SPECTRAL SELECTORS ON LENS SPACES AND APPLICATIONS TO THE GEOMETRY OF THE GROUP OF CONTACTOMORPHISMS

SIMON ALLAIS, PIERRE-ALEXANDRE ARLOVE, AND SHEILA SANDON

Abstract. Using Givental's non-linear Maslov index we define a sequence of spectral selectors on the universal cover of the identity component of the contactomorphism group of any lens space. As applications, we prove that the standard Reeb flow is a geodesic for the discriminant and oscillation norms, and we define a stably unbounded conjugation invariant spectral pseudonorm.

1. Introduction

For any integer $k \geq 2$ and n-tuple $\underline{w} = (w_1, \cdots, w_n)$ of positive integers relatively prime to k, the lens space $L_k^{2n-1}(\underline{w})$ is the quotient of the unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ by the free $\mathbb{Z}_k$-action generated by the map

$$(z_1, \cdots, z_n) \mapsto \left(e^{\frac{2\pi i}{k} \cdot w_1} z_1, \cdots, e^{\frac{2\pi i}{k} \cdot w_n} z_n\right).$$

Since the weights $\underline{w}$ do not play a particular role in the discussion, we denote $L_k^{2n-1}(\underline{w})$ simply by L_k^{2n-1}. We endow L_k^{2n-1} with its canonical contact structure ξ_0, the kernel of the contact form α_0 whose pullback $\bar{\alpha}_0$ by the projection $\mathbb{S}^{2n-1} \to L_k^{2n-1}$ is equal to the pullback of $\sum_{j=1}^{n} x_j dy_j - y_j dx_j$ by the inclusion $\mathbb{S}^{2n-1} \hookrightarrow \mathbb{R}^{2n}$. We denote by $\widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0)$ the universal cover of the identity component $\mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$ of the contactomorphism group. The non-linear Maslov index is a quasimorphism

$$\mu : \widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0) \to \mathbb{Z},$$

defined by Givental [14] for real projective spaces and extended to general lens spaces in [16]. Roughly speaking, it counts with multiplicity the number of intersections of contact isotopies with (a certain subspace of) the space of contactomorphisms that have at least one discriminant point.

Recall that a point p of a contact manifold (M, ξ) is said to be a discriminant point of a contactomorphism ϕ if $\phi(p) = p$ and $(\phi^*\alpha)_p = \alpha_p$ for some (hence any) contact form α for ξ, and is said to be a translated point of ϕ with respect to a contact form α if there exists a real number T (in general not unique) such that p is a discriminant point of $r_{-T}^{\alpha} \circ \phi$, where $\{r_t^{\alpha}\}$ denotes the Reeb flow; such T is then said to be a translation of the translated point p. Discriminant and translated points play a key role in certain proofs of several global rigidity results in contact topology, related in particular to contact non-squeezing [19, 13, 1], orderability [10, 6, 19, 20, 16, 1], and bi-invariant metrics on the contactomorphism group [18, 9, 5]. In particular, Givental's non-linear Maslov index for projective spaces has been used in [10], [21] and [9] respectively to prove that real projective spaces are orderable, satisfy a contact analogue of the Arnold conjecture and have unbounded discriminant and oscillation norms. All these results have then been generalized to lens spaces in [16] (recovering for orderability a result also obtained in [17] and [20]). In the original article of Givental [14], the non-linear Maslov index on projective spaces and a Legendrian version of it have been applied in particular to prove the Weinstein and chord conjectures, and a result on existence of Reeb chords between Legendrian submanifolds Legendrian isotopic to each other. Moreover, an analogue of the non-linear Maslov index for complex projective spaces has been used by Givental [14] and Théret [22] to prove the Arnold conjectures on fixed points of Hamiltonian symplectomorphisms and Lagrangian intersections.

In the present article we use the non-linear Maslov index to define spectral selectors on the universal cover of the identity component of the contactomorphism group of lens spaces, i.e. maps

$$c_j : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \mathbb{R}$$

that associate to every element of $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$ a real number belonging to its action spectrum. Recall that the action spectrum of a contactomorphism ϕ of a contact manifold (M, ξ) with respect to a contact form α is the set $\mathcal{A}_\alpha(\phi)$ of real numbers T that are translations of translated points of ϕ with respect to α. We denote by

$$\Pi : \widetilde{\mathrm{Cont}_0}(M, \xi) \to \mathrm{Cont}_0(M, \xi)$$

the standard projection, which sends an element $\widetilde{\phi} = [\{\phi_t\}_{t\in[0,1]}]$ of $\widetilde{\mathrm{Cont}_0}(M, \xi)$ to ϕ_1, and define the action spectrum of an element $\widetilde{\phi}$ of $\widetilde{\mathrm{Cont}_0}(M, \xi)$ by $\mathcal{A}_\alpha(\widetilde{\phi}) = \mathcal{A}_\alpha\big(\Pi(\widetilde{\phi})\big)$. Let

$$\mathcal{L} : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \widetilde{\mathrm{Cont}_0}\big(\mathbb{S}^{2n-1}, \bar{\xi}_0 = \ker(\bar{\alpha}_0)\big)$$

be the map that sends $\widetilde{\phi} = [\{\phi_t\}_{t\in[0,1]}]$ to the element of $\widetilde{\mathrm{Cont}_0}(\mathbb{S}^{2n-1}, \bar{\xi}_0)$ represented by the lift of $\{\phi_t\}_{t\in[0,1]}$ to $(\mathbb{S}^{2n-1}, \bar{\xi}_0)$. For $\widetilde{\phi} \in \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$ we denote

$$\mathcal{A}(\widetilde{\phi}) = \mathcal{A}_{\alpha_0}(\widetilde{\phi})$$

and

$$\bar{\mathcal{A}}(\widetilde{\phi}) = \mathcal{A}_{\bar{\alpha}_0}\big(\mathcal{L}(\widetilde{\phi})\big) \subset \mathcal{A}(\widetilde{\phi})\,.$$

Since the Reeb flow of α_0 on L_k^{2n-1} is periodic of period $\frac{2\pi}{k}$ and the Reeb flow of $\bar{\alpha}_0$ on $\mathbb{S}^{2n-1}$ is periodic of period 2π, $\mathcal{A}(\widetilde{\phi})$ and $\bar{\mathcal{A}}(\widetilde{\phi})$ are invariant by translation by $\frac{2\pi}{k}$ and 2π respectively. For a real number T we denote

$$\lceil T \rceil_{\frac{2\pi}{k}} = \frac{2\pi}{k}\left\lceil \frac{k}{2\pi}\,T \right\rceil \qquad \text{and} \qquad \lfloor T \rfloor_{\frac{2\pi}{k}} = \frac{2\pi}{k}\left\lfloor \frac{k}{2\pi}\,T \right\rfloor,$$

thus $\lceil T \rceil_{\frac{2\pi}{k}}$ and $\lfloor T \rfloor_{\frac{2\pi}{k}}$ are respectively the smallest multiple of $\frac{2\pi}{k}$ greater or equal than T and the greatest multiple of $\frac{2\pi}{k}$ smaller or equal than T.

Before stating our main result we recall that, since (L_k^{2n-1}, ξ_0) is orderable, the relation $\leq$ on $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$ defined by posing $\widetilde{\phi} \leq \widetilde{\psi}$ if there is a non-negative contact isotopy representing $\widetilde{\psi} \cdot \widetilde{\phi}^{-1}$ is a bi-invariant partial order; this is the partial order that appears in point (vii) below. Recall also that a translated point p of a contactomorphism ϕ of a contact manifold (M, ξ) with respect to a contact form α is said to be non-degenerate for a translation T if there is no vector $X \in T_pM \smallsetminus \{0\}$ such that $(r^\alpha_{-T} \circ \phi)_*(X) = X$ and $dg(X) = 0$, where g is the conformal factor of ϕ, i.e. the function defined by the relation $\phi^*\alpha = e^g\alpha$. In the case of (L_k^{2n-1}, ξ_0) or $(\mathbb{S}^{2n-1}, \bar{\xi}_0)$, if a translated point of a contactomorphism with respect to α_0 or $\bar{\alpha}_0$ is non-degenerate for a certain translation then it is non-degenerate for all the translations; we then just say that it is non-degenerate. For any $T \in \mathbb{R}$ we denote

$$\widetilde{r_T} = [\{r_{Tt}\}_{t\in[0,1]}]\,,$$

where $\{r_t\}$ is the Reeb flow on L_k^{2n-1} with respect to α_0. Moreover, we denote by $\widetilde{\mathrm{id}}$ the identity on $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$.

Our main result is the following theorem.

Theorem 1.1 (Spectral selectors)**.** *There exists a non-decreasing sequence of maps*

$$\big\{\, c_j : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \mathbb{R}\,,\ j \in \mathbb{Z} \,\big\}$$

satisfying the following properties:

(i) Spectrality*:*

$$c_j(\widetilde{\phi}) \in \bar{\mathcal{A}}(\widetilde{\phi})\,.$$

(ii) Normalization*:*

$$c_{-2n+1}(\widetilde{\mathrm{id}}) = c_0(\widetilde{\mathrm{id}}) = 0\,.$$

(iii) Relation with translated points*: if all the translated points of* $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ *are non-degenerate then the spectral selectors* $\{\, c_j(\widetilde{\phi})\,, j \in \mathbb{Z}\,\}$ *are all distinct. On the other hand, if*

$$c_{j-1}(\widetilde{\phi}) < c_j(\widetilde{\phi}) = c_{j+1}(\widetilde{\phi}) = \cdots = c_{j+m}(\widetilde{\phi}) = T < c_{j+m+1}(\widetilde{\phi})$$

for some j *and* $1 \leq m \leq 2n-1$ *and either* k *is even or* j *is odd or* $m > 1$ *then* $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ *has infinitely many translated points of translation* T*.*

(iv) Non-degeneracy*: if*

$$c_{-2n+1}(\widetilde{\phi}) = c_0(\widetilde{\phi}) = 0$$

then $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ *is the identity.*

(v) Composition with the Reeb flow*: for every* $T \in \mathbb{R}$ *we have*

$$c_j(\widetilde{r_T} \cdot \widetilde{\phi}) = c_j(\widetilde{\phi}) + T\,;$$

in particular, $c_{-2n+1}(\widetilde{r_T}) = c_0(\widetilde{r_T}) = T$*.*

(vi) Periodicity*:*

$$c_{j+2n}(\widetilde{\phi}) = c_j(\widetilde{\phi}) + 2\pi\,.$$

(vii) Monotonicity*: if* $\widetilde{\phi} \leq \widetilde{\psi}$ *then* $c_j(\widetilde{\phi}) \leq c_j(\widetilde{\psi})$*.*

(viii) Continuity*: if* $\widetilde{\phi} \cdot \widetilde{\psi}^{-1}$ *is represented by a contact isotopy with Hamiltonian function* $H_t : L_k^{2n-1} \to \mathbb{R}$ *with respect to* α_0 *then*

$$\int_0^1 \min H_t\, dt \leq c_j(\widetilde{\phi}) - c_j(\widetilde{\psi}) \leq \int_0^1 \max H_t\, dt\,.$$

Moreover, each c_j *is continuous with respect to the* $\mathcal{C}^1$*-topology.*

(ix) Triangle inequality*: if either* k *is even or* j *is even then*

$$c_{j+l}\big(\widetilde{\phi} \cdot \widetilde{\psi}\big) \leq c_j\big(\widetilde{\phi}\big) + \Big\lceil c_l\big(\widetilde{\psi}\big)\Big\rceil_{\frac{2\pi}{k}}\,,$$

in particular

$$\Big\lceil c_{j+l}\big(\widetilde{\phi} \cdot \widetilde{\psi}\big)\Big\rceil_{\frac{2\pi}{k}} \leq \Big\lceil c_j\big(\widetilde{\phi}\big)\Big\rceil_{\frac{2\pi}{k}} + \Big\lceil c_l\big(\widetilde{\psi}\big)\Big\rceil_{\frac{2\pi}{k}}\,.$$

(x) Conjugation invariance*:*

$$\Big\lceil c_j\big(\widetilde{\psi} \cdot \widetilde{\phi} \cdot \widetilde{\psi}^{-1}\big)\Big\rceil_{\frac{2\pi}{k}} = \Big\lceil c_j\big(\widetilde{\phi}\big)\Big\rceil_{\frac{2\pi}{k}}\,.$$

(xi) Poincaré duality*:*

$$\Big\lceil c_j\big(\widetilde{\phi}\big)\Big\rceil_{\frac{2\pi}{k}} = -\Big\lfloor c_{-j-(2n-1)}\big(\widetilde{\phi}^{-1}\big)\Big\rfloor_{\frac{2\pi}{k}}\,.$$

Using the Hamiltonian version of the non-linear Maslov index for complex projective spaces ([14, 22, 8]) it is possible to define also spectral invariants

$$\big\{\, c_j : \widetilde{\mathrm{Ham}}(\mathbb{CP}^n, \omega_0) \to \mathbb{R}\,,\ j \in \mathbb{Z}\,\big\}$$

satisfying properties analogue to those of Theorem 1.1, with stronger statements for (ix), (x) and (xi) not involving the $\frac{2\pi}{k}$-floors and ceilings. Such spectral invariants coincide with the ones defined by the first author in [2], and their projections to $\mathbb{S}^1$ coincide with the rotation numbers defined by Théret in [22]. Moreover, their properties are analogue to those satisfied by the spectral invariants defined with Floer homology by Entov and Polterovich in [11]. The fact that in the contact case the statements of the triangle inequality, conjugation invariance and Poincaré duality properties are weaker than in the symplectic case and involve the $\frac{2\pi}{k}$-floors and ceilings is similar to what happens for the spectral selectors of compactly supported contactomorphisms of $(\mathbb{R}^{2n} \times \mathbb{S}^1, \xi_0)$ defined by the third author in [19]. Indeed, these spectral selectors are contact analogues of the spectral selectors of compactly supported Hamiltonian symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ defined by Viterbo in [24], but they satisfy weaker versions of the triangle inequality, conjugation invariance and Poincaré duality properties involving their (integral) floors and ceilings. Roughly speaking, this can be explained as follows. In both cases the contact spectral selectors are generalizations of the symplectic ones, in the sense that the symplectic spectral selectors of Hamiltonian isotopies of $(\mathbb{CP}^{n-1}, \omega_0)$ and $(\mathbb{R}^{2n}, \omega_0)$ coincide with the contact spectral selectors of their lifts to (L_k^{2n-1}, ξ_0) and $(\mathbb{R}^{2n} \times \mathbb{S}^1, \xi_0)$ respectively. The fact that the contact spectral selectors satisfy weaker versions of the triangle inequality, conjugation invariance and Poincaré duality properties involving their floors and ceilings with respect to the period of the Reeb flow is due to the fact that, while the lifts of Hamiltonian isotopies of $(\mathbb{CP}^{n-1}, \omega_0)$ and $(\mathbb{R}^{2n}, \omega_0)$ are exactly the contact isotopies that commute with the standard Reeb flows, general contact isotopies commute with the Reeb flow at time t only when t is a multiple of the period of the Reeb flow. For conjugation invariance, for instance, while in the symplectic case the action spectrum is invariant by conjugation, in the contact case this is in general not true: the translated points of a contactomorphism are in general not in bijection with those of a conjugation. However, if the Reeb flow is periodic then the translated points of translation equal to the period of the Reeb flow are discriminant points, which are invariant by conjugation, and this fact can be used to prove that the corresponding floor and ceiling of the spectral selectors are invariant by conjugation (see also the discussion in [19]).

In [4] the first and second authors have defined invariants c_+ and c_- for elements of the universal cover of any closed orderable contact manifold and for contactomorphisms of any closed contact manifold with orderable contactomorphism group. In the universal cover case, these invariants satisfy all the properties in Theorem 1.1 (including conjugation invariance if the Reeb flow is periodic, and with stronger versions for the triangle inequality and Poincaré duality properties not involving floors and ceilings) except for periodicity (there are only two invariants c_+ and c_-, while we have a sequence c_j related by periodicity), spectrality and (iii). These properties are important for us to obtain the applications discussed below. In particular, spectrality is crucial to obtain Corollary 1.5 and the relation between the pseudonorm ν of Corollary 1.6 and the oscillation norm, while periodicity is used in Corollary 1.6 to show that the induced norm ν_* is bounded (see also Remarks 1.2 and 1.3 below for two more consequences of these properties). The first and second authors also defined in [4] invariants for Legendrian submanifolds and Legendrian isotopies (when the involved spaces are orderable) that do satisfy a spectrality property. Using the Legendrian version of the non-linear Maslov index defined in [14] it should be possible to obtain also a Legendrian version of our spectral selectors, with properties similar to those in Theorem 1.1. However, as far as we can see, the only new application of these spectral selectors with respect to those in [4] would be a better lower bound for the number of Reeb chords between Legendrian submanifolds Legendrian isotopic to each other, but (at least in the case of real projective space) such bound is already given by Givental in [14] just using the non-linear Maslov index.

Remark 1.2. *Properties (vi), (ix) and (xi) imply that each c_j is a quasimorphism.*

Remark 1.3. *Properties (i), (iii) and (vi) imply that every contactomorphism of (L_k^{2n-1}, ξ_0) contact isotopic to the identity has at least n translated points with respect to α_0, and at least $2n$ if k is even or if all the translated points are non-degenerate. We thus recover the corresponding result of* [16]*, but not the optimal bound obtained by the first author in* [3]*, where it is proved that every contactomorphism of (L_k^{2n-1}, ξ_0) contact isotopic to the identity has at least $2n$ translated points.*

Remark 1.4. *Suppose that k is prime. Recall that the cohomological index* $\mathrm{ind}(A)$ *of a subset A of L_k^{2n-1} is the dimension over $\mathbb{Z}_k$ of the image of the map $\check{H}^*(L_k^{2n-1}; \mathbb{Z}_k) \to \check{H}^*(A; \mathbb{Z}_k)$ on Čech cohomology induced by the inclusion $A \hookrightarrow L_k^{2n-1}$. As we will see, property (iii) can be refined in this case as follows: if*

$$c_{j-1}(\widetilde{\phi}) < c_j(\widetilde{\phi}) = c_{j+1}(\widetilde{\phi}) = \cdots = c_{j+m}(\widetilde{\phi}) = T < c_{j+m+1}(\widetilde{\phi})$$

for some j and $1 \leq m \leq 2n-1$ then the set of translated points of translation T of $\Pi(\widetilde{\phi})$ has cohomological index greater or equal than m, and greater or equal than $m+1$ if either $k=2$ or j is odd.

As a first application of Theorem 1.1 we prove that the standard Reeb flow on (L_k^{2n-1}, ξ_0) is a geodesic for the discriminant and oscillation norms introduced in [9]. The definition of the discriminant norm ν_{dis} and of the oscillation pseudonorm ν_{osc} on the universal cover of the identity component of the contactomorphism group of a closed contact manifold (M, ξ) are recalled in Section 4 below, as well as the definition of the discriminant and oscillation lengths of contact isotopies. Recall also from [9, Proposition 3.2] that the oscillation pseudonorm is non-degenerate if and only if (M, ξ) is orderable; in particular, it is thus a norm for lens spaces. As in [5], we say that a contact isotopy of a closed orderable contact manifold is a geodesic for the discriminant or for the oscillation norm if its discriminant or oscillation length is equal to the discriminant or oscillation norm of the element of the universal cover it represents. In other words, a contact isotopy is a geodesic for the discriminant or oscillation norm if it minimizes the discriminant or oscillation length in its homotopy class with fixed endpoints. In [5] it is proved that certain contact isotopies of $(\mathbb{R}^{2n} \times \mathbb{S}^1, \xi_0)$ are geodesics for the discriminant and oscillation norms. We obtain here a similar result for lens spaces, answering a question in [9].

In [9] and [16] respectively it is proved that the discriminant and oscillation norms on real projective spaces and on general lens spaces are unbounded, by showing that the classes in the universal cover represented by higher iterations of the Reeb flow have bigger and bigger discriminant and oscillation norms. More precisely, it is proved in [16] that for every N the discriminant and oscillation norms on $\widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0)$ of $\widetilde{r_{6\pi N}}$ and $\widetilde{r_{20\pi N}}$ respectively are at least equal to $N+1$. Since the discriminant length of $\{r_{6\pi Nt}\}_{t\in[0,1]}$ is $3Nk+1$ and the oscillation length of $\{r_{20\pi Nt}\}_{t\in[0,1]}$ is $10Nk+1$, the results in [16] (as well as the previous ones in [9]) left open the question of whether there exist contact isotopies in the same homotopy class with fixed endpoints as $\{r_{6\pi Nt}\}_{t\in[0,1]}$ or $\{r_{20\pi Nt}\}_{t\in[0,1]}$ having shorter discriminant or oscillation lengths. Note that this is what happens for the sphere $(\mathbb{S}^{2n-1}, \bar{\xi}_0)$: the N-th iteration $\{r_{2\pi Nt}\}_{t\in[0,1]}$ of the Reeb flow $\{r_{2\pi t}\}_{t\in[0,1]}$ of $\bar{\alpha}_0$ has discriminant and oscillation length $N+1$, but by [9, Proposition 4.3] the discriminant norm and the oscillation pseudonorm of $[\{r_{2\pi Nt}\}_{t\in[0,1]}]$ are smaller or equal than 4; in other words, there exist contact isotopies of $(\mathbb{S}^{2n-1}, \bar{\xi}_0)$ in the same homotopy class with fixed endpoints as certain iterations the Reeb flow having strictly shorter discriminant and oscillation length. As an application of Theorem 1.1, in Section 4 we show that for lens spaces this is not possible. More precisely, we show that Theorem 1.1 implies the following result.

Corollary 1.5 (Non-shortening of the standard Reeb flow). *For every real number T, the Reeb flow $\{r_{Tt}\}_{t\in[0,1]}$ on (L_k^{2n-1}, ξ_0) is a geodesic for the discriminant and oscillation norms. In particular,*

$$\nu_{\mathrm{dis}}(\widetilde{r_T}) = \nu_{\mathrm{osc}}(\widetilde{r_T}) = \left\lfloor \frac{k}{2\pi} T \right\rfloor + 1 \,.$$

Using the spectral selectors of Theorem 1.1 we also define a stably unbounded conjugation invariant pseudonorm on $\widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0)$. More precisely, posing $c_- = c_{-2n+1}$ and $c_+ = c_0$ we prove the following result.

Corollary 1.6 (Spectral pseudonorm). *The map $\nu : \widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0) \to \frac{2\pi}{k} \cdot \mathbb{Z}$ defined by*

$$\nu(\widetilde{\phi}) = \max \left\{ \left\lceil c_+(\widetilde{\phi}) \right\rceil_{\frac{2\pi}{k}} \,, \, - \left\lfloor c_-(\widetilde{\phi}) \right\rfloor_{\frac{2\pi}{k}} \right\}$$

is a stably unbounded conjugation invariant pseudonorm, which is compatible with the partial order $\leq$ and satisfies $\nu(\widetilde{\phi}) \leq \frac{2\pi}{k} \cdot \nu_{\mathrm{osc}}(\widetilde{\phi})$ for every $\widetilde{\phi} \in \widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0)$. The induced pseudonorm ν_* on $\mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$ is non-degenerate and bounded.

Finally, we remark that the spectral selectors of Theorem 1.1 can also be used as in [4] to define a time function on $\widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0)$, i.e. a function

$$\tau : \widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0) \to \mathbb{R}$$

that is continuous with respect to the $\mathcal{C}^1$-topology and satisfies $\tau(\widetilde{\phi}) < \tau(\widetilde{\psi})$ whenever $\widetilde{\phi} \leq \widetilde{\psi}$ with $\widetilde{\phi} \neq \widetilde{\psi}$. Such function can be defined by

$$\tau(\widetilde{\phi}) = \left(\sum_j \frac{1}{2^j \max(1, |c_0(\widetilde{\psi}_j)|)} \right)^{-1} \sum_j \frac{c_0(\widetilde{\phi} \cdot \widetilde{\psi}_j)}{2^j \max(1, |c_0(\widetilde{\psi}_j)|)} \,,$$

where $(\widetilde{\psi}_j)_{j \geq 1}$ is any dense sequence in $\widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0)$ with respect to the $\mathcal{C}^1$-topology. As in [4], the time function τ satisfies moreover $\tau(\widetilde{r_T} \cdot \widetilde{\phi}) = T + \tau(\widetilde{\phi})$ for all T and $\widetilde{\phi}$.

The article is organized as follows. In Section 2 we recall the definition of the non-linear Maslov index and discuss the properties that are needed for the construction of the spectral selectors. In Section 3 we define the spectral selectors and prove Theorem 1.1. In Section 4 we prove Corollary 1.5, and in Section 5 we prove Corollary 1.6.

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2. The non-linear Maslov index

In this section we recall the definition of the non-linear Maslov index

$$\mu : \widetilde{\mathrm{Cont}}_0(L_k^{2n-1}, \xi_0) \to \mathbb{Z}$$

following the presentation in [16], to which we refer for more details. We also discuss the properties of the non-linear Maslov index that are needed for the construction of the spectral selectors. Several of these properties do not appear in [16], and so we include detailed proofs.

As in [16], we first define the non-linear Maslov index assuming that k is prime and then obtain the general case (Proposition 2.12) by pullback. Assume thus for now that k is prime.

The construction of the non-linear Maslov index is based on generating functions. Recall that a function $F : E \to \mathbb{R}$ defined on the total space of a fibre bundle $p : E \to B$ is said to be a generating function if the differential $dF : E \to T^*E$ is transverse to the fibre conormal bundle N_E^*, the space of points (e, η) of T^*E such that η vanishes on the kernel of $dp(e)$. Then the set $\Sigma_F = (dF)^{-1}(N_E^*)$ of fibre critical points of F is a submanifold of E, and the map

$$i_F : \Sigma_F \to T^*B \,,\ e \mapsto \big(p(e), v^*(e)\big)$$

defined by posing $v^*(e)(X) = dF(\widehat{X})$ for $X \in T_{p(e)}B$, where $\widehat{X}$ is any vector in T_eE with $dp(e)(\widehat{X}) = X$, is a Lagrangian immersion with respect to the canonical symplectic form ω_{can} on T^*B. If i_F is an embedding then F is said to be a generating function of the Lagrangian submanifold $i_F(\Sigma_F)$ of $(T^*B, \omega_{\mathrm{can}})$. A function F is said to be a generating function of a symplectomorphism Φ of $(\mathbb{R}^{2n}, \omega_0)$ if it is a generating function of the Lagrangian submanifold of $(T^*\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ that is the image of the graph of Φ by the symplectomorphism $\tau : \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \to T^*\mathbb{R}^{2n}$ defined by

$$\tau(x, y, X, Y) = \Big(\frac{x + X}{2}, \frac{y + Y}{2}, Y - y, x - X\Big) \,.$$

Any contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of (L_k^{2n-1},ξ_0) starting at the identity can be uniquely lifted to a $\mathbb{Z}_k$-equivariant contact isotopy $\{\bar\phi_t\}_{t\in[0,1]}$ of $(\mathbb{S}^{2n-1},\bar\xi_0)$ starting at the identity, which in turn can be uniquely extended to a conical Hamiltonian isotopy $\{\Phi_t\}_{t\in[0,1]}$ of $(\mathbb{R}^{2n},\omega_0)$, i.e. a 1-parameter family of $(\mathbb{Z}_k\times\mathbb{R}_{>0})$-equivariant homeomorphisms of $\mathbb{R}^{2n}$ that is a Hamiltonian isotopy on $\mathbb{R}^{2n}\smallsetminus\{0\}$. If M is a multiple of n then we say that a function $F:\mathbb{R}^{2M}\to\mathbb{R}$ is conical if it is $\mathcal{C}^1$ with Lipschitz differential, homogeneous of degree 2 with respect to the radial action of $\mathbb{R}_{>0}$ on $\mathbb{R}^{2M}$, and invariant by the diagonal action of $\mathbb{Z}_k$ on $\mathbb{R}^{2M}$. We say that a conical function $F:E\to\mathbb{R}$ defined on the total space of a trivial vector bundle $E=\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\to\mathbb{R}^{2n}$ is a conical generating functions of a conical symplectomorphism Φ of $(\mathbb{R}^{2n},\omega_0)$, i.e. a $(\mathbb{Z}_k\times\mathbb{R}_{>0})$-equivariant homeomorphism of $\mathbb{R}^{2n}$ that is a symplectomorphism on $\mathbb{R}^{2n}\smallsetminus\{0\}$, if it is smooth near its fibre critical points other than the origin, $dF:E\to T^*E$ is transverse to the fibre conormal bundle N_E^* except possibly at the origin, and i_F is a homeomorphism between Σ_F and the image of the graph of Φ by τ. We say that $F_t:\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\to\mathbb{R}$, $t\in[0,1]$, is a family of conical generating functions for a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of (L_k^{2n-1},ξ_0) starting at the identity if for every t the function F_t is a conical generating function of Φ_t, where $\{\Phi_t\}$ denotes the conical Hamiltonian isotopy of $(\mathbb{R}^{2n},\omega_0)$ lifting $\{\phi_t\}$, and the map $(e,t)\mapsto F_t(e)$ is $\mathcal{C}^1$ with locally Lipschitz differential and smooth near (e,t) whenever e is a fibre critical point of F_t other than the origin. We say that F_t, $t\in[0,1]$, is a based family of conical generating functions for $\{\phi_t\}_{t\in[0,1]}$ if moreover F_0 is equivalent to the zero function on $\mathbb{R}^{2n}$, where we consider on the set of conical generating functions the smallest equivalence relation under which two such functions are equivalent if they differ by a stabilization (i.e. replacing a conical generating function $F:\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\to\mathbb{R}$ by $F\oplus Q:\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\times\mathbb{R}^{2nN'}\to\mathbb{R}$ for a non-degenerate $\mathbb{Z}_k$-invariant quadratic form Q on $\mathbb{R}^{2nN'}$) or by a fibre preserving conical homeomorphism (i.e. a $(\mathbb{Z}_k\times\mathbb{R}_{>0})$-equivariant homeomorphism of $\mathbb{R}^{2n}\times\mathbb{R}^{2nN}$ that takes each fibre $\{z\}\times\mathbb{R}^{2nN}$ to itself) that restricts to a diffeomorphism between neighborhoods of fibre critical points other than the origin. It is proved in [16, Proposition 2.14] that any contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of (L_k^{2n-1},ξ_0) starting at the identity has a based family $F_t:\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\to\mathbb{R}$ of conical generating functions.

A conical function $F:\mathbb{R}^{2M}\to\mathbb{R}$ induces uniquely a function $f:L_k^{2M-1}\to\mathbb{R}$, which is $\mathcal{C}^1$ with Lipschitz differential. All the critical points of F have critical value zero and come in $(\mathbb{Z}_k\times\mathbb{R}_{>0})$-families; moreover, there is a 1–1 correspondence between the $(\mathbb{Z}_k\times\mathbb{R}_{>0})$-families of critical points of F and the critical points of critical value zero of f. If F is a conical generating function of a conical symplectomorphism Φ whose restriction $\bar\phi$ to $\mathbb{S}^{2n-1}$ projects to a contactomorphism ϕ of (L_k^{2n-1},ξ_0) then there is a 1–1 correspondence between the critical points of critical value zero of f and the discriminant points of ϕ that lift to discriminant points of $\bar\phi$. In order to detect discriminant points of contactomorphisms of (L_k^{2n-1},ξ_0) we thus study the topology of the sublevel set at zero of the functions on (possibly higher dimensional) lens spaces induced by the corresponding conical generating functions. The topological invariant that we use for this is the cohomological index for subsets of lens spaces: for a subset A of L_k^{2M-1} such index, which we denote by $\mathrm{ind}(A)$, is the dimension over $\mathbb{Z}_k$ of the image of the map $\check H^*(L_k^{2M-1};\mathbb{Z}_k)\to\check H^*(A;\mathbb{Z}_k)$ on Čech cohomology induced by the inclusion $A\hookrightarrow L_k^{2M-1}$. For any conical function $F:\mathbb{R}^{2M}\to\mathbb{R}$ we thus define

$$\mathrm{ind}(F)=\mathrm{ind}(\{f\leq 0\})\,.$$

The following property is proved in [16, Corollary 3.15] (cf [16, Proposition 3.14, Proposition 3.9 (v) and Remark 3.11] for the case $k=2$).

Lemma 2.1. *For any two conical functions F and G we have*

$$|\,\mathrm{ind}(F\oplus G)-\mathrm{ind}(F)-\mathrm{ind}(G)\,|\leq 1\,,$$

and

$$\mathrm{ind}(F\oplus G)=\mathrm{ind}(F)+\mathrm{ind}(G)$$

if either $k=2$ or $\mathrm{ind}(F)$ is even or $\mathrm{ind}(G)$ is even.

For a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of (L_k^{2n-1},ξ_0) we define

$$\mu(\{\phi_t\}_{t\in[0,1]}) = \mathrm{ind}(F_0) - \mathrm{ind}(F_1)\,,$$

where F_t, $t\in[0,1]$, is any based family of conical generating functions for $\{\phi_t\}_{t\in[0,1]}$. It is proved in [16, Proposition 2.20] that any two based families of conical generating functions for $\{\phi_t\}_{t\in[0,1]}$ are equivalent, where we consider on the set of based families of conical generating functions the smallest equivalence relation under which two such families are equivalent if they differ by a stabilization (i.e. replacing $F_t:\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\to\mathbb{R}$ by $F_t\oplus Q:\mathbb{R}^{2n}\times\mathbb{R}^{2nN}\times\mathbb{R}^{2nN'}\to\mathbb{R}$ for a non-degenerate $\mathbb{Z}_k$-invariant quadratic form Q on $\mathbb{R}^{2nN'}$) or by a 1-parameter family of fibre preserving conical homeomorphism that restrict to diffeomorphisms between neighborhoods of fibre critical points other than the origin. Since, for $k>2$, $\mathrm{ind}(Q)$ is even for every $\mathbb{Z}_k$-invariant quadratic form Q ([16, Remark 3.13]), it thus follows from Lemma 2.1 that $\mu(\{\phi_t\}_{t\in[0,1]})$ is well-defined, i.e. it does not depend on the choice of a based family of conical generating functions. Moreover, it is proved in [16] (as a consequence of [16, Proposition 2.21]) that μ descends to a map

$$\mu:\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)\to\mathbb{Z}\,.$$

Example 2.2. *By definition we have $\mu(\widetilde{\mathrm{id}})=0$. By* [16, Example 4.1]*, if $\widetilde{\phi}$ is small enough in the $\mathcal{C}^1$-topology then $0\le\mu(\widetilde{\phi})\le 2n$, and if moreover $\widetilde{\phi}$ is positive then $\mu(\widetilde{\phi})=2n$. In particular, for $\epsilon>0$ small enough we have $\mu(\widetilde{r_\epsilon})=2n$. By* [16, Example 4.13]*, for every integer m we have $\mu(\widetilde{r_{2\pi m}})=2nm$.*

It is proved in [16, Theorem 1.4 (i) and Remark 1.7] that for any two elements $\widetilde{\phi}$ and $\widetilde{\psi}$ of $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$ we have

$$|\,\mu(\widetilde{\phi}\cdot\widetilde{\psi})-\mu(\widetilde{\phi})-\mu(\widetilde{\psi})\,|\le 2n+1\,, \tag{1}$$

and

$$|\,\mu(\widetilde{\phi}\cdot\widetilde{\psi})-\mu(\widetilde{\phi})-\mu(\widetilde{\psi})\,|\le 2n \tag{2}$$

if $k=2$; in particular, $\mu:\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)\to\mathbb{Z}$ is a quasimorphism. The non-linear Maslov index also satisfies the following triangle inequality, which is not proved in [16].

Proposition 2.3. *For any two elements $\widetilde{\phi}$ and $\widetilde{\psi}$ of $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$ we have*

$$\mu(\widetilde{\phi}\cdot\widetilde{\psi})\le\mu(\widetilde{\phi})+\mu(\widetilde{\psi})+1\,,$$

and

$$\mu(\widetilde{\phi}\cdot\widetilde{\psi})\le\mu(\widetilde{\phi})+\mu(\widetilde{\psi})$$

if either $k=2$ or $\mu(\widetilde{\phi})$ is even or $\mu(\widetilde{\psi})$ is even.

Before proving Proposition 2.3, recall ([16, Proposition 2.10]) that if $F\colon\mathbb{R}^{2n}\times\mathbb{R}^{2nN_1}\to\mathbb{R}$ and $G\colon\mathbb{R}^{2n}\times\mathbb{R}^{2nN_2}\to\mathbb{R}$ are conical generating functions for conical symplectomorphisms Φ and Ψ respectively, then the function $F\sharp G\colon\mathbb{R}^{2n}\times(\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2nN_1}\times\mathbb{R}^{2nN_2})\to\mathbb{R}$ defined by

$$F\sharp G\,(q;\zeta_1,\zeta_2,\nu_1,\nu_2)=F(\zeta_1,\nu_1)+G(\zeta_2,\nu_2)-2\,\langle\zeta_2-q,i(\zeta_1-q)\rangle$$

is a conical generating function of $\Psi\circ\Phi$, and ([16, Proposition 2.26]) there is a linear $(\mathbb{Z}_k\times\mathbb{R}_{>0})$-equivariant injection

$$\iota\colon\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2nN_1}\times\mathbb{R}^{2nN_2}\to\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2n}\times\mathbb{R}^{2nN_1}\times\mathbb{R}^{2nN_2}$$

such that $(F\sharp G)\circ\iota=F\oplus G$. Since the cohomological index is monotone, i.e. $\mathrm{ind}(A)\le\mathrm{ind}(B)$ if $A\subset B$ [16, Proposition 3.9 (i)], we deduce that

$$\mathrm{ind}(F\oplus G)\le\mathrm{ind}(F\sharp G)\,. \tag{3}$$

Recall also ([16, Lemma 4.2]) that if F and G are equivalent to the zero function then

$$\mathrm{ind}(F\sharp G)=\mathrm{ind}(F)+\mathrm{ind}(G)\,.$$

Proof of Proposition 2.3. Let F_t and G_t be based families of conical generating functions for contact isotopies $\{\phi_t\}_{t\in[0,1]}$ and $\{\psi_t\}_{t\in[0,1]}$ representing $\widetilde{\phi}$ and $\widetilde{\psi}$ respectively. Then, by [16, Proposition 2.10 and Remark 2.14], $G_t \sharp F_t$ is a based family of conical generating functions for $\{\phi_t \circ \psi_t\}_{t\in[0,1]}$, and thus

$$\mu(\widetilde{\phi}\cdot\widetilde{\psi}) = \mathrm{ind}(G_0 \sharp F_0) - \mathrm{ind}(G_1 \sharp F_1)\,.$$

Since F_0 and G_0 are equivalent to the zero function, we have $\mathrm{ind}(G_0 \sharp F_0) = \mathrm{ind}(G_0) + \mathrm{ind}(F_0)$. Moreover, by (3) we have

$$-\,\mathrm{ind}(G_1 \sharp F_1) \leq -\,\mathrm{ind}(G_1 \oplus F_1)\,.$$

Using Lemma 2.1 we thus deduce that

$$\mu(\widetilde{\phi}\cdot\widetilde{\psi}) \leq \mathrm{ind}(F_0) + \mathrm{ind}(G_0) - \mathrm{ind}(F_1) - \mathrm{ind}(G_1) + 1 = \mu(\widetilde{\phi}) + \mu(\widetilde{\psi}) + 1\,,$$

and that

$$\mu(\widetilde{\phi}\cdot\widetilde{\psi}) \leq \mu(\widetilde{\phi}) + \mu(\widetilde{\psi})$$

if either $k=2$ or $\mathrm{ind}(F_1)$ is even or $\mathrm{ind}(G_1)$ is even, hence (since, for $k>2$, $\mathrm{ind}(F_0)$ and $\mathrm{ind}(G_0)$ are even by [16, Remark 3.13]) if either $k=2$ or $\mu(\widetilde{\phi})$ is even or $\mu(\widetilde{\psi})$ is even. $\square$

It is proved in [14] that

$$\mu(\widetilde{\phi}\,\cdot\,\widetilde{\psi}) = \mu(\widetilde{\phi}) + \mu(\widetilde{\psi})$$

if either $\widetilde{\phi}$ or $\widetilde{\psi}$ are in $\pi_1\big(\,\mathrm{Cont}_0(\mathbb{RP}^{2n-1},\xi_0)\big)$. For our applications we only need this property in the case when one of the factors is the Reeb flow.

Proposition 2.4. *For every element $\widetilde{\phi}$ of $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$ and every integer m we have*

$$\mu(\widetilde{\phi}\,\cdot\,\widetilde{r_{2\pi m}}) = \mu(\widetilde{r_{2\pi m}}\,\cdot\,\widetilde{\phi}) = 2nm + \mu(\widetilde{\phi})\,.$$

Proof. Since $\widetilde{\phi}\,\cdot\,\widetilde{r_{2\pi m}} = \widetilde{r_{2\pi m}}\,\cdot\,\widetilde{\phi}$, it is enough to prove that $\mu(\widetilde{\phi}\,\cdot\,\widetilde{r_{2\pi m}}) = 2nm + \mu(\widetilde{\phi})$. We represent $\widetilde{\phi}\,\cdot\,\widetilde{r_{2\pi m}}$ by the concatenation

$$\{\varphi_t\}_{t\in[0,1]} = \{r_{4\pi mt}\}_{t\in[0,\frac{1}{2}]} \sqcup \{\phi_{2t-1}\}_{t\in[\frac{1}{2},1]}\,,$$

where $\{\phi_t\}_{t\in[0,1]}$ is a contact isotopy representing $\widetilde{\phi}$, and we consider a based family F_t, $t\in[0,1]$, of conical generating functions for $\{\varphi_t\}_{t\in[0,1]}$ so that F_t, $t\in[0,\frac{1}{2}]$, is a family of quadratic forms generating $\{r_{4\pi mt}\}_{t\in[0,\frac{1}{2}]}$ (cf. [16, Proposition 4.9]). Since $\{r_{4\pi mt}\}_{t\in[0,\frac{1}{2}]}$ is a loop, by [16, Lemma 4.10] the quadratic form $F_{\frac{1}{2}}$ is equivalent to the zero function, and so

$$\mu(\{\phi_t\}_{t\in[0,1]}) = \mathrm{ind}(F_{\frac{1}{2}}) - \mathrm{ind}(F_1)\,.$$

We thus have

$$\mu(\widetilde{\phi}\,\cdot\,\widetilde{r_{2\pi m}}) = \mathrm{ind}(F_0) - \mathrm{ind}(F_1) = \mathrm{ind}(F_0) - \mathrm{ind}(F_{\frac{1}{2}}) + \mathrm{ind}(F_{\frac{1}{2}}) - \mathrm{ind}(F_1)$$

$$= \mu(\widetilde{r_{2\pi m}}) + \mu(\widetilde{\phi}) = 2nm + \mu(\{\phi_t\}_{t\in[0,1]})\,,$$

where the last equality follows from Example 2.2. $\square$

In the next section we also need the following result.

Proposition 2.5. *For every $\widetilde{\phi}\in\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$ and every $T_0\in\mathbb{R}$ there exists $\epsilon>0$ such that $\mu(\widetilde{r_{-T}}\cdot\widetilde{\phi}) = \mu(\widetilde{r_{-T_0}}\cdot\widetilde{\phi})$ for every $T\in[T_0,T_0+\epsilon)$.*

Before proving Proposition 2.5, recall that the cohomological index is continuous ([16, Proposition 3.9 (ii)]): every closed subset A of L_k^{2M-1} has a neighborhood $\mathcal{U}$ such that if $A\subset\mathcal{V}\subset\mathcal{U}$ then $\mathrm{ind}(\mathcal{V}) = \mathrm{ind}(A)$.

Proof of Proposition 2.5. Let $\{\phi_t\}_{t\in[0,1]}$ be a contact isotopy representing $\widetilde{\phi}$, and let F_t, $t \in [0,1]$, be a based family of conical generating functions for the concatenation

$$\{\phi_{2t}\}_{t\in[0,\frac{1}{2}]} \sqcup \{r_{-(2t-1)(T_0+\epsilon')}\}_{t\in[\frac{1}{2},1]}$$

for some $\epsilon' > 0$. Then for every $T \in [T_0, T_0+\epsilon')$ we have

$$\mu(\widetilde{r_{-T}} \cdot \widetilde{\phi}) = \mathrm{ind}\left(\{f_0 \leq 0\}\right) - \mathrm{ind}\left(\left\{f_{\frac{1}{2}\left(\frac{T}{T_0+\epsilon'}+1\right)} \leq 0\right\}\right).$$

By monotonicity of generating functions ([16, Proposition 2.23]) we can assume that $\frac{df_t}{dt} \leq 0$ for all $t \in [\frac{1}{2}, 1]$, and so

$$\left\{f_{\frac{1}{2}(\frac{T}{T_0+\epsilon'}+1)} \leq 0\right\} \subset \left\{f_{\frac{1}{2}(\frac{T'}{T_0+\epsilon'}+1)} \leq 0\right\}$$

for $T, T' \in [T_0, T_0+\epsilon')$ with $T \leq T'$. By continuity of the cohomological index, there is a neighborhood $\mathcal{U}$ of $\left\{f_{\frac{1}{2}\left(\frac{T_0}{T_0+\epsilon'}+1\right)} \leq 0\right\}$ such that if $\left\{f_{\frac{1}{2}\left(\frac{T_0}{T_0+\epsilon'}+1\right)} \leq 0\right\} \subset \mathcal{V} \subset \mathcal{U}$ then

$$\mathrm{ind}(\mathcal{V}) = \mathrm{ind}\left(\left\{f_{\frac{1}{2}\left(\frac{T_0}{T_0+\epsilon'}+1\right)} \leq 0\right\}\right).$$

For every $T \in [T_0, T_0+\epsilon)$ with $\epsilon < \epsilon'$ small enough we have

$$\left\{f_{\frac{1}{2}\left(\frac{T_0}{T_0+\epsilon'}+1\right)} \leq 0\right\} \subset \left\{f_{\frac{1}{2}\left(\frac{T}{T_0+\epsilon'}+1\right)} \leq 0\right\} \subset \mathcal{U},$$

and so

$$\mathrm{ind}\left(\left\{f_{\frac{1}{2}\left(\frac{T}{T_0+\epsilon'}+1\right)} \leq 0\right\}\right) = \mathrm{ind}\left(\left\{f_{\frac{1}{2}\left(\frac{T_0}{T_0+\epsilon'}+1\right)} \leq 0\right\}\right),$$

i.e. $\mu(\widetilde{r_{-T}} \cdot \widetilde{\phi}) = \mu(\widetilde{r_{-T_0}} \cdot \widetilde{\phi})$ as we wanted. □

It is proved in [16, Proposition 4.21] that if $\{\phi_t\}_{t\in[0,1)}$ is a non-negative (respectively non-positive) contact isotopy then $\mu([\{\phi_t\}_{t\in[0,1)}]) \geq 0$ (respectively $\mu([\{\phi_t\}_{t\in[0,1)}]) \leq 0$). We actually have the following result.

Proposition 2.6. *If $\widetilde{\phi} \leq \widetilde{\psi}$ then $\mu(\widetilde{\phi}) \leq \mu(\widetilde{\psi})$.*

Proof. Assume that $\widetilde{\phi} \leq \widetilde{\psi}$. Then $\widetilde{\psi}$ can be represented by the concatenation

$$\{\psi_t\}_{t\in[0,1]} = \{\phi_{2t}\}_{t\in[0,\frac{1}{2}]} \sqcup \{\chi_{2t-1}\}_{t\in[\frac{1}{2},1]}$$

of a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ representing $\widetilde{\phi}$ and a non-negative contact isotopy $\{\chi_t\}_{t\in[0,1]}$. Let $F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ be a based family of generating functions for $\{\psi_t\}_{t\in[0,1]}$. By monotonicity of generating function ([16, Proposition 2.23]) we can assume that $F_1 \geq F_{\frac{1}{2}}$ and so, by monotonicity of the cohomological index ([16, Proposition 3.9 (i)]), $\mathrm{ind}(F_1) \leq \mathrm{ind}(F_{\frac{1}{2}})$. We thus have

$$\mu(\widetilde{\psi}) = \mu(\widetilde{\phi}) + \mathrm{ind}(F_{\frac{1}{2}}) - \mathrm{ind}(F_1) \geq \mu(\widetilde{\phi}).$$

□

Remark 2.7. *It follows from Proposition 2.5 and Proposition 2.6 that the map $T \mapsto \mu(\widetilde{r_{-T}} \cdot \widetilde{\phi})$ is lower semi-continuous, i.e. $\{T \in \mathbb{R} \,|\, \mu(\widetilde{r_{-T}} \cdot \widetilde{\phi}) \leq y\}$ is closed for every $y \in \mathbb{R}$.*

If F_t is a based family of conical generating functions for a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of (L_k^{2n-1}, ξ_0) then for every t there is a 1–1 correspondence between the critical points of critical value zero of f_t and the discriminant points of ϕ_t that lift to discriminant points of $\bar{\phi}_t$, where $\{\bar{\phi}_t\}_{t\in[0,1]}$ is the lift of $\{\phi_t\}_{t\in[0,1]}$ to $(\mathbb{S}^{2n-1}, \bar{\xi}_0)$. Since the non-linear Maslov index $\mu(\{\phi_t\}_{t\in[0,1]})$ counts, with multiplicity given by the change in the cohomological index of the sublevel sets $\{f_t \leq 0\}$, the critical points of f_t with critical value zero as t varies in $[0,1]$, its value is related to the presence of discriminant points of $\bar{\phi}_t$ for $t \in [0,1]$. More precisely, we have the following result ([16, Theorem 1.4 (iii)]).

Proposition 2.8. *Let $\{\phi_t\}_{t\in[0,1]}$ be a contact isotopy of (L_k^{2n-1},ξ_0) starting at the identity, and let $[t_0,t_1]$ be a subinterval of $[0,1]$. If $\mu(\{\phi_t\}_{t\in[0,t_0]})\neq\mu(\{\phi_t\}_{t\in[0,t_1]})$ then there is $\underline{t}\in[t_0,t_1]$ such that $\bar{\phi}_{\underline{t}}$ has discriminant points. If moreover*[1] *the map $s\mapsto\mu(\{\phi_t\}_{t\in[0,s]})$ is constant on $[t_0,\underline{t})$ and on $(\underline{t},t_1]$ and all the discriminant points of $\bar{\phi}_{\underline{t}}$ are non-degenerate then*

$$|\,\mu(\{\phi_t\}_{t\in[0,t_0]})-\mu(\{\phi_t\}_{t\in[0,t_1]})\,|\leq 1\,.$$

In particular, it follows from Example 2.2, Proposition 2.4 and the first statement of Proposition 2.8 that for every real number T we have

$$\mu(\widetilde{r_T})=2n\left\lceil\frac{T}{2\pi}\right\rceil\,. \tag{4}$$

We also have the following result.

Proposition 2.9. *Let $\{\phi_t\}_{t\in[0,1]}$ be a contact isotopy of (L_k^{2n-1},ξ_0) starting at the identity, and let $[t_0,t_1]$ be a subinterval of $[0,1]$. Assume that there is $\underline{t}\in[t_0,t_1]$ such that $\bar{\phi}_{\underline{t}}$ has discriminant points, and denote by $\Delta(\phi_{\underline{t}})\subset L_k^{2n-1}$ the set of discriminant points of $\phi_{\underline{t}}$. Assume also that the map $s\mapsto\mu(\{\phi_t\}_{t\in[0,s]})$ is constant on $[t_0,\underline{t})$ and on $(\underline{t},t_1]$. Then*

$$|\,\mu(\{\phi_t\}_{t\in[0,t_0]})-\mu(\{\phi_t\}_{t\in[0,t_1]})\,|\leq\operatorname{ind}\big(\Delta(\phi_{\underline{t}})\big)+1\,,$$

and

$$|\,\mu(\{\phi_t\}_{t\in[0,t_0]})-\mu(\{\phi_t\}_{t\in[0,t_1]})\,|\leq\operatorname{ind}\big(\Delta(\phi_{\underline{t}})\big)$$

if either $k=2$ or if $\{\phi_t\}_{t\in[0,1]}$ is negative and $\mu(\{\phi_t\}_{t\in[0,t_0]})$ is even.

Before proving Proposition 2.9, recall that the cohomological index is subadditive in the following sense ([16, Proposition 3.9 (iv)]): for any two closed subsets A and B of L_k^{2M-1} we have

$$\operatorname{ind}(A\cup B)\leq\operatorname{ind}(A)+\operatorname{ind}(B)+1\,,$$

and

$$\operatorname{ind}(A\cup B)\leq\operatorname{ind}(A)+\operatorname{ind}(B)$$

if either $k=2$ or $\operatorname{ind}(A)$ is even or $\operatorname{ind}(B)$ is even.

Proof of Proposition 2.9. The first inequality and the second one in the case $k=2$ are proved in [16, Proposition 4.15]. Suppose thus that $\{\phi_t\}_{t\in[0,1]}$ is negative and $\mu(\{\phi_t\}_{t\in[0,t_0]})$ is even. By monotonicity of generating functions ([16, Proposition 2.23]), there is a based family of conical generating functions F_t for $\{\phi_t\}_{t\in[0,1]}$ with $\frac{df_t}{dt}\leq 0$. The proof of [16, Proposition 2.23] actually shows that for every t either $\frac{df_t}{dt}<0$ or $\frac{df_t}{dt}=0$. We can thus assume that either $\{f_{t_0}\leq 0\}=\{f_{\underline{t}}\leq 0\}$ or $\{f_{t_0}\leq 0\}$ is included in the interior of $\{f_{\underline{t}}\leq 0\}$. In the first case, by continuity of the cohomological index and since the map $s\mapsto\mu(\{\phi_t\}_{t\in[0,s]})$ is constant on $(\underline{t},t_1]$ we have $\mu(\{\phi_t\}_{t\in[0,t_0]})=\mu(\{\phi_t\}_{t\in[0,t_1]})$, which implies the desired inequality

$$|\,\mu(\{\phi_t\}_{t\in[0,t_0]})-\mu(\{\phi_t\}_{t\in[0,t_1]})\,|\leq\operatorname{ind}\big(\Delta(\phi_{\underline{t}})\big)\,.$$

In the second case, take $\epsilon,\epsilon'>0$ such that

$$\{f_{t_0}\leq 0\}\subset\{f_{\underline{t}}\leq-\epsilon\}\subset\{f_{\underline{t}-\epsilon'}\leq 0\}\subset\{f_{\underline{t}}\leq 0\}\,.$$

By monotonicity of the cohomological index and since $s\mapsto\mu(\{\phi_t\}_{t\in[0,s]})$ is constant on $[t_0,\underline{t})$ we then have

$$\operatorname{ind}\big(\{f_{\underline{t}}\leq-\epsilon\}\big)=\operatorname{ind}\big(\{f_{t_0}\leq 0\}\big)\,. \tag{5}$$

As in the proof of [16, Proposition 4.15] we have

$$|\,\mu(\{\phi_t\}_{t\in[0,t_0]})-\mu(\{\phi_t\}_{t\in[0,t_1]})\,|\leq\operatorname{ind}\big(\{f_{\underline{t}}\leq\epsilon\}\big)-\operatorname{ind}\big(\{f_{\underline{t}}\leq-\epsilon\}\big) \tag{6}$$

---

[1] In [16, Theorem 1.4 (iii)] it is assumed that there is only one $\underline{t}\in[0,1]$ such that $\bar{\phi}_{\underline{t}}$ has discriminant points. However, the proof works in the same way also if we only assume that $s\mapsto\mu(\{\phi_t\}_{t\in[0,s]})$ is constant on $[t_0,\underline{t})$ and on $(\underline{t},t_1]$. Similarly for Proposition 2.9 below.

and
$$\operatorname{ind}\big(\{f_{\underline{t}} \leq \epsilon\}\big) \leq \operatorname{ind}\big(\{f_{\underline{t}} \leq -\epsilon\} \cup \mathcal{U}\big)\,, \tag{7}$$
where $\mathcal{U}$ is a neighborhood of the set C of critical points of $f_{\underline{t}}$ of critical value zero that has the same cohomological index as C. Since
$$\mu(\{\phi_t\}_{t\in[0,t_0]}) = \operatorname{ind}\big(\{f_0 \leq 0\}\big) - \operatorname{ind}\big(\{f_{t_0} \leq 0\}\big)$$
is even (by assumption) and $\operatorname{ind}\big(\{f_0 \leq 0\}\big)$ is even (by [16, Remark 3.13]), using (5) we see that $\operatorname{ind}\big(\{f_{\underline{t}} \leq -\epsilon\}\big)$ is even and so, by subadditivity of the cohomological index,
$$\operatorname{ind}\big(\{f_{\underline{t}} \leq -\epsilon\} \cup \mathcal{U}\big) \leq \operatorname{ind}\big(\{f_{\underline{t}} \leq -\epsilon\}\big) + \operatorname{ind}(\mathcal{U}) = \operatorname{ind}\big(\{f_{\underline{t}} \leq -\epsilon\}\big) + \operatorname{ind}(C)\,.$$
Since $\operatorname{ind}(C) = \operatorname{ind}\big(\Delta(\phi_{\underline{t}})\big)$ by [16, Proposition 2.22], using (6) and (7) we thus conclude that
$$|\,\mu(\{\phi_t\}_{t\in[0,t_0]}) - \mu(\{\phi_t\}_{t\in[0,t_1]})\,| \leq \operatorname{ind}\big(\Delta(\phi_{\underline{t}})\big)\,.$$
□

The following Poincaré duality property for the cohomological index is proved in [8, Proposition 4.1.15] for subsets of complex projective space. We adapt here the proof to the case of lens spaces.

Lemma 2.10. *Assume that zero is a regular value of a function $f : L_k^{2M-1} \to \mathbb{R}$. Then*
$$\operatorname{ind}\big(\{f \leq 0\}\big) + \operatorname{ind}\big(\{f \geq 0\}\big) = 2M\,.$$

Proof. Let $A = \{f \leq 0\}$ and $B = \{f \geq 0\}$. Since zero is a regular value of f, A and B are smooth submanifolds with boundary and thus deformation retract to sets A' and B' that are strictly included in $\{f < 0\}$ and $\{f > 0\}$ respectively. By continuity of the cohomological index, there are thus open subsets $\mathcal{U}_A$ and $\mathcal{U}_B$ of L_k^{2M-1} strictly included in $\{f < 0\}$ and $\{f > 0\}$ respectively with $\operatorname{ind}(\mathcal{U}_A) = \operatorname{ind}(A)$ and $\operatorname{ind}(\mathcal{U}_B) = \operatorname{ind}(B)$. Assume first that $\operatorname{ind}(A)$ is even. Since $A \cup B = L_k^{2M-1}$, by subadditivity of the cohomological index we have
$$\operatorname{ind}(A) + \operatorname{ind}(B) \geq 2M\,.$$
Assume by contradiction that
$$\operatorname{ind}(\mathcal{U}_A) + \operatorname{ind}(\mathcal{U}_B) = \operatorname{ind}(A) + \operatorname{ind}(B) \geq 2M + 1\,. \tag{8}$$
Let $\operatorname{ind}(A) = \operatorname{ind}(\mathcal{U}_A) = 2a$. By definition of the cohomological index (cf. [16, Lemma 3.3]) and since Čech cohomology agrees with singular cohomology on open sets, the homomorphism $H^{2a-1}(L_k^{2M-1};\mathbb{Z}_k) \to H^{2a-1}(\mathcal{U}_A;\mathbb{Z}_k)$ induced by the inclusion $\mathcal{U}_A \hookrightarrow L_k^{2M-1}$ is injective. Since k is prime, the coefficient ring $\mathbb{Z}_k$ is a field and so cohomology is the dual of homology, thus the homomorphism $H_{2a-1}(\mathcal{U}_A;\mathbb{Z}_k) \to H_{2a-1}(L_k^{2M-1};\mathbb{Z}_k)$ induced by the inclusion $\mathcal{U}_A \hookrightarrow L_k^{2M-1}$ is surjective. Consider the commutative square
$$\begin{tikzcd}
H_{2a-1}\big(L_k^{2M-1} \setminus \operatorname{int}(B);\mathbb{Z}_k\big) \arrow[r] & H_{2a-1}(L_k^{2M-1};\mathbb{Z}_k) \\
H^{2(M-a)}(L_k^{2M-1}, B;\mathbb{Z}_k) \arrow[u, "\cong"] \arrow[r, "j_B"] & H^{2(M-a)}(L_k^{2M-1};\mathbb{Z}_k) \arrow[u, "\cong"]
\end{tikzcd} \tag{9}$$
where the horizontal arrows are induced by the inclusions and the vertical ones are Poincaré duality isomorphisms (composed with excision $H^{2(M-a)}(L_k^{2M-1}, B;\mathbb{Z}_k) \to H^{2(M-a)}(L_k^{2M-1} \setminus \operatorname{int}(B), \partial B;\mathbb{Z}_k)$ for the vertical arrow on the left hand side). Since $\mathcal{U}_A \subset L_k^{2M-1} \setminus \operatorname{int}(B)$, surjectivity of the inclusion homomorphism $H_{2a-1}(\mathcal{U}_A;\mathbb{Z}_k) \to H_{2a-1}(L_k^{2M-1};\mathbb{Z}_k)$ implies that the homomorphism on the top horizontal line of the diagram is also surjective. Thus j_B is surjective, and so there exists a class
$$u \in H^{2(M-a)}(L_k^{2M-1}, B;\mathbb{Z}_k)$$
such that $j_B(u) = \beta^{M-a}$, where β denotes a generator of $H^2(L_k^{2M-1};\mathbb{Z}_k)$. By (8) we have $\operatorname{ind}(\mathcal{U}_B) \geq 2M + 1 - 2a$, and so by a similar argument there exists a class
$$v \in H^{2a-1}(L_k^{2M-1}, A;\mathbb{Z}_k)$$

such that $j_A(v) = \alpha\beta^{a-1}$, where

$$j_A : H_{2a-1}(L_k^{2M-1}, A; \mathbb{Z}_k) \to H_{2a-1}(L_k^{2M-1}; \mathbb{Z}_k)$$

is the homomorphism induced by the inclusion and where α denotes a generator of $H^1(L_k^{2M-1}; \mathbb{Z}_k)$. We then obtain a contradiction: on the one hand $v \cup u \in H^{2M-1}(L_k^{2M-1}, A \cup B; \mathbb{Z}_k)$ is zero since $A \cup B = L_k^{2M-1}$, on the other hand by naturality of the cup product we have

$$j_{A\cup B}(v \cup u) = j_A(v) \cup j_B(u) = \alpha\beta^{M-1} \neq 0\,.$$

This finishes the proof in the case when $\operatorname{ind}(A)$ is even. If $\operatorname{ind}(B)$ is even a similar argument also gives the result. Suppose thus that $\operatorname{ind}(A)$ and $\operatorname{ind}(B)$ are odd. Then $\operatorname{ind}(A) + \operatorname{ind}(B)$ is even, and so it is enough to prove

$$2M - 1 \leq \operatorname{ind}(A) + \operatorname{ind}(B) \leq 2M\,.$$

The first inequality follows from subadditivity of the cohomological index. For the second one, suppose by contradiction that

$$\operatorname{ind}(\mathcal{U}_A) + \operatorname{ind}(\mathcal{U}_B) = \operatorname{ind}(A) + \operatorname{ind}(B) \geq 2M + 1\,,$$

and let $\operatorname{ind}(A) = 2a - 1$. By a similar argument as above, there exist cohomology classes u in $H^{2M-2a+1}(L_k^{2M-1}, B; \mathbb{Z}_k)$ and v in $H^{2a-2}(L_k^{2M-1}, A; \mathbb{Z}_k)$ such that $j_B(u) = \alpha\beta^{M-a}$ and $j_A(v) = \beta^{a-1}$. Since $j_B(u) \cup j_A(v) = \alpha\beta^{M-1} \neq 0$, this leads to the same contradiction as above. □

Applying Lemma 2.10 we obtain the following result.

Proposition 2.11. *For every $\widetilde{\phi} \in \widetilde{\operatorname{Cont}_0}(L_k^{2n-1}, \xi_0)$ such that $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ does not have discriminant points we have*

$$\mu(\widetilde{\phi}) + \mu(\widetilde{\phi}^{-1}) = 2n\,.$$

Proof. Let $F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$ be a based family of conical generating functions for a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ representing $\widetilde{\phi}$. Then $-F_t$ is a based family of conical generating functions for $\{\phi_t^{-1}\}_{t\in[0,1]}$, which represent $\widetilde{\phi}^{-1}$. Thus

$$\mu(\widetilde{\phi}) + \mu(\widetilde{\phi}^{-1}) = \operatorname{ind}(F_0) - \operatorname{ind}(F_1) + \operatorname{ind}(-F_0) - \operatorname{ind}(-F_1)\,.$$

Since F_0 is equivalent to the zero function, up to a fibre preserving conical homeomorphism it is equal to a $\mathbb{Z}_k$-invariant quadratic form Q_0. Since F_0 generates the identity, by [16, Proposition 2.22] the nullity of Q_0 is equal to $2n$. Let ι be the dimension of the maximal subspace on which Q_0 is negative definite. Then

$$\operatorname{ind}(F_0) + \operatorname{ind}(-F_0) = \operatorname{ind}(Q_0) + \operatorname{ind}(-Q_0) = (2n + \iota) + \big(2n + (2nN - \iota)\big) = 4n + 2nN\,.$$

On the other hand, since $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ has no discriminant points we have that zero is a regular value of F_1 and so we can apply Lemma 2.10 to obtain

$$\operatorname{ind}(F_1) + \operatorname{ind}(-F_1) = 2n + 2nN\,.$$

We conclude that $\mu(\widetilde{\phi}) + \mu(\widetilde{\phi}^{-1}) = 2n$. □

So far we have assumed that k is prime. Suppose now that k is not prime, and let k' be the smallest prime that divides k. As in [16, Remark 1.4], we define the non-linear Maslov index

$$\mu : \widetilde{\operatorname{Cont}_0}(L_k^{2n-1}, \xi_0) \to \mathbb{Z}$$

by pulling back $\mu : \widetilde{\operatorname{Cont}_0}(L_{k'}^{2n-1}, \xi_0) \to \mathbb{Z}$ by the natural map $\widetilde{\operatorname{Cont}_0}(L_k^{2n-1}, \xi_0) \to \widetilde{\operatorname{Cont}_0}(L_{k'}^{2n-1}, \xi_0)$. This general non-linear Maslov index then satisfies the following properties.

Proposition 2.12 (Non-linear Maslov index for general k)**.** *For any integer $k \geq 2$ the non-linear Maslov index*

$$\mu : \widetilde{\operatorname{Cont}_0}(L_k^{2n-1}, \xi_0) \to \mathbb{Z}$$

satisfies the following properties:

(i) Identity and small elements*: if $\widetilde{\phi}$ is small enough in the $\mathcal{C}^1$-topology then $0 \leq \mu(\widetilde{\phi}) \leq 2n$, and if moreover $\widetilde{\phi}$ is positive then $\mu(\widetilde{\phi}) = 2n$. Moreover, $\mu(\widetilde{\mathrm{id}}) = 0$.*

(ii) Quasimorphism property*:*

$$|\, \mu(\widetilde{\phi} \cdot \widetilde{\psi}) - \mu(\widetilde{\phi}) - \mu(\widetilde{\psi}) \,| \leq 2n + 1\,,$$

and

$$|\, \mu(\widetilde{\phi} \cdot \widetilde{\psi}) - \mu(\widetilde{\phi}) - \mu(\widetilde{\psi}) \,| \leq 2n$$

if k is even; in particular, $\mu : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \mathbb{Z}$ is a quasimorphism.

(iii) Triangle inequality*:*

$$\mu(\widetilde{\phi} \cdot \widetilde{\psi}) \leq \mu(\widetilde{\phi}) + \mu(\widetilde{\psi}) + 1\,,$$

and

$$\mu(\widetilde{\phi} \cdot \widetilde{\psi}) \leq \mu(\widetilde{\phi}) + \mu(\widetilde{\psi})$$

if either k is even or $\mu(\widetilde{\phi})$ is even or $\mu(\widetilde{\psi})$ is even.

(iv) Monotonicity*: if $\widetilde{\phi} \leq \widetilde{\psi}$ then $\mu(\widetilde{\phi}) \leq \mu(\widetilde{\psi})$.*

(v) Relation with discriminant points*: for any contact isotopy $\{\phi_t\}_{t\in[0,1]}$ of (L_k^{2n-1}, ξ_0) starting at the identity and any subinterval $[t_0, t_1]$ of $[0,1]$, if $\mu(\{\phi_t\}_{t\in[0,t_0]}) \neq \mu(\{\phi_t\}_{t\in[0,t_1]})$ then there is $\underline{t} \in [t_0, t_1]$ such that $\bar{\phi}_{\underline{t}}$ has discriminant points. Assume that the map $s \mapsto \mu(\{\phi_t\}_{t\in[0,s]})$ is constant on $[t_0, \underline{t})$ and on $(\underline{t}, t_1]$, and denote by $\Delta_{k'}(\phi_{\underline{t}}) \subset L_{k'}^{2n-1}$ the set of discriminant points at time $\underline{t}$ of the lift of $\{\phi_t\}$ to $L_{k'}^{2n-1}$, where k' is the smallest prime dividing k. Then*

$$|\, \mu(\{\phi_t\}_{t\in[0,t_0]}) - \mu(\{\phi_t\}_{t\in[0,t_1]}) \,| \leq \mathrm{ind}\left(\Delta_{k'}(\phi_{\underline{t}})\right) + 1\,,$$

and

$$|\, \mu(\{\phi_t\}_{t\in[0,t_0]}) - \mu(\{\phi_t\}_{t\in[0,t_1]}) \,| \leq \mathrm{ind}\left(\Delta_{k'}(\phi_{\underline{t}})\right)$$

if either k is even or if $\{\phi_t\}_{t\in[0,1]}$ is negative and $\mu(\{\phi_t\}_{t\in[0,t_0]})$ is even. If moreover all the discriminant points of $\bar{\phi}_{\underline{t}}$ are non-degenerate then

$$|\, \mu(\{\phi_t\}_{t\in[0,t_0]}) - \mu(\{\phi_t\}_{t\in[0,t_1]}) \,| \leq 1\,.$$

(vi) Reeb flow*: for every real number T we have*

$$\mu(\widetilde{r_T}) = 2n \left\lceil \frac{T}{2\pi} \right\rceil\,.$$

(vii) Composition with the Reeb flow*: for every integer m we have*

$$\mu(\widetilde{\phi} \cdot \widetilde{r_{2\pi m}}) = \mu(\widetilde{r_{2\pi m}} \cdot \widetilde{\phi}) = 2nm + \mu(\widetilde{\phi})\,.$$

(viii) Lower semi-continuity*: the map $T \mapsto \mu(\widetilde{r_{-T}} \cdot \widetilde{\phi})$ is lower semi-continuous.*

(ix) Poincaré duality*: for every $\widetilde{\phi}$ such that $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ has no discriminant points we have*

$$\mu(\widetilde{\phi}) + \mu(\widetilde{\phi}^{-1}) = 2n\,.$$

Proof. In the case when k is prime all the properties have been discussed above in Example 2.2, (1), (2), Proposition 2.3, Proposition 2.6, Proposition 2.8, Proposition 2.9, (4), Proposition 2.4, Remark 2.7 and Proposition 2.11. If k is not prime and k' is the smallest prime that divides k then the properties of $\mu : \widetilde{\mathrm{Cont}_0}(L_{k'}^{2n-1}, \xi_0) \to \mathbb{Z}$ imply the corresponding properties for the pullback $\mu : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \mathbb{Z}$ by the natural map $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \widetilde{\mathrm{Cont}_0}(L_{k'}^{2n-1}, \xi_0)$. □

3. Spectral selectors

For any $j \in \mathbb{Z}$ we define the j-th spectral selector on $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$ by

$$c_j(\widetilde{\phi}) = \inf\left\{ T \in \mathbb{R} \;\middle|\; \mu(\widetilde{r_{-T}} \cdot \widetilde{\phi}) \leq -j \right\}.$$

By Proposition 2.12 (iv), (ii) and (vi), the function $T \mapsto \mu(\widetilde{r_{-T}} \cdot \widetilde{\phi})$ is non-increasing and tends to $\mp\infty$ as $T \to \pm\infty$, thus $c_j(\widetilde{\phi})$ is a well-defined real number. By Proposition 2.12 (viii) the infimum is in fact a minimum, in particular

$$\left\lceil c_j(\widetilde{\phi}) \right\rceil_{\frac{2\pi}{k}} = \min\left\{ N \in \frac{2\pi}{k} \cdot \mathbb{Z} \;\middle|\; \mu(\widetilde{r_{-N}} \cdot \widetilde{\phi}) \leq -j \right\}. \tag{10}$$

It follows from the definition that the sequence $\{c_j\}$ is non-decreasing. In the rest of this section we prove the other properties listed in Theorem 1.1.

We say that a contactomorphism of (L_k^{2n-1}, ξ_0) is non-degenerate with respect to α_0 if all its translated points with respect to α_0 are non-degenerate. We then have the following lemma.

Lemma 3.1. *The set of contactomorphisms of (L_k^{2n-1}, ξ_0) contact isotopic to the identity that are non-degenerate with respect to α_0 is dense in $\mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$ for the $\mathcal{C}^1$-topology.*

Proof. We first prove the following result. Let Λ_0 be a closed Legendrian submanifold of a contact manifold $\big(M, \xi = \ker(\alpha)\big)$, and denote by $\mathcal{L}eg(\Lambda_0)$ its Legendrian isotopy class. Then for any submanifold N of M the set $\mathcal{L}eg_N(\Lambda_0)$ of elements of $\mathcal{L}eg(\Lambda_0)$ transverse to N is dense in $\mathcal{L}eg(\Lambda_0)$ for the $\mathcal{C}^1$-topology. Indeed, let Λ be an element of $\mathcal{L}eg(\Lambda_0)$ and let $\mathcal{U}(\Lambda) \subset M$ be a Weinstein neighborhood of Λ. Fix a diffeomorphism Ψ from $\mathcal{U}(\Lambda)$ to an open neighborhood $\mathcal{U}(j^1 0)$ of the zero section of $J^1\Lambda$ such that $\Psi(\Lambda) = j^1 0$ and $\Psi^*(dz - \lambda_{\mathrm{can}}) = \alpha$. Denote the submanifold $\Psi\big(N \cap \mathcal{U}(\Lambda)\big)$ of $\mathcal{U}(j^1 0)$ by N'. Since the map $j^1 : \mathcal{C}^\infty(\Lambda) \to \mathcal{L}eg(j^1 0)$ that associates to a function its 1-jet is a local homeomorphism with respect to the C^2-topology on $\mathcal{C}^\infty(\Lambda)$ and the $\mathcal{C}^1$-topology on $\mathcal{L}eg(j^1 0)$ (cf. for instance [23, Section 3]), for a sufficiently small $\mathcal{C}^2$-neighborhood $\mathcal{U}(0)$ of the zero function in $\mathcal{C}^\infty(\Lambda)$ the 1-jet of any $f \in \mathcal{U}(0)$ is in $\mathcal{U}(j^1 0)$ and the map $\mathcal{U}(0) \to \mathcal{L}eg(\Lambda_0)$ that sends f to $\Psi^{-1}(j^1 f)$ is a local homeomorphism. By Thom's transversality theorem (see for instance [15, Corollary 4.10]), the subset $T^1_{N'}$ of $\mathcal{U}(0)$ consisting of functions with 1-jet transverse to N' is dense in $\mathcal{U}(0)$ for the $\mathcal{C}^\infty$-topology, hence also for the $\mathcal{C}^2$-topology. Thus for any $\mathcal{C}^1$-neighborhood $\mathcal{U}$ of Λ in $\mathcal{L}eg(\Lambda_0)$ there exists $f \in T^1_{N'}$ such that $\Psi^{-1}(j^1 f) \in \mathcal{U}$. The Legendrian $\Psi^{-1}(j^1 f)$ intersects N transversely, and so belongs to $\mathcal{L}eg_N(\Lambda_0)$. This shows that $\mathcal{L}eg_N(\Lambda_0)$ is $\mathcal{C}^1$-dense in $\mathcal{L}eg(\Lambda_0)$.

Using this, we now prove that the set of contactomorphisms of (L_k^{2n-1}, ξ_0) contact isotopic to the identity that are non-degenerate with respect to α_0 is $\mathcal{C}^1$-dense in $\mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$. Consider the product $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$ endowed with the contact structure given by the kernel of the contact form $\pi_2^*\alpha_0 - e^\theta \pi_1^*\alpha_0$, where π_1 and π_2 denote the projections on the first and second factor respectively and where θ is the coordinate in $\mathbb{R}$. Consider the closed submanifold $N = \bigcup_{t \in [0, \frac{2\pi}{k}]} r_t(\Delta \times \{0\})$, where $\{r_t\}$ denotes the Reeb flow of $\pi_2^*\alpha_0 - e^\theta \pi_1^*\alpha_0$ and Δ denotes the diagonal. A contactomorphism ϕ of (L_k^{2n-1}, ξ_0) is non-degenerate with respect to α_0 if and only if its graph

$$\mathrm{gr}_{\alpha_0}(\phi) = \left\{ \big(p, \phi(p), g(p)\big) \;\middle|\; p \in L_k^{2n-1} \right\},$$

where g is the conformal factor of ϕ with respect to α_0, is transverse to N. In other words, using the notation of the first part of the proof and since $\Delta \times \{0\} = \mathrm{gr}_{\alpha_0}(\mathrm{id})$, ϕ is non-degenerate with respect to α_0 if and only if $\mathrm{gr}_{\alpha_0}(\phi) \in \mathcal{L}eg_N(\mathrm{gr}_{\alpha_0}(\mathrm{id}))$. Our result thus follows from the first part of the proof and the fact that the map $\mathrm{gr}_{\alpha_0} : \mathrm{Cont}_0(L_k^{2n-1}, \xi_0) \to \mathcal{L}eg(\mathrm{gr}_{\alpha_0}(\mathrm{id}))$ that associates to a contactomorphism its graph is a local homeomorphism with respect to the $\mathcal{C}^1$-topologies. $\square$

We also remark the following fact.

Lemma 3.2. *For any $\widetilde{\phi} \in \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$, $\mathcal{A}(\widetilde{\phi})$ and $\bar{\mathcal{A}}(\widetilde{\phi})$ are closed and nowhere dense in $\mathbb{R}$.*

Proof. Let $F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \to \mathbb{R}$, $t \in [0,1]$, be a based family of conical generating functions for the concatenation

$$\{\phi_{2t}\}_{t\in[0,\frac{1}{2}]} \sqcup \{r_{-2\pi(2t-1)}\}_{t\in[\frac{1}{2},1]}\,.$$

For every $t \in [\frac{1}{2},1]$, the $\mathbb{Z}_k$-orbits of translated points of $\bar{\phi}_1$ of translation $2\pi(2t-1)$ are in 1–1 correspondence with the critical points of critical value zero of $f_t : L_k^{2n(N+1)-1} \to \mathbb{R}$. Consider the function

$$f : L_k^{2n(N+1)-1} \times \left[\frac{1}{2},1\right] \to \mathbb{R}\,,\ (x,t) \mapsto f_t(x)\,.$$

As in [22, Section 5.2 and Lemma 4.4], zero is a regular value of f and so $f^{-1}(0)$ is a submanifold of $L_k^{2n(N+1)-1} \times [\frac{1}{2},1]$. Let $p : f^{-1}(0) \to \left[\frac{1}{2},1\right]$ be the composition of the inclusion $f^{-1}(0) \hookrightarrow L_k^{2n(N+1)-1} \times [\frac{1}{2},1]$ with the projection on the second factor. Then $(x,t) \in f^{-1}(0)$ is a critical point of p (of critical value t) if and only if x is a critical point of f_t (of critical value zero). Thus the $\mathbb{Z}_k$-orbits of translated points of $\bar{\phi}_1$ of translation $2\pi(2t-1)$ are in 1–1 correspondence with the critical points of p of critical value t. It follows that

$$\bar{\mathcal{A}}(\widetilde{\phi}) = 2\pi\Big(2\,p\big(\operatorname{Crit}(p)\big) - 1\Big) + 2\pi\cdot\mathbb{Z}\,,$$

and so $\bar{\mathcal{A}}(\widetilde{\phi})$ is closed and nowhere dense. This implies that $\mathcal{A}(\widetilde{\phi}) = \bar{\mathcal{A}}(\widetilde{\phi}) + \frac{2\pi}{k}\cdot\mathbb{Z}$ is also closed and nowhere dense. □

We now prove that the spectral selectors satisfy the properties listed in Theorem 1.1.

Spectrality. We have to show that

$$c_j(\widetilde{\phi}) \in \bar{\mathcal{A}}(\widetilde{\phi})$$

for every $\widetilde{\phi}$ in $\widetilde{\operatorname{Cont}_0}(L_k^{2n-1},\xi_0)$. Suppose by contradiction that this is not the case. Since $\bar{\mathcal{A}}(\widetilde{\phi})$ is closed (by Lemma 3.2), there is then $\epsilon > 0$ such that $[c_j(\widetilde{\phi}) - \epsilon, c_j(\widetilde{\phi}) + \epsilon] \subset \mathbb{R} \smallsetminus \bar{\mathcal{A}}(\widetilde{\phi})$. By the first statement of Proposition 2.12 (v) we thus have

$$\mu\big(\widetilde{r_{-(c_j(\widetilde{\phi})-\epsilon)}}\cdot\widetilde{\phi}\big) = \mu\big(\widetilde{r_{-(c_j(\widetilde{\phi})+\epsilon)}}\cdot\widetilde{\phi}\big)\,,$$

but this contradicts the definition of $c_j(\widetilde{\phi})$.

Normalization. It follows from Proposition 2.12 (vi) that

$$c_j(\widetilde{\mathrm{id}}) = \begin{cases} \cdots & \\ -4\pi & \text{for } j = -6n+1,\cdots,-4n \\ -2\pi & \text{for } j = -4n+1,\cdots,-2n \\ 0 & \text{for } j = -2n+1,\cdots,0 \\ 2\pi & \text{for } j = 1,\cdots,2n \\ 4\pi & \text{for } j = 2n+1,\cdots,4n \\ \cdots & \end{cases}$$

In particular, $c_{-2n+1}(\widetilde{\mathrm{id}}) = c_0(\widetilde{\mathrm{id}}) = 0$.

Relation with translated points. If all the translated points of $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ are non-degenerate, it follows from the last statement of Proposition 2.12 (v) that the spectral selectors $\{c_j(\widetilde{\phi})\,,\, j\in\mathbb{Z}\}$ are all distinct. Suppose now that

$$c_{j-1}(\widetilde{\phi}) < c_j(\widetilde{\phi}) = c_{j+1}(\widetilde{\phi}) = \cdots = c_{j+m}(\widetilde{\phi}) = T < c_{j+m+1}(\widetilde{\phi})$$

for some j and $1 \le m \le 2n-1$. Then for $\epsilon > 0$ small enough we have

$$|\,\mu\big(\widetilde{r_{-(T-\epsilon)}}\cdot\widetilde{\phi}\big) - \mu\big(\widetilde{r_{-(T+\epsilon)}}\cdot\widetilde{\phi}\big)\,| = m+1$$

and $\mu\big(\widetilde{r_{-(T-\epsilon)}}\cdot\widetilde{\phi}\big)=-j+1$. It thus follows from Proposition 2.12 (v) that if either k is even or j is odd or $m>1$ then $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ has infinitely many translated points of translation T. If moreover k is prime then Proposition 2.12 (v) also implies that the set of translated points of translation T of $\Pi(\widetilde{\phi})$ has cohomological index greater or equal than m, and greater or equal than $m+1$ if either $k=2$ or j is odd.

Non-degeneracy. Assume that

$$c_{-2n+1}(\widetilde{\phi})=c_0(\widetilde{\phi})=0\,.$$

Then for $\epsilon>0$ small enough we have

$$\mid\mu\big(\widetilde{r_\epsilon}\cdot\widetilde{\phi}\big)-\mu\big(\widetilde{r_{-\epsilon}}\cdot\widetilde{\phi}\big)\mid=2n$$

and $\mu\big(\widetilde{r_\epsilon}\cdot\widetilde{\phi}\big)=2n$. By Proposition 2.12 (v) we then conclude that $\Pi(\widetilde{\phi})$ is the identity. Since, by spectrality, $0\in\bar{\mathcal{A}}(\widetilde{\phi})$, we have in fact that $\Pi\big(\mathcal{L}(\widetilde{\phi})\big)$ is the identity.

Composition with the Reeb flow. By definition of the spectral selectors, for every $\widetilde{\phi}$ and every $T\in\mathbb{R}$ we have

$$\begin{aligned}c_j(\widetilde{r_T}\cdot\widetilde{\phi})&=\inf\{\,\underline{T}\in\mathbb{R}\;\mid\mu(\widetilde{r_{-\underline{T}}}\cdot\widetilde{r_T}\cdot\widetilde{\phi})\leq -j\,\}\\&=\inf\{\,\underline{T}-T\in\mathbb{R}\;\mid\mu(\widetilde{r_{-\underline{T}+T}}\cdot\widetilde{\phi})\leq -j\,\}+T\\&=c_j(\widetilde{\phi})+T\,.\end{aligned}$$

Periodicity. Using the previous property and Proposition 2.12 (vii) we have

$$c_j(\widetilde{\phi})+2\pi=c_j(\widetilde{r_{2\pi}}\cdot\widetilde{\phi})=c_{j+2n}(\widetilde{\phi})\,.$$

Monotonicity. Suppose that $\widetilde{\phi}\leq\widetilde{\psi}$. Then $\widetilde{r_{-T}}\cdot\widetilde{\phi}\leq\widetilde{r_{-T}}\cdot\widetilde{\psi}$ and so, by Proposition 2.12 (iv),

$$\mu(\widetilde{r_{-T}}\cdot\widetilde{\phi})\leq\mu(\widetilde{r_{-T}}\cdot\widetilde{\psi})\,.$$

This implies that $c_j(\widetilde{\phi})\leq c_j(\widetilde{\psi})$.

Continuity. Suppose that $\widetilde{\phi}\cdot\widetilde{\psi}^{-1}$ is represented by a contact isotopy with Hamiltonian function $H_t:L_k^{2n-1}\to\mathbb{R}$ with respect to α_0. Let $m(t)=\min H_t$ and $M(t)=\max H_t$. The flows of m and M are respectively $\big\{r_{\int_0^t m}\big\}$ and $\big\{r_{\int_0^t M}\big\}$, thus

$$\widetilde{r_{\int_0^1 m}}\leq\widetilde{\phi}\cdot\widetilde{\psi}^{-1}\leq\widetilde{r_{\int_0^1 M}}$$

and so

$$\widetilde{r_{\int_0^1 m}}\cdot\widetilde{\psi}\leq\widetilde{\phi}\leq\widetilde{r_{\int_0^1 M}}\cdot\widetilde{\psi}\,.$$

By the composition with the Reeb flow property and monotonicity we thus have

$$\int_0^1\min H_t\,dt\leq c_j(\widetilde{\phi})-c_j(\widetilde{\psi})\leq\int_0^1\max H_t\,dt\,. \tag{11}$$

We now show that (11) implies that each c_j is continuous with respect to the $\mathcal{C}^1$-topology.

Notice first that the Shelukhin–Hofer norm $\nu_\alpha:\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)\to\mathbb{R}$, which is defined by

$$\nu_\alpha(\widetilde{\phi})=\inf\int_0^1\max|H_t|\,dt$$

with the infimum taken over all contact Hamiltonian functions H_t whose flow represents $\widetilde{\phi}$, is continuous with respect to the $\mathcal{C}^1$-topology. Indeed, this can be seen as follows. Since $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$ is a topological group, it is enough to show that ν_α is $\mathcal{C}^1$-continuous at the identity, i.e. that for every $\epsilon>0$ there is a $\mathcal{C}^1$-neighborhood $\widetilde{\mathcal{U}}$ of $\widetilde{\mathrm{id}}$ in $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$ such that for every $\widetilde{\phi}\in\widetilde{\mathcal{U}}$ we have $\nu_\alpha(\widetilde{\phi}) < \epsilon$. As in the proof of Lemma 3.1 we consider the product $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$ endowed with the contact structure given by the kernel of the contact form $\pi_2^*\alpha_0 - e^\theta \pi_1^*\alpha_0$. Applying the Weinstein theorem we can find a neighborhood $\mathcal{U}(\Delta \times \{0\})$ of the Legendrian $\Delta \times \{0\} \cong L_k^{2n-1}$ of $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$, a neighborhood $\mathcal{U}(j^1 0)$ of the zero section of $J^1 L_k^{2n-1}$ of the form $\mathcal{U}(j^1 0) = \underline{\mathcal{U}}(j^1 0) \times (-\epsilon', \epsilon')$ with $\epsilon' < \epsilon$ and a diffeomorphism Ψ from $\mathcal{U}(\Delta \times \{0\})$ to $\mathcal{U}(j^1 0)$ with $\Psi(\Delta \times \{0\}) = j^1 0$ and $\Psi^*(dz - \lambda_{\text{can}}) = \pi_2^*\alpha_0 - e^\theta \pi_1^*\alpha_0$. Since the map $j^1 : \mathcal{C}^\infty(L_k^{2n-1}) \to \mathcal{L}eg(j^1 0)$ that associates to a function its 1-jet is a local homeomorphism with respect to the $\mathcal{C}^2$-topology on $\mathcal{C}^\infty(L_k^{2n-1})$ and the $\mathcal{C}^1$-topology on $\mathcal{L}eg(j^1 0)$, we can find a convex $\mathcal{C}^2$-neighborhood $\mathcal{U}(0)$ of the zero function in $\mathcal{C}^\infty(L_k^{2n-1})$ such that $j^1 f \in \mathcal{U}(j^1 0)$ for any $f \in \mathcal{U}(0)$, and the map $\mathcal{U}(0) \mapsto \mathcal{L}eg(\Delta \times \{0\})$ that sends f to $\Psi^{-1}(j^1 f)$ is a local homeomorphism. Since $\Delta \times \{0\} = \text{gr}_{\alpha_0}(\text{id})$ and the map $\text{gr}_{\alpha_0} : \text{Cont}_0(L_k^{2n-1}, \xi_0) \to \mathcal{L}eg\big(\text{gr}_{\alpha_0}(\text{id})\big)$ that associates to a contactomorphism its graph is a local homeomorphism with respect to the $\mathcal{C}^1$-topologies, we obtain a map $\mathcal{U}(0) \to \text{Cont}_0(L_k^{2n-1}, \xi_0)$ that associates to a function f a contactomorphism ϕ with $j^1 f = \Psi(\text{gr}_{\alpha_0}(\phi))$, which is a homeomorphism on its image $\mathcal{U}$. Since $\mathcal{U}(0)$ is convex, $\mathcal{U}$ is simply connected; let thus $\widetilde{\mathcal{U}}$ be the open neighborhood of $\widetilde{\text{id}}$ in $\widetilde{\text{Cont}_0}(L_k^{2n-1}, \xi_0)$ that projects homeomorphically to $\mathcal{U}$. Consider now any $\widetilde{\phi}$ in $\widetilde{\mathcal{U}}$. Let $f : L_k^{2n-1} \to (-\epsilon, \epsilon)$ be the function in $\mathcal{U}(0)$ such that $j^1 f = \Psi\big(\text{gr}_{\alpha_0}\big(\Pi(\widetilde{\phi})\big)\big)$. Since $\mathcal{U}(0)$ is convex, $f_t := tf$ is in $\mathcal{U}(0)$ for every $t \in [0,1]$, and so for every $t \in [0,1]$ there is $\phi_t \in \mathcal{U}$ with $j^1 f_t = \Psi\big(\text{gr}_{\alpha_0}(\phi_t)\big)$. Consider the two Legendrian isotopies $j^1 f_t$ and $\text{gr}_{\alpha_0}(\phi_t)$, with parametrizations given respectively by

$$i_1 : [0,1] \times j^1 0 \to J^1 L_k^{2n-1}\,,\ i_1\big(t, (x,0,0)\big) = j^1 f_t(x)$$

and

$$i_2 : [0,1] \times \big(\Delta \times \{0\}\big) \to L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}\,,\ i_2\big(t,(x,x,0)\big) = \big(x, \phi_t(x), g_t(x)\big)\,,$$

where g_t is the conformal factor of ϕ_t. Let H_t be the contact Hamiltonian function of the contact isotopy $\{\phi_t\}$. Then

$$H_t\big(\phi_t(x)\big) = (\pi_2^*\alpha_0 - e^\theta\pi_1^*\alpha_0)\Big(\frac{d}{dt}\, i_2\big(t,(x,x,0)\big)\Big) = (dz - \lambda_{\text{can}})\Big(\frac{d}{dt}\, i_1\big(t,(x,0,0)\big)\Big) = f(x)\,,$$

and so $|H_t| < \epsilon$. Moreover $[\{\phi_t\}_{t\in[0,1]}] = \widetilde{\phi}$, because $\widetilde{\phi} \cdot [\{\phi_t\}_{t\in[0,1]}]^{-1} \in \pi_1\big(\text{Cont}_0(L_k^{2n-1}, \xi_0)\big)$ can be represented by a loop in $\mathcal{U}$, which is simply connected. We thus conclude that $\nu_\alpha(\widetilde{\phi}) < \epsilon$, as we wanted.

Using (11) and the fact that ν_α is $\mathcal{C}^1$-continuous we now deduce that each c_j is $\mathcal{C}^1$-continuous. Let $\widetilde{\phi} \in \widetilde{\text{Cont}_0}(L_k^{2n-1}, \xi_0)$. By $\mathcal{C}^1$-continuity of ν_α, for any $\epsilon > 0$ there is a $\mathcal{C}^1$-neighborhood $\widetilde{\mathcal{U}}$ of $\widetilde{\text{id}}$ in $\widetilde{\text{Cont}_0}(L_k^{2n-1}, \xi_0)$ such that $\nu_\alpha|_{\widetilde{\mathcal{U}}} < \epsilon$. Then $\widetilde{\mathcal{V}} := \widetilde{\mathcal{U}} \cdot \widetilde{\phi}$ is a $\mathcal{C}^1$-neighborhood of $\widetilde{\phi}$ such that for every $\widetilde{\psi} \in \widetilde{\mathcal{V}}$ we have

$$\nu_\alpha(\widetilde{\phi} \cdot \widetilde{\psi}^{-1}) = \nu_\alpha(\widetilde{\psi} \cdot \widetilde{\phi}^{-1}) < \epsilon\,.$$

This implies that there is a contact Hamiltonian function H_t whose flow represents $\widetilde{\phi} \cdot \widetilde{\psi}^{-1}$ and satisfies $\int_0^1 \max |H_t|\, dt < \epsilon$. Using (11) we thus conclude that for every $\widetilde{\psi} \in \widetilde{\mathcal{V}}$ we have $\big|\, c_j(\widetilde{\phi}) - c_j(\widetilde{\psi})\,\big| < \epsilon$, and so that c_j is $\mathcal{C}^1$-continuous.

Triangle inequality. We have to prove that if either k is even or j is even then

$$c_{j+l}(\widetilde{\phi} \cdot \widetilde{\psi}) \leq c_j(\widetilde{\phi}) + \Big\lceil c_l(\widetilde{\psi}) \Big\rceil_{\frac{2\pi}{k}}\,.$$

Let $T = c_j(\widetilde{\phi})$ and $N = \lceil c_l(\widetilde{\psi}) \rceil_{\frac{2\pi}{k}}$. Since $\Pi : \widetilde{\text{Cont}_0}(L_k^{2n-1}, \xi_0) \to \text{Cont}_0(L_k^{2n-1}, \xi_0)$ is a local homeomorphism, by the continuity property of the spectral selectors and Lemma 3.1 we can assume that $\Pi(\widetilde{\phi})$ is non-degenerate with respect to α_0. By the relation with translated points property, the spectral selectors of $\widetilde{\phi}$ are thus all distinct, and so $\mu\big(\widetilde{r_{-T}} \cdot \widetilde{\phi}\big) = -j$. Using the fact that $\widetilde{r_{-N}}$ commutes with $\widetilde{\phi}$ (since $\{r_t\}$ is $\frac{2\pi}{k}$-periodic) and the triangle inequality for the nonlinear Maslov index (Proposition 2.12 (iii)) we thus have

$$\mu\big(\widetilde{r_{-(T+N)}}\cdot\widetilde{\phi}\cdot\widetilde{\psi}\big)=\mu\big((\widetilde{r_{-T}}\cdot\widetilde{\phi})\cdot(\widetilde{r_{-N}}\cdot\widetilde{\psi})\big)\leq\mu\big(\widetilde{r_{-T}}\cdot\widetilde{\phi}\big)+\mu\big(\widetilde{r_{-N}}\cdot\widetilde{\psi}\big)\leq-(j+l)\,.$$

By definition of c_{j+l} we conclude that

$$c_{j+l}\big(\widetilde{\phi}\cdot\widetilde{\psi}\big)\leq T+N=c_j(\widetilde{\phi})+\Big\lceil c_l(\widetilde{\psi})\Big\rceil_{\frac{2\pi}{k}}\,.$$

Conjugation invariance. We have to prove that

$$\Big\lceil c_j(\widetilde{\psi}\cdot\widetilde{\phi}\cdot\widetilde{\psi}^{-1})\Big\rceil_{\frac{2\pi}{k}}=\Big\lceil c_j(\widetilde{\phi})\Big\rceil_{\frac{2\pi}{k}}\,. \tag{12}$$

Assume first that $\Pi(\widetilde{\phi})$ does not have discriminant points. Let $\{\psi_t\}_{t\in[0,1]}$ be a contact isotopy representing $\widetilde{\psi}$, and consider the homotopy $\widetilde{\psi}_s=[\{\psi_{st}\}_{t\in[0,1]}]$ from $\widetilde{\psi}_0=\widetilde{\mathrm{id}}$ to $\widetilde{\psi}_1=\widetilde{\psi}$. By the continuity property, the map

$$s\mapsto c_j(\widetilde{\psi}_s\cdot\widetilde{\phi}\cdot\widetilde{\psi}_s^{-1})\in\bar{\mathcal{A}}(\widetilde{\psi}_s\cdot\widetilde{\phi}\cdot\widetilde{\psi}_s^{-1})$$

is continuous. Moreover, $c_j(\widetilde{\psi}_s\cdot\widetilde{\phi}\cdot\widetilde{\psi}_s^{-1})\in\mathbb{R}\smallsetminus\frac{2\pi}{k}\cdot\mathbb{Z}$ for all $s\in[0,1]$. Indeed, if we had $c_j(\widetilde{\psi}_{\underline{s}}\cdot\widetilde{\phi}\cdot\widetilde{\psi}_{\underline{s}}^{-1})\in\frac{2\pi}{k}\cdot\mathbb{Z}$ for some $\underline{s}$ then, by the spectrality property, $\Pi(\widetilde{\psi}_{\underline{s}}\cdot\widetilde{\phi}\cdot\widetilde{\psi}_{\underline{s}}^{-1})$ would have discriminant points. But this is absurd, because the discriminant points of $\Pi(\widetilde{\psi}_{\underline{s}}\cdot\widetilde{\phi}\cdot\widetilde{\psi}_{\underline{s}}^{-1})$ are in bijection with the discriminant points of $\Pi(\widetilde{\phi})$. We thus obtain (12) in this case.

The general case can be obtained as follows. Given any $\widetilde{\phi}\in\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$, since (by Lemma 3.2) $\mathcal{A}(\widetilde{\phi})$ is nowhere dense, there is a sequence (ϵ_n) of positive real numbers with $\epsilon_n\to0$ such that, for every n, $\Pi(\widetilde{r_{-\epsilon_n}}\cdot\widetilde{\phi})$ does not have discriminant points. Pose $\widetilde{\chi_n}=\widetilde{r_{-\epsilon_n}}\cdot\widetilde{\phi}$. By the first part of the proof we have

$$\Big\lceil c_j(\widetilde{\psi}\cdot\widetilde{\chi_n}\cdot\widetilde{\psi}^{-1})\Big\rceil_{\frac{2\pi}{k}}=\Big\lceil c_j(\widetilde{\chi}_n)\Big\rceil_{\frac{2\pi}{k}} \tag{13}$$

for all n. Since $(\widetilde{\chi}_n)$ converges to $\widetilde{\phi}$ in the $\mathcal{C}^1$-topology and $\widetilde{\chi}_n\leq\widetilde{\phi}$ for all n, by the continuity and monotonicity properties of the spectral selectors for n big enough we have $\Big\lceil c_j(\widetilde{\phi})\Big\rceil_{\frac{2\pi}{k}}=\lceil c_j(\widetilde{\chi}_n)\rceil_{\frac{2\pi}{k}}$ and $\Big\lceil c_j(\widetilde{\psi}\cdot\widetilde{\phi}\cdot\widetilde{\psi}^{-1})\Big\rceil_{\frac{2\pi}{k}}=\Big\lceil c_j(\widetilde{\psi}\cdot\widetilde{\chi}_n\cdot\widetilde{\psi}^{-1})\Big\rceil_{\frac{2\pi}{k}}$. Equation (13) thus gives the desired result (12).

Poincaré duality. We first notice that if $\Pi\big(\widetilde{\phi}\big)$ does not have discriminant points then

$$\Big\lfloor c_j\big(\widetilde{\phi}\big)\Big\rfloor_{\frac{2\pi}{k}}=\max\left\{N\in\frac{2\pi}{k}\cdot\mathbb{Z}\;\Big|\;\mu(\widetilde{r_{-N}}\cdot\widetilde{\phi})>-j\right\}\,. \tag{14}$$

Indeed, by spectrality we have $c_j(\widetilde{\phi})\notin\frac{2\pi}{k}\cdot\mathbb{Z}$ and thus $\underline{N}:=\lfloor c_j\big(\widetilde{\phi}\big)\rfloor_{\frac{2\pi}{k}}<c_j(\widetilde{\phi})$. This implies that $\mu(\widetilde{r_{-\underline{N}}}\cdot\widetilde{\phi})>-j$, and so the inequality $\leq$ in (14). On the other hand, the opposite inequality follows (without any assumption on $\widetilde{\phi}$) from (10) and the fact that $\lfloor c_j\big(\widetilde{\phi}\big)\rfloor_{\frac{2\pi}{k}}+\frac{2\pi}{k}\geq\lceil c_j\big(\widetilde{\phi}\big)\rceil_{\frac{2\pi}{k}}$.

We now prove the Poincaré duality property, i.e. that

$$\Big\lceil c_j(\widetilde{\phi})\Big\rceil_{\frac{2\pi}{k}}=-\Big\lfloor c_{-j-(2n-1)}(\widetilde{\phi}^{-1})\Big\rfloor_{\frac{2\pi}{k}} \tag{15}$$

for any $\widetilde{\phi}\in\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$. Assume first that $\Pi\big(\widetilde{\phi}\big)$ does not have discriminant points. Then Proposition 2.12 (ix) implies that

$$\mu(\widetilde{r_N}\cdot\widetilde{\phi})+\mu(\widetilde{\phi}^{-1}\cdot\widetilde{r_{-N}})=2n \tag{16}$$

for every N that is a multiple of $\frac{2\pi}{k}$. The Poincaré duality (15) then follows from (10), (14), (16) and the fact that $\widetilde{\phi}^{-1}\cdot\widetilde{r_{-N}}=\widetilde{r_{-N}}\cdot\widetilde{\phi}^{-1}$ for every N that is a multiple of $\frac{2\pi}{k}$.

For a general $\widetilde{\phi}$, as in the proof of conjugation invariance we can find a sequence $(\widetilde{\chi}_n)$ that converges to $\widetilde{\phi}$ in the $\mathcal{C}^1$-topology and such that, for all n, $\widetilde{\chi}_n \leq \widetilde{\phi}$ and $\Pi(\widetilde{\chi}_n)$ does not have discriminant points. By the first part of the proof we have $\lceil c_j(\widetilde{\chi}_n)\rceil_{\frac{2\pi}{k}} = -\lfloor c_{-j-(2n-1)}(\widetilde{\chi}_n^{-1})\rfloor_{\frac{2\pi}{k}}$. By monotonicity and continuity of the spectral selectors we thus obtain (15) also in this case.

4. Non-shortening of the standard Reeb flow with respect to the discriminant and oscillation norms

Recall from [9] that the discriminant norm ν_{dis} on the universal cover $\widetilde{\mathrm{Cont}_0}(M,\xi)$ of the identity component of the contactomorphism group of a closed contact manifold (M,ξ) is the word norm associated to the generating set $\mathcal{D} \subset \widetilde{\mathrm{Cont}_0}(M,\xi)$ formed by elements $\widetilde{\phi}$ that can be represented by an embedded contact isotopy, i.e. a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ such that $\phi_t \circ \phi_s^{-1}$ has no discriminant points for all $s \neq t \in [0,1]$. Recall also from [5] that the discriminant length of a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ is the minimal N such that there is a decomposition $0 = t_0 < \cdots < t_N = 1$ of the time interval $[0,1]$ with $\{\phi_t\}_{t\in[t_j,t_{j+1}]}$ embedded for all $j = 0, \cdots, N-1$.

Consider the discriminant norm on $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$. For every positive real number T we have

$$\nu_{\text{dis}}(\widetilde{r_T}) \geq \frac{2n\lceil \frac{T}{2\pi}\rceil + 1}{2n+1}\,.$$

Indeed, let $N = \nu_{\text{dis}}(\widetilde{r_T})$ and write $\widetilde{r_T} = \prod_{j=1}^N \widetilde{\phi}_j$ with $\widetilde{\phi}_j \in \mathcal{D}$. Then, by Proposition 2.12 (vi), (iii), (i) and the first statement of (v) we have

$$2n\left\lceil \frac{T}{2\pi}\right\rceil = \mu(\widetilde{r_T}) \leq \sum_{j=1}^N \mu(\widetilde{\phi_j}) + N - 1 \leq 2nN + N - 1\,. \tag{17}$$

Similarly, in the case of projective space we have

$$\nu_{\text{dis}}(\widetilde{r_T}) \geq \left\lceil \frac{T}{2\pi}\right\rceil\,. \tag{18}$$

The estimates (17) and (18) are better than those obtained in [9] and [16], since in those references just the quasimorphism property of the non-liner Maslov index (Proposition 2.12 (ii)) is used and not the triangle inequality (Proposition 2.12 (iii)). However, they are still not optimal. Indeed, writing

$$0 < T_0 := \frac{T}{\lfloor \frac{k}{2\pi}T\rfloor + 1} < \frac{2\pi}{k}$$

we have

$$\{r_{Tt}\}_{t\in[0,1]} = \{\underbrace{r_{T_0t}\circ\cdots\circ r_{T_0t}}_{\lfloor \frac{k}{2\pi}T\rfloor+1}\}_{t\in[0,1]} \tag{19}$$

and so, since $\{r_{T_0t}\}_{t\in[0,1]} \in \mathcal{D}$, $\{r_{Tt}\}_{t\in[0,1]}$ has discriminant length smaller or equal than $\lfloor \frac{k}{2\pi}T\rfloor + 1$; this length is actually equal to $\lfloor \frac{k}{2\pi}T\rfloor + 1$, because for any interval $[t_0,t_1]$ of length $t_1 - t_0 \geq \frac{2\pi}{k}$ the contact isotopy $\{r_t\}_{t\in[t_0,t_1]}$ is not embedded. For instance, in the case of projective space the discriminant length of $\{r_{2\pi m t}\}_{t\in[0,1]}$ is thus $4m+1$, while (18) only gives $\nu_{\text{dis}}(\widetilde{r_{2\pi m}}) \geq m$. In this section we prove the optimal estimates for the discriminant and oscillation lengths of the Reeb flow of lens spaces using the spectral selectors defined in Section 3. The main advantage of using the spectral selectors is that while (by Proposition 2.12 (v)) the non-linear Maslov index only jumps in the presence of discriminant points of the lift of a contact isotopy of (L_k^{2n-1},ξ_0) to the sphere, so that in particular for instance $\mu(\widetilde{r_T}) = 2n\left\lceil\frac{T}{2\pi}\right\rceil$, the spectral selectors allow to distinguish $\widetilde{r_T}$ also for different values of T in $[0,2\pi]$, indeed by Theorem 1.1 (v) we have for instance $c_0(\widetilde{r_T}) = T$.

We start with the following lemma.

Lemma 4.1. *For any element $\widetilde{\phi}$ of $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1},\xi_0)$, if $\widetilde{\phi} \in \mathcal{D}$ then $c_0(\widetilde{\phi}) < \frac{2\pi}{k}$.*

Proof. If $\widetilde{\phi} \in \mathcal{D}$ then $\widetilde{\phi}$ can be represented by a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ such that ϕ_t does not have discriminant points for all $t \in (0,1]$. Suppose by contradiction that $c_0(\widetilde{\phi}) \geq \frac{2\pi}{k}$, and for $s \in [0,1]$ let $\widetilde{\phi}_s = [\{\phi_{st}\}_{t\in[0,1]}]$. Since $c_0(\widetilde{\phi}_1) = c_0(\widetilde{\phi}) \geq \frac{2\pi}{k}$ and, by Theorem 1.1 (ii), $c_0(\widetilde{\phi}_0) = c_0(\widetilde{\mathrm{id}}) = 0$, by continuity of c_0 (Theorem 1.1 (viii)) there is a value of s in $(0,1]$ such that $c_0(\widetilde{\phi}_s) = \frac{2\pi}{k}$. But then, by spectrality (Theorem 1.1 (i)), $\frac{2\pi}{k}$ belongs to $\bar{\mathcal{A}}(\widetilde{\phi}_s)$. This means that ϕ_s has discriminant points, which is a contradiction. $\square$

We have seen above that $\{r_{Tt}\}_{t\in[0,1]}$ has discriminant length $\left\lfloor \frac{k}{2\pi}\, T \right\rfloor + 1$. In order to prove that it is a geodesic we thus have to show that

$$\nu_{\mathrm{dis}}(\widetilde{r_T}) \geq \left\lfloor \frac{k}{2\pi}\, T \right\rfloor + 1\,. \tag{20}$$

Let $\nu_{\mathrm{dis}}(\widetilde{r_T}) = N$, and write $\widetilde{r_T} = \prod_{j=1}^{N} \widetilde{\phi_j}$ with $\widetilde{\phi_j} \in \mathcal{D}$ for all j. By Theorem 1.1 (v), (ix) and Lemma 4.1 we then have

$$T = c_0(\widetilde{r_T}) \leq c_0(\widetilde{\phi_1}) + \sum_{j=2}^{N} \left\lceil c_0(\widetilde{\phi_j}) \right\rceil_{\frac{2\pi}{k}} < N\, \frac{2\pi}{k}\,.$$

This implies that $\nu_{\mathrm{dis}}(\widetilde{r_T}) \geq \left\lfloor \frac{k}{2\pi}\, T \right\rfloor + 1$, as we wanted.

We now show that $\{r_{Tt}\}_{t\in[0,1]}$ is a geodesic with respect to the oscillation norm. Recall from [9] that the oscillation pseudonorm ν_{osc} on the universal cover $\widetilde{\mathrm{Cont}_0}(M,\xi)$ of the identity component of the contactomorphism group of a closed contact manifold (M,ξ) is defined as follows. Let $\mathcal{D}_+$ and $\mathcal{D}_-$ be the sets of elements of $\widetilde{\mathrm{Cont}_0}(M,\xi)$ that can be represented respectively by an embedded non-negative or non-positive contact isotopy. It is proved in [9] that every element $\widetilde{\phi}$ of $\widetilde{\mathrm{Cont}_0}(M,\xi)$ can be written as $\widetilde{\phi} = \prod_{j=1}^{N} \widetilde{\phi_j}$ with $\widetilde{\phi_j} \in \mathcal{D}_+$ or $\widetilde{\phi_j} \in \mathcal{D}_-$ for every j. We denote by $\nu_+(\widetilde{\phi})$ and $\nu_-(\widetilde{\phi})$ respectively the minimal number of elements of $\mathcal{D}_+$ and minus the minimal number of elements of $\mathcal{D}_-$ in such a decomposition. The oscillation pseudonorm is then defined by

$$\nu_{\mathrm{osc}}(\widetilde{\phi}) = \nu_+(\widetilde{\phi}) - \nu_-(\widetilde{\phi})$$

for $\widetilde{\phi} \neq \widetilde{\mathrm{id}}$, and $\nu_{\mathrm{osc}}(\widetilde{\mathrm{id}}) = 0$. By [9, Proposition 3.2], the oscillation pseudonorm on $\widetilde{\mathrm{Cont}_0}(M,\xi)$ is non-degenerate if and only if (M,ξ) is orderable; it is thus a norm for lens spaces. Recall also from [5] that the oscillation length of a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ is the sum of $\mathcal{L}_+\big(\{\phi_t\}_{t\in[0,1]}\big)$ and $\mathcal{L}_-\big(\{\phi_t\}_{t\in[0,1]}\big)$, where $\mathcal{L}_+\big(\{\phi_t\}_{t\in[0,1]}\big)$ is the minimal N_+ for which there is $N \geq N_+$ and a decomposition $0 = t_0 < \cdots < t_N = 1$ with each $\{\phi_t\}_{t\in[t_j,t_{j+1}]}$ embedded and non-negative or non-positive and exactly N_+ of them non-negative, and $\mathcal{L}_-\big(\{\phi_t\}_{t\in[0,1]}\big)$ is the minimal N_- for which there is $N \geq N_-$ and a decomposition $0 = t_0 < \cdots < t_N = 1$ with each $\{\phi_t\}_{t\in[t_j,t_{j+1}]}$ embedded and non-negative or non-positive and exactly N_- of them non-positive.

Consider now as above the Reeb flow $\widetilde{r_T} = [\{r_{Tt}\}_{t\in[0,1]}]$ on (L_k^{2n-1},ξ_0). The decomposition (19) shows that $\nu_-(\widetilde{r_T}) = 0$ and $\nu_+(\widetilde{r_T}) \leq \left\lfloor \frac{k}{2\pi}\, T \right\rfloor + 1$, thus

$$\nu_{\mathrm{osc}}(\widetilde{r_T}) = \nu_+(\widetilde{r_T}) \leq \left\lfloor \frac{k}{2\pi}\, T \right\rfloor + 1\,.$$

In order to show that $\{r_{Tt}\}_{t\in[0,1]}$ is a geodesic with respect to the oscillation norm we thus have to show that

$$\nu_+(\widetilde{r_T}) \geq \left\lfloor \frac{k}{2\pi}\, T \right\rfloor + 1\,.$$

Let $\nu_+(\widetilde{r_T}) = N_+$, and write $\widetilde{r_T} = \prod_{j=1}^{N} \widetilde{\phi}_j$ with $\widetilde{\phi}_j \in \mathcal{D}_\pm$ for all j and with exactly N_+ of the $\widetilde{\phi_j}$ in $\mathcal{D}_+$. Denote such elements by $\widetilde{\phi_{\sigma(1)}}, \cdots, \widetilde{\phi_{\sigma(N_+)}}$. Then $\widetilde{r_T} \leq \prod_{j=1}^{N_+} \widetilde{\phi}_{\sigma(j)}$, and so by Theorem 1.1

(v), (vii), (ix) and Lemma 4.1 we have

$$T = c_0(\widetilde{r_T}) \leq c_0\Big(\prod_{j=1}^{N_+} \widetilde{\phi}_{\sigma(j)}\Big) \leq c_0(\widetilde{\phi}_{\sigma(1)}) + \sum_{j=2}^{N_+} \Big\lceil c_0(\widetilde{\phi}_{\sigma(j)})\Big\rceil_{\frac{2\pi}{k}} < \frac{2\pi}{k} \cdot N_+ \,.$$

This implies that $\nu_+(\widetilde{r_T}) \geq \big\lfloor \frac{k}{2\pi}\, T\big\rfloor + 1$, as we wanted.

5. A spectral pseudonorm

Let $c_- = c_{-2n+1}$ and $c_+ = c_0$, and define $\nu : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \frac{2\pi}{k} \cdot \mathbb{Z}$ by

$$\nu(\widetilde{\phi}) = \max\left\{ \Big\lceil c_+(\widetilde{\phi})\Big\rceil_{\frac{2\pi}{k}} \,, - \Big\lfloor c_-(\widetilde{\phi})\Big\rfloor_{\frac{2\pi}{k}} \right\}.$$

In this section we prove that ν is a pseudonorm satisfying the properties stated in Corollary 1.6.

Recall that a pseudonorm ν on a group G is said to be stably unbounded if there is an element σ of G such that $\lim_{m\to\infty} \frac{\nu(\sigma^m)}{m} \neq 0$, and is said to be compatible with a bi-invariant partial order $\leq$ if $\mathrm{id} \leq \sigma_1 \leq \sigma_2$ implies $\nu(\sigma_1) \leq \nu(\sigma_2)$.

Proposition 5.1. *The map $\nu : \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0) \to \frac{2\pi}{k} \cdot \mathbb{Z}$ is a stably unbounded conjugation invariant pseudonorm compatible with the partial order $\leq$.*

Proof. We first show that for every $\widetilde{\phi}$ we have $\nu(\widetilde{\phi}) \geq 0$. Suppose by contradiction that $\nu(\widetilde{\phi}) < 0$. Then $\Big\lceil c_+(\widetilde{\phi})\Big\rceil_{\frac{2\pi}{k}} < 0$, thus $c_+(\widetilde{\phi}) < 0$, and $- \Big\lfloor c_-(\widetilde{\phi})\Big\rfloor_{\frac{2\pi}{k}} < 0$, thus $c_-(\widetilde{\phi}) > 0$. But this contradicts the fact that, since the sequence c_j is non-decreasing, $c_-(\widetilde{\phi}) \leq c_+(\widetilde{\phi})$. The triangle inequality $\nu(\widetilde{\phi}\cdot\widetilde{\psi}) \leq \nu(\widetilde{\phi}) + \nu(\widetilde{\psi})$ follows from Theorem 1.1 (ix) and (xi), and symmetry, i.e. $\nu(\widetilde{\phi}) = \nu(\widetilde{\phi}^{-1})$, from Theorem 1.1 (xi). This shows that ν is a pseudonorm. Invariance by conjugation follows from Theorem 1.1 (x) and (xi). The pseudonorm ν is stably unbounded, indeed Theorem 1.1 (v) implies that

$$\nu\Big(\widetilde{r_{\frac{2\pi}{k}}}^{\,m}\Big) = \nu\big(\widetilde{r_{m\frac{2\pi}{k}}}\big) = m$$

for every positive integer m, thus posing $\sigma = \widetilde{r_{\frac{2\pi}{k}}}$ we have $\lim_{m\to\infty} \frac{\nu(\sigma^m)}{m} = 1 \neq 0$. Finally, the fact that ν is compatible with the partial order $\leq$ follows from Theorem 1.1 (vii) and (xi). □

It would be interesting to know if ν is equivalent to the oscillation norm ν_{osc}. In this direction, we prove the following inequality.

Proposition 5.2. *For every element $\widetilde{\phi}$ of $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$ we have*

$$\nu(\widetilde{\phi}) \leq \frac{2\pi}{k} \cdot \nu_{\mathrm{osc}}(\widetilde{\phi})\,.$$

Proof. Let $\nu_+(\widetilde{\phi}) = N_+$, and write $\widetilde{\phi} = \prod_{j=1}^{N} \widetilde{\phi}_j$ with all the $\widetilde{\phi}_j$ in $\mathcal{D}_+$ or $\mathcal{D}_-$ and exactly N_+ of them in $\mathcal{D}_+$. Denote such elements by $\widetilde{\phi_{\sigma(1)}}, \cdots, \widetilde{\phi_{\sigma(N_+)}}$. Then $\widetilde{\phi} \leq \prod_{j=1}^{N_+} \widetilde{\phi}_{\sigma(j)}$, and thus by Theorem 1.1 (vii), (ix) and Lemma 4.1 we have

$$c_+(\widetilde{\phi}) \leq c_+\Big(\prod_{j=1}^{N_+} \widetilde{\phi}_{\sigma(j)}\Big) \leq \sum_{j=1}^{N_+} \Big\lceil c_+(\widetilde{\phi}_{\sigma(j)})\Big\rceil_{\frac{2\pi}{k}} \leq \frac{2\pi}{k} \cdot N_+ \,.$$

Similarly, setting $\nu_-(\widetilde{\phi}) = -N_-$ we have $c_+(\widetilde{\phi}^{-1}) \leq \frac{2\pi}{k} \cdot N_-$ and so, by Theorem 1.1 (xi),

$$- \Big\lfloor c_-(\widetilde{\phi})\Big\rfloor_{\frac{2\pi}{k}} = \Big\lceil c_+\left(\widetilde{\phi}^{-1}\right)\Big\rceil_{\frac{2\pi}{k}} \leq \frac{2\pi}{k} \cdot N_- \,.$$

We deduce that

$$\nu(\widetilde{\phi}) \leq \frac{2\pi}{k} \max\left\{\nu_+(\widetilde{\phi}) \,,\, -\nu_-(\widetilde{\phi})\right\} \leq \frac{2\pi}{k}\big(\nu_+(\widetilde{\phi}) - \nu_-(\widetilde{\phi})\big) = \frac{2\pi}{k} \cdot \nu_{\mathrm{osc}}(\widetilde{\phi})\,.$$

□

We do not know whether the pseudonorm ν is non-degenerate, i.e. whether $\nu(\widetilde{\phi}) = 0$ if and only if $\widetilde{\phi} = \widetilde{\mathrm{id}}$. Indeed, by the definition of ν we have that $\nu(\widetilde{\phi}) = 0$ if and only if $c_+(\widetilde{\phi}) = c_-(\widetilde{\phi}) = 0$, which by Theorem 1.1 (iv) only implies that $\Pi(\widetilde{\phi})$ is the identity. On the other hand, the induced conjugation invariant pseudonorm on $\mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$, i.e. the pseudonorm ν_* defined by

$$\nu_*(\phi) = \inf\{\, \nu(\widetilde{\phi}) \mid \Pi(\widetilde{\phi}) = \phi\,\}\,,$$

is non-degenerate, hence a norm. However, this norm is bounded (hence equivalent to the trivial norm, since it is discrete), as shown in the following proposition.

Proposition 5.3. *For every $\phi \in \mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$ we have $\nu_*(\phi) \leq 2\pi + \frac{2\pi}{k}$.*

Proof. We show that

$$\nu_*\big(\Pi(\widetilde{\phi})\big) \leq \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}} - \left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}} \leq 2\pi + \frac{2\pi}{k} \tag{21}$$

for every $\widetilde{\phi} \in \widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$. Using periodicity of the spectral selectors (Theorem 1.1 (vi)) and the fact that the sequence of spectral selectors c_j is non-decreasing we have $c_+(\widetilde{\phi}) \leq c_-(\widetilde{\phi}) + 2\pi$, which implies the second inequality in (21). For the first inequality, it is enough to find $N \in \frac{2\pi}{k}\cdot\mathbb{Z}$ such that $\nu(\widetilde{r_{-N}}\cdot\widetilde{\phi}) = \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}} - \left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}}$. Suppose first that $\nu(\widetilde{\phi}) = \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}}$, and pose $N = \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}}$. By Theorem 1.1 (v) we then have

$$\begin{aligned}\nu(\widetilde{r_{-N}}\cdot\widetilde{\phi}) &= \max\left\{\left\lceil c_+(\widetilde{r_{-N}}\cdot\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}} \,,\, -\left\lfloor c_-(\widetilde{r_{-N}}\cdot\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}}\right\}\\ &= \max\left\{0\,,\, \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}} - \left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}}\right\} = \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}} - \left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}}\,.\end{aligned}$$

Similarly, if $\nu(\widetilde{\phi}) = \left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}} =: N$ then $\nu(\widetilde{r_{-N}}\cdot\widetilde{\phi}) = \left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}} - \left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}}$. □

Remark 5.4. *It follows from* [4, Corollary 4.12] *that on the universal cover of the identity component of the contactomorphism group of the unit cotangent bundle of the torus $\mathbb{T}^n$ for $n \geq 2$ the difference of the invariants c_+ and c_- defined in* [4] *is unbounded. This difference with respect to* (21) *might be related to the fact that the identity component of the contactomorphism group of the unit cotangent bundle of the torus does not contain positive loops. It would be interesting to investigate if on the other hand the difference of the invariants c_+ and c_- of* [4] *is bounded on $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$. This would then imply as in Proposition 5.3 that the induced norm on $\mathrm{Cont}_0(L_k^{2n-1}, \xi_0)$ is bounded, and therefore answer partially a question in* [12, Example 2.21].

Remark 5.5. *If $\nu : G \to \mathbb{R}_{\geq 0}$ is a pseudonorm on a group G then, for any $c > 0$, the map $\nu' : G \to \mathbb{R}_{\geq 0}$ defined by*

$$\nu'(g) := \begin{cases} \max\{\nu(g), c\} & \text{if } g \neq \mathrm{id} \\ 0 & \text{if } g = \mathrm{id}\end{cases}$$

is a norm. Moreover, ν' is invariant by conjugation if and only if so is ν. This trick (which is similar to one used in [7]*) can be applied to our pseudonorm ν, with $c = \frac{2\pi}{k}$, to obtain a stably unbounded conjugation invariant norm ν' on $\widetilde{\mathrm{Cont}_0}(L_k^{2n-1}, \xi_0)$. Since ν takes values in $\frac{2\pi}{k}\cdot\mathbb{Z}$, if ν is already a norm then $\nu' \equiv \nu$. Proposition 5.2 holds also for ν', indeed for any element $\widetilde{\phi} \neq \widetilde{\mathrm{id}}$ we have*

$$\nu'(\widetilde{\phi}) = \max\left\{\left\lceil c_+(\widetilde{\phi})\right\rceil_{\frac{2\pi}{k}}, -\left\lfloor c_-(\widetilde{\phi})\right\rfloor_{\frac{2\pi}{k}}, \frac{2\pi}{k}\right\} \leq \frac{2\pi}{k}\cdot\max\left\{\nu_{\mathrm{osc}}(\widetilde{\phi}), 1\right\} = \frac{2\pi}{k}\cdot\nu_{\mathrm{osc}}(\widetilde{\phi})\,.$$

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S. Allais, Université de Strasbourg, IRMA UMR 7501, F-67000 Strasbourg, France

Email address: simon.allais@math.unistra.fr

URL: https://irma.math.unistra.fr/~allais/

P.-A. Arlove, Ruhr-Universität Bochum, Fakultät für Mathematik, 44780 Bochum, Germany

Email address: pierre-alexandre.arlove@ruhr-uni-bochum.de

S. Sandon, Université de Strasbourg, CNRS, IRMA UMR 7501, F-67000 Strasbourg, France

Email address: sandon@math.unistra.fr