Solving cubic equation by paper folding

Simon Allais

November 29, 2015

Abstract

In 1936, Margherita Piazzola Beloch showed that paper folding allows to solve every cubic equation, thus extending the field of constructible numbers. In this article we give an elementary proof of this result based on analytic geometry. This proof gives a means to solve cubic equations that we applied to doubling the cube and trisecting the angle.

1 Introduction

Famous classical problems play an important role in the development of mathematics. Some of them required more than two thousand years to be solved. Doubling the cube and trisecting the angle with ruler and compass were among them.

The first serious answer to these ancient problems was given by Pierre-Laurent Wantzel who proved in 1837 that doubling the cube and trisecting the angle can't be done with ruler and compass. [7] As a matter of fact, he showed that cubic equations can't generally be solved with these tools which would be necessary to obtain the desired constructions. On the other hand, in 1853, Tandalam Sundara Rao suggested that paper folding provides at least the same results as ruler and compass. [6] Moreover, in 1936, Margherita Piazzola Beloch showed that doubling and trisecting the angle can be done by paper folding. [1] Her proof was based on the fact that paper folding does allow to solve every cubic equation.

However, these constructions aren't well known. Thus, the purpose of this article is to introduce paper folding, which is an elementary but under recognised subject. We want to give a simple proof of an easy algorithm to solve any cubic equation and, thus, to give a means to double the cube and to trisect the angle by paper folding.

2 Paper folding construction

Paper folding constructions are described as a sequence of basic folds, also called axioms. Here is a list of them (according to Huzita [4]):

- O_1: Given two points p_1 and p_2, we can fold a line connecting them.
- O_2: Given two points p_1 and p_2, we can fold p_1 onto p_2.
- O_3: Given two lines l_1 and l_2, we can fold line l_1 onto l_2.

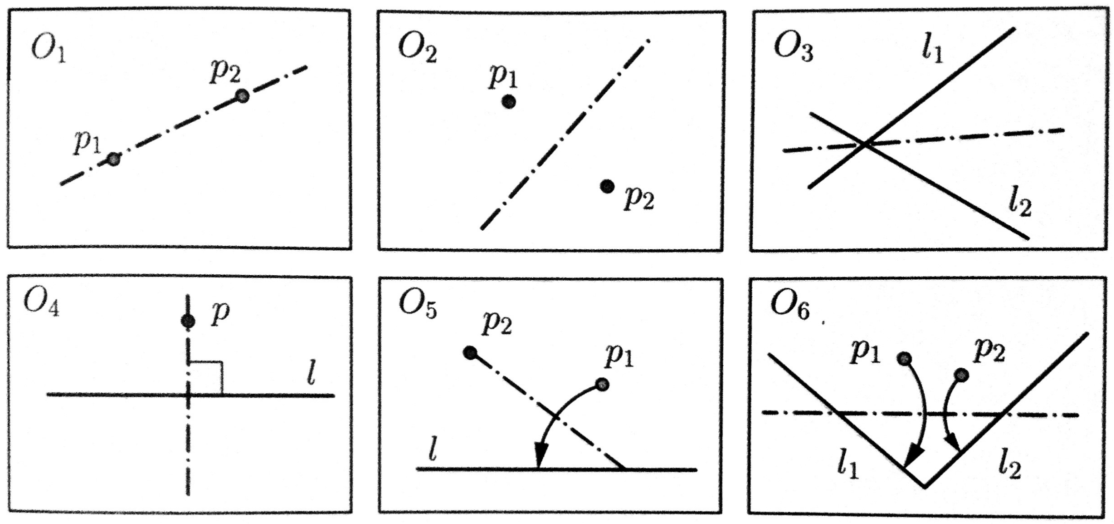

Figure 1: the six basic folds (from [2])

- O_4: Given a point p and a line l, we can make a fold perpendicular to l passing through the point p.
- O_5: Given two points p_1 and p_2 and a line l, we can make a fold that places p_1 onto l and passes through the point p_2.
- O_6: Given two points p_1 and p_2 and two lines l_1 and l_2, we can make a fold that places p_1 onto line l_1 and places p_2 onto l_2.

It is well known that the first five basic folds are equivalent to the ruler and compass constructions (see [5] for instance). Actually, the sixth basic fold is the one which extends the field of constructible numbers. This basic fold is equivalent to finding a common tangent of two parabolas of the plane. [3] As a matter of fact, (p_1, l_1) and (p_2, l_2) can be seen as the pairs focus-directrix which determine parabolas. Then one can remark that folding p_1 onto a point of l_1 gives a tangent of the (p_1, l_1)-parabola.

Actually, finding common tangents of two parabolas is equivalent to solving every cubic equation. [3] We will now give an elementary proof of this using the “paper folding” approach.

3 The link with cubic equations

Given two points A and B and two lines D_0 and D of the plane, we will show that applying O_6 is equivalent to solving a certain cubic equation. The whole construction is illustrated in figure 2.

We can choose an orthogonal frame $(O; \vec{\imath}, \vec{\jmath})$ such that $A = (0, 1)$ and $D_0 = (Ox)$, then $B = (x_B, y_B)$ and $D : ax + by + c = 0$. Let $C(t) = (t, 0)$ (where $t \in \mathbb{R}$) be a moving point of (Ox) and $\Delta(t)$ be the fold leading A onto C (i.e. the perpendicular bisector of $[AC]$). Let $H(t)$ be the

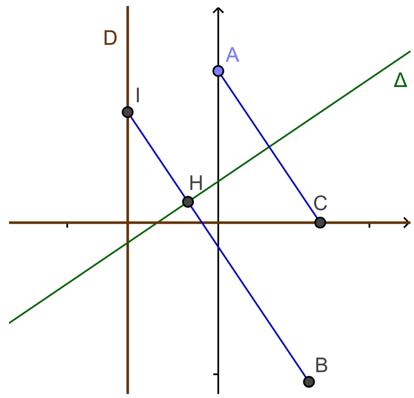

Figure 2: the construction

orthographic projection of B onto $\Delta(t)$ and $I(t)$ be the intersection of (BH) and D (if it exists). We want to know for which t, $\overline{BH} = \overline{HI}$.

Simple calculus gives that $\overline{BH} = \overline{HI}$ is equivalent to

$$at^3 + (b(y_B - 1) + c - ax_B)t^2 + (2(bx_B + ay_B) - a)t + ax_B + b(y_B + 1) + c = 0 \quad (1)$$

which is a cubic equation.

4 Solving cubic equations by paper folding

Given a polynomial on $\mathbb{R}$: $P = aX^3 + bX^2 + cX + d$ with $a \neq 0$, finding its roots is equivalent to finding roots of $Q = \frac{1}{a}P\left(X - \frac{b}{3a}\right) = X^3 + pX + q$. Now we choose to take $b = 0$ in (1), then, with

$$D : x = -\frac{q}{2} \quad \text{and} \quad B = \left(\frac{q}{2}, \frac{p+1}{2}\right),$$

$C(t, 0)$ given by O_6 verifies

$$t^3 + pt + q = Q(t) = 0.$$

Thus, as $\frac{q}{2}$ and $\frac{p+1}{2}$ are easily constructible, we have a simple construction of the roots of Q by paper folding.

5 Doubling the cube and trisecting the angle

Doubling the cube of vertices c is equivalent to constructing the number x such that the cube of vertices xc has a doubled volume. That is to say

$$(xc)^3 = 2c^3 \quad \text{which is equivalent to} \quad x^3 = 2.$$

Thus applying our algorithm with $p = 0$ and $q = 2$, the problem is solved.

Trisecting the angle θ is equivalent to constructing $\cos\frac{\theta}{3}$ which is a solution of $4X^3 - 3X - \cos\theta = 0$. Thus, applying our algorithm with $p = -\frac{3}{4}$ and $q = -\frac{\cos\theta}{4}$, the problem is also solved.

6 Conclusion

Thus we've given a proof which provides a simple means to extend the field of constructible numbers using paper folding in addition to ruler and compass. We can notice that some studies extend these constructions to more complex origami. In this case, basic moves no longer consist in one fold but in multiple folds. These new means of construction extend the field a bit more (see [5] for instance).

References

[1] Margherita Piazzola Beloch. Sul metodo del ripiegamento della carta per la risoluzione dei problemi geometrici. *Periodico di Mathematiche*, pages 104–108, 1936.

[2] Pascal Boyer. *Algèbre et géométrie*. Calvage & Mounet, 2015.

[3] Thomas Hull. Solving cubics with creases: The work of beloch and lill. *American Mathematical Monthly*, pages 307–315, 2011.

[4] Humiaki Huzita. Axiomatic development of origami geometry. *Proceedings of the First International Meeting of Origami Science and Technology*, pages 143–158, 1989.

[5] Robert Lang. One-, two-, and multi-fold origami axioms. 2006.

[6] Tandalam Sundara Rao. *Geometric exercises in paper folding*. 1853.

[7] Pierre-Laurent Wantzel. Recherches sur les moyens de reconnaître si un problème de géométrie peut se résoudre avec la règle et le compas. *Journal de mathématiques pures et appliquées*, 2:366–372, 1837.