \mathbb{R})$ of μ is defined by

$$\langle [\nu], A(\mu) \rangle = \int \langle [\nu], A(x) \rangle d\mu(x).$$

We have

Proposition 31. *Let $(R_{t\alpha})_{t \in \mathbb{R}}$ the flow of rotations of $\mathbb{T}^n$ with vector $\alpha \in \mathbb{R}^n$ that is defined by*

$$R_{t\alpha}(\theta) = \theta + t\alpha.$$

We identify $H_1(\mathbb{T}^n)$ with $\mathbb{R}^n$ in the usual way. Then every point of $\mathbb{T}^n$ is quasi-regular, and the asymptotic cycle of every point of $\mathbb{T}^n$ and of every invariant probability measure is α.

Observe that when two flows $(f_t) : M \circlearrowleft$ and $(g_t) : N \circlearrowleft$ are conjugated via some homeomorphism $h : M \to N$, then the quasi-regular points of (g_t) are the h-images of the quasi-regular points of (f_t) and that when $x \in M$ is quasi-regular, we have

$$h_* A(x) = A(h(x)).$$

This allows us to introduce a notion of *rotational torus* $\mathcal{T}$ for a flow $(f_t) : M \circlearrowleft$. A rotational torus is a C^0-embedded torus $j : \mathbb{T}^m \hookrightarrow M$ such that $(j^{-1} \circ f_t \circ j)$ is a flow of rotation. When j is a C^1-embedding, $\mathcal{T}$ is a C^1-rotational torus. Thanks to Proposition 31, all the points of a rotational torus are quasi-regular with the same asymptotic cycle that we denote by $A(\mathcal{T}) \in H_1(M)$ and every measure with support in $\mathcal{T}$ has also the same asymptotic cycle.

Let us prove a result that is reminiscent of a result of Herman in the symplectic setting [6].

Proposition 32. *Assume that $j(\mathbb{T}^m) = \mathcal{T}$ is a C^1-rotational torus for a conformal Hamiltonian flow (φ_t) of (M, η, ω). Then*

- *if the flow restricted to $\mathcal{T}$ is minimal, $\omega = \mathrm{d}_\eta\lambda$ is η-exact and $j^*\eta$ is exact, then $\mathcal{T}$ is isotropic;*
- *if $\mathcal{T}$ is not isotropic, then $\langle[\eta], A(\mathcal{T})\rangle = 0$.*

In particular, when the cohomological class of η is rational and non zero and when the flow restricted to $\mathcal{T}$ is minimal then $\mathcal{T}$ is isotropic.

Proof. We use the notation $R_{t\alpha} = j^{-1} \circ \varphi_t \circ j$.
Let us prove the first point. As $j^*\eta = \mathrm{d}f$ is exact, we have

$$\mathrm{d}(e^{-f}j^*\lambda) = e^{-f}(j^*\mathrm{d}\lambda - \mathrm{d}f \wedge j^*\lambda) = e^{-f}j^*(\mathrm{d}_\eta\lambda) = e^{-f}j^*\omega.$$

Hence $e^{-f}j^*\omega$ is exact. Observe that $j^*X_H = \alpha$. Hence $\forall x \in \mathbb{T}^m$, $\forall t \in \mathbb{R}$,

$$r_t(j(x)) = \int_0^t \eta(\varphi_s(j(x)))X_H(\varphi_s(j(x)))\mathrm{d}s = \int_0^t \mathrm{d}f(R_{s\alpha}(x))\alpha\mathrm{d}s = f(x+t\alpha) - f(x).$$

Because $(\varphi_t^H)^*\omega = e^{r_t}\omega$ and $\varphi_t^H \circ j = j \circ R_{t\alpha}$, we deduce

$$R_{t\alpha}^*(j^*\omega) = e^{r_t \circ j}j^*\omega$$

and then

$$R_{t\alpha}^*(e^{-f}j^*\omega) = e^{-f}j^*\omega.$$

If we write $e^{-f}j^*\omega = \sum_{1 \leqslant i < j \leqslant m} a_{i,j}\mathrm{d}x_i \wedge \mathrm{d}x_j$, we deduce that every continuous function $a_{i,j}$ is invariant by $(R_{t\alpha})$, and then constant because the flow is minimal. The form $e^{-f}j^*\omega$ is constant and exact, it is then the zero form and $\mathcal{T}$ is isotropic.

Let us prove the second point. We assume that $\mathcal{T}$ is not isotropic. Hence $j^*\omega$ is not zero. There exists a sequence (t_n) of real numbers that tends to $+\infty$ and satisfies

$$\lim_{n\to\infty} t_n\alpha = 0 \quad \text{in} \quad \mathbb{T}^m.$$

Then $(R_{t_n\alpha})$ tends to $\mathrm{id}_{\mathbb{T}^m}$ in topology C^1. We deduce that $(R_{t_n\alpha}^*(j^*\omega))$ tends to $j^*\omega$. Moreover, we also have

$$R_{t_n\alpha}^*(j^*\omega) = e^{r_{t_n} \circ j}j^*\omega$$

where, as $\varphi_s \circ j = j \circ R_{s\alpha}$ (so $X \circ j = \mathrm{d}j \cdot \alpha$),

$$r_{t_n} \circ j(x) = \int_0^{t_n} \eta(\mathrm{d}j(R_{s\alpha}(x))\alpha)\mathrm{d}s = t_n\langle[j^*\eta], \alpha\rangle + o(1).$$

As $j^*\omega$ is not the zero form, we have then $0 = \lim_{n\to\infty} r_{t_n}(x)$ which implies that $[j^*\eta]$ is orthogonal to α i.e. $\langle[\eta], A(\mathcal{T})\rangle = 0$. $\qquad\square$

If we assume that the invariant torus is C^3, we can relax the hypothesis on the dynamics for the second point of Proposition 32, asking only a C^0-conjugacy. The main argument that we use is very similar to part 3 of [1].

Proposition 33. *Assume that $\mathcal{T} \subset M$ is a C^3-submanifold of M that is a C^0-rotational torus of a conformal Hamiltonian flow (φ_t) of (M, η, ω) such that $\langle[\eta], A(\mathcal{T})\rangle \neq 0$. Then $\mathcal{T}$ is isotropic.*

In particular, when the cohomological class of η is rational and non zero and when the flow restricted to $\mathcal{T}$ is minimal, then $\mathcal{T}$ is isotropic.

Proof of Proposition 33. We denote the canonical injection $\mathcal{T} \hookrightarrow M$ by j. We endow $\mathcal{T}$ with a Riemannian metric and denote by $d_\mathcal{G}$ the distance along the leaves

of the characteristic foliation $\mathcal{G}$ of $j^*\omega$. We assume that $\mathcal{T}$ is not isotropic. We denote the maximum rank of $j^*\omega$ by r and by U the open set

$$U = \{x \in \mathcal{T} \mid \mathrm{rank}\,(j^*\omega(x)) = r\}$$

As $\varphi_t^*(j^*\omega) = e^{r_t}j^*\omega$, this set is invariant by the flow.

A result of Proposition 31 is

(12) $$r_n(x) = n\left(\langle[\eta], A(\mathcal{T})\rangle + o_{n\to\infty}(1)\right), \quad \forall x \in \mathcal{T}.$$

There are two cases:

- either $U = \mathcal{T}$ is compact; we choose $x \in U$ and $\mathcal{K} := U$;
- or $U \neq \mathcal{T}$. Then the closure of the orbit of a fixed point $x \in U$ is homeomorphic to a torus with dimension $k < m$. As $(\varphi_t|_{\mathcal{T}})$ is conjugate to a flow of rotation, there exists a compact invariant neighbourhood $\mathcal{K}$ of x in U that is homeomorphic to $\mathbb{T}^k \times [-1, 1]^{m-k}$.

In $\mathcal{K}$, we consider the characteristic foliation $\mathcal{G}$ of ω. We now follow the arguments and notation of [1] (except that $\mathcal{F}$ and the $\mathcal{F}_i$'s are here denoted $\mathcal{G}$ and $\mathcal{G}_i$). We use a finite covering of $\mathcal{K}$ by foliated charts $\mathcal{W}_1, \ldots, \mathcal{W}_I$ in $\mathcal{U}$ and denote by $\mathcal{G}_i$ the foliation restricted to $\mathcal{W}_i$. Then there exists a constant $\mu > 0$ such that every $(m - r)$-submanifold $\mathcal{S}$ of $\mathcal{W}_i$ that intersects every leaf of $\mathcal{G}_i$ at most once satisfies $|\omega^{\frac{r}{2}}(\mathcal{S})| \leqslant \mu$.

Moreover, we may assume that there exists $\varepsilon > 0$ such that:

(i) if x, y are in some $\mathcal{W}_i$, and such that $d_{\mathcal{G}}(x, y) < \varepsilon$, then x and y are in the same leaf of $\mathcal{W}_i$

where $d_{\mathcal{G}}$ is the distance along the leaves.

We also have the existence of $\nu \in (0, \varepsilon)$ such that

(13) $$d_{\mathcal{G}}(x, y) < \nu \Rightarrow d_{\mathcal{G}}(\varphi_{-1}(x), \varphi_{-1}(y)) < \varepsilon, \quad \forall x, y \in \mathcal{K}.$$

We then use a decomposition $(Q_j)_{1\leqslant j\leqslant J}$ of $\mathcal{K}$ into submanifolds with corners that may intersect only along their boundary such that every Q_j is contained in at least one $\mathcal{W}_i$ that satisfies:

(ii) if $Q_j \subset \mathcal{W}_i$, then if $x, y \in Q_j$ are in the same leaf of $\mathcal{W}_i$, we have $d_{\mathcal{G}}(x, y) < \nu$.

If $\mathcal{S}$ is a piece of r-dimensional submanifold contained in some $Q_{j_0} \subset \mathcal{W}_{i_0}$ that is transverse to $\mathcal{G}$ and intersects every leaf of $\mathcal{G}_{i_0}$ at most once, let us consider $\mathcal{S}' = \varphi_1(\mathcal{S} \cap Q_{j_0}) \cap Q_{j_1}$ for some j_1. Then $\mathcal{S}'$ is also transverse to $\mathcal{G}$. Let $\mathcal{W}_{i_1}$ that contains Q_{j_1} and let us assume that $x, y \in \mathcal{S}'$ are in a same leaf of $\mathcal{G}_{i_1}$. Because of (ii) and (13), $d_{\mathcal{G}}(\varphi_{-1}(x), \varphi_{-1}(y)) < \varepsilon$ and by (i), we have $\varphi_{-1}(x) = \varphi_{-1}(y)$ and $x = y$. Iterating this argument, we deduce that all the sets

$$\mathcal{S}' = \varphi_k(\mathcal{S} \cap Q_{j_0}) \cap \varphi_{k-1}(Q_{j_1}) \cap \cdots \cap Q_{j_k}$$

are such that if $Q_{j_k} \subset \mathcal{W}_{i_k}$, $\mathcal{S}'$ intersects every leaf of $\mathcal{F}_{i_k}$ at most once and then $|\omega^{\frac{r}{2}}(\mathcal{S}')| \leqslant \mu$.

If now N_k is the number of k-uples $(j_1, \ldots, j_k)$ such that $\varphi_k(\mathcal{S} \cap Q_{j_0}) \cap \varphi_{k-1}(Q_{j_1}) \cap \cdots \cap Q_{j_k} \neq \varnothing$, then we have

(14) $$|\omega^{\frac{r}{2}}(\varphi_k(\mathcal{S}))| \leqslant N_k\mu.$$

By (12), we have

(15) $$\omega^{\frac{r}{2}}(\varphi_k(\mathcal{S})) = \exp\left(k\frac{r}{2}\left(\langle[\eta], A(\mathcal{T})\rangle + o_{k\to\infty}(1)\right)\right)\omega^{\frac{r}{2}}(\mathcal{S}).$$

Combining (14) and (15), we deduce that

$$\limsup_{k\to\infty} \frac{1}{k}\log(N_k) \geqslant |\langle[\eta], A(\mathcal{T})\rangle| > 0$$

is a lower bound for the topological entropy of $(\varphi_t|_{\mathcal{T}})$. But this contradicts the fact that $(\varphi_t|_{\mathcal{T}})$ is C^0-conjugate to a flow of rotation and has zero entropy. $\qquad\square$

APPENDIX A. ISOTROPIC SUBMANIFOLDS

A.1. **Isotropic embeddings.**

Lemma 34. *Given a manifold N endowed with a closed 1-form β, there exists a Legendrian embedding of N in a contact manifold (V, α) endowed with a closed 1-form, the pull-back to N of which is β. This contact manifold is closed if N is closed.*

Proof. Given a submanifold M' of a Riemannian manifold (M, g), we denote $\nu^1 M' \subset T^1 M$ its unit normal bundle. Let us endow N and S^1 with Riemannian metrics and let us consider unit tangent bundle V of the product Riemannian manifold $(N \times S^1, g)$ endowed with its standard contact form. Let us recall that unit normal bundles of submanifolds of $N \times S^1$ are Legendrian submanifolds of V. Therefore, $\nu^1(N \times \{x\})$, $x \in S^1$ fixed, is a Legendrian submanifold, it is the disjoint union of two copies of N. We lift the closed 1-form β to V by pulling it back by the canonical projection $T^1(N \times S^1) \to N$. $\qquad\square$

Lemma 35. *Given a manifold N endowed with a closed 1-form β, there exists a Lagrangian embedding of $N \times S^1$ in a conformal exact symplectic manifold (M, η, ω) such that the pull-back of η to $N \times S^1$ is $\pi^*\beta - \mathrm{d}\theta$ where π is the projection on the first factor. This manifold M is closed if N is closed.*

Proof. Let (V, α) be the contact manifold of Lemma 34, we assume $N \subset V$ and identify β with its pull-back to V. Then the β-twisted symplectization (M, η, ω) of V endowed with its standard Lee form satisfies the statement. $\qquad\square$

A.2. **Weinstein neighborhood of isotropic submanifolds.** We extend the usual Weinstein neighborhood theorem [13, Lecture 5] to the conformal setting following and adapting [4, Section 2.5.2]. The special case of Lagrangian submanifolds was already treated by Chantraine-Murphy [2, Theorem 2.11].

Let us first describe the conformal structure of the local model of a neighborhood of an isotropic manifold $Q^k \subset (M^{2n}, \eta, \omega)$. Let us denote $T_Q M$ the restriction of the tangent bundle of M to Q and $TQ^\omega \subset T_Q M$ the ω-orthogonal bundle of TQ. Then the normal bundle $\pi : \nu Q \to Q$ can be non-canonically decomposed as

(16) $$\nu Q = T_Q M / TQ \simeq TQ^\omega / TQ \oplus T_Q M / TQ^\omega.$$

In order to fix this decomposition, let us fix a complex structure J compatible with ω, *i.e.* such that $g := \omega(\cdot, J\cdot)$ defines a Riemannian metric. With respect to g, νQ is canonically isomorphic to the orthogonal vector bundle $TQ^\perp$, TQ^ω / TQ is isomorphic to $(TQ \oplus J(TQ))^\omega$ and $T_Q M / TQ^\omega$ to $J(TQ)$ so that the decomposition (16) takes the concrete form

$$TQ^\perp = (TQ \oplus J(TQ))^\omega \oplus J(TQ).$$

We will work through this identification. Fibers of TQ^ω / TQ are symplectic vector spaces of dimension $2(n - k)$ for the structure induced by ω. Let β be the pull-back of the Lee form to Q. The fiber bundle $T_Q M / TQ^\omega$ is diffeomorphic to $T_\beta^* Q$ under $(q, [v]) \mapsto \omega_q(v, \cdot)$. One can check in local coordinates that the direct sum of the fibered symplectic form and the conformal symplectic form of $T_\beta^* Q$ defines a $\pi^*\beta$-conformal symplectic form $\omega_{\nu Q}$ such that $(\nu Q, \eta_{\nu Q} = \pi^*\beta, \omega_{\nu Q})$ is a well-defined conformal symplectic manifold. Moreover, for all $q \in Q$, $(\omega_{\nu Q})_{(q,0)} = \omega_q$ by definition: both structures agree along Q (seeing νQ as $TQ^\perp$, we recall that $T_{(q,0)}(TQ^\perp)$ is naturally identified with $T_q Q$). The structure of νQ depends on the

isotropic embedding $Q \subset M$, when Q is Lagrangian $\nu Q \simeq T_\beta^* Q$ only depends on $\eta|_Q$.

Theorem 36 (Weinstein neighborhood). *Let $Q \subset (M, \eta, \omega)$ be an isotropic submanifold, there exist a neighborhood $U_1 \subset M$ of Q, a neighborhood $U_2 \subset \nu Q$ of the 0-section and a conformal symplectomorphism $\varphi : (U_1, \eta, \omega) \to (U_2, \eta_{\nu Q}, \omega_{\nu Q})$ sending Q to the 0-section canonically.*

The proof is an adaptation of the symplectic case. We will use the following stability theorem proven by Chantraine-Murphy.

Theorem 37 ([2, Theorem 2.10]). *Let (M, η) be a closed manifold endowed with a closed 1-form and let $(\omega_t)_{t \in [0,1]}$ be a path of η-conformal symplectic forms such that $\omega_t = \omega_0 + \mathrm{d}_\eta \lambda_t$. There exists an isotopy $\varphi_t : M \to M$ and functions $f_t : M \to \mathbb{R}$, $t \in [0, 1]$, such that $\varphi_t^* \eta = \eta + \mathrm{d} f_t$ and $\varphi_t^* \omega_t = e^{f_t} \omega_0$.*

Let us first prove the following conformal extension of [9, Lemma 3.14].

Lemma 38. *Let (M, η) be a manifold endowed with a closed 1-form and $Q \subset M$ be a compact submanifold. Let us assume that there exist two η-conformal symplectic forms ω_0 and ω_1 agreeing on $T_q M$ for all $q \in Q$. There exist two neighborhoods U_0 and U_1 of Q, a diffeomorphism $\psi : U_0 \to U_1$ and a map $f : U_0 \to \mathbb{R}$ vanishing on Q such that $\psi|_Q = \mathrm{id}$, $\psi^* \omega_1 = e^f \omega_0$ and $\psi^* \eta = \eta + \mathrm{d} f$.*

Proof of Lemma 38. Let us endow M with a Riemannian metric and let us define a tubular neighborhood U of Q as the image under the diffeomorphism $(q, v) \mapsto \exp_q(v)$ of a neighborhood of the 0-section of the normal bundle of Q. Let $\pi : U \to Q$ be the orthogonal projection. Since π is a retraction by deformation, $\eta|_U$ is cohomologous to $\pi^* \beta$ where β is the pull-back of η to Q. One can assume $\eta = \pi^* \beta$: indeed, $\eta = \pi^* \beta + \mathrm{d} g$ with $g : U \to \mathbb{R}$ vanishing on Q so one can do a gauge transformation.

The remainder of the proof closely follows the non-conformal one [9, Lemma 3.14]. We set $\tau := \omega_1 - \omega_0$ in order to show that $\tau = \mathrm{d}_\eta \sigma$ for some σ and apply Theorem 37 with $\lambda_t = t\sigma$. In order to prove that, we consider the map $\phi_t : \exp_q(v) \mapsto \exp_q(tv)$ defined on U for $t \in [0, 1]$ so that $\phi_0 = \pi$, $\phi_1 = \mathrm{id}$ and $\tau = \phi_1^* \tau - \phi_0^* \omega$ ($\pi^* \omega_0 = \pi^* \omega_1$ by assumption). Let us set $X_t = \partial_t \phi_t \circ \phi_t^{-1}$, well-defined for $t \in (0, 1]$. Then

$$\frac{\mathrm{d}}{\mathrm{d} t}(\phi_t^* \tau) = \phi_t^*(\mathrm{d}_\eta(\iota_{X_t} \tau) + \pi^* \beta(X_t)\tau) = \phi_t^*(\mathrm{d}_\eta(\iota_{X_t} \tau)),$$

since $\mathrm{d}\pi \cdot X_t = 0$. We remark that $\sigma_t := \phi_t^*(\iota_{X_t} \tau)$ is smoothly defined for $t \in [0, 1]$ (X_0 is not well-defined but $X_0 \circ \phi_0$ is). We conclude by setting $\sigma = \int_0^1 \sigma_t \mathrm{d} t$. $\square$

Proof of Theorem 36. Let us restrict ourselves to a tubular neighborhood $V \simeq \nu Q$ of $Q \subset M$. Under the gauge change $\eta = \pi^* \beta$ for the structure $(\eta, \omega)|_V$, we are under the assumptions of Lemma 38 with $(V, \pi^* \beta)$, $(\eta, \omega)|_V$, and $(\eta_{\nu Q}, \omega_{\nu Q})$. $\square$

REFERENCES

[1] M.-C. Arnaud and J. Fejoz, *Invariant submanifolds of conformal symplectic dynamics*, https://hal.archives-ouvertes.fr/hal-03372201, 2021.

[2] B. Chantraine and E. Murphy, *Conformal symplectic geometry of cotangent bundles*, J. Symplectic Geom. **17** (2019), no. 3, 639–661, https://arxiv.org/abs/1606.00861.

[3] Ivar Ekeland, *Convexity methods in Hamiltonian mechanics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 19, Springer-Verlag, Berlin, 1990. MR 1051888

[4] Hansjörg Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008.

[5] André Haefliger, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **16** (1962), 367–397. MR 189060

[6] Michael-R. Herman, *Inégalités "a priori" pour des tores lagrangiens invariants par des difféomorphismes symplectiques*, Inst. Hautes Études Sci. Publ. Math. (1989), no. 70, 47–101 (1990). MR 1067380

[7] Anatole Katok and Boris Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza.

[8] C. Liverani and M. P. Wojtkowski, *Conformally symplectic dynamics and symmetry of the Lyapunov spectrum*, Comm. Math. Phys. **194** (1998), no. 1, 47–60, https://www.esi.ac.at/static/esiprpr/esi427.pdf.

[9] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.

[10] David Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 27–58.

[11] Sol Schwartzman, *Asymptotic cycles*, Ann. of Math. (2) **66** (1957), 270–284. MR 88720

[12] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. **24** (1976), no. 3-4, 338–351.

[13] Alan Weinstein, *Lectures on symplectic manifolds*, Regional Conference Series in Mathematics, No. 29, American Mathematical Society, Providence, R.I., 1977, Expository lectures from the CBMS Regional Conference held at the University of North Carolina, March 8–12, 1976.

SIMON ALLAIS, UNIVERSITÉ PARIS CITÉ AND SORBONNE UNIVERSITÉ, CNRS, IMJ-PRG, F-75006 PARIS, FRANCE.
Email address: `simon.allais@imj-prg.fr`
URL: `https://webusers.imj-prg.fr/~simon.allais/`

MARIE-CLAUDE ARNAUD, UNIVERSITÉ PARIS CITÉ AND SORBONNE UNIVERSITÉ, CNRS, IMJ-PRG, F-75006 PARIS, FRANCE.
Email address: `marie-Claude.Arnaud@imj-prg.fr`