SPECTRAL SELECTORS AND CONTACT ORDERABILITY

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Abstract. We study the notion of orderability of isotopy classes of Legendrian submanifolds and their universal covers, with some weaker results concerning spaces of contactomorphisms. Our main result is that orderability is equivalent to the existence of spectral selectors analogous to the spectral invariants coming from Lagrangian Floer Homology. A direct application is the existence of Reeb chords between any closed Legendrian submanifolds of a same orderable isotopy class. Other applications concern the Sandon conjecture, the Arnold chord conjecture, Legendrian interlinking, the existence of time-functions and the study of metrics due to Hofer-Chekanov-Shelukhin, Colin-Sandon, Fraser-Polterovich-Rosen and Nakamura.

1. Introduction

1.1. **Historical background.** The abundance of interactions between Hofer geometry on the group of Hamiltonian diffeomorphisms and symplectic geometry [43, 46, 51, 61] led Eliashberg and Polterovich in 2000 [30] to introduce the notion that will be later called *orderability* on the universal cover $\widetilde{\mathcal{G}}$ of the group of contactomorphisms isotopic to the identity. To do so they define a bi-invariant binary relation $\preceq$ on this group: $\varphi \preceq \psi$ if there exists a non-negative contact Hamiltonian generating an isotopy from φ to ψ. They show that for some contact manifolds this binary relation turns out to be a partial order: we now say that the group $\widetilde{\mathcal{G}}$ is orderable in this case. Some years later together with Kim [29] they show how orderability can be used to detect some squeezing and non-squeezing phenomena. They show in particular that there really is a dichotomy: for some contact manifolds such as the standard contact sphere $\widetilde{\mathcal{G}}$ is unorderable.

The study of orderability naturally extends to the group of contactomorphisms isotopic to the identity $\mathcal{G}$ and to the isotopy class $\mathcal{L}$ of a closed Legendrian submanifold Λ_* as well as its universal cover $\widetilde{\mathcal{L}}$. The notion of orderability has been quite influential in the study of contact topology: we refer to our paragraph Examples 2.10 in Section 2.4 for an account on the study of the dichotomy orderable/unorderable space throughout the last two decades. Nevertheless, rather little is known about the properties that are specifically implied by orderability. One of the most significant achievements in that direction is the discovery of Albers-Fuchs-Merry that the unorderability of $\mathcal{G}(M,\xi)$ implies the Weinstein conjecture in (M,ξ) [2]. The goal of this article is to establish an equivalence between orderability and the existence of (hopefully spectral) contact selectors.

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The notion of spectral selectors in contact geometry is a natural generalization of spectral selectors defined in symplectic geometry. Given a closed symplectic manifold (M,ω) satisfying some mild hypothesis (*e.g.* symplectic asphericality), one can define spectral selectors on the space of Hamiltonian diffeomorphisms $\mathrm{Ham}(M,\omega)$ or its universal cover. In this context a spectral selector is a continuous map $c : \mathrm{Ham}(M,\omega) \to \mathbb{R}$ associating to φ a real number $c(\varphi)$ belonging to its spectrum, which is the set of action values of its fixed points. Such selectors were first constructed by Viterbo for compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ [73] then generalized by the works of Schwarz [67] and Oh [60]. Likewise, given an isotopy class of Lagrangian submanifolds of (M,ω) satisfying some mild hypothesis, one can associate spectral selectors ℓ such that $\ell(\Lambda_1,\Lambda_0)$ is in the spectrum of the couple of Lagrangian submanifolds (Λ_1,Λ_0) (which corresponds to symplectic areas associated to couples of points of $\Lambda_0\cap\Lambda_1$) [73, 59, 47, 48]. Among the spectral selectors c or ℓ (a specific construction being given), one can usually distinguish two specific selectors $c_-\leq c_+$ or $\ell_-\leq\ell_+$ (in the case of Hamiltonian Floer theory, they correspond to the fundamental class and the class of a point) which satisfy additional property, *e.g.*

1. (triangular inequality) $c_+(\varphi\psi)\leq c_+(\varphi)+c_+(\psi)$,
2. (Poincaré duality) $c_+(\psi)=-c_-(\psi^{-1})$,
3. (conjugation invariance) $c_\pm(\varphi\psi\varphi^{-1})=c_\pm(\psi)$,
4. (non-degeneracy) $c_+(\psi)=c_-(\psi)$ implies $\psi=\mathrm{id}$.

The number of applications of the existence of spectral selectors, especially c_+, is quite large. Among other ones, on the dynamical side, it has recently been used for the study of the Hofer-Zehnder conjecture [37, 69] or the C^∞-closing lemma [24], it also has more topological applications: the study of non-squeezing phenomena [73], the geometry of the group of Hamiltonian diffeomorphisms [11, 44] or C^0-symplectic geometry [12].

This notion of spectral selectors has been extended to compactly supported contactomorphisms of $\mathbb{R}^{2n}\times S^1$ isotopic to the identity by Sandon in [64]. She discovered that the notion of translated points was key in order to define the spectrum of contactomorphisms (see Section 1.3 below for the definition). These notions of spectral selectors were then applied in multiple contexts in order to address numerous questions of contact topology: study of bi-invariant metrics on the group of compactly supported contactomorphisms isotopic to identity [63, 26], contact non-squeezing phenomena [64, 4], orderability [65]. Albers-Fuchs-Merry use the Rabinowitz Floer theory to define contact selectors in a class of Liouville fillable contact manifolds [2, 3] (see in Section 1.4 some applications of their work). Our main result is that orderability is the necessary and sufficient condition in order to define contact analogues of selectors $c_\pm$ and $\ell_\pm$ in full generality (although our $c_\pm$'s are not spectral *a priori*) whose signs only are invariant by contactomorphisms (*cf.* paragraph just above Corollary 1.3).

1.2. **Conventions.** Whenever no precision regarding regularity is given, maps and manifolds considered are C^∞-smooth. Every contact manifold (M,ξ) considered are assumed to be connected and cooriented (*i.e.* TM/ξ is an oriented line bundle). We write that a contact form α is supporting ξ if $\ker\alpha=\xi$ and α respects the coorientation. A contactomorphism of (M,ξ) will always preserve the coorientation. Given a contact manifold (M,ξ) and a closed Legendrian submanifold $\Lambda_*\subset M$, one denotes $\mathcal{L}(\Lambda_*)$ the isotopy class of Λ_* and $\widetilde{\mathcal{L}}(\Lambda_*)$ its universal cover. One denotes $\mathcal{G}(M)$ the group $\mathrm{Cont}_0^c(M,\xi)$ of compactly supported contactomorphisms of (M,ξ) isotopic to id through compactly supported contactomorphisms while $\widetilde{\mathcal{G}}(M)$ will denote its universal cover.

Usually, we will only write $\mathcal{L}$ for $\mathcal{L}(\Lambda_*)$, $\widetilde{\mathcal{L}}$ for $\widetilde{\mathcal{L}}(\Lambda_*)$ etc. implicitly meaning that Λ_* and M are fixed. By a slight abuse of notation:

1. id $\in \widetilde{\mathcal{G}}$ will denote the class of the constant isotopy $s \mapsto$ id. Given a contact form α supporting ξ and $t \in \mathbb{R}$, let $\phi_t^\alpha \in \mathrm{Cont}(M,\xi)$ denote the α-Reeb flow at time $t \in \mathbb{R}$ (see *e.g.* Section 2.1);
2. when it is understood that ϕ_t^α must be an element of the universal cover of the identity component of $\mathrm{Cont}(M,\xi)$ (*e.g.* when acting on $\widetilde{\mathcal{G}}$ or $\widetilde{\mathcal{L}}$), it will denote the isotopy $[0,1] \to \mathrm{Cont}_0(M,\xi)$, $s \mapsto \phi_{st}^\alpha$;
3. the cover maps $\widetilde{\mathcal{G}} \to \mathcal{G}$ and $\widetilde{\mathcal{L}} \to \mathcal{L}$ will both be denoted Π.

On O being either $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$, we write $x \preceq y$, or equivalently $y \succeq x$, if there exists a non-negative isotopy from x to y. O is called orderable if and only if $\preceq$ defines a partial order (see Section 2.4 for details). In particular, the orderability of $\mathcal{L}$ (resp. $\mathcal{G}$) implies the orderability of $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{G}}$). We refer to Examples 2.10 for a list of known orderable spaces.

1.3. **Order spectral selectors.** Let (M,ξ) be a closed cooriented contact manifold. For any contact form α supporting ξ, let us define c_-^α and c_+^α as the maps $\mathcal{G} \to \mathbb{R}\cup\{\pm\infty\}$ (resp. $\widetilde{\mathcal{G}} \to \mathbb{R}\cup\{\pm\infty\}$) defined by

$$c_-^\alpha(\psi) := \sup\{t \in \mathbb{R} \mid \psi \succeq \phi_t^\alpha\} \quad \text{and} \quad c_+^\alpha(\psi) := \inf\{t \in \mathbb{R} \mid \psi \preceq \phi_t^\alpha\}, \qquad (1)$$

for $\psi \in \mathcal{G}$ (resp. $\psi \in \widetilde{\mathcal{G}}$). These maps are highly inspired by constructions of Fraser-Polterovich-Rosen [35] and were already considered by the second author in his PhD-thesis [8] and very recently by Nakamura [58] from a metrical point of view.

Let us now withdraw the closeness assumption on M and let $\mathcal{L} = \mathcal{L}(\Lambda_*)$ for some closed Legendrian $\Lambda_* \subset M$. For any complete contact form α supporting ξ (*i.e.* a contact form the Reeb vector field of which is complete), let us define ℓ_-^α and ℓ_+^α as the maps $\mathcal{L}\times\mathcal{L} \to \mathbb{R}\cup\{\pm\infty\}$ (resp. $\widetilde{\mathcal{L}}\times\widetilde{\mathcal{L}} \to \mathbb{R}\cup\{\pm\infty\}$) given by

$$\ell_-^\alpha(\Lambda_1,\Lambda_0) := \sup\{t \in \mathbb{R} \mid \Lambda_1 \succeq \phi_t^\alpha\Lambda_0\} \quad \text{and} \quad \ell_+^\alpha(\Lambda_1,\Lambda_0) := \inf\{t \in \mathbb{R} \mid \Lambda_1 \preceq \phi_t^\alpha\Lambda_0\},$$

for $\Lambda_0,\Lambda_1 \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$).

For $\psi \in \mathcal{G}$ and α supporting ξ, the α-spectrum of ψ is the set of time-shifts of its α-translated points that is

$$\mathrm{Spec}^\alpha(\psi) := \{t \in \mathbb{R} \mid \exists p \in M, (\psi^*\alpha)_p = \alpha_p \text{ and } \psi(x) = \phi_t^\alpha(x)\}.$$

For $\psi \in \widetilde{\mathcal{G}}$, the α-spectrum is defined as the α-spectrum of $\Pi\psi$. For $\Lambda_0,\Lambda_1 \in \mathcal{L}$, the α-spectrum of (Λ_1,Λ_0) is the set of (positive and non-positive) lengths of α-Reeb chords joining Λ_0 to Λ_1 that is

$$\mathrm{Spec}^\alpha(\Lambda_1,\Lambda_0) := \{t \in \mathbb{R} \mid \Lambda_1 \cap \phi_t^\alpha\Lambda_0 \neq \emptyset\}.$$

For $\Lambda_0,\Lambda_1 \in \widetilde{\mathcal{L}}$, the α-spectrum of (Λ_1,Λ_0) is defined as the α-spectrum of $(\Pi\Lambda_1,\Pi\Lambda_0)$. The latter notion of spectrum can be seen as a generalization of the former. Indeed, if $\psi \in \mathrm{Cont}(M,\ker\alpha)$, its contact graph

$$\mathrm{gr}^\alpha(\psi) := \{(x,\psi(x),g(x)) \mid x \in M\}, \text{ where } \psi^*\alpha = e^g\alpha,$$

is a Legendrian submanifold of the contact manifold $M \times M \times \mathbb{R}$ endowed with the contact form $\tilde{\alpha} := \alpha_2 - e^\theta \alpha_1$ defined in Example 2.10.6. Then,

$$\mathrm{Spec}^\alpha(\psi) = \mathrm{Spec}^{\tilde{\alpha}}(\mathrm{gr}^\alpha(\psi), \mathrm{gr}^\alpha(\mathrm{id})).$$

Theorem 1.1 (Legendrian order spectral selectors)**.** *Let Λ_* be a closed Legendrian submanifold of (M, ξ) such that $\mathcal{L}(\Lambda_*)$ (resp. $\widetilde{\mathcal{L}}(\Lambda_*)$) is orderable and let α be a complete contact form supporting ξ. The selectors $\ell^\alpha_\pm$ are real-valued and satisfy the following properties for every $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$),*

1. *(normalization) $\ell^\alpha_\pm(\Lambda_0, \Lambda_0) = 0$ and $\ell^\alpha_\pm(\phi^\alpha_t \Lambda_1, \Lambda_0) = t + \ell^\alpha_\pm(\Lambda_1, \Lambda_0)$, $\forall t \in \mathbb{R}$,*
2. *(monotonicity) $\Lambda_2 \preceq \Lambda_1$ implies $\ell^\alpha_\pm(\Lambda_2, \Lambda_0) \leq \ell^\alpha_\pm(\Lambda_1, \Lambda_0)$,*
3. *(triangle inequalities) $\ell^\alpha_+(\Lambda_2, \Lambda_0) \leq \ell^\alpha_+(\Lambda_2, \Lambda_1) + \ell^\alpha_+(\Lambda_1, \Lambda_0)$ and $\ell^\alpha_-(\Lambda_2, \Lambda_0) \geq \ell^\alpha_-(\Lambda_2, \Lambda_1) + \ell^\alpha_-(\Lambda_1, \Lambda_0)$,*
4. *(Poincaré duality) $\ell^\alpha_+(\Lambda_1, \Lambda_0) = -\ell^\alpha_-(\Lambda_0, \Lambda_1)$,*
5. *(compatibility) $\ell^\alpha_\pm(\varphi(\Lambda_1), \varphi(\Lambda_0)) = \ell^{\varphi^*\alpha}_\pm(\Lambda_1, \Lambda_0)$, for every $\varphi \in \mathrm{Cont}(M, \xi)$ (resp. $\widetilde{\mathrm{Cont}}(M, \xi)$),*
6. *(non-degeneracy) $\ell^\alpha_+(\Lambda_1, \Lambda_0) = \ell^\alpha_-(\Lambda_1, \Lambda_0) = t$ for some $t \in \mathbb{R}$ implies $\Lambda_1 = \phi^\alpha_t \Lambda_0$ (resp. it only implies the equality $\Pi\Lambda_1 = \phi^\alpha_t \Pi\Lambda_0$ in $\mathcal{L}$).*
7. *(spectrality) $\ell^\alpha_\pm(\Lambda_1, \Lambda_0) \in \mathrm{Spec}^\alpha(\Lambda_1, \Lambda_0)$.*

Theorem 1.2 (Contactomorphism order selectors)**.** *Let (M, ξ) be a closed contact manifold such that $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) is orderable and let α be a contact form supporting ξ. The selectors $c^\alpha_\pm$ are real-valued and satisfy the following properties for $\varphi, \psi \in \mathcal{G}$ (resp. $\varphi, \psi \in \widetilde{\mathcal{G}}$):*

1. *(normalization) $c^\alpha_\pm(\mathrm{id}) = 0$ and $c^\alpha_\pm(\phi^\alpha_t \psi) = t + c^\alpha_\pm(\psi)$, $\forall t \in \mathbb{R}$,*
2. *(monotonicity) $\varphi \preceq \psi$ implies $c^\alpha_\pm(\varphi) \leq c^\alpha_\pm(\psi)$,*
3. *(triangle inequalities) $c^\alpha_+(\varphi\psi) \leq c^\alpha_+(\varphi) + c^\alpha_+(\psi)$ and $c^\alpha_-(\varphi\psi) \geq c^\alpha_-(\varphi) + c^\alpha_-(\psi)$,*
4. *(Poincaré duality) $c^\alpha_+(\psi) = -c^\alpha_-(\psi^{-1})$,*
5. *(compatibility) $c^\alpha_\pm(\varphi\psi\varphi^{-1}) = c^{\varphi^*\alpha}_\pm(\psi)$, which extends to every $\varphi \in \mathrm{Cont}(M, \xi)$ (resp. $\widetilde{\mathrm{Cont}}(M, \xi)$),*
6. *(non-degeneracy) $c^\alpha_-(\psi) = c^\alpha_+(\psi) = t$ for some $t \in \mathbb{R}$ implies $\psi = \phi^\alpha_t$ (resp. it only implies the equality $\Pi\psi = \phi^\alpha_t$ in $\mathcal{G}$).*

The major results in these statements concern the non-degeneracy property and the spectrality. The non-degeneracy property has also been recently proven by Nakamura with a different approach [58].

We emphasize the fact that we do not know that the $c^\alpha_\pm(\psi)$'s do actually select spectral values of ψ. Nevertheless, we strongly believe it should be the case. The following direct corollary states that orderability is equivalent to the existence of selectors in a rather weak sense (a similar statement for spaces $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ also holds).

In contrast with their symplectic counterparts, these selectors are not actually invariant: identities $\ell(\varphi\Lambda_1, \varphi\Lambda_0) = \ell(\Lambda_1, \Lambda_0)$ and $c(\varphi\psi\varphi^{-1}) = c(\psi)$ are not verified for every contactomorphism φ (resp. lift in the universal cover), Λ_0, Λ_1 and ψ being fixed. In that sense, the α-selectors are only invariant under α-strict contactomorphisms: $\ell^\alpha_\pm(\varphi\Lambda_1, \varphi\Lambda_0) = \ell^\alpha_\pm(\Lambda_1, \Lambda_0)$ and $c^\alpha_\pm(\varphi\psi\varphi^{-1}) = c^\alpha_\pm(\psi)$ when $\varphi^*\alpha = \alpha$. Of course, this is compatible with the fact that the α-spectrum is only invariant under strict contactomorphisms: as in general $\varphi^{-1}\phi^\alpha_t\varphi = \phi^{\varphi^*\alpha}_t$ for all $t \in \mathbb{R}$, one has

$$\mathrm{Spec}^\alpha(\varphi\Lambda_1, \varphi\Lambda_0) = \mathrm{Spec}^{\varphi^*\alpha}(\Lambda_1, \Lambda_0) \text{ and } \mathrm{Spec}^\alpha(\varphi\psi\varphi^{-1}) = \mathrm{Spec}^{\varphi^*\alpha}(\psi).$$

Nonetheless, the sign of these selectors is invariant (see Lemmata 3.2 and 3.8)

Corollary 1.3. *Let (M,ξ) be a closed cooriented contact manifold. The space $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) is orderable if and only if there exists a non-decreasing map $c:(\mathcal{G},\preceq)\to(\mathbb{R},\leq)$ (resp. $(\widetilde{\mathcal{G}},\preceq)\to(\mathbb{R},\leq)$) such that*
1. *$c(\mathrm{id})=0$,*
2. *$\psi\succeq\mathrm{id}$ and $\psi\neq\mathrm{id}$ implies $c(\psi)>0$.*

Proof of Corollary 1.3. The direct implication is a consequence of Theorem 1.2 (*cf.* Corollary 3.13 below). Conversely, assuming the existence of c, if $\varphi,\psi\in\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) are such that $\varphi\preceq\psi$ and $\psi\preceq\varphi$ while $\varphi\neq\psi$, then $c(\varphi\psi^{-1})>0$ by the positivity property of c while $\varphi\psi^{-1}\preceq\mathrm{id}$ implies $c(\varphi\psi^{-1})\leq c(\mathrm{id})=0$ by monotonicity of c, a contradiction. □

When $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable and $\Lambda_0\in\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), any non-decreasing map $\ell(\cdot,\Lambda_0):\mathcal{L}\to\mathbb{R}$ (resp. $\widetilde{\mathcal{L}}\to\mathbb{R}$) that is normalized with respect to the contact form α in the sense of Theorem 1.1 satisfies

$$\ell_-^\alpha(\cdot,\Lambda_0)\leq\ell(\cdot,\Lambda_0)\leq\ell_+^\alpha(\cdot,\Lambda_0),$$

so that ℓ_-^α can be thought of as the minimal α-spectral selector and ℓ_+^α can be thought as the maximal one. A similar statement can be asserted for $c_\pm^\alpha$.

Remark 1.4 (Removing the closeness assumption on M). When studying $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$), we have added the additional hypothesis of closeness for the contact space M. Following Fraser-Polterovich-Rosen [35], one can remove this hypothesis by asking orderability not only for the group $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) but for the group $\mathcal{G}_\alpha$ (resp. $\widetilde{\mathcal{G}}_\alpha$) generated by $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) and the elements of the Reeb flow (ϕ_t^α), once fixed a complete supporting contact form α. This way the relation $\psi\preceq\phi_t^\alpha$ makes sense for $\psi\in\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) and $t\in\mathbb{R}$, and Theorem 1.2 still holds.

1.4. Applications to the existence of Reeb chords and translated points.

A direct consequence of the existence of spectral selectors in orderable Legendrian isotopy classes, is the non-triviality of the spectrum.

Corollary 1.5 (Existence of Reeb chords)**.** *Let Λ_* be a closed Legendrian submanifold of a cooriented contact manifold (M,ξ) such that $\widetilde{\mathcal{L}}(\Lambda_*)$ is orderable. Then any two Legendrian submanifolds $\Lambda_0,\Lambda_1\in\mathcal{L}(\Lambda_*)$ are joined by two distinct (positive or non-positive) α-Reeb chords, given any complete α supporting ξ.*
In particular, if (M,ξ) is a closed contact manifold such that $\widetilde{\mathcal{L}}(\Delta\times\{0\})$ as defined in Example 2.10.6 is orderable, any contactomorphism of M isotopic to id *has an α-translated point (and at least two distinct couples (p,t) where p is a translated point of time-shift t), given any α supporting ξ.*

The existence of Reeb chords between two isotopic Legendrian submanifolds has been widely studied. When Legendrian submanifolds are isotopic to $\mathbb{RP}^n$ in the standard $\mathbb{RP}^{2n+1}$ (*cf.* Example 2.10.1), the non-linear Maslov index developed by Givental [38] directly implies that the number of chords whose images are not overlapping each other is at least $n+1$ for the standard contact form. When Legendrian submanifolds are Legendrian isotopic to the zero-section of the standard J^1N where N is closed (*cf* Example 2.10.3), Chekanov gave lower bounds on the number of chords based on Morse theory (sum of the Betti numbers plus twice torsion numbers in the generic case, cuplength in the general case) for the standard contact form

[19]. With the exception of these two cases and the question of translated points discussed below, most of the known results concern the case of a *small* Legendrian isotopy in various senses (*e.g.* regarding C^1-metric, C^0-metric or Hofer type metrics), we refer to [27, 49] and references therein where bounds similar to Chekanov's are obtained. Of course, one can easily construct examples of isotopic Legendrian submanifolds without any Reeb chord joining them (*e.g.* a Legendrian circle in $J^1\mathbb{R}$ pushed horizontally). In contrast with these local studies, Corollary 1.5 asserts that orderability of $\widetilde{\mathcal{L}}(\Lambda_*)$ is enough for the existence of two chords in between any two Legendrian submanifolds isotopic to Λ_*, no matter how far they are from each other. In addition to this global statement, it is remarkable that one gets the existence of two distinct Reeb chords without any transversality assumption. Indeed, sharper estimates on the number of Reeb chords involving Morse type estimates are known in the aforementioned local studies but most of them require additional transversality hypothesis. Without these hypothesis, it is not clear that one could keep two distinct Reeb chords: when passing from a generic case to the general one, one usually looses the multiplicity (*e.g.* the classical Lefschetz fixed-point theorem or more recently the critical point theory of closed 1-forms [33]).

The existence of translated points is an important question that had been asked by Sandon in [66]. Originally, Sandon asked whether the number of translated points of any element of $\mathcal{G}(M,\xi)$ was at least the minimal number of critical points of a map $M \to \mathbb{R}$, by analogy with the Arnold conjecture bounding from below the number of fixed points a Hamiltonian diffeomorphism can have. This original statement was proven in the case of projective spaces and lens spaces endowed with their standard contact form [66, 6]. Lower bound related to Morse theory have been discovered in multiple other cases including the whole class of non-degenerate contactomorphisms of hypertight contact manifolds (*i.e.* admitting a Reeb flow without any contractible closed orbit) [2, 53, 5]. However, Cant proved recently that some elements of $\mathcal{G}(\mathbb{S}^{2n+1})$ do not admit any translated point for the standard contact form of the sphere of dimension $2n+1 \geq 3$, answering negatively to Sandon's question. As the space $\widetilde{\mathcal{G}}(\mathbb{S}^{2n+1})$ is unorderable, this result is compatible with our conjecture that the selectors $c^\alpha_\pm$ are spectral (which implies the existence of translated points when $\widetilde{\mathcal{G}}$ is orderable). Concerning the lower bound, the original conjecture of Sandon might be too optimistic but variants exist: see the introduction of [5].

Corollary 1.5 can in fact be generalized to a statement about positive Legendrian paths.

Theorem 1.6 (Intersection of a positive Legendrian path)**.** *Let Λ_* be a closed Legendrian submanifold of a cooriented contact manifold such that $\widetilde{\mathcal{L}}(\Lambda_*)$ is orderable. Given any uniformly positive isotopy $\underline{\Lambda}$ in $\mathcal{L}(\Lambda_*)$ (*c.f. *the beginning of Section 3), any Legendrian submanifold $\Lambda \in \mathcal{L}(\Lambda_*)$ intersect $\underline{\Lambda}_t$ for at least two distinct $t \in \mathbb{R}$ unless $\Lambda = \underline{\Lambda}_t$ for some $t \in \mathbb{R}$.*

Another consequence of our spectral selectors relating to Reeb chords is the following theorem addressing the Arnold chord conjecture (see a discussion of the conjecture below).

Theorem 1.7 (About the Arnold chord conjecture)**.** *Let $\Lambda_* \subset (M,\xi)$ be a Legendrian submanifold such that $\mathcal{L}(\Lambda_*)$ is unorderable and $\widetilde{\mathcal{L}}(\Lambda_*)$ is orderable. Then for every Legendrian submanifold $\Lambda \in \mathcal{L}(\Lambda_*)$ and every complete contact form α*

supporting ξ, there exist two distinct non-constant Reeb chords $\gamma_i : [0,1] \to M$, $i \in \{1,2\}$, such that $\gamma_i(\{0,1\}) \subset \Lambda$.

Having a periodic Reeb flow for some contact form supporting ξ is a sufficient condition to ensure the unorderability of $\mathcal{L}$ (in this case Theorem 1.7 is relevant when applied to other contact forms supporting ξ). This can be seen as a Legendrian analogue of the theorem of Albers-Fuchs-Merry addressing the Weinstein conjecture: every closed contact manifold such that $\mathcal{G}$ is unorderable admits a closed Reeb chord for any supporting contact form [2]. In our setting, we still ask for the orderability of the universal cover of $\mathcal{L}$. Nonetheless, let us remark that this theorem can be applied to non closed contact manifolds, only the Legendrian submanifolds need to be closed, and the statement gives at least two distinct Reeb chords. Therefore, it implies the Weinstein conjecture on (M,ξ) in the case $\Lambda_* := \Delta \times \{0\} \subset M \times M \times \mathbb{R}$ satisfies the hypothesis of Theorem 1.7 for the contact structure of Example 2.10.6; this set of (M,ξ) could possibly include cases not treated by the aforementioned theorem of Albers-Fuchs-Merry.

In its original setting, the Arnold chord conjecture asked about the existence of a non-trivial Reeb chord the endpoints of which lie on a same Legendrian circle of the standard contact 3-sphere, given any Legendrian circle [9, §8]. In this setting Mohnke proved a generalization to closed contact manifolds arising as the boundary of subcritical Stein manifolds (Legendrian circles are replaced by closed Legendrian submanifolds) [57]. Hutching and Taubes extended it to any closed contact 3-manifolds [45]. Very recent works still discuss generalized forms of the original conjecture: Chantraine proved a version for a specific kind of fillable contact manifolds with Lagrangian slices taking the place of Legendrian submanifolds [17], the case of Legendrian submanifolds lying in $P \times \mathbb{R}$, for a Liouville manifold P, has also received a lot of attention (see [49] and references therein).

Soon before releasing this article, Leonid Polterovich brought to our attention another potential class of applications: the study of robust Legendrian interlinking that had been introduced by Entov-Polterovich [31]. It turns out that our spectral selectors do bring new insights to this notion. We refer directly to Section 4.2 for more details.

1.5. **Applications to the metric structures of $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$.** Since the discovery by Hofer of a bi-invariant metric on the group of Hamiltonian diffeomorphisms [43], multiple attempts to obtain a contact counterpart where made. On the one hand, the obvious generalization of the Hofer metric, often referred to as the Shelukhin-Hofer metric, is not bi-invariant. On the other hand, one can define interesting bi-invariant metrics on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ but they are discrete, as was discovered by Colin-Sandon [26] and Fraser-Polterovich-Rosen [35]. Our spectral selectors allow us to define a contact analogue of the spectral metric discovered by Viterbo in [73]. This metric has applications to the study of both Hofer type contact metrics and discrete bi-invariant metrics of Colin-Sandon and Fraser-Polterovich-Rosen on the questions of non-degeneracy, unboundedness, metric-equivalence and geodesics. We directly refer to Section 4 for details.

We point out that the spectral metric has also been introduced by the very recent work of Nakamura [58] (without reference to spectrality) in order to show the metrizability of the interval topology and to study the Hofer type contact metrics. Our results and his do overlap on these two subjects.

A last metric application of our spectral selectors concerns the recent Lorentz-Finsler structure introduced by Abbondandolo-Benedetti-Polterovich [1]. A natural question arising from their study was the existence of a so-called *time function*, a natural object coming from the theory of relativity [1, Question K.1]. The following theorem is a positive answer in a broader setting to their question.

Theorem 1.8 (Existence of time functions). *Let O be either $\mathcal{L}, \widetilde{\mathcal{L}}, \mathcal{G}$ or $\widetilde{\mathcal{G}}$ associated with a cooriented contact manifold (M,ξ) (which is closed if $O = \mathcal{G}$ or $\widetilde{\mathcal{G}}$). If O is orderable and α is supporting ξ, then there exists a continuous non-decreasing map $\tau^\alpha : (O, \preceq) \to (\mathbb{R}, \leq)$ such that $x \preceq y$ and $x \neq y$ implies $\tau^\alpha(x) < \tau^\alpha(y)$ and $\tau^\alpha(\phi_t^\alpha x) = t + \tau^\alpha(x)$ for all $t \in \mathbb{R}$ and $x, y \in O$.*

The construction is highly non-canonical and one cannot expect much more natural properties satisfied by our time functions. Nevertheless, it is natural to ask whether there exists a time function on the orderable group $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ that can be conjugation invariant. We show that such a map does not exist (Theorem 5.4).

Organization of the paper. In Section 2, we provide the background on contact geometry needed throughout the article. In Section 3, we study the order spectral selectors defined in the introduction and prove Theorems 1.1 and 1.2 as well as the results introduced in Section 1.4. In Section 4, we define the spectral metric and use it to study Hofer type metrics, Colin-Sandon metrics and the Fraser-Polterovich-Rosen metric. In Section 5, we show Theorem 1.8.

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2. Preliminaries

2.1. **Legendrian and contact isotopies and their Hamiltonian maps.** Throughout this section $I \subset \mathbb{R}$ will always be an interval containing 0. Let (M,ξ) be a cooriented contact manifold. A contact form α supporting ξ (*i.e.* $\ker\alpha = \xi$) is called *complete* if its Reeb vector field R^α induces a complete Reeb flow. We recall that the Reeb vector field R^α is uniquely defined by the equations $\alpha(R^\alpha) \equiv 1$ and $\iota_{R^\alpha}\mathrm{d}\alpha = 0$. Let us fix a complete contact form α supporting ξ.

Let us recall that for any contact isotopy (that is a smooth isotopy of contactomorphisms) $(\varphi_t)_{t\in I}$ with $\varphi_0 = \mathrm{id}$, there exists a unique smooth map $H : I \times M \to \mathbb{R}$

related to the time-dependent contact vector field $(X_t)_{t\in I}$ generating the isotopy by

$$\begin{cases}\alpha(X_t) = H_t,\\ \iota_{X_t}\mathrm{d}H_t = (\mathrm{d}H_t \cdot R^\alpha)\alpha - \mathrm{d}H_t,\end{cases} \tag{2}$$

where R^α denotes the Reeb vector field of α and $H_t : M \to \mathbb{R}$ is induced by the restriction of H to $\{t\} \times M$. (see *e.g.* [36, §2.3]). The smooth map H is called the α-contact Hamiltonian map of (φ_t) (Hamiltonian map for short). Conversely, to any smooth map $H : I \times M \to \mathbb{R}$ is associated a unique time-dependent contact vector field (X_t) by the system (2). When the contact vector field is complete (*e.g.* when the differential of H is compactly supported), we say that H generates the contact isotopy obtained by integrating (X_t).

The notion of Hamiltonian map naturally extends to Legendrian isotopies. Throughout the paper, we only consider closed Legendrian submanifolds. A Legendrian isotopy $(\Lambda_t)_{t\in I}$ is a family of Legendrian submanifolds $\Lambda_t \subset M$, such that there exists a smooth map $j : I \times \Lambda_0 \to M$ whose restriction j_t to $\{t\} \times \Lambda_0$ induces a diffeomorphism $\Lambda_0 \simeq \Lambda_t$ for every $t \in I$. Given such an isotopy (Λ_t) and such a map j, let us define the family of maps $H_t : \Lambda_t \to \mathbb{R}$, $t \in I$, by

$$H_t \circ j(t,x) := \alpha\left(\frac{\partial j}{\partial t}(t,x)\right), \quad \forall (t,x) \in I \times \Lambda_0. \tag{3}$$

As $j_t^*\alpha = 0$ for all $t \in I$, one can deduce that (H_t) only depends on the isotopy (Λ_t) and not the specific choice of parametrization j. The family (H_t) is called the α-contact Hamiltonian map of (Λ_t). The contact Hamiltonian map can be seen as a smooth map $H : N \to \mathbb{R}$ defined on the submanifold $N := \bigcup_t \{t\} \times \Lambda_t$ of $I \times M$. If $\Lambda_t = \varphi_t(\Lambda_0)$ for some contact isotopy (φ_t) generated by the Hamiltonian K, the Hamiltonian of (Λ_t) is the restriction of K to N.

We will use the following unparametrized version of the Legendrian isotopy extension theorem.

Lemma 2.1. *Let $I \subset \mathbb{R}$ be an interval containing 0 and let $(\Lambda_t)_{t\in I}$ be a Legendrian isotopy on $(M, \ker\alpha)$, the α-contact Hamiltonian map of which is $H : \bigcup_t \{t\} \times \Lambda_t \to \mathbb{R}$. Let $K : I \times M \to \mathbb{R}$ be an α-contact Hamiltonian smoothly extending H and generating a contact isotopy (φ_t), then $\varphi_t(\Lambda_0) = \Lambda_t$ for all $t \in I$.*

Before proving Lemma 2.1, let us recall the link between Legendrian isotopies and the Hamilton-Jacobi equation. Let (Λ_t) be a Legendrian isotopy. According to the Legendrian Weinstein neighborhood theorem, there exists an open subset $W \subset M$ containing Λ_0 that is contactomorphic to a neighborhood of the zero-section of the 1-jet space $J^1\Lambda_0 := T^*\Lambda_0 \times \mathbb{R}$, through this identification Λ_0 is sent to the zero-section. Moreover, our chosen contact form α is sent to the standard contact form $\alpha_0 := \mathrm{d}z - \lambda$, z being the $\mathbb{R}$-coordinate and λ being the pull-back of the tautological form of $T^*\Lambda_0$. With this local identification, for $t \in I$ close to 0, Λ_t is a Legendrian graph above the zero section Λ_0, so it is the 1-jet of a smooth map $f_t : \Lambda_0 \to \mathbb{R}$,

$$j^1 f_t := \{(q, \mathrm{d}f_t(q), f_t(q)) \mid q \in \Lambda_0\} \subset J^1\Lambda_0.$$

Applying the parametrization $j(t,q) := (q, \mathrm{d}f_t(q), f_t(q))$ of (Λ_t), t small enough, to Equation (3), one then derives the Hamilton-Jacobi equation

$$H_t(q, \mathrm{d}f_t(q), f_t(q)) = \frac{\partial f_t}{\partial t}(q), \tag{4}$$

for all $q \in \Lambda_0$ and $t \in I$ close to 0 (we have used that α is identified to α_0 in W).

Proof of Lemma 2.1. By connectivity of the interval I, it is enough to prove this statement for $t \in I$ close enough to 0. By considering a Weinstein neighborhood of Λ_0 as above, one can then identify Λ_t and $\varphi_t(\Lambda_0)$ to the 1-jet of the respective maps $f_t : \Lambda_0 \to \mathbb{R}$ and $g_t : \Lambda_0 \to \mathbb{R}$, for all $t \in I$, shrinking I if necessary. Let us remark that the contact Hamiltonian map of $(\varphi_t(\Lambda_0))$ is the restriction of K to $\bigcup_t \{t\} \times \varphi_t(\Lambda_0)$. Therefore, by the above discussion, both families of maps (f_t) and (g_t) satisfy the same Hamilton-Jacobi equation

$$\frac{\partial u_t}{\partial t}(q) = K_t(q, \mathrm{d}u_t(q), u_t(q)), \quad \forall q \in \Lambda_0, \forall t \in I,$$

with the initial condition $u_0 \equiv 0$. By unicity of the smooth solutions of the Hamilton-Jacobi equation in a small interval of time (see *e.g.* [52, Exercice 3.5.17]), $f_t = g_t$ for all $t \in I$ and $\Lambda_t = \varphi_t(\Lambda_0)$. □

Corollary 2.2. *Let $(M, \ker\alpha)$ be a contact manifold endowed with a complete contact form α. Let $I \subset \mathbb{R}$ be an interval containing 0 and let $(\Lambda_t)_{t\in I}$ be a (closed) Legendrian isotopy on $(M, \ker\alpha)$, the α-contact Hamiltonian map of which is H. Then there exist contact Hamiltonians $K, G : I \times M \to \mathbb{R}$ such that $\inf K = \inf H$ and $\sup G = \sup H$, the respective contact flows of which (φ_t) and (ψ_t) satisfy $\varphi_t(\Lambda_0) = \psi_t(\Lambda_0) = \Lambda_t$ for all $t \in I$.*

Proof. Let $U \subset I \times M$ be a tubular neighborhood of $N := \bigcup_t \{t\} \times \Lambda_t$ and let $\pi : U \to N$ be the associated smooth retraction. Let $\chi : I \times M \to [0, 1]$ be a smooth map such that $\chi|_N \equiv 1$ and $\chi|_{(I\times M)\setminus U} \equiv 0$. The statement is now a consequence of Lemma 2.1 applied to the following smooth extensions of H:

$$K := \chi \cdot H \circ \pi + (1 - \chi) \cdot \sup H \text{ and } G := \chi \cdot H \circ \pi + (1 - \chi) \cdot \inf H. \qquad \square$$

2.2. **The C^1-topology of Legendrian and contact isotopy classes.** Let Λ_* be a closed Legendrian submanifold of (M, ξ). Let us briefly describe the C^1-topology of the space $\mathcal{L}(\Lambda_*)$ of Legendrian submanifolds isotopic to Λ_* and the space $\mathcal{G}(M, \xi)$ of compactly supported contactomorphisms isotopic to the identity. The C^1-topology on $\mathcal{G}$ is understood as the topology induced by the Whitney C^1-topology on diffeomorphisms as it is described in [42, Chapter 2] (both weak and strong topologies coincide for compactly supported diffeomorphisms). One way to define the C^1-topology on $\mathcal{L}(\Lambda_*)$ is to consider $\mathcal{L}(\Lambda_*)$ as a subset of the quotient $\mathrm{Emb}(\Lambda_*, M)/\mathrm{Diff}(\Lambda_*)$ of smooth embedding $\Lambda_* \hookrightarrow M$ by the right-action by diffeomorphisms of Λ_*. This quotient being endowed with the topology induced by the Whitney C^1-topology on $\mathrm{Emb}(\Lambda_*, M)$, the C^1-topology of $\mathcal{L}(\Lambda_*)$ is the induced topology as a subset of this quotient. Equivalently, considering $\mathcal{L}(\Lambda_*)$ as the homogeneous space $\mathcal{G}(M, \xi)/\mathrm{Stab}(\Lambda_*)$, its C^1-topology is the quotient topology induced by the C^1-topology of $\mathcal{G}$.

Let us fix $\Lambda_0 \in \mathcal{L}(\Lambda_*)$. According to the Legendrian Weinstein neighborhood theorem, there exists a neighborhood $W \subset M$ of Λ_0 that is contactomorphic to a neighborhood of the 0-section of $J^1\Lambda_0$ endowed with the standard contact structure and that identifies Λ_0 with the 0-section. Every Legendrian submanifold Λ that is C^1-close to Λ_0 is the 1-jet of a map $f : \Lambda_0 \to \mathbb{R}$. Therefore, a base of neighborhoods of the point Λ_0 in its isotopy class $\mathcal{L}(\Lambda_*)$ is identifiable with the set described by the balls $B_r := \{f \in C^\infty(\Lambda_0, \mathbb{R}) \mid \|f\|_{C^1} < r\}$, $0 < r < \varepsilon$, for a small $\varepsilon > 0$. In other words, the topological space $\mathcal{L}(\Lambda_*)$ is locally modeled on the space of smooth maps on Λ_* endowed with the C^1-topology. Therefore, it is a locally contractible space and it admits a genuine universal cover $\widetilde{\mathcal{L}}(\Lambda_*)$ that can be formally described as a set of equivalent classes of paths $\gamma : [0,1] \to \mathcal{L}(\Lambda_*)$, with $\gamma(0) = \Lambda_*$, with the equivalence relation given by homotopy relative to endpoints. With this description of $\widetilde{\mathcal{L}}(\Lambda_*)$, the cover $\Pi : \widetilde{\mathcal{L}} \to \mathcal{L}$ is the evaluation $\gamma \mapsto \gamma(1)$.

Following a similar argument applied to the contact graphs $\mathrm{gr}^\alpha(\psi) \subset M \times M \times \mathbb{R}$ of contactomorphisms $\psi \in \mathcal{G}(M)$, the space $\mathcal{G}(M)$ endowed with the C^1-topology is locally modeled on the space of compactly supported smooth maps $C^\infty_c(M,\mathbb{R})$ endowed with the C^1-topology (see *e.g.* [66, §1]). It is therefore a locally contractible space and admits a universal cover $\widetilde{\mathcal{G}}(M)$ that can be described as a space of homotopy classes. The space $\widetilde{\mathcal{G}}$ has a natural group structure making the cover $\Pi : \widetilde{\mathcal{G}} \to \mathcal{G}$ a morphism (coming from composing homotopies (φ_t) and (ψ_t) timewise: $(\varphi_t \circ \psi_t)$). The action by conjugation of contactomorphisms of (M,ξ) on $\mathcal{G}$ naturally lift to an action on $\widetilde{\mathcal{G}}$: for $g, x \in \widetilde{\mathcal{G}}$, gxg^{-1} only depends on Πg and x. In particular, $\Pi^{-1}\{\mathrm{id}\}$ lies in the center of $\widetilde{\mathcal{G}}$.

One has a natural map $\mathrm{Cont}_0(M) \times \mathcal{L}(\Lambda_*) \to \mathcal{L}(\Lambda_*)$ given by $(\varphi,\Lambda) \mapsto \varphi(\Lambda)$. Let I be an interval containing 0, if $(\varphi_t)_{t\in I}$ is a contact isotopy with $\varphi_0(\Lambda_*) = \Lambda_*$, its elements act naturally on $\widetilde{\mathcal{L}}(\Lambda_*)$ by defining $\varphi_t(\Lambda)$ for $\Lambda \in \widetilde{\mathcal{L}}(\Lambda_*)$ and $t \in I$ as the endpoint of the path lifting $s \mapsto \varphi_{st}(\Pi\Lambda)$, $s \in [0,1]$, and starting at Λ. It induces in particular a continuous map $\widetilde{\mathcal{G}}(M) \times \widetilde{\mathcal{L}}(\Lambda_*) \to \widetilde{\mathcal{L}}(\Lambda_*)$.

Section 2.1 can be naturally extended to Legendrian isotopies of $\widetilde{\mathcal{L}}(\Lambda_*)$. A family (Λ_t) of elements of $\widetilde{\mathcal{L}}(\Lambda_*)$ will be called a Legendrian isotopy if $(\Pi\Lambda_t)$ is a Legendrian isotopy (conversely any Legendrian isotopy of $\mathcal{L}$ gives rise to a Legendrian isotopy of $\widetilde{\mathcal{L}}$ once a starting point is fixed). The contact Hamiltonian map of a Legendrian isotopy of $\widetilde{\mathcal{L}}(\Lambda_*)$ will designate the contact Hamiltonian map of the projected Legendrian isotopy on $\mathcal{L}(\Lambda_*)$ and similarly for isotopies of $\widetilde{\mathcal{G}}$. The lifting property of the universal cover implies the following extension of Lemma 2.1 (where we use the natural action of contact flows on $\widetilde{\mathcal{L}}$ discussed above).

Lemma 2.3. *The statement of Lemma 2.1 is still true if the Legendrian isotopy (Λ_t) belongs to $\widetilde{\mathcal{L}}(\Lambda_*)$ for some closed Legendrian submanifold $\Lambda_* \subset M$ instead of $\mathcal{L}(\Lambda_*)$. In particular, the action of $\widetilde{\mathcal{G}}(M)$ on $\widetilde{\mathcal{L}}(\Lambda_*)$ is transitive.*

The topology of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ can be made into the topology of a length metric space. An auxiliary contact form α supporting ξ and an auxiliary Riemannian metric g on M being fixed, one can define the length metric

$$d_{C^1}(\Lambda_0,\Lambda_1) := \inf_{(\Lambda_t)} \int_0^1 \|H_t\|_{C^1(\Lambda_t)}\mathrm{d}t, \quad \forall \Lambda_0,\Lambda_1 \in \mathcal{L} \text{ (resp. } \widetilde{\mathcal{L}}), \tag{5}$$

where the infimum is taken over every isotopy (Λ_t) from Λ_0 to Λ_1, the Hamiltonian of which is denoted H_t, and $\|\cdot\|_{C^1(\Lambda)}$ is the usual C^1-norm on maps of the Riemannian submanifold $\Lambda \subset (M,g)$. The equivalence between the topology induced by d_{C^1} and the previously described topology is a consequence of the equivalence of their base of neighborhoods which comes from the Hamilton-Jacobi equation (4).

One defines similarly a length metric on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ inducing the C^1-topology:

$$d_{C^1}(\varphi,\psi) := \inf_{(H_t)} \int_0^1 \|H_t\|_{C^1(M,g)}\mathrm{d}t, \quad \forall \varphi,\psi \in \mathcal{G} \text{ (resp. } \widetilde{\mathcal{G}}), \tag{6}$$

where the infimum is taken over every Hamiltonian map $(H_t)_{t\in[0,1]}$ generating a flow from id to $\psi\circ\varphi^{-1}$.

Remark 2.4 (from C^1 to C^0). As they induce C^1-topology on their respective space, these C^1-distances are indeed non-degenerate (*i.e.* only vanishing on the diagonal). However, it is not clear for their natural C^0-analogues obtained by putting C^0-norms instead of C^1-norms in their respective definition. In fact, in the case of $\mathcal{L}$, it can not be the case (see Section 2.3 below). The reason is that it is not clear which topology is associated with these metrics. Indeed, when *e.g.* a Legendrian Λ is C^0-close to Λ_0, there is no reason for it to be a graph in any Weinstein neighborhood of Λ_0. Therefore, one cannot state that the C^0-topology of $\mathcal{L}$ is locally modelled on the C^0-topology of $C^\infty_c(\Lambda_0,\mathbb{R})$. Without this local identification to a normed vector space, the induced length pseudo-metric can be degenerate or even vanish identically. Nevertheless, this is not always the case and these pseudo-distances are relevant by themselves since their non-degeneracy and geodesics are intimately linked to contact and symplectic rigidity phenomena see Section 2.3 and 4.3 below.

The following lemma will be useful in order to construct time-functions.

Lemma 2.5. *Let (M,ξ) be a contact manifold and $\Lambda_*\subset M$ be a closed Legendrian submanifold. The induced topological spaces $\widetilde{\mathcal{G}}(M)$ and $\widetilde{\mathcal{L}}(\Lambda_*)$ are separable.*

Proof. Being a subspace of the separable metrizable space of compactly supported C^1-diffeomorphisms of M, the space $\mathcal{G}(M)$ is a separable metrizable space. So $\mathcal{G}(M)$ admits a countable basis of open sets, which implies that $\widetilde{\mathcal{G}}(M)$ admits a countable basis of open sets by the Poincaré-Volterra theorem. Since $\widetilde{\mathcal{G}}(M)$ has a countable basis of open sets, we conclude that it is separable.

The fact that the $\widetilde{\mathcal{L}}(\Lambda_*)$ is separable is now a consequence of Lemma 2.3. □

2.3. **Hofer type pseudo-metrics.** In this section, we recall the definitions of the standard generalization of the Hofer metric of symplectic geometry to the contact setting.

Let us first recall that a *pseudo-distance* d on a set X is a symmetric map $d: X\times X\to[0,+\infty)$ vanishing on the diagonal and satisfying the triangle inequality. The pseudo-distance d is *non-degenerate* if d is a genuine distance, *i.e.* it only vanishes on the diagonal.

We denote $\mathrm{d}^\alpha_{\mathrm{SCH}}$ the Shelukhin-Chekanov-Hofer pseudo-distance on $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$):

$$\mathrm{d}^\alpha_{\mathrm{SCH}}(\Lambda_0,\Lambda_1) := \inf_{(H_t)}\int_0^1 \max|H_t|\mathrm{d}t, \tag{7}$$

where the infimum is taken over Hamiltonian maps $H_t:\Lambda_t\to\mathbb{R}$, $t\in[0,1]$, generating a Legendrian isotopy $(\Lambda_t)_{t\in[0,1]}$ from Λ_0 to Λ_1. The Hofer oscillation pseudo-distance is defined similarly with $\mathrm{osc}(H_t) := \max H_t - \min H_t$ in place of $|H_t|$:

$$\mathrm{d}^\alpha_{\mathrm{H,osc}}(\Lambda_0,\Lambda_1) := \inf_{(H_t)}\int_0^1 \mathrm{osc}(H_t)\mathrm{d}t. \tag{8}$$

Let us remark that the Hofer oscillation pseudo-distance is clearly degenerate as $\mathrm{d}^\alpha_{\mathrm{H,osc}}(\Lambda,\phi^\alpha_t\Lambda)=0$ for all Λ and all $t\in\mathbb{R}$.

The pseudo-distance $\mathrm{d}^\alpha_{\mathrm{SCH}}$ was first studied by Rosen-Zhang in [62] where it was more generally defined on non-Legendrian subsets (the equivalence of their definition with ours comes from Corollary 2.2). This pseudo-distance is known to be non-degenerate in multiple cases including when $\mathcal{L}$ is orderable [41, Theorem 5.2] (see also Corollary 4.1). However, this pseudo-distance can also be degenerate on unorderable $\mathcal{L}$ [15]. In the case it is degenerate on $\mathcal{L}$, $\mathrm{d}^{\alpha}_{\mathrm{SCH}}$ is actually identically zero, according to the generalization of the Chekanov's dichotomy proven by Rosen-Zhang [62, Theorem 1.10].

Given a group G, a *pseudo-(group-)norm* ν on G is a map $G \to [0, +\infty)$ satisfying the triangle inequality $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$ and the invariance under the inversion: $\nu(g) = \nu(g^{-1})$, $g \in G$. It is non-degenerate if it only vanishes at the neutral element, in which case ν is called a *(group-)norm*. Any group pseudo-norm ν induces a right-invariant and a left-invariant pseudo-distance on G by defining either $(g, h) \mapsto \nu(gh^{-1})$ or $(g, h) \mapsto \nu(h^{-1}g)$. The non-degeneracy of the induced metrics is equivalent to the non-degeneracy of ν while the bi-invariance of the induced pseudo-distance (which are then equal) is equivalent to the conjugation-invariance of ν.

Given a contact form α supporting the contact structure, we denote $|\cdot|^{\alpha}_{\mathrm{SH}}$ the associated Shelukhin-Hofer pseudo-norm on $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$):

$$|\varphi|^{\alpha}_{\mathrm{SH}} := \inf_{(H_t)} \int_0^1 \max |H_t| \mathrm{d}t, \quad \forall \varphi \in \mathcal{G} \text{ (resp. } \widetilde{\mathcal{G}}), \tag{9}$$

where the infimum is taken over Hamiltonian maps generating a contact flow $(\varphi_t)_{t\in[0,1]}$ joining id to φ. The Hofer oscillation pseudo-norm is defined similarly with $\mathrm{osc}(H_t) := \max H_t - \min H_t$ in place of $|H_t|$:

$$|\varphi|^{\alpha}_{\mathrm{osc}} := \inf_{(H_t)} \int_0^1 \mathrm{osc}(H_t) \mathrm{d}t. \tag{10}$$

As before, the Hofer oscillation pseudo-norm is clearly degenerate as it vanishes on $\{\phi^{\alpha}_t\}_t$. We denote $\mathrm{d}^{\alpha}_{\mathrm{SH}}$ and $\mathrm{d}^{\alpha}_{\mathrm{H,osc}}$ the respectively induced right-invariant pseudo-distances. These pseudo-norms were studied in depth by Shelukhin in [68]. He proved that $|\cdot|^{\alpha}_{\mathrm{SH}}$ is always non-degenerate on $\mathcal{G}$ using an energy-capacity inequality intimately linked to non-squeezing phenomena.

In contrast to their symplectic counterparts, neither of these pseudo-distances is invariant by the left-action of contactomorphisms, but the following compatibility property for $g \in \mathrm{Cont}(M, \xi)$, $\Lambda_0, \Lambda_1 \in \mathcal{L}$ and $\varphi, \psi \in \mathcal{G}$ holds:

$$\mathrm{d}^{\alpha}_{\mathrm{SCH}}(g\Lambda_0, g\Lambda_1) = \mathrm{d}^{g^*\alpha}_{\mathrm{SCH}}(\Lambda_0, \Lambda_1) \text{ and } \mathrm{d}^{\alpha}_{\mathrm{SH}}(g\varphi, g\psi) = \mathrm{d}^{g^*\alpha}_{\mathrm{SH}}(\varphi, \psi).$$

The same identities hold by considering $g, \Lambda_0, \Lambda_1, \varphi$ and ψ in the universal cover of their respective spaces. We refer to Remark 4.6 below for a discussion on the apparent lack of invariance of these contact metrics. Topologies induced by these pseudo-distances are coarser than the C^1-topology, as they are dominated by the metric defined at (5) and (6).

2.4. **Contact orderability.** An isotopy (Λ_t) in $\mathcal{L}$ or $\widetilde{\mathcal{L}}$ is said to be non-negative (resp. positive) if its contact Hamiltonian for some, and thus for any, supporting contact form is non-negative (resp. positive). Given $\Lambda_0, \Lambda_1 \in \mathcal{L}$ or $\widetilde{\mathcal{L}}$, we write $\Lambda_0 \preceq \Lambda_1$ (resp. $\Lambda_0 \lll \Lambda_1$) if there exists a non-negative (resp. positive) isotopy (Λ_t) joining Λ_0 and Λ_1. One defines similarly relations $\preceq$ and $\lll$ on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ by considering contact Hamiltonian maps of compactly supported contact isotopies (φ_t), one also defines these relations on the whole space of contactomorphisms and its universal cover by considering contact isotopies.

The relation $\preceq$ is reflexive and transitive (*i.e.* it is a pre-order), whereas $\lll$ is only transitive. These relations satisfy properties of invariance with respect to the action of contactomorphisms: we have for all $x_1, y_1, x_2, y_2 \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$, resp. $\mathrm{Cont}(M,\xi)$, resp. $\widetilde{\mathrm{Cont}_0}(M,\xi)$)

$$\begin{cases} x_1 \preceq y_1 \text{ and } x_2 \preceq y_2 \Rightarrow x_1x_2 \preceq y_1y_2, \\ x_1 \lll y_1 \text{ and } x_2 \preceq y_2 \Rightarrow x_1x_2 \lll y_1y_2, \\ x_1 \preceq y_1 \text{ and } x_2 \lll y_2 \Rightarrow x_1x_2 \lll y_1y_2, \end{cases} \tag{11}$$

and the same is formally true when replacing x_2 and y_2 with elements in $\mathcal{L}$ (resp. in $\widetilde{\mathcal{L}}$).

A space $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ is said to be *orderable* if and only if $\preceq$ is antisymmetric (*i.e.* $\preceq$ is a partial order) which is equivalent that they do not contain a non-negative and non constant loop. Obviously, the orderability of $\mathcal{L}$ (resp. $\mathcal{G}$) implies the orderability of $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{G}}$), but the orderability of the latter is much more common (see Examples 2.10 below). The notion of orderability has been introduced by Eliashberg-Polterovich [30] and has then been investigated by numerous authors. A short account on these investigations is given by the paragraph Examples 2.10 below.

Remark 2.6 (On the terminology). Depending on the authors, the actual meaning of orderability may differ. When a contact manifold is referred to as orderable, it seems to always mean that either $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ is orderable. Most of the time, it means that $\widetilde{\mathcal{G}}$ is orderable, following the initial focus on $\widetilde{\mathcal{G}}$ put by Eliashberg-Polterovich [64, 18, 58]. In [18, 50], a contact manifold (M,ξ) is called strongly orderable if $\widetilde{\mathcal{L}}(\Delta \times \{0\})$ is orderable (see Example 2.10.6) but in [16, 58] it means that $\mathcal{G}$ is orderable. Chernov-Nemirovski call $\mathcal{L}$ (resp. $\mathcal{G}$) universally orderable when $\widetilde{\mathcal{L}}$ (resp. $\widetilde{\mathcal{G}}$) is orderable [21].

The following lemmata will be useful.

Lemma 2.7 ([30, Proposition 2.1.B], [22, Proposition 4.5])**.** *Let O be either $\mathcal{L}$ or $\widetilde{\mathcal{L}}$. The space O is orderable if and only if there does not exist any positive loop among isotopies of O. The same is true for O being either $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ for a* closed *contact manifold.*

Lemma 2.8. *Let O be either $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ for a given contact manifold (M,ξ) and possibly a closed Legendrian submanifold Λ_*. Then*

$$\begin{cases} x \preceq y \text{ and } y \lll z \Rightarrow x \lll z, \\ x \lll y \text{ and } y \preceq z \Rightarrow x \lll z, \end{cases} \qquad \forall x,y,z \in O.$$

Proof. Let us prove the statement for $O = \mathcal{L}$, the proof being similar for the other cases. Let us assume $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathcal{L}$ are such that $\Lambda_0 \preceq \Lambda_1$ and $\Lambda_1 \lll \Lambda_2$; we want to show that $\Lambda_0 \lll \Lambda_2$. By smoothly concatenating isotopies, one can assume the Λ_i's, $i \in \{0,1,2\}$, to be part of a non-negative isotopy $(\Lambda_t)_{t\in[0,2]}$ that is positive for $t \in (1,2]$. Let us now adapt a construction of Fraser-Polterovich-Rosen in the proof of [35, Proposition 2.6] in order to find a positive isotopy from Λ_0 to Λ_2. Let $v : [0,2] \to \mathbb{R}$ be a smooth map such that $v(0) = v(2) = 0$ with v' positive on $[0,3/2]$. For $\varepsilon > 0$, the isotopy $\underline{\Lambda}^\varepsilon := (\phi^\alpha_{\varepsilon v(t)}\Lambda_t)$ from Λ_0 to Λ_2 is then positive for $t \in [0,3/2]$. Indeed, by Lemma 2.1, one can write $\Lambda_t = \psi_t\Lambda_0$ with (ψ_t) contact flow generated by a non-negative Hamiltonian map, so $(\phi^{\alpha}_{\varepsilon v(t)}\psi_t)$ has a α-contact Hamiltonian $H_t \geq \varepsilon v'(t)$ for all $t \in [0,2]$ and so does $\underline{\Lambda}^{\varepsilon}$. Since the positivity of an isotopy is a C^1-open condition, the isotopy $\underline{\Lambda}^{\varepsilon}$ is positive on $[3/2,2]$ for $\varepsilon > 0$ small enough, which brings the conclusion. The proof of the case $\Lambda_0 \lll \Lambda_1$ and $\Lambda_1 \preceq \Lambda_2$ is similar. □

Corollary 2.9. *If $(x_t)_{t\in[0,1]}$ is a non-negative isotopy in either $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$, the Hamiltonian (H_t) of which is positive at some $t_0 \in [0,1]$, then $x_0 \lll x_1$.*

Proof. By continuity of (H_t), there is an interval $[a,b] \subset [0,1]$ containing t_0 in its interior such that H_t is positive for $t \in [a,b]$. Therefore, $x_a \lll x_b$ and the conclusion follows from Lemma 2.8. □

Examples 2.10 (Orderable and unorderable spaces).

1. $\widetilde{\mathcal{G}}(\mathbb{RP}^{2n+1})$ is orderable for the standard contact structure pulled-back from the sphere. Moreover $\widetilde{\mathcal{L}}(\mathbb{RP}^n)$ is orderable, where $\mathbb{RP}^n \subset \mathbb{RP}^{2n+1}$ is the projectivization of the Lagrangian subspace $\mathbb{R}^{n+1} \times \{0\}$ of $\mathbb{R}^{2(n+1)}$ [38, 30]. It generalizes to lens spaces [54, 65, 39].
2. $\widetilde{\mathcal{G}}(\mathbb{S}^{2n+1})$ is not orderable for $n \geq 1$ [29].
3. Let $J^1N = T^*N \times \mathbb{R}$ be the 1-jet space of some manifold N endowed with the standard contact structure $\ker(\mathrm{d}z - \lambda)$, z being the $\mathbb{R}$-coordinate and λ the pull-back of the Liouville form on T^*N. This contact structure induces a contact structure on the quotient space $J^1N/\mathbb{Z}\partial_z = T^*N \times S^1$. Let $0_N \subset J^1N$ denote the zero-section and $p : J^1N \to J^1N/\mathbb{Z}\partial_z$ the quotient map. Then $\mathcal{L}(0_N)$ and $\widetilde{\mathcal{L}}(p(0_N))$ are orderable when N is closed (it also generalizes to compactly supported Legendrian isotopies when N is open) [10, 25, 76].
4. Given any contact manifold, there always are many Legendrian submanifolds for which $\widetilde{\mathcal{L}}$ is unorderable. Such submanifolds can be obtain by "stabilizing" any Legendrian submanifold. For any loose Legendrian submanifold of dimension ≥ 2, $\widetilde{\mathcal{L}}$ is unorderable [25, 50].
5. Given a Riemannian manifold (N,g), let SN denote the unit tangent bundle of N endowed with the contact form $\alpha_{(x,v)} \cdot \eta := g(v, \mathrm{d}\pi \cdot \eta)$, where $\pi : SN \to N$ is the bundle map. For any closed N, $\widetilde{\mathcal{L}}(S_xN)$ is orderable for any fiber S_xN of π [22]. In particular, $\widetilde{\mathcal{G}}(SN)$ is orderable. Moreover, $\mathcal{L}(S_xN)$ is orderable, and thus so is $\mathcal{G}(SN)$, when the universal cover of N is open and $\dim N \geq 2$ [25, 21].
6. Given a closed contact manifold (M,ξ) and a supporting contact form α, let us consider the manifold $M \times M \times \mathbb{R}$ endowed with the contact structure $\tilde{\xi}$ induced by the contact form $\tilde{\alpha} := \alpha_2 - e^{\theta}\alpha_1$, where α_i is the pull-back of α under the projection on the i-th factor and θ is the $\mathbb{R}$-coordinate. The contact form $\tilde{\alpha}$ is complete and its Reeb flow is given by
$$\phi^{\tilde{\alpha}}_t(x_1,x_2,\theta) = (x_1, \phi^{\alpha}_t(x_2), \theta), \quad \forall (x_1,x_2,\theta) \in M \times M \times \mathbb{R}, \forall t \in \mathbb{R}.$$
The contact structure $\tilde{\xi}$ is independent of the choice of α supporting ξ, up to isomorphism. Let $\Delta \subset M \times M$ be the diagonal. If (M,ξ) is the boundary of a Liouville domain the symplectic homology of which does not vanish for some choice of coefficients, then $\widetilde{\mathcal{L}}(\Delta \times \{0\})$ is orderable. In particular, $\widetilde{\mathcal{G}}(M)$ is orderable [75, 3, 18].
7. A contact manifold (M,ξ) is called hypertight if it admits a contact form having no contractible periodic Reeb orbit, the latter contact form is called hypertight

as well. A Legendrian submanifold $\Lambda \subset (M,\xi)$ is called hypertight if there is a hypertight contact form for which Λ has no contractible Reeb chord, *i.e.* no Reeb chord in the null homotopy class of $\pi_1(M,\Lambda)$. If Λ is a closed hypertight Legendrian of a closed contact manifold, $\widetilde{\mathcal{L}}(\Lambda)$ is orderable. If (M,ξ) is a closed hypertight contact manifold, $\widetilde{\mathcal{L}}(\Delta\times\{0\})$ (hence $\widetilde{\mathcal{G}}(M)$) is orderable for the contact manifold $M \times M \times \mathbb{R}$ defined just above [3, 18].

3. Order spectral selectors

3.1. **The Legendrian case.** Let (M,ξ) be a cooriented contact manifold. Let $\underline{\Lambda} := (\underline{\Lambda}_t)_{t\in\mathbb{R}}$ be a uniformly positive path in $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), *i.e.* a positive path, the α-Hamiltonian of which satisfies $\inf_t \min H_t > 0$ for some complete α supporting the contact structure. Given $\Lambda \in \mathcal{L}$ (resp. in $\widetilde{\mathcal{L}}$), let us define the spectrum of $(\Lambda, \underline{\Lambda})$ by

$$\operatorname{Spec}(\Lambda,\underline{\Lambda}) := \{t \in \mathbb{R} \mid \Lambda \cap \underline{\Lambda}_t \neq \emptyset\},$$

(resp. $\operatorname{Spec}(\Pi\Lambda, \Pi\underline{\Lambda})$). The associated order spectral selectors are defined by

$$\ell_-(\Lambda,\underline{\Lambda}) := \sup\{t \in \mathbb{R} \mid \Lambda \succeq \underline{\Lambda}_t\} \quad \text{and} \quad \ell_+(\Lambda,\underline{\Lambda}) := \inf\{t \in \mathbb{R} \mid \Lambda \preceq \underline{\Lambda}_t\}.$$

According to Lemma 2.8, since $\underline{\Lambda}$ is positive, one could replace $\preceq$ (resp. $\succeq$) in the definition of the selectors with $\lll$ (resp. $\ggg$). The selectors $\ell^\alpha_\pm$ correspond to the special case where $\underline{\Lambda} = (\phi^\alpha_t \Lambda_0)$.

Proposition 3.1. *If $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable, the order spectral selectors are real-valued.*

Proof. Let us prove the statement for Λ and elements in $\underline{\Lambda}$ in the universal cover $\widetilde{\mathcal{L}}$ assumed orderable. Since $\preceq$ is a partial order, it amounts to proving $\underline{\Lambda}_s \preceq \Lambda \preceq \underline{\Lambda}_t$ for some real numbers $s < t$. Let us fix a complete α supporting ξ such that $\varepsilon := \inf_t \min H_t$ is positive.

Then according to the Legendrian isotopy extension theorem as expressed at Lemma 2.3 (see also Corollary 2.2), for every $t \geq 0$, there exists $g_t \in \widetilde{\operatorname{Cont}}_0(M,\xi)$ sending $\underline{\Lambda}_0$ on $\underline{\Lambda}_t$ that is the time-one map of a (non necessarily compactly supported) contact flow, the Hamiltonian map $h : [0,1] \times M \to \mathbb{R}$ of which satisfies $\inf h = t\varepsilon$. Let us pick an isotopy from Λ to $\underline{\Lambda}_0$ and extend its Hamiltonian to a compactly supported map $[0,1] \times M \to \mathbb{R}$. The induced flow (φ_t) is compactly supported and $\varphi_1\Lambda = \underline{\Lambda}_0$.

The composition formula of Hamiltonian maps implies that $g_t\varphi_1$ is the time-one map of a flow generated by a positive Hamiltonian when t is taken large enough. Since $\underline{\Lambda}_t = g_t\varphi_1\Lambda$, $\Lambda \preceq \underline{\Lambda}_t$ when t is taken large enough. One proceeds similarly to prove that $\underline{\Lambda}_s \preceq \Lambda$ for $-s > 0$ large enough and the proof in $\mathcal{L}$ is formally identical. $\square$

From now on, we always assume that $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable.

The properties of normalization, monotonicity, triangle inequalities, Poincaré duality stated in Theorem 1.1 directly follow from the invariance and transitivity of $\preceq$ and the definition of the selectors $\ell^\alpha_\pm$'s. The compatibility property is a direct consequence of the fact that for any complete contact form α supporting ξ,

$$g^{-1}\phi^\alpha_t g = \phi^{g^*\alpha}_t, \quad \forall g \in \operatorname{Cont}(M,\xi), \forall t \in \mathbb{R}. \tag{12}$$

The following consequence of the invariance of $\preceq$ will be useful to restrict ourself to the spectral selectors $\ell^\alpha_\pm$ when needed.

Lemma 3.2 (Sign invariance)**.** *For every complete contact form α supporting ξ and every $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$),*

$$\ell_\pm(\Lambda, \underline{\Lambda}) < 0 \text{ (resp. } = 0, \text{ resp. } > 0) \Leftrightarrow \ell_\pm^\alpha(\Lambda, \underline{\Lambda_0}) < 0 \text{ (resp. } = 0, \text{ resp. } > 0).$$

Moreover, if $\beta = e^f\alpha$ for some smooth $f : M \to \mathbb{R}$, $\forall \Lambda_0, \Lambda_1 \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$),

$$\begin{cases} e^{\inf f}\ell_\pm^\alpha(\Lambda_1,\Lambda_0) \leq \ell_\pm^\beta(\Lambda_1,\Lambda_0) \leq e^{\sup f}\ell_\pm^\alpha(\Lambda_1,\Lambda_0) & \text{when } \ell_\pm^\beta(\Lambda_1,\Lambda_0) \geq 0, \\ e^{\sup f}\ell_\pm^\alpha(\Lambda_1,\Lambda_0) \leq \ell_\pm^\beta(\Lambda_1,\Lambda_0) \leq e^{\inf f}\ell_\pm^\alpha(\Lambda_1,\Lambda_0) & \text{when } \ell_\pm^\beta(\Lambda_1,\Lambda_0) \leq 0. \end{cases}$$

Proof. One has $\ell_+(\Lambda, \underline{\Lambda}) < 0$ if and only if $\Lambda \lll \underline{\Lambda_0}$ so we have the first equivalence for ℓ_+. Let us assume $\ell_+(\Lambda, \underline{\Lambda}) > \varepsilon > 0$. If $\ell_+^\alpha(\Lambda, \underline{\Lambda_0}) = 0$, then $\Lambda \preceq \phi_t^\alpha \underline{\Lambda_0}$ for all $t > 0$. As $\underline{\Lambda}_{\varepsilon/2} \ggg \underline{\Lambda_0}$, there exists $t > 0$ such that $\underline{\Lambda}_{\varepsilon/2} \ggg \phi_t^\alpha \underline{\Lambda_0}$ so $\Lambda \preceq \underline{\Lambda}_{\varepsilon/2}$, contradicting $\ell_+(\Lambda, \underline{\Lambda}) > \varepsilon$. Conversely, if $\ell_+(\Lambda, \underline{\Lambda}) = 0$ while $\ell_+^\alpha(\Lambda, \underline{\Lambda_0}) > \varepsilon > 0$, one gets a contradiction by using $\phi_{\varepsilon/2}^\alpha \underline{\Lambda_0} \ggg \underline{\Lambda}_t$ for small $t > 0$. This implies the two other equivalences of the statement for ℓ_+. The proof for ℓ_- is similar.

Let us prove the second statement regarding the comparison of $\ell_\pm^\alpha$ and $\ell_\pm^\beta$ for $\beta := e^f\alpha$. The β-Hamiltonian map of (ϕ_t^α) is $\beta(R^\alpha) = e^f$ while the β-Hamiltonian map of (ϕ_t^β) is the constant $\equiv 1$. The comparison of Hamiltonian maps $e^{\inf f}1 \leq e^f \leq e^{\sup f}1$ implies

$$\begin{cases} \phi^\beta_{e^{\inf f}t} \preceq \phi_t^\alpha \preceq \phi^\beta_{e^{\sup f}t} & \text{for } t \geq 0, \\ \phi^\beta_{e^{\sup f}t} \preceq \phi_t^\alpha \preceq \phi^\beta_{e^{\inf f}t} & \text{for } t \leq 0, \end{cases}$$

which easily brings the conclusion. □

Lemma 3.3. *A contact form α supporting ξ being fixed, if $(\Lambda_t)_{t\in[0,1]}$ is a Legendrian isotopy of $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), the Hamiltonian maps of which are denoted $H_t : \Lambda_t \to \mathbb{R}$, $t \in [0,1]$, one has*

$$\int_0^1 \min H_t \mathrm{d}t \leq \ell_-^\alpha(\Lambda_1,\Lambda_0) \leq \ell_+^\alpha(\Lambda_1,\Lambda_0) \leq \int_0^1 \max H_t \mathrm{d}t.$$

Proof. By the isotopy extension theorem, one can find a contact Hamiltonian flow $(g_t)_{t\in[0,1]}$ of (M,ξ) sending Λ_0 on Λ_1, the Hamiltonian map (K_t) of which satisfies $\max K_t = \max H_t$ for all $t \in [0,1]$. Let $I(t) := \int_0^t \max H_s \mathrm{d}s$, the time derivative I' of which generates the reparametrized Reeb flow $(\phi_{I(t)}^\alpha)$. By compatibility of $\preceq$ with contactomorphisms (*cf.* properties (11)), $g_1 \preceq \phi_{I(1)}^\alpha$ implies $\Lambda_1 \preceq \phi_{I(1)}^\alpha \Lambda_0$ and the monotonicity of ℓ_+^α together with its normalization property under the Reeb flow implies the last inequality of the statement. The first inequality is the consequence of a similar argument or the application of the "Poincaré duality" property. □

Corollary 3.4 (Continuity)**.** *A complete contact form α being fixed,*

$$|\ell_\pm^\alpha(\Lambda_1,\Lambda) - \ell_\pm^\alpha(\Lambda_0,\Lambda)| \leq \mathrm{d}_{\mathrm{SCH}}(\Lambda_0,\Lambda_1), \quad \forall \Lambda, \Lambda_0, \Lambda_1 \in \mathcal{L} \text{ (resp. } \widetilde{\mathcal{L}}).$$

In particular, the maps $\Lambda \mapsto \ell_\pm(\Lambda, \underline{\Lambda})$ are continuous with respect to the C^1-topology.

This corollary is somehow reinterpreted in Corollary 4.1 (see also inequality (14)).

Proof. Given any Legendrian isotopy (Λ_t) joining Λ_0 and Λ_1, the Hamiltonian map of which is (H_t), the triangular identity together with Lemma 3.3 imply

$$\ell_+^\alpha(\Lambda_1,\Lambda) - \ell_+^\alpha(\Lambda_0,\Lambda) \leq \ell_+^\alpha(\Lambda_1,\Lambda_0) \leq \int_0^1 \max H_t \mathrm{d}t.$$

Intertwining Λ_0 and Λ_1 one then gets

$$|\ell_+^\alpha(\Lambda_1,\Lambda) - \ell_+^\alpha(\Lambda_0,\Lambda)| \leq \max\left(-\int_0^1 \min H_t \mathrm{d}t, \int_0^1 \max H_t \mathrm{d}t\right) \leq \int_0^1 \max|H_t|\mathrm{d}t,$$

which brings the desired inequality by taking the infimum over all isotopies. The analogous inequality for ℓ_-^α follows by Poincaré duality or a similar proof.

Since $\mathrm{d}_{\mathrm{SCH}}$ is C^1-continuous, the maps $\Lambda \mapsto \ell_\pm^\alpha(\Lambda,\Lambda')$ are C^1-continuous, Λ' being fixed. In order to prove that a map $f: X \to \mathbb{R}$ is continuous, it is enough to prove that $f^{-1}(-\infty,x)$ and $f^{-1}(x,+\infty)$ are open for all $x \in \mathbb{R}$. Let $f := \ell_\pm(\cdot,\underline{\Lambda})$ for a fixed $\underline{\Lambda}$. Given a fixed $x \in \mathbb{R}$, let $\underline{\Lambda}' := (\underline{\Lambda}_{t+x})_{t\in\mathbb{R}}$, then

$$f^{-1}(-\infty,x) = \{\Lambda \mid \ell_+(\Lambda,\underline{\Lambda}') < 0\} = \{\Lambda \mid \ell_+^\alpha(\Lambda,\underline{\Lambda}_x) < 0\},$$

where we applied the sign invariance property to some supporting contact form α at the last step. By C^1-continuity of $\ell_+^\alpha(\cdot,\underline{\Lambda}_x)$, this last set is C^1-open. The other cases are treated similarly. ☐

Let us remark that the C^1-continuity is also a direct consequence of the C^1-openness of the relations $\lll$ and $\ggg$.

Proposition 3.5 (Spectrality). *For every $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), both $\ell_\pm(\Lambda,\underline{\Lambda})$ belong to* $\mathrm{Spec}(\Lambda,\underline{\Lambda})$*.*

Proof. Let us prove the statement in $\widetilde{\mathcal{L}}$. Let us prove that $\Pi\Lambda \cap \Pi\underline{\Lambda}_t = \emptyset$ and $\ell_+(\Lambda,\underline{\Lambda}) \leq t$ implies $\ell_+(\Lambda,\underline{\Lambda}) < t$; it would imply the result for ℓ_+ and the result for ℓ_- will follow from a similar argument. Since $\Pi\Lambda\cap\Pi\underline{\Lambda}_t = \emptyset$, by compactness $\exists\varepsilon > 0$, $\forall s \in (-\varepsilon,\varepsilon)$, $\Pi\Lambda \cap \Pi\underline{\Lambda}_{t+s} = \emptyset$. We now essentially apply the trick employed by Chernov-Nemirovski in [21, Lemma 2.2]. By compactly extending the Hamiltonian generated by the isotopy $s \mapsto \underline{\Lambda}_{t+s\varepsilon/2}$, $s \in [0,1]$, in $[0,1] \times (M \setminus \Pi\Lambda)$, one finds a compactly supported $g \in \widetilde{\mathcal{G}}$ sending $\underline{\Lambda}_{t+\varepsilon/2}$ to $\underline{\Lambda}_{t-\varepsilon/2}$ and fixing Λ (according to the isotopy extension theorem as expressed in Lemma 2.3). By definition of ℓ_+ and positivity of $\underline{\Lambda}$, $\Lambda \preceq \underline{\Lambda}_{t+\varepsilon/2}$. By invariance of $\preceq$ under the action of $\widetilde{\mathcal{G}}$, applying g, $\Lambda \preceq \underline{\Lambda}_{t-\varepsilon/2}$ so $\ell_+(\Lambda,\underline{\Lambda}) \leq t - \varepsilon/2$.

The proof in $\mathcal{L}$ is formally the same (removing Π's and tildes). ☐

Lemma 3.6. *Given any $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), if $\Lambda \succeq \underline{\Lambda}_0$ and $\Lambda \neq \underline{\Lambda}_0$, then $\ell_+(\Lambda,\underline{\Lambda}) > 0$.*

Proof. Let us prove the statement in $\widetilde{\mathcal{L}}$. Let $\Lambda_0 := \underline{\Lambda}_0$ and let α be a complete contact form supporting ξ. By sign invariance, it is enough to prove $\ell_+^\alpha(\Lambda,\Lambda_0) > 0$. Let us consider a non-negative path (Λ_t) from Λ_0 to $\Lambda_1 = \Lambda$. By replacing Λ with Λ_t for some smaller $t \in (0,1]$, one can moreover assume that $\Pi\Lambda$ belongs to a Weinstein neighborhood of $\Pi\Lambda_0$. Therefore, one can assume that (M,ξ) is an open neighborhood of the zero-section of $J^1\Pi\Lambda_0$, $\Pi\Lambda_0$ being identified with the zero-section and $\Pi\Lambda$ being C^1-close to $\Pi\Lambda_0$. Let (H_t) be a non-negative Hamiltonian, the flow (ψ^t) of which satisfies $\psi^t\Lambda_0 = \Lambda_t$ for all $t \in [0,1]$. Since $\Pi\Lambda_0 \neq \Pi\Lambda$, there exists $t_0 \in [0,1)$, and a non-empty neighborhood $U \subset \Pi\Lambda_0$, such that $H_{t_0} \circ \psi^{t_0}(q) > 0$ for $q \in U$. One can furthermore assume without loss of generality $t_0 = 0$ (by replacing Λ_0 with $\psi^{t_0}\Lambda_0$, U with $\psi^{t_0}U$ etc.) so that $H_0(q) > 0$ for all $q \in U$.

We will adapt a procedure due to Eliashberg-Polterovich (see the proof of [30, Proposition 2.1.B]). Let $(\varphi_i)_{1\leq i\leq n}$ be a finite family of diffeomorphisms of $\Pi\Lambda_0$ isotopic to the identity such that $(\varphi_i(U))$ covers $\Pi\Lambda_0$ (it exists by closeness of $\Pi\Lambda_0$).

It lifts to $J^1\Pi\Lambda_0$ as a family of contactomorphisms isotopic to identity and preserving the zero-section: associating to a diffeomorphism φ, the contactomorphism $(q,p,z) \mapsto (\varphi(q), p\circ \mathrm{d}\varphi^{-1}, z)$. By cutting-off their Hamiltonian maps away from the zero section, one gets contactomorphisms $(g_i)_{1\leq i\leq n}$ of (M,ξ) fixing Λ_0 such that $(g_i(U))$ covers $\Pi\Lambda_0$. Let $\psi_i^t := g_i\psi^t g_i^{-1}$ for $t\in[0,1]$ and $1\leq i\leq n$.

The key point is that $\psi_n\cdots\psi_1\Lambda_0 \ggg \Lambda_0$, where $\psi_k := \psi_k^1$. Indeed, it is enough to prove that the Hamiltonian map (K_t) of the flow $(\psi_n^t\psi_{n-1}^t\cdots\psi_1^t)_t$ is positive along $\Pi\Lambda_0$ at time $t=0$ (see Lemma 2.8). But for all $q\in\Pi\Lambda_0$,

$$K_0(q) = \alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_n^t\cdots\psi_1^t(q)\right)\Big|_{t=0}\right) = \sum_{i=1}^n \alpha\left(\dot\psi_i^0(q)\right),$$

where $\dot\psi_i^0$ stands for the time-derivative of ψ_i^t taken at time $t=0$. As $g_i^*\alpha = \lambda_i\alpha$ for some positive $\lambda_i : M\to(0,+\infty)$, $\alpha(\dot\psi_i^0(q))$ has the sign of $\alpha(\dot\psi^0(g_i^{-1}(q))) = H_0(g_i^{-1}(q))$. So each term of the summand is non-negative and the i-th term is positive when $q\in g_i(U)$. As $(g_i(U))$ covers $\Pi\Lambda_0$, one concludes that K_0 is positive along $\Pi\Lambda_0$.

Therefore $\psi_n\cdots\psi_1\Lambda_0\ggg\Lambda_0$ so $\ell_+^\alpha(\psi_n\cdots\psi_1\Lambda_0,\Lambda_0) > 0$. Now, by the triangular inequality,

$$0 < \ell_+^\alpha(\psi_n\cdots\psi_1\Lambda_0, \psi_n\cdots\psi_2\Lambda_0) + \ell_+^\alpha(\psi_n\cdots\psi_2\Lambda_0, \psi_n\cdots\psi_3\Lambda_0) + \cdots + \ell_+^\alpha(\psi_n\Lambda_0,\Lambda_0),$$

so $\ell_+^\alpha(\psi_n\cdots\psi_k\Lambda_0,\psi_n\cdots\psi_{k+1}\Lambda_0) > 0$ for some k. By sign invariance under the contactomorphism $\psi_n\cdots\psi_{k+1}$, $\ell_+^\alpha(\psi_k\Lambda_0,\Lambda_0)>0$. Since $g_k\Lambda_0=\Lambda_0$, sign invariance implies $\ell_+^\alpha(\psi^1\Lambda_0,\Lambda_0)>0$, but $\psi^1\Lambda_0=\Lambda$.

The proof in $\mathcal{L}$ is similar. □

Proposition 3.7 (Non-degeneracy). *Given any $\Lambda\in\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), if $\ell_-(\Lambda,\underline{\Lambda}) = \ell_+(\Lambda,\underline{\Lambda}) = t$, then $\Lambda = \underline{\Lambda}_t$ (resp. for $\widetilde{\mathcal{L}}$ it only implies $\Pi\Lambda = \Pi\underline{\Lambda}_t$).*

Proof. Let us prove the statement in $\widetilde{\mathcal{L}}$, the case of $\mathcal{L}$ being similar. By contradiction, let us assume that $\ell_\pm(\Lambda,\underline{\Lambda})\equiv t$ and $\Pi\Lambda\neq\Pi\Lambda_t$, where $\Lambda_t := \underline{\Lambda}_t$. By shifting the parametrization of $\underline{\Lambda}$ and fixing a complete contact form α, Lemma 3.2 implies $\ell_\pm^\alpha(\Lambda,\Lambda_t)\equiv 0$. Let $p\in\Pi\Lambda\setminus\Pi\Lambda_t$ and let us consider a non-negative Hamiltonian map $H : M\to\mathbb{R}$ supported outside $\Pi\Lambda_t$ and such that $H(p)>0$, the flow of which is denoted (g_s). Then $g_s\Lambda\succeq\Lambda$ with $g_s\Lambda\neq\Lambda$ when $s>0$ is sufficiently small (as $g_s(p)\notin\Pi\Lambda$) whereas $g_s\Lambda_t=\Lambda_t$. By Lemma 3.6, $\ell_+^\alpha(g_s\Lambda,\Lambda)>0$ whereas sign invariance under g_s implies $\ell_+^\alpha(g_s\Lambda,\Lambda_t) = \ell_+^\alpha(\Lambda,\Lambda_t) = 0$. This contradicts the triangle inequality $\ell_+^\alpha(g_s\Lambda,\Lambda)\leq\ell_+^\alpha(g_s\Lambda,\Lambda_t)+\ell_+^\alpha(\Lambda_t,\Lambda)$ as $\ell_+^\alpha(\Lambda_t,\Lambda) = -\ell_-^\alpha(\Lambda,\Lambda_t) = 0$. □

As a corollary of Proposition 3.5 and 3.7, one gets Theorem 1.7 stated in the introduction (Section 1.4).

Proof of Theorem 1.7. According to Lemma 2.7, there exists a positive isotopy (Λ_t) in $\mathcal{L}$ with $\Lambda_1=\Lambda_0$. Let $\Lambda\in\mathcal{L}$. By applying a contactomorphism sending Λ on Λ_0, one can assume $\Lambda_0=\Lambda$. Let $(\widetilde{\Lambda}_t)$ be a lift of (Λ_t) in $\widetilde{\mathcal{L}}$. By positivity, $\widetilde{\Lambda}_1\ggg\widetilde{\Lambda}_0$, which implies $\ell_\pm^\alpha(\widetilde{\Lambda}_1,\widetilde{\Lambda}_0)>0$. If $\ell_+^\alpha(\widetilde{\Lambda}_1,\widetilde{\Lambda}_0) > \ell_-^\alpha(\widetilde{\Lambda}_1,\widetilde{\Lambda}_0)$, one then gets two Reeb chords of distinct lengths by spectrality (Proposition 3.5). Otherwise, one gets infinitely many distinct chords of the same length by non-degeneracy of the spectral selectors (Proposition 3.7). □

3.2. **The case of contactomorphisms.** Let us assume that (M, ξ) is a closed cooriented contact manifold such that $\mathcal{G}$ (reps. $\widetilde{\mathcal{G}}$) is orderable. Following Equations (1), one defines maps $c_\pm^\alpha$ for any supporting contact form α. Similarly to the Legendrian case, compactness of M and orderability imply that these maps are real-valued and the basic properties of normalization, monotonicity, triangle inequalities, duality and compatibility are straightforward consequences of the compatibility of the partial order $\preceq$ with the composition of contactomorphisms and (12). They also obey a sign invariance property analogous to Lemma 3.2.

Lemma 3.8 (Sign invariance). *For all contact forms α and β supporting ξ (it is always assumed that such forms preserve the coorientation) and all $\varphi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$),*

$$c_\pm^\alpha(\varphi) < 0 \text{ (resp. } = 0\text{, resp. } > 0\text{)} \Leftrightarrow c_\pm^\beta(\varphi) < 0 \text{ (resp. } = 0\text{, resp. } > 0\text{)}.$$

More precisely, if $f : M \to \mathbb{R}$ is such that $\beta = e^f\alpha$, $\forall \varphi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$),

$$\begin{cases} e^{\inf f} c_\pm^\alpha(\varphi) \leq c_\pm^\beta(\varphi) \leq e^{\sup f} c_\pm^\alpha(\varphi) & \text{when } c_\pm^\beta(\varphi) \geq 0, \\ e^{\sup f} c_\pm^\alpha(\varphi) \leq c_\pm^\beta(\varphi) \leq e^{\inf f} c_\pm^\alpha(\varphi) & \text{when } c_\pm^\beta(\varphi) \leq 0. \end{cases}$$

The following results are consequence of the basic properties of the maps $c_\pm^\alpha$ and are proved in a way similar to their Legendrian counterparts.

Lemma 3.9. *A contact form α supporting ξ being fixed, if (φ_t) is an isotopy of $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) with $\varphi_0 = \mathrm{id}$, the Hamiltonian map of which is $H_t : M \to \mathbb{R}$, $t \in [0, 1]$, one has*

$$\int_0^1 \min H_t \mathrm{d}t \leq c_-^\alpha(\varphi_1) \leq c_+^\alpha(\varphi_1) \leq \int_0^1 \max H_t \mathrm{d}t.$$

Corollary 3.10 (Continuity). *A supporting contact form α being fixed,*

$$|c_\pm^\alpha(\varphi) - c_\pm^\alpha(\psi)| \leq \mathrm{d}_{\mathrm{SH}}^\alpha(\varphi, \psi), \quad \forall \varphi, \psi \in \mathcal{G} \text{ (resp. } \widetilde{\mathcal{G}}\text{)}.$$

In particular, the maps $c_\pm^\alpha$ are continuous with respect to the C^1-topology.

Remark 3.11 (Extensions of the selectors to completions). Corollaries 3.4 and 3.10 allow us to naturally extend the selectors to C^1, Hofer or spectral-completions of the spaces $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$. The Hofer-1-Lipchitzness of the selectors implies their uniform continuity with respect to the C^1-metrics defined in Section 2.2. But we actually do not need this remark to extend the spectral selectors to C^1-contactomorphisms or Legendrian C^1-submanifolds: the definition of $\preceq$ and $\lll$ naturally extends to these spaces and the orderability of the C^1-completion is equivalent to the orderability of the smooth space.

On the other hand, it could be interesting to study the Hofer-completion or the spectral-completion of orderable $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ (see Section 4.1 and Remark 4.6 below), as was initiated by Humilière [44] and recently revitalized by Viterbo [74] in the symplectic setting. It would also be interesting to compare these completions to the respective C^0-completions, which in the case of $\mathcal{G}$ correspond to its C^0-closure inside the group of homeomorphism [72]. In the symplectic setting, such a comparison can be done in some special cases thanks to the C^0-continuity of the selectors [12].

Contrary to the Legendrian case, we only conjecture that the maps $c_\pm^\alpha$ are indeed spectral selectors while the non-degeneracy will follow from the theorem of Tsuboi on the simplicity of the C^1-contactomorphisms isotopic to the identity [71].

Proposition 3.12. *Let $\psi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) be such that $c_-^\alpha(\psi) = c_+^\alpha(\psi) = t$ for some $t \in \mathbb{R}$. Then $\psi = \phi_t^\alpha$ (resp. $\Pi\psi = \phi_t^\alpha$).*

Proof. Let us first deal with the case of $\mathcal{G}$. We want to apply the theorem of Tsuboi asserting that the group of C^1-contactomorphisms isotopic to identity $\mathcal{G}^1$ is a simple group. The group of C^1-contactomorphisms is defined by Tsuboi in [71, §2], it contains $\mathcal{G}$ and its identity component is contained in the C^1-completion of $\mathcal{G}$: φ is a C^1-contactomorphism if it is a C^1-diffeomorphism such that $\varphi^*\alpha = e^f\alpha$ for a C^1-map $f : M \to \mathbb{R}$. As $\mathcal{G}^1$ is contained in the C^1-completion of $\mathcal{G}$, we endow it with the C^1-topology and the maps $c_\pm^\alpha$ naturally extend to C^1-continuous maps $\bar{c}_\pm^\alpha : \mathcal{G}^1 \to \mathbb{R}$ (see Remark 3.11 just above). Let us define the subset $Z^1 \subset \mathcal{G}^1$

$$Z^1 := \{\varphi \in \mathcal{G}^1 \mid \bar{c}_-^\alpha(\varphi) = \bar{c}_+^\alpha(\varphi) = 0\},$$

and denote $Z := Z^1 \cap \mathcal{G}$. The subset Z^1 is in fact a normal subgroup of $\mathcal{G}^1$ (resp. $\widetilde{\mathcal{G}}$). It is indeed a subgroup: $\mathrm{id} \in Z^1$ by the normalization property, if $\varphi, \psi \in Z^1$, then $\varphi^{-1} \in Z^1$ by the "Poincaré duality" property, while $\varphi\psi \in Z^1$ by applying both triangle inequalities and $\bar{c}_-^\alpha \leq \bar{c}_+^\alpha$.

In order to prove that Z^1 is normal, let us prove that $\bar{c}_\pm^\alpha(\varphi) = 0$ implies $\bar{c}_\pm^\alpha(g\varphi g^{-1}) = 0$ (for $\bar{c}_+^\alpha$ or $\bar{c}_-^\alpha$ independently) for all $g, \varphi \in \mathcal{G}^1$. Since $c_\pm^\alpha(g\varphi g^{-1}) = c_\pm^{g^*\alpha}(\varphi)$ for $g, \varphi \in \mathcal{G}$, by Lemma 3.8,

$$e^{-\sup|f_g|} c_\pm^\alpha(\varphi) \leq c_\pm^\alpha(g\varphi g^{-1}) \leq e^{\sup|f_g|} c_\pm^\alpha(\varphi), \tag{13}$$

where $f_g : M \to \mathbb{R}$ is the smooth function such that $g^*\alpha = e^{f_g}\alpha$. Let us recall that $g^*\alpha = e^{f_g}\alpha$ for f_g of class C^1 when $g \in \mathcal{G}^1$ so that $h \mapsto \sup|f_h|$ is a continuous map $\mathcal{G} \to \mathbb{R}$ that extends to $\mathcal{G}^1$. Therefore the identity (13) extends to those g, φ in $\mathcal{G}^1$ by replacing $c_\pm^\alpha$ with its extension $\bar{c}_\pm^\alpha$. This extended identity implies that Z^1 is normal. The theorem of Tsuboi [71] then implies that $Z^1 = \{\mathrm{id}\}$ as $\phi_t^\alpha \notin Z^1$ for $t \neq 0$. So $Z = \{\mathrm{id}\}$ which brings the conclusion.

In the case of $\widetilde{\mathcal{G}}$, one needs to also consider the universal cover $\widetilde{\mathcal{G}}^1$ of $\mathcal{G}^1$ (it is a genuine universal cover since $\mathcal{G}^1$ is locally contractible [71, §3]). We denote Π^1 the cover map (extending the cover map Π) and define $Z^1 \subset \widetilde{\mathcal{G}}^1$ as the intersection of the zero sets of $\bar{c}_\pm^\alpha$ as above. As Before, Z^1 is a normal subgroup of $\widetilde{\mathcal{G}}^1$. As Π^1 is a surjective group morphism, $\Pi^1 Z^1$ is a normal subgroup of $\mathcal{G}^1$ which does not contain ϕ_t^α for $t \neq 0$ small enough. The theorem of Tsuboi then implies $\Pi^1 Z^1 = \{\mathrm{id}\}$, which allows us to conclude. ☐

There is a counterpart to Lemma 3.6 to the case of contactomorphisms that can be proved using the same procedure inspired by Eliashberg-Polterovich. However, in the current case, it can also be seen as a consequence of Proposition 3.12.

Corollary 3.13. *Given $\varphi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$), if $\varphi \succeq \mathrm{id}$ and $\varphi \neq \mathrm{id}$, then $c_+^\alpha(\varphi) > 0$ for any supporting α.*

Proof. In the case $\varphi \in \mathcal{G}$, by monotonicity of $c_\pm^\alpha$ and normalization, $c_+^\alpha(\varphi) \geq c_-^\alpha(\varphi) \geq 0$. Therefore, $c_+^\alpha(\varphi) = 0$ would imply $c_-^\alpha(\varphi) = 0$ so $\varphi = \mathrm{id}$ by Proposition 3.12. In the case $\varphi \in \widetilde{\mathcal{G}}$, since $\varphi \succeq \mathrm{id}$, there exists a non-negative isotopy (φ_t) from id to $\varphi_1 = \varphi$. Assuming furthermore $\varphi \neq \mathrm{id}$, this isotopy is non-constant and $\Pi\varphi_t \neq \Pi\mathrm{id}$ for some $t \in (0, 1]$. By a similar argument as above, Proposition 3.12 then implies $c_+^\alpha(\varphi_t) > 0$ so $c_+^\alpha(\varphi) > 0$ by monotonicity. ☐

4. Metrics and pseudo-metrics

4.1. **The spectral metrics.** In this section, we define the natural pseudo-distance associated with our spectral selectors, following a classical process originated in Viterbo's seminal work [73], where he defined a norm γ on the compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$. During the writing of this paper, the article of Nakamura [58] was prepublished. In this article, Nakamura defines $\mathrm{d}^\alpha_{\mathrm{spec}}$ with the aim of generating the interval topology (see Proposition 4.4). Although the link with the α-spectrum is out of the scope of his work, the results of this section can also be found there.

Given a supporting contact form α, let us define the following pseudo-distances on $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$): $\forall\Lambda,\Lambda'\in\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$),

$$\begin{aligned}&\text{the spectral distance} && \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda,\Lambda') := \max(\ell^\alpha_+(\Lambda,\Lambda'),\ell^\alpha_+(\Lambda',\Lambda)),\\ &\text{the gamma distance} && \gamma^\alpha(\Lambda,\Lambda') := \ell^\alpha_+(\Lambda,\Lambda') + \ell^\alpha_+(\Lambda',\Lambda).\end{aligned}$$

These pseudo-distance are symmetric, non-negative and satisfy the triangle inequality, according to the basic properties of the spectral selectors (the non-negativity follows from the Poincaré duality property). The compatibility properties of the selectors also imply

$$\mathrm{d}^\alpha_{\mathrm{spec}}(\varphi(\Lambda),\varphi(\Lambda')) = \mathrm{d}^{\varphi^*\alpha}_{\mathrm{spec}}(\Lambda,\Lambda') \text{ and } \gamma^\alpha(\varphi(\Lambda),\varphi(\Lambda')) = \gamma^{\varphi^*\alpha}(\Lambda,\Lambda'),$$

for all $\varphi\in\mathrm{Cont}(M,\xi)$, making these pseudo-distances invariant under strict contactomorphisms of (M,α) (see Remark 4.6 below for a discussion on the apparent lack of invariance of contact metrics). The normalization property regarding the Reeb flow also implies the following invariance:

$$\gamma^\alpha(\phi^\alpha_t\Lambda,\Lambda') = \gamma^\alpha(\Lambda,\Lambda'),\quad \forall t\in\mathbb{R},$$

so that γ^α is clearly degenerate. Concerning non-degeneracy, Proposition 3.7 and Lemma 3.3 imply the following less direct properties.

Corollary 4.1 ([58, Theorems 2.17 and 2.26])**.** *On either $\mathcal{L}$ or $\widetilde{\mathcal{L}}$ orderable, one has*

$$\mathrm{d}^\alpha_{\mathrm{spec}} \leq \mathrm{d}^\alpha_{\mathrm{SCH}} \textit{ and } \gamma^\alpha \leq \mathrm{d}^\alpha_{\mathrm{H,osc}},$$

for any supporting contact form α. Moreover, $\mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda,\Lambda')=0$ implies $\Lambda=\Lambda'$ on $\mathcal{L}$ (resp. $\Pi\Lambda=\Pi\Lambda'$ on $\widetilde{\mathcal{L}}$) while $\gamma^\alpha(\Lambda,\Lambda')=0$ implies $\Lambda=\phi^\alpha_t\Lambda'$ for some $t\in\mathbb{R}$ on $\mathcal{L}$ (resp. $\Pi\Lambda=\phi^\alpha_t\Pi\Lambda'$ on $\widetilde{\mathcal{L}}$).

Nakamura's proof of the non-degeneracy properties of the Legendrian spectral metrics follows a different path than ours: it is inspired by Rosen-Zhang work [62] and provides a version of Chekanov's dichotomy.

The triangular inequality satisfied by ℓ^α_+ implies that the maps $\ell^\alpha_\pm(\cdot,\Lambda_0)$ are $\mathrm{d}^\alpha_{\mathrm{spec}}$-1-Lipschitz:

$$|\ell^\alpha_\pm(\Lambda,\Lambda_0) - \ell^\alpha_\pm(\Lambda',\Lambda_0)| \leq \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda,\Lambda'),\quad \forall\Lambda,\Lambda',\Lambda_0\in\mathcal{L} \text{ (resp. } \widetilde{\mathcal{L}}). \tag{14}$$

In particular, Corollary 3.4 can be seen as a consequence of Corollary 4.1.

Similarly to the Legendrian case, given a supporting contact form α, one can define pseudo-distances $\mathrm{d}^\alpha_{\mathrm{spec}}$ and γ^α on $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$). Since our space of interest is a group, it is customary to first define the associated pseudo-norms (see Section 2.3): $\forall\varphi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$),

$$\text{the spectral norm} \qquad |\varphi|^\alpha_{\text{spec}} := \max(c^\alpha_+(\varphi), c^\alpha_+(\varphi^{-1})),$$
$$\text{the gamma norm} \qquad \gamma^\alpha(\varphi) := c^\alpha_+(\varphi) + c^\alpha_+(\varphi^{-1}),$$

and then define the right-invariant pseudo-distances by $\mathrm{d}^\alpha_{\text{spec}}(\varphi,\psi) := |\varphi\psi^{-1}|^\alpha_{\text{spec}}$ and $\gamma^\alpha(\varphi,\psi) := \gamma^\alpha(\varphi\psi^{-1})$ (although the term "spectral" would suggest it actually comes from spectral selectors, which we only conjecture for $c^\alpha_\pm$). Although they are not left-invariant (see however Remark 4.6), they satisfy a compatibility property:

$$\mathrm{d}^\alpha_{\text{spec}}(g\varphi, g\psi) = \left|g\varphi\psi^{-1}g^{-1}\right|^\alpha_{\text{spec}} = \left|\varphi\psi^{-1}\right|^{g^*\alpha}_{\text{spec}} = \mathrm{d}^{g^*\alpha}_{\text{spec}}(\varphi,\psi).$$

These pseudo-norms and pseudo-distances share properties similar to their Legendrian counterparts, among which

$$|c^\alpha_\pm(\varphi) - c^\alpha_\pm(\psi)| \leq \mathrm{d}^\alpha_{\text{spec}}(\varphi,\psi). \tag{15}$$

Corollary 4.2 ([58, Theorem 2.6])**.** *On either $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ orderable, one has*

$$|\cdot|^\alpha_{\text{spec}} \leq |\cdot|^\alpha_{\text{SH}} \text{ and } \gamma^\alpha \leq |\cdot|^\alpha_{\text{osc}},$$

for any supporting contact form α. Moreover, $|\varphi|^\alpha_{\text{spec}} = 0$ implies $\varphi = \mathrm{id}$ on $\mathcal{G}$ (resp. $\Pi\varphi = \mathrm{id}$ on $\widetilde{\mathcal{G}}$) while $\gamma^\alpha(\varphi) = 0$ implies $\varphi = \phi^\alpha_t$ for some $t \in \mathbb{R}$ on $\mathcal{G}$ (resp. $\Pi\varphi = \phi^\alpha_t$ on $\widetilde{\mathcal{G}}$)

Let us recall that a partially ordered metric space $(Z, d, \leq)$ is a metric space (Z, d) endowed with a partial order $\leq$ such that $a \leq b \leq c$ implies $d(a,b) \leq d(a,c)$. The following statement directly follows from the definitions.

Proposition 4.3. *Let O be either $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$, associated to some contact manifold $(M, \ker\alpha)$ (and possibly some closed Legendrian submanifold of M). If O is orderable, then $(O, \mathrm{d}^\alpha_{\text{spec}}, \preceq)$ is a partially ordered metric space.*

The topology induced by the spectral distance is already known and has been introduced by Chernov-Nemirovski [23]. It was the motivation of the recent work of Nakamura for introducing this distance independently [58].

Proposition 4.4 ([58, Proposition 2.3])**.** *Let O be either $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$, associated to some contact manifold $(M, \ker\alpha)$ (and possibly some closed Legendrian submanifold of M) and let us assume that O is orderable. The topology induced by the spectral distance $\mathrm{d}^\alpha_{\text{spec}}$ is the interval topology, that is the topology generated by the basis of open subsets*

$$(a,b) := \{x \in O \mid a \lll x \lll b\},$$

where $a, b \in O$ satisfies $a \preceq b$.

Proof. Indeed, given $r \geq 0$ and $x \in O$, the open $\mathrm{d}^\alpha_{\text{spec}}$-ball $B_r(x)$ centered at x of radius r in O is exactly $(\phi^\alpha_{-r}x, \phi^\alpha_r x)$ and these subsets form a basis of open neighborhoods of both the interval topology and the $\mathrm{d}^\alpha_{\text{spec}}$-topology.

Let us show precisely that $B_r(x) = (\phi^\alpha_{-r}x, \phi^\alpha_r x)$ in the case $O = \widetilde{\mathcal{L}}$, the other cases being similar. Let $\Lambda \in \widetilde{\mathcal{L}}$, then $\Lambda \in B_r(x)$ is equivalent to $\ell^\alpha_+(\Lambda, x) < r$ and $\ell^\alpha_-(\Lambda, x) > -r$ (applying $\ell^\alpha_- \leq \ell^\alpha_+$ and the Poincaré duality). Now, by definition of ℓ^α_+, $\ell^\alpha_+(\Lambda, x) < r$ is equivalent to $\Lambda \lll \phi^\alpha_r x$ while $\ell^\alpha_-(\Lambda, x) > -r$ is equivalent to $\Lambda \ggg \phi^\alpha_{-r} x$, bringing the conclusion.

The fact that the intervals $(\phi^\alpha_{-r} x, \phi^\alpha_r x)$ form a basis of the interval topology is a consequence of the fact that if $x \lll y$ then $x \lll \phi^\alpha_\varepsilon x \lll \phi^\alpha_{-\varepsilon} y \lll y$ for $\varepsilon > 0$ small enough, which comes from the openness of the relation $\lll$. □

This proposition incidentally shows that spectral metrics induce a same topology, any complete contact form supporting ξ being given. It can be quantified as a corollary of Lemmata 3.2 and 3.8.

Corollary 4.5. *Let (M, ξ) be a contact manifold and α be a complete contact form supporting ξ. Let $f : M \to \mathbb{R}$ be a bounded map and $\beta := e^f \alpha$. Then, on either $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ orderable for which selectors are well-defined,*

$$e^{\inf f} g^\alpha \leq g^\beta \leq e^{\sup f} g^\alpha,$$

where g^δ either stands for $|\cdot|^\delta_{\mathrm{spec}}$ or $\mathrm{d}^\delta_{\mathrm{spec}}$ for any complete contact form δ. In particular, on the non-vanishing set of g^α,

$$\Big| \log |g^\alpha| - \log |g^\beta| \Big| \leq d_{C^0}(\alpha, \beta),$$

with $d_{C^0}(\alpha, \beta) = \sup |f|$ where $\alpha = e^f \beta$.

Remark 4.6 (Invariant uniform structures and invariant metrics). Let us assume (M, ξ) is closed, for simplicity. Corollary 4.5 implies that one can not only endow orderable spaces $\mathcal{L}$, $\widetilde{\mathcal{L}}$, $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ with a topology that is invariant by the natural action by contactomorphisms (left and right actions in the case of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$) but that this topology is induced by an invariant uniformity structure. Indeed, one can make sense of Cauchy sequences or uniform continuity by using any α-spectral metric, given any auxiliary contact form α. And since these spectral metrics are sent to one-another by the actions induced by contactomorphisms, the uniformity structure they induce is invariant. A consequence is that the spectral completion of these spaces will not depend on the choice of α either. In fact, a bit more is invariant: the notion of Lipschitz maps is also preserved as well as the notion of boundedness. The same story could also be applied to the Hofer-type pseudo-distance.

One could ask whether one could have more: a natural invariant metric (bi-invariant in the case of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$). According to Fraser-Polterovich-Rosen [35, Theorem 3.1], such metric would essentially be discrete because of the possibility to squeeze a Darboux ball of (M, ξ) into an arbitrarily small open subset. Therefore, the induced topology will not be that interesting, although discrete bi-invariant metrics could carry interesting information, especially regarding their asymptotic behaviors (see Sections 4.3, 4.4 and 4.5 below).

4.2. **Spectrally robust Legendrian interlinkings.** The notion of interlinked Legendrian submanifolds was introduced by Entov and Polterovich in [31]. We recall below the definition of interlinked Legendrian submanifolds given in [32]. Let (M, ξ) be a cooriented contact manifold and α be a complete contact form supporting ξ. An ordered pair (Λ_0, Λ_1) of disjoint Legendrian submanifolds is μ-interlinked for some positive number μ if for every Hamiltonian map $H : \mathbb{R} \times M \to \mathbb{R}$ generating the contact flow (g_t) and satisfying $H \geq c$ for some $c > 0$ there exist $x \in \Lambda_0$ and $t \in (0, \mu/c]$ such that $g_t(x) \in \Lambda_1$. The number μ does depend on the specific choice of supporting α. A pair (Λ_0, Λ_1) is called interlinked if it is μ-interlinked for some $\mu > 0$. A pair (Λ_0, Λ_1) is called C^1-robustly interlinked if any pair (Λ_0', Λ_1') obtained from a sufficiently C^1-small Legendrian isotopy is interlinked. We define spectral-robustness and Hofer-robustness by replacing C^1-smallness with $\mathrm{d}^\alpha_{\mathrm{spec}}$ and $\mathrm{d}^\alpha_{\mathrm{SH}}$-smallness respectively.

Theorem 4.7. *Suppose there exists a closed Legendrian $\Lambda_* \subset M$ such that $\widetilde{\mathcal{L}}(\Lambda_*)$ is orderable. If $\Lambda_0, \Lambda_1 \in \widetilde{\mathcal{L}}(\Lambda_*)$ satisfies $\mu := \ell^\alpha_+(\Lambda_1, \Lambda_0) > 0$, then $(\Pi\Lambda_0, \Pi\Lambda_1)$ is μ-interlinked with respect to α. In particular, $(\Pi\Lambda_0, \Pi\Lambda_1)$ is spectral-robustly interlinked (so Hofer and C^1-robustly interlinked).*

In particular the hypothesis of Theorem 4.7 is satisfied whenever Λ_0 and Λ_1 are two different elements in $\widetilde{\mathcal{L}}$ satisfying $\Lambda_0 \preceq \Lambda_1$ thanks to Lemma 3.6. Moreover for any $\varphi \in \widetilde{\mathcal{G}}$ the sign invariance guarantees that $\ell^\alpha_+(\Lambda_1, \Lambda_0) > 0$ if and only if $\ell^\alpha_+(\varphi(\Lambda_1), \varphi(\Lambda_0)) > 0$ (Lemma 3.2). Therefore, Theorem 4.7 generalizes [32, Theorem 1.5 (i)].

Let us first prove the following lemma.

Lemma 4.8. *Let α be a complete contact form supporting the structure of (M, ξ) and $\Lambda_* \subset M$ be a closed Legendrian submanifold such that $\widetilde{\mathcal{L}}(\Lambda_*)$ is orderable. Let $H : \mathbb{R} \times M \to \mathbb{R}$ be a Hamiltonian map such that it generates a contact flow $(g_t)_{t\in\mathbb{R}}$ and $c := \inf H > 0$. If $\Lambda_0, \Lambda_1 \in \widetilde{\mathcal{L}}(\Lambda_*)$ satisfy $\ell^\alpha_+(\Lambda_1, \Lambda_0) > 0$, then*

$$0 < \ell_+\left(\Lambda_1, (g_t\Lambda_0)_{t\in\mathbb{R}}\right) \leq \frac{1}{c}\ell^\alpha_+(\Lambda_1, \Lambda_0).$$

Proof. The first inequality comes from the sign invariance property of Lemma 3.2.

For the second inequality, let us remark that $\frac{1}{c}\ell^\alpha_+ = \ell^{\frac{1}{c}\alpha}_+$ since $\phi^{\frac{1}{c}\alpha}_t = \phi^\alpha_{ct}$ for all $t \in \mathbb{R}$. Moreover thanks to our hypothesis on the sign of $\ell^\alpha_+(\Lambda_1, \Lambda_0)$ we get the following equality

$$\frac{1}{c}\ell^\alpha_+(\Lambda_1, \Lambda_0) = \ell^{\frac{1}{c}\alpha}_+(\Lambda_0, \Lambda_1) = \inf\left\{t \geq 0 \mid \phi^{ct}_\alpha\Lambda_0 \preceq \Lambda_1\right\}.$$

Since $\inf H = c > 0$ we deduce that $\phi^{ct}_\alpha\Lambda_0 \preceq g_t\Lambda_0$ for all $t \geq 0$. Therefore, we get the desired inequality $\ell_+(\Lambda_1, \underline{\Lambda}_0) \leq \frac{1}{c}\ell^\alpha_+(\Lambda_1, \Lambda_0)$ by transitivity of $\preceq$. ☐

Proof of Theorem 4.7. If $\Lambda_0, \Lambda_1 \in \widetilde{\mathcal{L}}(\Lambda_*)$ satisfies $\mu := \ell^\alpha_+(\Lambda_1, \Lambda_0) > 0$, for a Hamiltonian $H \geq c > 0$ generating a contact flow (g_t), Lemma 4.8 implies that $0 < \ell_+\left(\Lambda_1, (g_t\Lambda_0)_{t\in\mathbb{R}}\right) \leq \mu/c$. By spectrality of ℓ_+ (Proposition 3.5), there exists $t \in (0, \mu/c]$ such that $\Pi\Lambda_1$ intersects $g_t\Pi\Lambda_0$, which brings the conclusion.

The spectral-robustness now follows from the 1-$\mathrm{d}^\alpha_{\mathrm{spec}}$-Lipchitzness of $\ell^\alpha_+(\Lambda_1, \cdot)$ and $\ell^\alpha_+(\cdot, \Lambda_0)$ (by (14) and the Poincaré duality property). It implies Hofer and C^1-robustness by Corollary 4.1. ☐

4.3. **Unboundedness of Hofer type pseudo-metrics.** The results of this section have also been derived by Nakamura [58] except for Proposition 4.14.

Corollary 4.9. *Let $\Lambda_* \subset (M, \xi)$ be a closed Legendrian submanifold of a contact manifold such that $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable and α be a complete contact form supporting ξ. Then $\mathrm{d}^\alpha_{\mathrm{SCH}}(\Lambda, \Lambda') = 0$ if and only if $\Lambda = \Lambda'$ (resp. $\Pi\Lambda = \Pi\Lambda'$), for any $\Lambda, \Lambda' \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$). Moreover, for all $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), $(\phi^\alpha_t\Lambda)_{t\in\mathbb{R}}$ is a geodesic for $\mathrm{d}^\alpha_{\mathrm{SCH}}$:*

$$\mathrm{d}^\alpha_{\mathrm{SCH}}(\phi^\alpha_t\Lambda, \phi^\alpha_s\Lambda) = |t - s|, \quad \forall t, s \in \mathbb{R}, \forall\Lambda \in \mathcal{L} \text{ (resp. } \widetilde{\mathcal{L}}).$$

In particular, $\mathrm{d}^\alpha_{\mathrm{SCH}}$ *is unbounded. When* M *is closed, the analogous statements for* $\mathrm{d}^\alpha_{\mathrm{SH}}$ *on* $\mathcal{G}$ *(resp.* $\widetilde{\mathcal{G}}$*) hold when the latter is orderable.*

Remark 4.10. Note that the maps $c^{\alpha,\infty}_\pm(\psi) := \lim_{n\to+\infty} \frac{c^\alpha_\pm(\psi^n)}{n}$ are well-defined and conjugation invariant for all $\psi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) orderable, since the sequences $(c^\alpha_+(\psi^n))$ and $(c^\alpha_-(\psi^n))$ are respectively subadditive and superadditive. Moreover, they satisfy $c^{\alpha,\infty}_\pm(\phi^\alpha_t) = t$ for all $t \in \mathbb{R}$ and $c^\alpha_- \leq c^{\alpha,\infty}_- \leq c^{\alpha,\infty}_+ \leq c^\alpha_+$. Therefore, even the maps $\psi \mapsto \inf_{\varphi\in\mathcal{G}} |\varphi\psi\varphi^{-1}|^\alpha_{\mathrm{spec}}$ and $\psi \mapsto \inf_{\varphi\in\mathcal{G}} |\varphi\psi\varphi^{-1}|^\alpha_{\mathrm{SH}}$ are unbounded.

The non-degeneracy of $\mathrm{d}^\alpha_{\mathrm{SCH}}$ when $\mathcal{L}$ is orderable was originally due to Hedicke [41, Theorem 5.2], while the non-degeneracy of $\mathrm{d}^\alpha_{\mathrm{SH}}$ was proven by Shelukhin without any condition on $\mathcal{G}$ [68], as mentioned earlier in Section 2.3. The unboundedness of $\mathrm{d}^\alpha_{\mathrm{SH}}$ and $\mathrm{d}^\alpha_{\mathrm{SCH}}$ had already been proven by Hedicke in the special case where (M,ξ) is a unit tangent bundle with open cover and Λ_* is a fiber of the bundle (*cf* Example 2.10.5) [41, Theorems 5.7 and 5.8]. For the open contact manifold $(\mathbb{R}^{2n}\times S^1, \ker\alpha_{\mathrm{st}})$ defined in Example 2.10.3 (here $N := \mathbb{R}^n$ and $\alpha_{\mathrm{st}} := \mathrm{d}z - \lambda$), some geodesics of $\mathrm{d}^{\alpha_{\mathrm{st}}}_{\mathrm{SH}}$ have also been characterized in [7] using, among other things, the spectrality of Sandon contact selectors [64].

Proof. According to Corollaries 4.1 and 4.2, one has $\mathrm{d}^\alpha_{\mathrm{spec}} \leq \mathrm{d}^\alpha_{\mathrm{SCH}}$ on $\mathcal{L}$ or $\widetilde{\mathcal{L}}$ orderable and $\mathrm{d}^\alpha_{\mathrm{spec}} \leq \mathrm{d}^\alpha_{\mathrm{SH}}$ on $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ orderable. The non-degeneracy statements directly follow from their counterpart relative to $\mathrm{d}^\alpha_{\mathrm{spec}}$.

Let us prove that $(\phi^\alpha_t\Lambda)$ is a geodesic for $\Lambda \in \mathcal{L}$ orderable. By the above inequality, one gets

$$\mathrm{d}^\alpha_{\mathrm{SCH}}(\phi^\alpha_t\Lambda, \phi^\alpha_s\Lambda) \geq \mathrm{d}^\alpha_{\mathrm{spec}}(\phi^\alpha_t\Lambda, \phi^\alpha_s\Lambda) = |t-s|, \quad \forall t,s \in \mathbb{R},$$

the equality coming from the normalization property of ℓ^α_+. On the other hand, since ϕ^α_{t-s} can be generated by the α-contact Hamiltonian $H \equiv t-s$, $\mathrm{d}^\alpha_{\mathrm{SCH}}(\phi^\alpha_t\Lambda, \phi^\alpha_s\Lambda) \leq |t-s|$, for any $t,s \in \mathbb{R}$. □

For $n \geq 2$, let us consider the unit tangent bundle of the flat torus $(\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{R}^n$, that can be seen as $\mathbb{T}^n \times \mathbb{S}^{n-1}$, that we endow with the contact form α defined in Example 2.10.5. In this context, Eliashberg and Polterovich [30] used the Shape invariant [70, 28] to construct maps $r_\pm(p,\cdot) : \widetilde{\mathcal{G}} \to \mathbb{R}$, for all $p \in \mathbb{S}^{n-1}$, which are compatible with the order: if $\varphi \preceq \psi$ then $r_\pm(p,\varphi) \leq r_\pm(p,\psi)$. They showed moreover that for Hamiltonian maps H of the form $H(q,p) := f(p)$, where $f : \mathbb{S}^{n-1} \to \mathbb{R}$, one has $r_\pm(p, \phi^H_1) = f(p)$, where $\phi^H_1 \in \widetilde{\mathcal{G}}$ is the lift of the path (ϕ^H_t) generated by H. The proposition was already noticed by the second author [8, §5.2.2] and Nakamura [58, Example 2.9].

Proposition 4.11. *For any* $\varphi \in \widetilde{\mathcal{G}}$ *and* $p \in \mathbb{S}^{n-1}$

$$c^\alpha_-(\varphi) \leq r_\pm(p,\varphi) \leq c^\alpha_+(\varphi).$$

Proof. Let us set $c := c^\alpha_+(\varphi)$. This implies that $\varphi \preceq \phi^\alpha_{c+\varepsilon}$ for any $\varepsilon > 0$ and so $r_\pm(p,\varphi) \leq r_\pm(p, \phi^\alpha_{c+\varepsilon})$. Since $\phi^\alpha_{c+\varepsilon}$ is generated by the constant function equal to $c+\varepsilon$ by the result of Eliashberg and Polterovich discussed above $r_\pm(p,\phi^\alpha_{c+\varepsilon}) = c+\varepsilon$. Letting ε going to 0 we get the inequality $r_\pm(p,\varphi) \leq c^\alpha_+(\varphi)$. The proof for the other inequality follows the same lines. □

Corollary 4.12. *Let $H : \mathbb{T}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ be a Hamiltonian map of the form $H(q,p) := f(p)$ for some smooth $f : \mathbb{S}^{n-1} \to \mathbb{R}$ generating the flow (ϕ_t^H). Then $c_-^\alpha(\phi_1^H) = \min H$ and $c_+^\alpha(\phi_1^H) = \max H$ so that*

$$\gamma^\alpha\left(\phi_1^H\right) = \left|\phi_1^H\right|_{\mathrm{osc}}^\alpha = \operatorname{osc} H.$$

In particular, the gamma pseudo-norms and the Hofer oscillation pseudo-norms are unbounded on $\widetilde{\mathcal{G}}(\mathbb{T}^n \times \mathbb{S}^{n-1}, \ker\alpha)$.

Proof. As $H \leq \max H$, one has $\phi_1^H \preceq \phi_{\max H}^\alpha$ so $c_+^\alpha(\phi_1^H) \leq \max H$. Conversely, Proposition 4.11 implies $c_+^\alpha(\phi_1^H) \geq r_+(p, \phi_1^H) = f(p)$ for all $p \in \mathbb{S}^{n-1}$ so that $c_+^\alpha(\phi_1^H) = \max H$. Similarly, $c_-^\alpha(\phi_1^H) = \min H$ so $\gamma^\alpha(\phi_1^H) = \operatorname{osc} H$. By definition, $\left|\phi_1^H\right|_{\mathrm{osc}}^\alpha \leq \operatorname{osc} H$ so the last equality is a consequence of Corollary 4.2. □

Remark 4.13. For any closed contact manifold $(M, \ker\alpha)$ a Hamiltonian map $H : M \to \mathbb{R}$ that is invariant under the Reeb flow (*i.e.* $H \circ \phi_t^\alpha = H$ for all $t \in \mathbb{R}$) generates a path (ϕ_t^H) of strict contactomorphisms: $(\phi_t^H)^*\alpha = \alpha$ for all $t \in \mathbb{R}$. In this case, for any critical point $p \in M$ of H one can easily check that $\phi_t^H(p) = \phi_{H(p)t}^\alpha(p)$. In particular $\{tH(p) \mid \mathrm{d}H(p) = 0\}$ is contained in $\mathrm{Spec}^\alpha(\phi_t^H)$. Since in $(\mathbb{T}^n \times \mathbb{S}^{n-1}, \ker\alpha)$ the Reeb flow is the geodesic flow, *i.e.* $\phi_t^\alpha(q,p) = (q + tp, p)$, any Hamiltonian map $H : \mathbb{T}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ of the form $H(q,p) = f(p)$ satisfies $H \circ \phi_t^\alpha \equiv H$. Therefore for such Hamiltonian maps, the previous corollary guarantees that $c_\pm^\alpha(\phi_1^H) \in \mathrm{Spec}^\alpha(\phi_1^H)$.

Since $|\cdot|_{\mathrm{spec}}^\alpha$ is not a conjugation invariant norm, a natural question is to ask whether

$$\varphi \mapsto \sup_{\psi \in \mathcal{G}} \left|\psi\varphi\psi^{-1}\right|_{\mathrm{spec}}^\alpha$$

is a well defined conjugation invariant norm, *i.e.* whether it takes finite values or not. This question was already asked by Shelukhin [68, Question 18] for the Hofer-Shelukhin norm $|\cdot|_{\mathrm{SH}}^\alpha$. We do not know the answer to this question, however if we extend the action by conjugation to the whole group of coorientation preserving contactomorphisms that are not necessarily isotopic to the identity we have the following proposition.

Proposition 4.14. *Seeing the time-one map of the Reeb flow ϕ_1^α as an element of $\widetilde{\mathcal{G}}(\mathbb{T}^n \times \mathbb{S}^{n-1}, \ker\alpha)$,*

$$\sup_{\psi \in \mathrm{Cont}} \left|\psi\phi_1^\alpha\psi^{-1}\right|_{\mathrm{spec}}^\alpha = \sup_{\psi \in \mathrm{Cont}} \left|\psi\phi_1^\alpha\psi^{-1}\right|_{\mathrm{SH}}^\alpha = +\infty.$$

Proof. We recall that any diffeomorphism ψ of the torus $\mathbb{T}^n$ can be lifted to a contactomorphism of $\mathbb{T}^n \times \mathbb{S}^{n-1}$

$$\Psi(q,p) := \left(\psi(q), \frac{\mathrm{d}\psi(q)^{-T} \cdot p}{\|\mathrm{d}\psi(q)^{-T} \cdot p\|}\right)$$

whose conformal factor is $g(q,p) := -\ln\left(\left\|\mathrm{d}\psi(q)^{-T} \cdot p\right\|\right)$, where $\mathrm{d}\psi(q)^{-T}$ denotes the adjoint of $\mathrm{d}\psi(q)^{-1}$. In particular, if $\psi \in \mathrm{GL}_n(\mathbb{Z})$ is a linear diffeomorphism of the torus, the conformal factor g of its lift is the function $g(q,p) := -\ln\left(\left\|\psi^{-T} \cdot p\right\|\right)$ depending only on $\mathbb{S}^{n-1}$.

Consider the sequence of linear diffeomorphisms $(\psi_k := \psi^k)_{k\in\mathbb{N}}$ where

$$\psi := \left(\begin{array}{c|c|c} -1 & 2 & 0 \\ \hline 1 & -1 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array}\right) \in \mathrm{GL}_n(\mathbb{Z})$$

and denote by $(\Psi_k)_{k\in\mathbb{N}}$ the lifted sequence of contactomorphisms. We note that for $k \geq 1$ the contactomorphism Ψ_k is not isotopic to the identity since its action on the fundamental group of $\mathbb{T}^n \times \mathbb{S}^{n-1}$ is given by the non trivial action of ψ_k on $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$. We will show that $\lim\limits_{k\to+\infty} c_+^\alpha\left(\Psi_k^{-1}\phi_1^\alpha\Psi_k\right) = +\infty$.

Indeed, $v_0 := (\sqrt{2}, 1, 0, \dots, 0)^T \in \mathbb{R}^n$ is an eigenvector of $\psi_k^{-T} = \left(\begin{array}{c|c|c} 1 & 2 & 0 \\ \hline 1 & 1 & 0 \\ \hline 0 & 0 & I_{n-2} \end{array}\right)^k$ associated to the eigenvalue $(1+\sqrt{2})^k$ for all $k \in \mathbb{N}$. Moreover the Hamiltonian map of the path $\left(\Psi_k^{-1}\phi_\alpha^t\Psi_k\right)$ is given by $(q,p) \mapsto e^{-g_k(q,p)} = \|\psi_k^{-T}\cdot p\|$ depending only on p. By the result of Eliashberg and Polterovich discussed above we deduce that $r_\pm\left(p_0, \Psi_k^{-1}\phi_1^\alpha\Psi_k\right) = (1+\sqrt{2})^k$ where $p_0 := \frac{v_0}{\|v_0\|}$. Therefore

$$\left|\Psi_k^{-1}\phi_1^\alpha\Psi_k\right|_{\mathrm{spec}}^\alpha = c_+^\alpha(\Psi_k^{-1}\phi_1^\alpha\Psi_k) \geq (1+\sqrt{2})^k$$

where the equality comes from the relation $\mathrm{id} \preceq \Psi_k^{-1}\phi_1^\alpha\Psi_k$ and the inequality from Proposition 4.11. Letting k go to infinity implies the result for both norms since $|\cdot|_{\mathrm{SH}}^\alpha \geq |\cdot|_{\mathrm{spec}}^\alpha$. □

Remarks 4.15.

1. Eliashberg and Polterovich showed that the maps $r_\pm(p,\cdot)$ are invariant under the action by conjugation of $\mathcal{G}$ on $\widetilde{\mathcal{G}}$ for all $p \in \mathbb{S}^{n-1}$. Together with Corollary 4.12 it implies in particular that the conjugation invariant map $\gamma_\infty^\alpha : \widetilde{\mathcal{G}} \to \mathbb{R}$ defined by $\gamma_\infty^\alpha(\phi) := \inf\limits_{\psi\in\mathcal{G}} \gamma^\alpha(\psi^{-1}\phi\psi)$ is also unbounded. However as shown in the previous proof the maps $r_\pm(p,\cdot)$ are not anymore invariant under the action of Cont by conjugation.
2. It is interesting to note that if $(M, \ker\alpha)$ is a closed contact manifold such that the Reeb flow of α is 1-periodic and $\widetilde{\mathcal{G}}(M)$ is orderable then

$$\sup_{\psi\in\mathrm{Cont}} \left|\psi\varphi\psi^{-1}\right|_{\mathrm{spec}}^\alpha \leq |\varphi|_{\mathrm{FPR}} < +\infty, \quad \forall\varphi \in \widetilde{\mathcal{G}},$$

   where the definition of $|\cdot|_{\mathrm{FPR}}$ is given in Section 4.5. It is not known whether there exists or not a closed contact manifold (M,ξ) such that $\mathcal{G}(M,\xi)$ is orderable and $\widetilde{\mathcal{G}}(M,\xi)$ is not bounded in the sense of [14], *i.e.* so that there exists an unbounded conjugation invariant norm on $\widetilde{\mathcal{G}}(M,\xi)$.

4.4. **Colin-Sandon discriminant and oscillation metrics.** Let Λ_* be a closed Legendrian submanifold of a cooriented contact manifold (M,ξ) (not necessarily closed) and let $\mathcal{L} := \mathcal{L}(\Lambda_*)$ (resp. $\widetilde{\mathcal{L}} := \widetilde{\mathcal{L}}(\Lambda_*)$). In this specific section, given $\Lambda_0, \Lambda_1 \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$), by a path $\gamma : \Lambda_0 \rightsquigarrow \Lambda_1$ we will mean a continuous map $\gamma : [0,1] \to \mathcal{L}$ (resp. $[0,1] \to \widetilde{\mathcal{L}}$) such that $\gamma(i) = \Lambda_i$ for $i \in \{0,1\}$. The concatenation $\gamma_1 \cdot \gamma_2$ of two paths $\gamma_1 : \Lambda_0 \rightsquigarrow \Lambda_1$ and $\gamma_2 : \Lambda_1 \rightsquigarrow \Lambda_2$ in $\mathcal{L}$ (resp. in $\widetilde{\mathcal{L}}$) is by definition the path $\Lambda_0 \rightsquigarrow \Lambda_2$,

$$\gamma_1 \cdot \gamma_2 : t \mapsto \begin{cases} \gamma_1(2t) & t \in [0, 1/2], \\ \gamma_2(2t-1) & t \in [1/2, 1]. \end{cases}$$

The reverse path $\bar{\gamma}$ of a path $\gamma : \Lambda_0 \rightsquigarrow \Lambda_1$ is the path $\Lambda_1 \rightsquigarrow \Lambda_0$, $t \mapsto \gamma(1-t)$. By definition of the topology defined on $\mathcal{L}$, for any path $\gamma : [0,1] \to \mathcal{L}$, there exists a continuous map $j_\gamma : [0,1] \times \Lambda_* \to M$ such that for every $t \in [0,1]$, $j_\gamma(t, \cdot)$ is a diffeomorphism between Λ_* and $\gamma(t)$. A path γ in $\mathcal{L}$ will be called *embedded* if there exists a continuous map j_γ as above that is a smooth embedding $[0,1] \times \Lambda_* \hookrightarrow M$. A path γ in $\widetilde{\mathcal{L}}$ will be called embedded if $\Pi\gamma : t \mapsto \Pi(\gamma(t))$ is an embedded path.

In [26, Section 8], Colin and Sandon defined the discriminant length of a path γ in $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) as the integral number

$$\ell_{\mathrm{disc}}(\gamma) := \min \left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exist embedded paths } \gamma_1, \ldots, \gamma_n \text{ such that} \\ \gamma_1 \cdots \gamma_n \text{ and } \gamma \text{ are in the same homotopy class with} \\ \text{fixed endpoints} \end{array} \right\},$$

with convention $\ell_{\mathrm{disc}}(\gamma) = 0$ if γ is a constant map. Colin-Sandon proved that ℓ_{disc} takes values in $\mathbb{N}$ (with convention $0 \in \mathbb{N}$) and that it induces an integral metric on $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) called the discriminant metric and defined by

$$\mathrm{d}_{\mathrm{disc}}(\Lambda_0, \Lambda_1) = \min_{\gamma : \Lambda_0 \rightsquigarrow \Lambda_1} \ell_{\mathrm{disc}}(\gamma), \quad \forall \Lambda_0, \Lambda_1 \in \mathcal{L} \text{ (resp. } \widetilde{\mathcal{L}}). \tag{16}$$

Moreover, this metric is invariant under the action by contactomorphisms of (M, ξ) since embedded paths are preserved by this action. Let us remark that in the case of the universal cover $\widetilde{\mathcal{L}}$, there is a unique homotopy class of paths $\Lambda_0 \rightsquigarrow \Lambda_1$ so that one can erase the minimum from Equation (16).

In addition to the discriminant length, Colin-Sandon defined an oscillation norm (to be distinguished from the Hofer oscillation norms). In order to properly define it, let us say that a path γ of $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is monotone if it is either a positive or a negative isotopy. Then, for a path γ, one defines the integral number $\nu^+_{\mathrm{osc}}(\gamma)$ by

$$\nu^+_{\mathrm{osc}}(\gamma) := \min \left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exist embedded monotone paths } \gamma_1, \ldots, \gamma_n,\, k \\ \text{of which are positive, such that } \gamma_1 \cdots \gamma_n \text{ and } \gamma \text{ are} \\ \text{in the same homotopy class with fixed endpoints} \end{array} \right\},$$

with convention $\nu^+_{\mathrm{osc}}(\gamma) = 0$ if γ is constant and defines $\nu^-_{\mathrm{osc}}(\gamma) := -\nu^+_{\mathrm{osc}}(\bar{\gamma})$. Colin-Sandon proved that these numbers are finite for all paths and they defined the oscillation norm of a path γ by

$$\nu_{\mathrm{osc}}(\gamma) := \nu^+_{\mathrm{osc}}(\gamma) - \nu^-_{\mathrm{osc}}(\gamma) \in \mathbb{N}.$$

Colin-Sandon proved that the induced distance $\mathrm{d}_{\mathrm{CS,osc}}$ on $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is non-degenerate if and only if $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable. It is also a metric invariant under the action by the contactomorphisms. We are primarily interested in the case of the universal cover $\widetilde{\mathcal{L}}$ for which $\mathrm{d}_{\mathrm{CS,osc}}(\Lambda_0, \Lambda_1)$ is simply defined as $\nu_{\mathrm{osc}}(\gamma)$ for any $\gamma : \Lambda_0 \rightsquigarrow \Lambda_1$. In the case of $\widetilde{\mathcal{L}}$, one can thus write $\mathrm{d}_{\mathrm{CS,osc}}$ as the difference $\mathrm{d}^+_{\mathrm{CS,osc}} - \mathrm{d}^-_{\mathrm{CS,osc}}$ with

$$\mathrm{d}^\pm_{\mathrm{CS,osc}}(\Lambda_0, \Lambda_1) := \nu^\pm_{\mathrm{osc}}(\gamma), \quad \forall \gamma : \Lambda_0 \rightsquigarrow \Lambda_1.$$

In [26], the unboundedness of these two distances was only proven in the case of $\widetilde{\mathcal{L}}(\mathbb{RP}^n)$ (see Example 2.10.1), some 1-dimensional Legendrian knots, and $\widetilde{\mathcal{L}}(p(0_N))$ defined in Example 2.10.3.

Theorem 4.16. *Let us assume there exists a contact form α supporting ξ the Reeb flow of which is T-periodic for some $T > 0$. If $\widetilde{\mathcal{L}}$ is orderable, then*

$$\begin{cases} \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda_0, \Lambda_1) < T\mathrm{d}_{\mathrm{disc}}(\Lambda_0, \Lambda_1), \\ \ell^\alpha_+(\Lambda_1, \Lambda_0) < T\mathrm{d}^+_{\mathrm{CS,osc}}(\Lambda_0, \Lambda_1), \end{cases} \qquad \forall \Lambda_0, \Lambda_1 \in \widetilde{\mathcal{L}} \textit{ with } \Lambda_0 \neq \Lambda_1.$$

In particular, the discriminant metric and the Colin-Sandon oscillation metric are unbounded.

Let us point out that the second inequality of Theorem 4.16 implies

$$\mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda_0, \Lambda_1) < T\mathrm{d}_{\mathrm{CS,osc}}(\Lambda_0, \Lambda_1), \quad \forall \Lambda_0, \Lambda_1 \in \widetilde{\mathcal{L}} \text{ with } \Lambda_0 \neq \Lambda_1, \tag{17}$$

as $\mathrm{d}^+_{\mathrm{CS,osc}}(\Lambda_1, \Lambda_0)$ equals $-\mathrm{d}^-_{\mathrm{CS,osc}}(\Lambda_0, \Lambda_1)$ and is always non-negative.

Proof. Let us first prove the inequality involving the discriminant metric. Let $\Lambda_0 \neq \Lambda$ be elements of $\widetilde{\mathcal{L}}$ and let $n := \mathrm{d}_{\mathrm{disc}}(\Lambda_0, \Lambda) \in \mathbb{N}^*$. By symmetry of the role of Λ_0 and Λ, it is enough to prove $\ell^\alpha_+(\Lambda, \Lambda_0) < nT$.

By definition, there exists embedded paths $\gamma_i : \Lambda_{i-1} \rightsquigarrow \Lambda_i$, $1 \leq i \leq n$, with $\Lambda_n = \Lambda$. Let us consider the maps $f_i : t \mapsto \ell^\alpha_+(\gamma_i(t), \gamma_i(0))$, $1 \leq i \leq n$. By the normalization property of ℓ^α_+, $f_i(0) = 0$ for all i. By continuity of ℓ^α_+, the f_i's are continuous maps. By spectrality of ℓ^α_+, if $f_i(t) = T$ for some $t \in (0, 1]$ and $1 \leq i \leq n$, it would imply that $\Pi(\gamma_i(t))$ intersects $\phi^\alpha_T \Pi(\gamma_i(0))$. But $\phi^\alpha_T = \mathrm{id}$ by assumption so γ_i would not be an embedded path. Therefore, f_i does not take the value T. By continuity, it implies that the f_i's take their values in $(-T, T)$. Now, by the triangular inequality,

$$\ell^\alpha_+(\Lambda, \Lambda_0) \leq f_1(1) + f_2(1) + \cdots + f_n(1) < nT,$$

the conclusion follows.

For the second inequality, one can apply the same strategy. Let $k := \mathrm{d}^+_{\mathrm{CS,osc}}(\Lambda_0, \Lambda)$. By definition, there exists embedded monotone paths $\gamma_i : \Lambda_{i-1} \rightsquigarrow \Lambda_i$, $1 \leq i \leq n$ for some $n \in \mathbb{N}$ such that k of them are positive, $n - k$ of them are negative and $\Lambda_n = \Lambda$. By the triangular inequality,

$$\ell^\alpha_+(\Lambda, \Lambda_0) \leq \ell^\alpha_+(\Lambda_n, \Lambda_{n-1}) + \cdots + \ell^\alpha_+(\Lambda_1, \Lambda_0). \tag{18}$$

By the previous discussion, since γ_i is embedded, $\ell^\alpha_+(\Lambda_i, \Lambda_{i-1}) < T$, $1 \leq i \leq n$. Moreover, if γ_i is negative, the monotonicity property implies that $\ell^\alpha_+(\Lambda_i, \Lambda_{i-1}) < 0$. As exactly $n-k$ of the γ_i's are negative, the inequality (18) implies $\ell^\alpha_+(\Lambda, \Lambda_0) < kT$.

The unboundedness of both metrics then follows from the unboundedness of the spectral metric (see also Equation (17)): we recall that $\mathrm{d}^\alpha_{\mathrm{spec}}(\phi^\alpha_t \Lambda, \Lambda) = |t|$ for all $t \in \mathbb{R}$ and $\Lambda \in \widetilde{\mathcal{L}}$. □

Corollary 4.17. *Let us assume that there exists a contact form α supporting ξ the Reeb flow of which is T-periodic for some $T > 0$. If $\Lambda \in \widetilde{\mathcal{L}}$ orderable is such that $\phi^\alpha_t \Pi\Lambda \cap \Pi\Lambda = \emptyset$ for all $t \notin T\mathbb{Z}$, then the isotopy $(\phi^\alpha_{tT}\Lambda)_{t\in\mathbb{R}}$ defines a geodesic of both the discriminant and the Colin-Sandon oscillation metrics, in the sense that*

$$d(\phi^\alpha_{tT}\Lambda, \phi^\alpha_{sT}\Lambda) = \big\lceil |t - s| \big\rceil, \quad \forall t, s \in \mathbb{R} \textit{ with } t - s \notin \mathbb{Z}^*,$$

for $d = \mathrm{d}_{\mathrm{disc}}$ or $\mathrm{d}_{\mathrm{CS,osc}}$ (when $t - s \in \mathbb{Z}^*$, the distance is $\lceil |t - s| \rceil + 1$).

For $\widetilde{\mathcal{G}}(\mathbb{R}^{2n}\times S^1,\xi_{\mathrm{st}})$, there is a characterization of some geodesics of the discriminant and oscillation norms of Colin-Sandon in [7].

Proof. Concerning the case $t-s\notin\mathbb{Z}^*$, by invariance of both metrics under contactomorphisms, it is enough to prove that $d(\phi^\alpha_{tT}\Lambda,\Lambda)=\lceil t\rceil$ for every $t\in\mathbb{R}_+\setminus\mathbb{N}$. Let us fix such a t and consider the case $d=\mathrm{d}_{\mathrm{disc}}$. Then Theorem 4.16 implies that this distance is at least $\lceil t\rceil$. The assumption on Λ implies that paths $s\mapsto\phi^\alpha_{t_0+s(T-\varepsilon)}\Lambda$, $s\in[0,1]$, are embedded for every $\varepsilon\in(0,T)$ and $t_0\in\mathbb{R}$. Let $\varepsilon:=T\left(1-\frac{t}{\lceil t\rceil}\right)\in(0,T)$. The path $s\mapsto\phi^\alpha_{stT}\Lambda$ is homotopic (with fixed endpoints) to the concatenation $\gamma_1\cdots\gamma_{\lceil t\rceil}$ of the paths

$$\gamma_i : s\mapsto\phi^\alpha_{(s+i-1)(T-\varepsilon)}\Lambda,\quad \forall i\in\{1,\dots,\lceil t\rceil\},$$

which are embedded paths, as was just remarked. By definition of the discriminant metric, it implies the reverse inequality: $\mathrm{d}_{\mathrm{disc}}(\phi^\alpha_{tT}\Lambda,\Lambda)\leq\lceil t\rceil$.

The argument to prove $\mathrm{d}_{\mathrm{CS,osc}}(\phi^\alpha_{tT}\Lambda,\Lambda)=\lceil t\rceil$ for $t\in\mathbb{R}_+\setminus\mathbb{N}$ is the same once remarked that $\mathrm{d}_{\mathrm{CS,osc}}(\phi^\alpha_{tT}\Lambda,\Lambda)=\mathrm{d}^+_{\mathrm{CS,osc}}(\phi^\alpha_{tT}\Lambda,\Lambda)$ in order to apply Theorem 4.16 (since the path $s\mapsto\phi^\alpha_{stT}\Lambda$ is positive).

Concerning the case $t-s\in\mathbb{Z}^*$, Theorem 4.16 implies the optimal lower bound since both metrics take integral values, while a similar decomposition of the isotopy gives the reverse inequality. ☐

Examples 4.18. The space $\widetilde{\mathcal{L}}(\mathbb{R}\mathrm{P}^n)$ described at Example 2.10.1 is a space for which Corollary 4.17 applies by taking $\Lambda:=\mathbb{R}\mathrm{P}^n$ and the contact form of $\mathbb{R}\mathrm{P}^{2n+1}$ induced by the Liouville form $\frac{1}{2}(y\mathrm{d}x-x\mathrm{d}y)$ on $\mathbb{R}^{2(n+1)}$. It extends to lens spaces as well.

When N is a closed manifold, $\widetilde{\mathcal{L}}(p(0_N))$ described at Example 2.10.3 also applies by taking $\Lambda:=p(0_N)$ and the standard contact form $\alpha=\mathrm{d}z-\lambda$.

Let us give another family of examples of such a situation. Consider the unit tangent bundle $M:=SN$ of a Riemannian manifold N all of whose geodesics are closed embedded curves (*e.g* $\mathbb{S}^n$, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, $\mathbb{C}\mathrm{aP}^2$ for the metric induced by the round metric on the sphere). For the standard choice of contact form on M, the Reeb flow corresponds to the geodesic flow and $\widetilde{\mathcal{L}}(S_xN)$ is orderable given any $x\in N$ (*cf.* Example 2.10.5). By the geometric assumption on the geodesics of N, $\Lambda:=S_xN$ satisfies the hypothesis of Corollary 4.17 for any $x\in N$.

4.5. **Equivalence of the Colin-Sandon oscillation metric and the Fraser-Polterovich-Rosen metric.** Let us study the natural generalization of the norm introduced by Fraser-Polterovich-Rosen [35] in the context of Legendrian isotopy classes. Let (M,ξ) be a contact manifold endowed with a contact form α the Reeb flow of which is 1-periodic. If $\widetilde{\mathcal{G}}$ is orderable, the Fraser-Polterovich-Rosen norm of $\varphi\in\widetilde{\mathcal{G}}$ is defined as

$$|\psi|_{\mathrm{FPR}}:=\min\left\{k\in\mathbb{N}\mid\phi^\alpha_{-k}\preceq\psi\preceq\phi^\alpha_k\right\}.$$

If $\widetilde{\mathcal{L}}$ is orderable for some closed Legendrian submanifold of M, one then naturally generalizes the Fraser-Polterovich-Rosen norm as the distance defined by

$$\mathrm{d}_{\mathrm{FPR}}(\Lambda,\Lambda'):=\min\left\{k\in\mathbb{N}\mid\phi^\alpha_{-k}\Lambda'\preceq\Lambda\preceq\phi^\alpha_k\Lambda'\right\},\quad\forall\Lambda,\Lambda'\in\widetilde{\mathcal{L}}.$$

This is indeed a non-degenerate distance by definition of orderability. Moreover, this distance is invariant under the action of the universal cover of $\mathrm{Cont}_0(M,\xi)$, as $\phi^\alpha_k\in\widetilde{\mathrm{Cont}_0}(M,\xi)$ belongs to the center of this group for every $k\in\mathbb{Z}$. This metric also endows the partially ordered space $(\widetilde{\mathcal{L}}, \preceq)$ with the structure of a partially ordered metric space (see above Proposition 4.3).

In the original setting of $\widetilde{\mathcal{G}}$, the domination of $\mathrm{d}_{\mathrm{FPR}}$ over the oscillation metric of Colin-Sandon has been remarked by Fraser-Polterovich-Rosen [35, Remark 3.8]. The close link between $\mathrm{d}_{\mathrm{FPR}}$ and the spectral invariant $\ell^\alpha_\pm$ allows us to prove even more in the Legendrian case.

Theorem 4.19. *Let (M,ξ) be a contact manifold endowed with a contact form α the Reeb flow of which is 1-periodic and let us assume that $\widetilde{\mathcal{L}}(\Lambda_*)$ is orderable for some closed Legendrian submanifold $\Lambda_* \subset M$. Then the associated Colin-Sandon oscillation metric and Fraser-Polterovich-Rosen metric satisfy*

$$\mathrm{d}_{\mathrm{FPR}} \leq \mathrm{d}_{\mathrm{CS,osc}} + 1 \leq 3A\mathrm{d}_{\mathrm{FPR}} + 1,$$

where $A := \mathrm{d}_{\mathrm{CS,osc}}(\Lambda_0, \phi^\alpha_1\Lambda_0)$ for some $\Lambda_0 \in \widetilde{\mathcal{L}}(\Lambda_*)$. In particular, the two metrics are equivalent.

Proof. The spectral distance satisfies

$$\mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda,\Lambda') = \inf\left\{t \in [0,+\infty) \mid \phi^\alpha_{-t}\Lambda' \preceq \Lambda \preceq \phi^\alpha_t\Lambda'\right\}, \quad \forall \Lambda,\Lambda' \in \widetilde{\mathcal{L}},$$

so

$$\left\lceil \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda,\Lambda')\right\rceil \leq \mathrm{d}_{\mathrm{FPR}}(\Lambda,\Lambda') \leq \left\lfloor \mathrm{d}^\alpha_{\mathrm{spec}}(\Lambda,\Lambda')\right\rfloor + 1, \quad \forall \Lambda,\Lambda' \in \widetilde{\mathcal{L}}.$$

In particular, when $\mathrm{d}^\alpha_{\mathrm{spec}}$ is not an integer, $\mathrm{d}_{\mathrm{FPR}} = \lceil \mathrm{d}^\alpha_{\mathrm{spec}}\rceil$. The inequality $\mathrm{d}_{\mathrm{FPR}} \leq \mathrm{d}_{\mathrm{CS,osc}} + 1$ is then a consequence of Theorem 4.16 (see also inequality (17)).

The domination of $\mathrm{d}_{\mathrm{CS,osc}}$ by $\mathrm{d}_{\mathrm{FPR}}$ had already been remarked by Fraser-Polterovich-Rosen [35, Remark 3.8]. It is a consequence of the fact that $\mathrm{d}_{\mathrm{CS,osc}}$ induces a partially ordered metric space on $(\widetilde{\mathcal{L}}, \preceq)$ (see [26, Proposition 3.4]). Let us fix $A := \mathrm{d}_{\mathrm{CS,osc}}(\Lambda_0, \phi^\alpha_1\Lambda_0)$ for some $\Lambda_0 \in \widetilde{\mathcal{L}}$ and let us show that

$$\mathrm{d}_{\mathrm{CS,osc}}(\Lambda, \phi^\alpha_k\Lambda) \leq A|k|, \quad \forall k \in \mathbb{Z}, \forall \Lambda \in \widetilde{\mathcal{L}}. \tag{19}$$

First, the left-hand side of the inequality does not depend on the choice of Λ: let $g \in \widetilde{\mathcal{G}}$ such that $g\Lambda = \Lambda_0$, then

$$\mathrm{d}_{\mathrm{CS,osc}}(\Lambda, \phi^\alpha_k\Lambda) = \mathrm{d}_{\mathrm{CS,osc}}(g\Lambda, g\phi^\alpha_k g^{-1}g\Lambda) = \mathrm{d}_{\mathrm{CS,osc}}(\Lambda_0, \phi^\alpha_k\Lambda_0),$$

where we have used the $\widetilde{\mathcal{G}}$-invariance of $\mathrm{d}_{\mathrm{CS,osc}}$ and the fact that ϕ^α_k commutes with $g^{\pm 1}$. The inequality (19) now follows from the triangular inequality associated with the invariance of the distance under the action of the Reeb flow. Now, let $\Lambda, \Lambda' \in \widetilde{\mathcal{L}}$ and let $k := \mathrm{d}_{\mathrm{FPR}}(\Lambda,\Lambda')$. By definition, $\phi^\alpha_{-k}\Lambda' \preceq \Lambda \preceq \phi^\alpha_k\Lambda'$ so $\mathrm{d}_{\mathrm{CS,osc}}(\Lambda, \phi^\alpha_{-k}\Lambda') \leq \mathrm{d}_{\mathrm{CS,osc}}(\phi^\alpha_{-k}\Lambda', \phi^\alpha_k\Lambda')$ by compatibility of $\mathrm{d}_{\mathrm{CS,osc}}$ with the partial order $\preceq$. Therefore,

$$\begin{aligned}\mathrm{d}_{\mathrm{CS,osc}}(\Lambda,\Lambda') &\leq \mathrm{d}_{\mathrm{CS,osc}}(\Lambda, \phi^\alpha_{-k}\Lambda') + \mathrm{d}_{\mathrm{CS,osc}}(\phi^\alpha_{-k}\Lambda', \Lambda') \\ &\leq \mathrm{d}_{\mathrm{CS,osc}}(\phi^\alpha_{-k}\Lambda', \phi^\alpha_k\Lambda') + Ak \\ &\leq 3Ak,\end{aligned}$$

which brings the conclusion as $k = \mathrm{d}_{\mathrm{FPR}}(\Lambda,\Lambda')$. □

Remark 4.20. With slight adaptations, one can loosen the hypothesis in both Theorems 4.16 and 4.19 by asking for the existence of a 1-periodic positive contact isotopy (ϕ_t) with $\phi_0 = \mathrm{id}$ instead of a 1-periodic Reeb flow. One should then replace the use of ℓ^α_+ with the use of

$$\ell^\phi_+(\Lambda_1,\Lambda_0) := \ell_+(\Lambda_1, (\phi_t\Lambda_0)_{t\in\mathbb{R}}), \quad \forall \Lambda_0,\Lambda_1 \in \widetilde{\mathcal{L}},$$

and make the necessary changes in the definition of $\mathrm{d}^\alpha_{\mathrm{spec}}$ as well as $\mathrm{d}_{\mathrm{FPR}}$. As (ϕ_t) is not an autonomous flow, the triangular inequality is not accessible anymore but one still has

$$\lceil \ell_+^\phi(\Lambda_2,\Lambda_0)\rceil \leq \lceil \ell_+^\phi(\Lambda_2,\Lambda_1)\rceil + \lceil \ell_+^\phi(\Lambda_1,\Lambda_0)\rceil.$$

In this case, the inequalities stated in Theorem 4.16 are not open anymore but it still allows to show the unboundedness of the metrics and Theorem 4.19. Moreover, for any $\Lambda \in \widetilde{\mathcal{L}}$ such that $\phi_t\Pi\Lambda \cap \Pi\Lambda = \emptyset$ for all $t \in \mathbb{R}\setminus\mathbb{Z}$, $(\phi_t\Lambda)$ is a geodesic in the sense of Corollary 4.17.

5. Lorentzian geometry and time functions

When $(M, \ker\alpha)$ is a closed cooriented contact manifold, the relations $\preceq$ and $\lll$ on $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) have the property to come from a closed proper cone structure, *i.e.* a distribution of closed sharp convex cones with non empty interiors [34, 55, 40]. Indeed, the Lie algebra $\mathfrak{g}$ of the infinite dimensional Lie group $\mathcal{G}$ is the space of contact vector fields. The subset $\mathfrak{g}_{\geq 0}$ of contact vector fields $X \in \mathfrak{g}\setminus\{0\}$ satisfying $\alpha(X) \geq 0$ is a closed proper cone whose interior $\mathfrak{g}_{>0}$ consists of vector fields for which the previous inequality is open. The closed proper cone structure used to define $\preceq$ and $\lll$ is then given by right translating these cones of the Lie algebra to the whole tangent space. Note that left translations would give rise to the same closed proper cone structure since $\mathfrak{g}_{\geq 0}$ is invariant under the adjoint action of $\mathcal{G}$ (which is the push forward). This point of view recently led Abbondandolo-Benedetti-Polterovich [1] and Hedicke [41] to introduce objects of Lorentzian geometry to study $(\mathcal{G}, \preceq, \lll)$ such as Lorentz-Finsler structure or Lorentzian distances. An open question in [1] was about the existence or not of a time function on $(\widetilde{\mathcal{G}}(\mathbb{R}\mathrm{P}^{2n-1}), \preceq)$, *i.e.* a function τ such that $x \preceq y$ and $x \neq y$ implies $\tau(x) < \tau(y)$. In this section, we give a positive answer to this question generalized to all orderable $\mathcal{G}, \widetilde{\mathcal{G}}, \mathcal{L}$ and $\widetilde{\mathcal{L}}$. We show moreover that time functions cannot be invariant.

Remarks 5.1.

1. In Lorentzian geometry, the existence of a time function is equivalent to stable causality. The notion of stable causality in this context is strictly stronger than the notions of causality and of strong causality [56, 55].
2. From a different perspective, Chernov-Nemirovski imported Lorentzian geometric notions to the study of some Legendrian isotopy classes [20, 23, 22]. The starting point of their study is that the space of null future pointing unparametrized geodesics of a globally hyperbolic Lorentzian manifold carries a canonical contact structure.
3. A consequence of a recent paper of Buhovsky-Stokić [13] is that the Lie algebra of the group of Hamiltonian symplectomorphisms of any closed symplectic manifold has no non-trivial invariant convex cone. It would be interesting to know if the only non-trivial invariant convex cones of $\mathfrak{g}$ are $\pm\mathfrak{g}_{\geq 0}$ and $\pm\mathfrak{g}_{>0}$ for any closed cooriented contact manifold (M, ξ).

Let us fix a contact form α supporting ξ. According to Lemma 2.5, $\widetilde{\mathcal{G}}$ is separable (and so is $\mathcal{G}$). Then, let us fix $(\psi_n)_{n\geq 1}$ a dense sequence on $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) and consider

$$\tau^\alpha(\varphi) := a\sum_n \frac{c_+^\alpha(\varphi\psi_n)}{2^n \max(1, |c_\pm^\alpha(\psi_n)|)} + b, \quad \forall \varphi \in \mathcal{G} \text{ (resp. } \widetilde{\mathcal{G}}),$$

where $a, b \in \mathbb{R}$ are normalization factors defined by the relations (assuming the series does converge)

$$a = \left[\sum_n \frac{1}{2^n \max(1, |c_\pm^\alpha(\psi_n)|)}\right]^{-1} \quad \text{and } \tau^\alpha(\mathrm{id}) = 0.$$

Theorem 5.2. *τ^α is a well-defined $\mathrm{d}^\alpha_{\mathrm{spec}}$-1-Lipschitz (so $\mathrm{d}^\alpha_{\mathrm{SH}}$-1-Lipschitz and C^1-continuous) map $\mathcal{G} \to \mathbb{R}$ (resp. $\widetilde{\mathcal{G}} \to \mathbb{R}$) satisfying*

1. $\tau^\alpha(\mathrm{id}) = 0$,
2. $\tau^\alpha(\phi^\alpha_t \psi) = \tau^\alpha(\psi) + t$ for all $t \in \mathbb{R}$ and $\psi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$),
3. $\varphi \preceq \psi$ with $\varphi \neq \psi$ implies $\tau^\alpha(\varphi) < \tau^\alpha(\psi)$.

Proof. In order to simplify the notation let $c := c^\alpha_+$. According to the triangular inequality,

$$c(\varphi) - c(\psi^{-1}) \leq c(\varphi\psi) \leq c(\varphi) + c(\psi),$$

where the left-hand side inequality comes from the decomposition $\varphi = (\varphi\psi)\psi^{-1}$. Therefore,

$$\Big||c(\varphi\psi)| - |c(\varphi)|\Big| \leq \max\Big(|c(\psi)|, |c(\psi^{-1})|\Big).$$

Applying this inequality with $\psi = \psi_n$, one gets that the sum defining τ^α is absolutely convergent. Since c is $\mathrm{d}^\alpha_{\mathrm{spec}}$-1-Lipschitz (Equation (15)), and $\mathrm{d}^\alpha_{\mathrm{spec}}$ is right-invariant, one gets

$$|\tau^\alpha(\varphi) - \tau^\alpha(\psi)| \leq a \sum_n \frac{\mathrm{d}_{\mathrm{spec}}(\varphi\psi_n, \psi\psi_n)}{2^n \max(1, |c_\pm^\alpha(\psi_n)|)} \leq \mathrm{d}_{\mathrm{spec}}(\varphi, \psi),$$

so τ^α is $\mathrm{d}_{\mathrm{spec}}$-1-Lipschitz which implies that it is Hofer-1-Lipschitz as well as C^1-continuous according to Corollary 4.2.

Property 1 is true by construction while property 2 is a direct consequence of the normalization property of c.

Finally, suppose $\varphi \preceq \psi$ with $\varphi \neq \psi$, then by Corollary 3.13, $c(\psi\varphi^{-1}) > 2\varepsilon$ for some $\varepsilon > 0$ while $c(\varphi\varphi^{-1}) = 0$. By C^1-density of (ψ_n) and C^1-continuity of c (Corollary 3.10), $c(\varphi \circ \psi_k) < \varepsilon < c(\psi \circ \psi_k)$ for some k such that ψ_k is close to φ^{-1}. Since c is non-decreasing, $c(\varphi \circ \psi_n) \leq c(\psi \circ \psi_n)$ for all n and property 3 follows. □

One defines time functions on $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) similarly. Keeping the previously fixed dense sequence (ψ_n) of $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) and fixing some $\Lambda_0 \in \mathcal{L}$ (resp. in $\widetilde{\mathcal{L}}$), let us consider

$$\tau^\alpha_{\Lambda_0}(\Lambda) := a \sum_n \frac{\ell^\alpha_+(\Lambda, \psi_n\Lambda_0)}{2^n \max(1, |c_\pm^\alpha(\psi_n)|)} + b, \quad \forall \Lambda \in \mathcal{L} \text{ (resp. } \widetilde{\mathcal{L}}),$$

where $a \in \mathbb{R}$ is defined as above and $b \in \mathbb{R}$ is such that $\tau^\alpha_{\Lambda_0}(\Lambda_0) = 0$. One proves similarly the Legendrian counterpart of Theorem 5.2

Theorem 5.3. *$\tau^\alpha_{\Lambda_0}$ is a well-defined $\mathrm{d}^\alpha_{\mathrm{spec}}$-1-Lipschitz (so $\mathrm{d}^\alpha_{\mathrm{SCH}}$-1-Lipschitz and C^1-continuous) map $\mathcal{L} \to \mathbb{R}$ (resp. $\widetilde{\mathcal{L}} \to \mathbb{R}$) satisfying*

1. $\tau^\alpha_{\Lambda_0}(\Lambda_0) = 0$,
2. $\tau^\alpha_{\Lambda_0}(\phi^\alpha_t \Lambda) = \tau^\alpha_{\Lambda_0}(\Lambda) + t$ for all $t \in \mathbb{R}$ and $\Lambda \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$),
3. $\Lambda \preceq \Lambda'$ with $\Lambda \neq \Lambda'$ implies $\tau^\alpha_{\Lambda_0}(\Lambda) < \tau^\alpha_{\Lambda_0}(\Lambda')$.

Since the binary relations considered on $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are invariant by the left action of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ respectively, and the ones considered on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are invariant by the action by conjugation, it is natural to ask if there exists a time function on these spaces that can be invariant with respect to these actions. We show that it is not possible.

Theorem 5.4 (Non existence of invariant time function)**.**

1. *Let $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) such that $\mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) is orderable and τ a time function on $(\mathcal{L}, \preceq)$ (resp. $(\widetilde{\mathcal{L}}, \preceq)$). Then there exist $\Lambda, \Lambda' \in \mathcal{L}$ (resp. $\widetilde{\mathcal{L}}$) and $g \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) such that the time difference between Λ and Λ' is not the same as the time difference between $g\Lambda$ and $g\Lambda'$,* i.e. $\tau(g\Lambda) - \tau(g\Lambda') \neq \tau(\Lambda) - \tau(\Lambda')$.
2. *Let O be either $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ such that O is orderable. Let τ be a time function. Then there exist $\varphi, g \in O$ such that $\tau(g^{-1}\varphi g) > \tau(\varphi)$.*

Proof of Theorem 5.4 part 1. Let us prove it for the case $\widetilde{\mathcal{L}}$. For any $\Lambda_0 \in \widetilde{\mathcal{L}}$, by applying a Reeb flow for a small timespan, one obtains $\Lambda_1 \in \widetilde{\mathcal{L}}$ such that $\Pi\Lambda_1 \cap \Pi\Lambda_0 = \emptyset$ and $\Lambda_0 \preceq \Lambda_1$. We now consider a point $p \in \Pi\Lambda_0$ and a neighborhood U of p such that U does not intersect $\Pi\Lambda_1$. Let $h : M \to [0, +\infty)$ be a Hamiltonian map compactly supported in U such that $h(p) > 0$. The induced contact flow (g_t) satisfies $\Lambda_0 \preceq g_t\Lambda_0$ and $\Lambda_0 \neq g_t\Lambda_0$ for $t > 0$ small enough. Moreover since $\mathrm{supp}(h) \cap \Pi\Lambda_1 = \emptyset$, we deduce that $g_t\Lambda_1 = \Lambda_1$. This implies that $\tau(g_t\Lambda_0) - \tau(g_t\Lambda_1) > \tau(\Lambda_0) - \tau(\Lambda_1)$ for $t > 0$ small enough. □

Let us remark that to prove the second part of Theorem 5.4 it is enough to construct two elements $g, \varphi \in \mathcal{G}$ (resp. $g \in \mathcal{G}$ and $\varphi \in \widetilde{\mathcal{G}}$) such that

$$\varphi \preceq g^{-1}\varphi g \text{ and } \varphi \neq g^{-1}\varphi g. \tag{20}$$

We will first construct such elements when the contact manifold is the standard Euclidean contact manifold $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ and then transport this construction to any cooriented contact manifold using Darboux charts.

Proof of Theorem 5.4 part 2. We will prove that there exist $g, \varphi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) such that (20) is satisfied. Let us first remark that it is enough to prove it in the standard contact vector space $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ (we recall that in an open manifold, $\mathcal{G}$ stands for the set of time-one maps of compactly supported contact flows). Indeed, there would exists contact isotopies (g_t), (φ_t) and (h_t) supported in some open ball $B \subset \mathbb{R}^{2n+1}$ such that $g_0 = \mathrm{id} = \varphi_0$, (h_t) is non-negative with $h_0 = \varphi_1$ and $h_1 = g_1^{-1}\varphi_1 g_1$, and $\varphi_1 \neq g_1^{-1}\varphi_1 g_1$. Now, given any contact $(2n+1)$-manifold (M, ξ), there exists a contact embedding of B inside (M, ξ) by the Darboux neighborhood theorem. Since the contact isotopies (g_t), (φ_t) and (h_t) are compactly supported in B, they naturally extend by the identity to contact isotopies of (M, ξ). Seen in (M, ξ), (h_t) is still non-negative so that $\varphi := \varphi_1$ and $g := g_1$ still satisfy (20) as needed to conclude.

Let us now prove the existence of such $g, \varphi \in \mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$) for $(\mathbb{R}^{2n+1}, \xi_{\mathrm{st}})$ to finish the proof. The standard contact form α_{st} is $\mathrm{d}z - \sum_i y_i \mathrm{d}x_i$ where (x, y, z) denotes the usual coordinate functions on $\mathbb{R}^{2n+1}$. Let $\rho : [0, +\infty) \to [0, +\infty)$ be a smooth non-increasing function supported in $[0, 1/4]$ such that $\rho(0) = 1$. Let $H : \mathbb{R}^{2n+1} \to \mathbb{R}$ be the Hamiltonian map defined by $H(p) := \rho(|p|^2)$ for all $p \in \mathbb{R}^{2n+1}$ and where $|\cdot|$ denotes the usual Euclidean norm. We denote by (φ_t) the contact flow generated by H. Finally for all $a \in \mathbb{R}$ we denote by Φ_a the non compactly supported contactomorphism of $\mathbb{R}^{2n+1}$ defined as $(x, y, z) \mapsto (e^a x, e^a y, e^{2a} z)$.

Let $a < 0$ be sufficiently close to 0 so that $\Phi_a^{-1}(B_0(1/2))$ is strictly included in $B_0(1)$, where $B_0(r)$ denotes the open ball centered at 0 of radius $r > 0$. Then the support of $\Phi_a^{-1} \circ \varphi_t \circ \Phi_a$, that is $\Phi_a^{-1}(\operatorname{supp} \varphi_t)$, is contained in $B_0(1)$ and contains strictly $\operatorname{supp} \varphi_t$ for all $t \in [0, 1]$. Moreover since the compactly supported Hamiltonian function

$$h_a : \mathbb{R}^{2n+1} \to [0, +\infty), \quad (x, y, z) \mapsto e^{-2a} h(e^a x, e^a y, e^{2a} z),$$

generates the contact flow $(\Phi_a^{-1}\varphi_t\Phi_a)$ and trivially satisfies $h_a \geq h$ we deduce that $(\Phi_a^{-1}\varphi_t\Phi_a)$ is a non-negative compactly supported contact isotopy, so

$$\varphi_1 \preceq \Phi_a^{-1}\varphi_1\Phi_a \text{ and } \varphi_1 \neq \Phi_a^{-1}\varphi_1\Phi_a.$$

In order to conclude, one needs to replace Φ_a with a compactly supported contactomorphism. Since $\Phi_a^{-1}\varphi_t\Phi_a = g^{-1}\varphi_t g$ for any $t \in \mathbb{R}$ and any diffeomorphism g agreeing with Φ_a on $\Phi_a^{-1}B_0(1/2)$, one can take any such g in $\mathcal{G}$. Such a g can be induced by a compactly supported Hamiltonian map obtained by cutting off the Hamiltonian map generating $(\Phi_{ta})_{t\in[0,1]}$. Finally, the elements g and φ_1 satisfy the relations (20), as needed. □

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