

On Mirković–Vilonen polytopes

Pierre Baumann

Abstract

Mirković–Vilonen polytopes encode in a geometrical way the numerical data present in the Kashiwara crystal $B(\infty)$ of a semisimple group G . We retrieve these polytopes from the coproduct of the Hopf algebra $\mathcal{O}(N)$ of regular functions on a maximal unipotent subgroup N of G . We bring attention to a remarkable behavior that the classical bases (dual canonical, dual semicanonical, Mirković–Vilonen) of $\mathcal{O}(N)$ manifest with respect to the extremal points of these polytopes, which extends the crystal operations. This study leans on a notion of stability for graded bialgebras.

1 Introduction

Let G be a split semisimple, or more generally a split reductive group over a field k of characteristic zero. The representation theory of G is well understood thanks to Weyl’s theory of the highest weight. Let B be a Borel subgroup of G , let T be a maximal torus of B , let P be the character lattice of T , and let N be the unipotent radical of B . Then all the irreducible rational G -modules can be embedded as N -modules in the algebra $\mathcal{O}(N)$ of regular functions on N , which thus plays a central role in the theory. Moreover, the images of these embeddings are invariant under the action of T on $\mathcal{O}(N)$ deduced from the action on N by conjugation.

It is therefore natural to pay attention to the N -submodules of $\mathcal{O}(N)$ that are invariant under the action of T . A basic piece of combinatorial information associated to such modules is the set of their weights. Conversely, given a convex polytope K in the vector space $P \otimes_{\mathbb{Z}} \mathbb{R}$, one can consider the largest T -invariant N -submodule $\mathcal{S}(K)$ of $\mathcal{O}(N)$ whose weights lie in K . The map $K \mapsto \mathcal{S}(K)$ then defines a filtration of $\mathcal{O}(N)$ indexed by the set of all convex polytopes, endowed with the partial order given by inclusion. The places in the filtration where the jumps occur, to wit the convex polytopes K such that $\mathcal{S}(K) \neq \sum_{K' \subsetneq K} \mathcal{S}(K')$, certainly reflect the structure of the N -module $\mathcal{O}(N)$.

It turns out that these polytopes are already known in the literature: they are the Mirković–Vilonen polytopes introduced by Anderson [1]. The name comes from the Geometric Satake Equivalence, where the irreducible rational G -modules are realized as the intersection cohomology of affine Schubert varieties. In this context, Mirković and Vilonen selected algebraic cycles whose fundamental classes form a basis of the cohomology [22], and the polytopes appear as the images of these cycles by the moment map of a natural Hamiltonian action of a torus.

Another occurrence of these polytopes was found in [7]. The context here is the representation theory of the preprojective algebra Λ build from the Dynkin diagram of G (assumed to be of simply-laced type). To a Λ -module M , one can associate its Harder–Narasimhan polytope, that is, the convex hull in the Grothendieck group $K(\Lambda\text{-mod}) \otimes_{\mathbb{Z}} \mathbb{R}$ of the dimension-vectors of all the submodules of M . Now the representation spaces $\text{rep}(\Lambda, \nu)$ of Λ (one for each dimension-vector ν) are algebraic varieties, and it makes sense to speak of a general point in an irreducible component of these varieties. The Mirković–Vilonen polytopes then appear as the Harder–Narasimhan polytopes of general Λ -modules.

The fact that the moment polytopes of Mirković and Vilonen’s cycles coincide with the Harder–Narasimhan polytopes of general Λ -modules seems miraculous. The initial proof of this coincidence was indeed rather circuitous. On the one hand, the moment polytopes were described using Berenstein, Fomin and Zelevinsky’s Chamber Ansatz and Braverman and Gaitsgory’s crystal structure on the set of Mirković and Vilonen’s cycles [13, 14]. On the other hand, the Harder–Narasimhan polytopes were studied with the help of tilting theory on Λ -modules [5, 7]. The two descriptions were then equated thanks to Saito’s work on Lusztig data and reflections in the crystal $B(\infty)$ [24].

In this paper we propose a more straightforward argument. Through the Geometric Satake Equivalence, the fundamental class in cohomology of a Mirković–Vilonen cycle Z can be viewed as a basis element b_Z in an irreducible rational G -module, which we view as embedded in $\mathcal{O}(N)$. Gathering the elements b_Z coming from all the irreducible rational G -modules, we obtain a basis of $\mathcal{O}(N)$, called the Mirković–Vilonen basis [6]. Similarly, the study of Λ -modules leads to a basis of $\mathcal{O}(N)$, called the dual semicanonical basis [11, 21]. This basis consists of elements ρ_Y , where Y denotes an irreducible component of a variety $\text{rep}(\Lambda, \nu)$. Lastly, to a function $f \in \mathcal{O}(N)$, homogeneous with respect to the weight grading, we associate $\underline{\text{Pol}}(f)$, the convex hull in $P \otimes_{\mathbb{Z}} \mathbb{R}$ of the set of weights of the N -submodule generated by f . Said differently, in terms of the Hopf algebra structure of $\mathcal{O}(N)$, the polytope $\underline{\text{Pol}}(f)$ is the convex hull of the weights of the elements b_i that appear in a minimal writing $\sum_{i=1}^n b_i \otimes c_i$ of the coproduct of f .

We will show that: (1) For each irreducible component Y of $\text{rep}(\Lambda, \nu)$, the Harder–Narasimhan polytope of a general point M of Y is equal to $-\underline{\text{Pol}}(\rho_Y)$. (2) For each Mirković–Vilonen cycle Z , the moment polytope of Z is equal to $\underline{\text{Pol}}(b_Z)$. (3) Both the dual semicanonical basis and the Mirković–Vilonen basis are compatible with the filtration $K \mapsto \mathcal{S}(K)$, in the sense that each subspace $\mathcal{S}(K)$ is spanned by subsets of either basis.

By construction, $\mathcal{S}(K)$ is the vector space spanned by the functions f such that $\underline{\text{Pol}}(f) \subset K$. Then (3) implies that the jumps in the filtration $K \mapsto \mathcal{S}(K)$ occur precisely at the polytopes $\underline{\text{Pol}}(\rho_Y)$ and $\underline{\text{Pol}}(b_Z)$, so these two families of polytopes are the same, up to the indexing.

The proof of (1) is almost contained in [7]: each vertex μ of the Harder–Narasimhan polytope of a Λ -module M is the dimension-vector of a unique submodule M_t of M . Moreover, if M is a general point of an irreducible component Y of $\text{rep}(\Lambda, \nu)$, then M_t and M/M_t are general points in irreducible components Y_t and Y_f of $\text{rep}(\Lambda, \mu)$ and $\text{rep}(\Lambda, \nu - \mu)$, respectively. One can then easily show that the term $\rho_{Y_t} \otimes \rho_{Y_f}$ occurs in the coproduct of ρ_Y , which implies that $-\mu$ belongs to $\underline{\text{Pol}}(\rho_Y)$. Assertion (2) is proved by a similar argument: one can cut a

Mirković–Vilonen cycle at each vertex of its moment polytope, thereby producing two other cycles Z_t and Z_s such that $b_{Z_t} \otimes b_{Z_s}$ occurs in the coproduct of b_Z (see section 4.3 for the accurate statement). These observations can also be put to good use to prove (3) (see the proofs of Proposition 3.11 and Theorem 4.8).

The dual semicanonical basis and the Mirković–Vilonen basis are examples of perfect bases of $\mathcal{O}(N)$. This notion, introduced by Berenstein and Kazhdan [8], codifies a specific behavior that a basis exhibits regarding the action of the Chevalley generators of the Lie algebra of N . The perfect bases of $\mathcal{O}(N)$ are parameterized by a common combinatorial object, namely Kashiwara’s crystal $B(\infty)$, and it is possible to recover the Mirković–Vilonen polytopes from the crystal operations on $B(\infty)$ [13]. Perfect bases enjoy a number of nice properties (see for instance [6] for a review), but are generally not compatible with the subspaces $\mathcal{S}(K)$. However, as we saw in (3) above, the dual semicanonical basis and Mirković and Vilonen’s basis are compatible with them, and as it happens, the dual of Lusztig’s canonical basis [20] (Kashiwara’s upper global basis [16]) also is. It then seems desirable to record this feature in a definition. With this aim in view, we shall introduce the notion of polite basis. Polite bases are perfect in the sense of Berenstein and Kazhdan, but the converse is not true.

As mentioned above, if f is a homogeneous element in $\mathcal{O}(N)$ with respect to the weight grading and if $\Delta(f) = \sum_{i=1}^n b_i \otimes c_i$ is a minimal writing of the coproduct of f , then $\underline{\text{Pol}}(f)$ is the convex hull of the degrees of the elements b_i . With this notation, a basis \mathbf{B} is polite if each subspace $\mathcal{S}(K)$ is spanned by its intersection with \mathbf{B} and if, for each $f \in \mathbf{B}$ and each vertex μ of $\underline{\text{Pol}}(f)$, there is just one term $b_i \otimes c_i$ in $\Delta(f)$ such that b_i has weight μ , and for this i both b_i and c_i belong to \mathbf{B} . The precise definition we adopt in section 3.3 combines the two conditions at the price of seeming intricate. One reason for this choice is that the filtration is more conveniently described when we regard polytopes as intersections of closed half-spaces. In other words, we look at the polytopes through a plane projection of the weight lattice.

The plan of this paper is the following. Section 2 adapts Harder and Narasimhan’s notion of stability to the case of a connected bialgebra graded by a submonoid Γ of the half-plane; its main result is Theorem 2.6. In section 3, we apply this theory to $\mathcal{O}(N)$ and define the notion of polite basis (Definition 3.1); we prove that the semicanonical basis is polite (Proposition 3.11) and that polite bases are perfect (Proposition 3.12). Section 4 deals with Mirković and Vilonen’s basis; in particular we explain how to cut a Mirković–Vilonen cycle at a vertex of its moment polytope (Proposition 4.4) and we prove that this basis is polite (Theorem 4.8).

Acknowledgements. The work reported in this paper originates from a stay at Osaka Central Advanced Mathematical Institute (OCAMI) in 2016. The author thanks this institution, especially Yoshiyuki Kimura for the invitation and the useful discussions which ensued. He is also grateful to Joel Kamnitzer for more than a decade of conversations on this topic and for comments which helped improve the writing of this paper. Thanks are also due to Frédéric Chapoton, JiaRui Fei and Dinakar Muthiah for relevant suggestions.

2 Stability for graded connected bialgebras

In this section we consider a discrete submonoid Γ of the right half-plane

$$\{(r, d) \in \mathbb{R}^2 \mid r > 0 \text{ or } (r = 0 \text{ and } d \geq 0)\}$$

and a connected Γ -graded bialgebra $A = \bigoplus_{\gamma \in \Gamma} A^\gamma$ over a field k . The connectedness assumption means that $A^0 = k$, where $0 = (0, 0)$ is the identity element of Γ .

Our aim is to explain a procedure to factorize A , patently inspired by the method that Harder and Narasimhan introduced to study vector bundles on a curve (see for instance [2] for a survey of slope filtrations and stability).

We denote the comultiplication and the augmentation maps of A by Δ and ε , respectively. If a is a homogeneous element in A , we write $|a| = \gamma$ to indicate that $a \in A^\gamma$. Given $(\alpha, \beta) \in \Gamma^2$, the notation $\alpha - \beta$ will be reserved to the case where the result of the subtraction in \mathbb{R}^2 actually belongs to Γ .

2.1 Semistable elements

For any homogeneous element $a \in A$, we denote by $L(a)$ (respectively, $R(a)$) the set of all β (respectively, γ) such that $\Delta(a)$ has a nonzero component along a summand $A^\beta \otimes A^\gamma$ in the decomposition

$$A \otimes A = \bigoplus_{(\beta, \gamma) \in \Gamma^2} (A^\beta \otimes A^\gamma).$$

The obvious relation $R(a) = |a| - L(a)$ connects these two sets. If we write $\Delta(a) = \sum_{i=1}^n b_i \otimes c_i$ with each b_i and c_i homogeneous and n minimal, then

$$L(a) = \{|b_1|, \dots, |b_n|\} \quad \text{and} \quad R(a) = \{|c_1|, \dots, |c_n|\}.$$

Since A is connected (and assuming that $a \neq 0$), we have $\{0, |a|\} \subset L(a)$; specifically the homogeneous components of bidegree $(0, |a|)$ (respectively, $(|a|, 0)$) is $1 \otimes a$ (respectively, $a \otimes 1$).

Lemma 2.1 *Let $a \in A$ be a homogeneous element and write $\Delta(a) = \sum_{i=1}^n b_i \otimes c_i$ as above. Then $L(b_i) \subset L(a)$ and $R(c_i) \subset R(a)$ for all $i \in \llbracket 1, n \rrbracket$.*

Proof. Both families $(b_i)_{i \in \llbracket 1, n \rrbracket}$ and $(c_i)_{i \in \llbracket 1, n \rrbracket}$ consist of linearly independent elements. Pick $i \in \llbracket 1, n \rrbracket$ and choose a linear form $f : A \rightarrow k$ that vanishes on c_j for all $j \neq i$ and that verifies $f(c_i) = 1$. The coassociativity axiom implies

$$\begin{aligned} \Delta(b_i) &= (\Delta \otimes f) \circ \Delta(a) = (\text{id}_A \otimes \text{id}_A \otimes f) \circ (\Delta \otimes \text{id}_A) \circ \Delta(a) \\ &= (\text{id}_A \otimes ((\text{id}_A \otimes f) \circ \Delta)) \circ \Delta(a) \\ &= \sum_{j=1}^n b_j \otimes ((\text{id}_A \otimes f) \circ \Delta(c_j)), \end{aligned}$$

which shows that $L(b_i) \subset L(a)$. The inclusion $R(c_i) \subset R(a)$ is proved in a similar way. \square

We set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$; here ∞ is regarded as larger than any real number. We define

$$\Gamma_\infty = \{(r, d) \in \Gamma \mid r = 0\} \quad \text{and} \quad \Gamma_\mu = \{(r, d) \in \Gamma \mid d = \mu r\}$$

for $\mu \in \mathbb{R}$. Given $\mu \in \overline{\mathbb{R}}$, we set

$$\Gamma_{\leq \mu} = \bigcup_{\nu \leq \mu} \Gamma_\nu \quad \text{and} \quad \Gamma_{\geq \mu} = \bigcup_{\nu \geq \mu} \Gamma_\nu.$$

Let $\mu \in \overline{\mathbb{R}}$. A homogeneous element $a \in A$ is said to be of slope μ if $|a| \in \Gamma_\mu$. It is said to be semistable of slope μ if moreover $L(a) \subset \Gamma_{\leq \mu}$, or equivalently $R(a) \subset \Gamma_{\geq \mu}$.

We denote by A_μ (respectively, $A_{[\mu]}$) the subspace of A spanned by homogeneous elements of slope μ (respectively, semistable elements of slope μ). We denote by $p_\mu : A \rightarrow A_\mu$ the projection according to the decomposition

$$A = A_\mu \oplus \left(\bigoplus_{\gamma \in \Gamma \setminus \Gamma_\mu} A^\gamma \right)$$

and we set $\Delta_\mu = (p_\mu \otimes p_\mu) \circ \Delta$.

Proposition 2.2 *The subspace $A_{[\mu]}$ is a subalgebra of A_μ . Further, $\Delta_\mu(A_{[\mu]}) \subset A_{[\mu]} \otimes A_{[\mu]}$ and $A_{[\mu]}$ becomes a bialgebra when endowed with the coproduct Δ_μ .*

Proof. Set $A_{\leq \mu} = \sum_{\nu \leq \mu} A_\nu$ and $A_{\geq \mu} = \sum_{\nu \geq \mu} A_\nu$. Then A_μ , $A_{\leq \mu}$ and $A_{\geq \mu}$ are subalgebras of A and p_μ restricts to morphisms of algebras $A_{\leq \mu} \rightarrow A_\mu$ and $A_{\geq \mu} \rightarrow A_\mu$. It follows that $A_{[\mu]} = A_\mu \cap \Delta^{-1}(A_{\leq \mu} \otimes A_{\geq \mu})$ is a subalgebra of A and that the composition

$$A_{[\mu]} \xrightarrow{\Delta} A_{\leq \mu} \otimes A_{\geq \mu} \xrightarrow{p_\mu \otimes p_\mu} A_\mu \otimes A_\mu$$

is a morphism of algebras.

Now let $a \in A$ be a homogeneous semistable element of slope μ and write $\Delta(a) = \sum_{i=1}^n b_i \otimes c_i$ with each b_i and c_i homogeneous and n minimal. By Lemma 2.1

$$L(b_i) \subset L(a) \subset \Gamma_{\leq \mu} \quad \text{and} \quad R(c_i) \subset R(a) \subset \Gamma_{\geq \mu}$$

for each $i \in \llbracket 1, n \rrbracket$, so b_i and c_i are semistable as soon as they are of slope μ . Therefore $\Delta_\mu(a) \in A_{[\mu]} \otimes A_{[\mu]}$. Routine verifications show then that $A_{[\mu]}$ equipped with the coproduct Δ_μ fulfills the requirements to be a bialgebra. \square

2.2 Ordered Monomials

For any homogeneous element $a \in A$ of nonzero degree, we denote the convex hull of $L(a)$ in \mathbb{R}^2 by $\text{Pol}(a)$. The (closed) region between the upper rim of $\text{Pol}(a)$ —which starts at 0 and ends at $|a|$ —and the straight line between these two points will be of particular interest; we denote it by $\text{Pol}^\wedge(a)$. Obviously a is semistable if and only if $\text{Pol}^\wedge(a)$ is just the straight line between 0 and $|a|$.

Definition 2.3. An *ordered monomial* is a product $a_1 a_2 \cdots a_p$ of homogeneous semistable elements of nonzero degree, ordered so that their slopes form a decreasing sequence.

Lemma 2.4 *Let $a = a_1 a_2 \cdots a_p$ be an ordered monomial.*

- (i) *The upper rim of the polytope $\text{Pol}(a)$ is the polygonal line $[0, |a_1|, |a_1| + |a_2|, \dots, |a|]$.*
- (ii) *Let $j \in \llbracket 0, p \rrbracket$ and let $\alpha = |a_1| + \cdots + |a_j|$ be the j -th extremal point on the upper rim of $\text{Pol}(a)$. Then the homogeneous component of $\Delta(a)$ of bidegree $(\alpha, |a| - \alpha)$ is $a_1 \cdots a_j \otimes a_{j+1} \cdots a_p$.*

Proof. As the element a_j is homogeneous and semistable, the upper rim of $\text{Pol}(a_j)$ is the line $[0, |a_j|]$. The condition on the slopes then entails that the upper rim of the Minkowski sum $\text{Pol}(a_1) + \cdots + \text{Pol}(a_p)$ is the polygonal line $[0, |a_1|, |a_1| + |a_2|, \dots, |a|]$. Furthermore, given $j \in \llbracket 0, p \rrbracket$, the only tuple $(\alpha_1, \dots, \alpha_p) \in \text{Pol}(a_1) \times \cdots \times \text{Pol}(a_p)$ such that $\alpha_1 + \cdots + \alpha_p = |a_1| + \cdots + |a_j|$ is $(\alpha_1, \dots, \alpha_p) = (|a_1|, \dots, |a_j|, 0, \dots, 0)$. The lemma follows from these geometric considerations, using the multiplicativity of the coproduct and the connectedness of A . \square

Ordered monomials span the algebra A . More precisely:

Proposition 2.5 *Any homogeneous element a of nonzero degree is the sum of ordered monomials a_1, \dots, a_ℓ such that $|a_i| = |a|$ and $\text{Pol}^\wedge(a_i) \subset \text{Pol}^\wedge(a)$ for each $i \in \llbracket 1, \ell \rrbracket$.*

The proof uses a method introduced by Schiffmann [25].

Proof. The statement is obviously true if a itself is already semistable. In the rest of the proof, we assume that a is not semistable (in particular it is not of slope ∞) and proceed by induction on the number $N(a)$ of points in $\Gamma \cap \text{Pol}^\wedge(a)$.

Let r be the abscissa of $|a|$. The upper rim of the polytope $\text{Pol}(a)$ is the graph of a piecewise linear concave function $f : [0, r] \rightarrow \mathbb{R}$ and has at least one angular point. We choose such a point, let us say α ; it belongs to $L(a)$.

We write $\Delta(a) = \sum_{i=1}^n b_i \otimes c_i$ with each b_i and c_i homogeneous and n minimal. Let $I \subset \llbracket 1, n \rrbracket$ be the set of all indices i such that $|b_i| = \alpha$ and set $a' = a - \sum_{i \in I} b_i c_i$. We will show that

$N(b_i) < N(a)$ and $N(c_i) < N(a)$ for all $i \in I$ and that $N(a') < N(a)$. Applying the induction hypothesis, we may then write these elements a' , b_i and c_i as sums of ordered monomials and obtain thereby the desired expression for a .

Let x denote the abscissa of α . Let us pick $i \in I$. By Lemma 2.1 we have $L(b_i) \subset L(a)$. Therefore $\text{Pol}(b_i)$ is contained in $\text{Pol}(a) \cap ([0, x] \times \mathbb{R})$. Noting that the line $\mathbb{R}|b_i|$ lies above $\mathbb{R}|a|$ in the band $[0, x] \times \mathbb{R}$, we obtain $\text{Pol}^\wedge(b_i) \subset \text{Pol}^\wedge(a)$ and conclude that $N(b_i) < N(a)$ (the inequality is strict because the point $|a|$ is in $\text{Pol}^\wedge(a)$ but not in $\text{Pol}^\wedge(b_i)$). Likewise, $R(c_i) \subset R(a)$. Since $|a| - |c_i| = |b_i| = \alpha$, this can be rewritten as $\alpha + L(c_i) \subset L(a)$. Therefore $\alpha + \text{Pol}(c_i)$ is contained in $\text{Pol}(a) \cap ([x, r] \times \mathbb{R})$. Noting that the line $\alpha + \mathbb{R}|c_i|$ lies above $\mathbb{R}|a|$ in the band $[x, r] \times \mathbb{R}$, we obtain $\alpha + \text{Pol}^\wedge(c_i) \subset \text{Pol}^\wedge(a)$ and conclude that $N(c_i) < N(a)$.

Now let $\beta \in L(b_i)$ and $\gamma \in L(c_i)$. Both β and $\alpha + \gamma$ belong to $\text{Pol}(a)$. Let y and z denote the abscissae of these two points; then $0 \leq y \leq x \leq z \leq r$. Also, α is the point $(x, f(x))$, β lies below the point $(y, f(y))$, and $\alpha + \gamma$ lies below the point $(z, f(z))$, so $\beta + \gamma$ lies below the point $(y + z - x, f(y) + f(z) - f(x))$. Since f is concave, we have $f(y) + f(z) \leq f(x) + f(y + z - x)$, and therefore $\beta + \gamma$ lies below $(y + z - x, f(y + z - x))$, that is, below the upper rim of $\text{Pol}(a)$. To sum up, all the points in $L(b_i) + L(c_i)$ lie below the upper rim of $\text{Pol}(a)$. Moreover the concavity inequality is strict if $(y, z) \neq (x, x)$ (because α is an extremal point of $\text{Pol}(a)$), so $(\alpha, 0)$ is the only pair $(\beta, \gamma) \in L(b_i) \times L(c_i)$ for which $\beta + \gamma$ lies on the upper rim of $\text{Pol}(a)$.

Since Δ is a morphism of algebras, we have $L(b_i c_i) \subset L(b_i) + L(c_i)$. From the previous paragraph, we deduce that for each $i \in I$, all the points in $L(b_i c_i)$ lie below the upper rim of $\text{Pol}(a)$, and that moreover the homogeneous component of degree $(\alpha, |a| - \alpha)$ in $\Delta(b_i c_i)$ is $b_i \otimes c_i$. We conclude that all the points in $L(a')$ lie below the upper rim of $\text{Pol}(a)$ and that $\Delta(a')$ does not contain any term of degree $(\alpha, |a| - \alpha)$. In other words $\text{Pol}^\wedge(a')$ is contained in $\text{Pol}^\wedge(a) \setminus \{\alpha\}$. It follows that $N(a') < N(a)$, as announced.

Now pick $i \in I$, and consider two concave polygonal lines, one that joins 0 to α and is contained in $\text{Pol}^\wedge(b_i)$, and another one that joins α to $|a|$ and is contained in $\alpha + \text{Pol}^\wedge(c_i)$. Arguments given earlier in the proof imply that the concatenation of these two lines lies in $\text{Pol}^\wedge(a)$; in addition, this concatenated line is concave because α is an extremal point of $\text{Pol}(a)$. These geometrical observations guarantee that, when we substitute in $a' + \sum_{i \in I} b_i c_i$ expansions of a' , b_i and c_i obtained by induction, we obtain an expression for a that satisfies the requirements. \square

Let $J \subset \overline{\mathbb{R}}$. For any finite subset $M \subset J$, we can form the vector space $\bigotimes_{\mu \in M} A_{[\mu]}$, where the tensor product is computed in the decreasing order. An inclusion $M_1 \subset M_2 \subset J$ gives rise to an injective linear map $\bigotimes_{\mu \in M_1} A_{[\mu]} \rightarrow \bigotimes_{\mu \in M_2} A_{[\mu]}$ defined by inserting the unit element 1 at each factor that occurs only in the second product. We denote the limit of this inductive system by $\bigotimes_{\mu \in J} A_{[\mu]}$. Inclusions $J_1 \subset J_2 \subset \overline{\mathbb{R}}$ give rise to injective linear maps $\bigotimes_{\mu \in J_1} A_{[\mu]} \rightarrow \bigotimes_{\mu \in J_2} A_{[\mu]} \rightarrow \bigotimes_{\mu \in \overline{\mathbb{R}}} A_{[\mu]}$.

Theorem 2.6 *The multiplication in A induces a bijective linear map $m : \bigotimes_{\mu \in \overline{\mathbb{R}}} A_{[\mu]} \rightarrow A$.*

Proof. The surjectivity of m is a direct consequence of Proposition 2.5.

Suppose that there exists a nonzero element

$$a = \sum_{i=1}^n a_{i,1} \otimes \cdots \otimes a_{i,p_i}$$

in the kernel of m . In this writing, each $a_{i,j}$ is homogeneous semistable of nonzero degree, and for each $i \in \llbracket 1, n \rrbracket$ the slope of $a_{i,j}$ decreases when j increases from 1 to p_i .

We assume that a and this writing have been chosen with n as small as possible, and for this n , with $\max(p_1, \dots, p_n)$ as small as possible. This minimality assumption implies that a is homogeneous and that all the terms in the sum above have the same degree. Also, at most one p_i is equal to 1 (all the terms with $p_i = 1$ have the same slope so they can be combined).

Let μ be the maximum value among the slopes of the elements $a_{i,1}$, let α be the point that is the farthest from the origin among those $|a_{i,1}|$ that lie on the line $\mathbb{R}\mu$, and let $I \subset \llbracket 1, n \rrbracket$ be the subset of indices i such that $|a_{i,1}| = \alpha$. Extracting the homogeneous component of degree $(\alpha, |a| - \alpha)$ from the equation

$$\Delta \left(\sum_{i=1}^n a_{i,1} \cdots a_{i,p_i} \right) = 0$$

with the help of Lemma 2.4, we obtain

$$\sum_{i \in I} a_{i,1} \otimes a_{i,2} \cdots a_{i,p_i} = 0.$$

Any linear form $f : A^\alpha \rightarrow k$ then gives birth to an element

$$\sum_{i \in I} f(a_{i,1}) a_{i,2} \otimes \cdots \otimes a_{i,p_i}$$

in the kernel of m . Our minimality requirement for a forces this shorter element to be zero.

Picking an f that does not annihilate all the $a_{i,1}$, we obtain a linear dependence relation between the elements $a_{i,2} \otimes \cdots \otimes a_{i,p_i}$. With the help of this relation, we can shorten the writing of a , and thereby contradict our minimality requirement. Consequently our assumption about the existence of a was incorrect, and we conclude that m is injective. \square

The next proposition encompasses the existence of straightening relations in A .

Proposition 2.7 *If J is an interval in $\overline{\mathbb{R}}$, then $m\left(\bigotimes_{\mu \in J} A_{[\mu]}\right)$ is a subalgebra of A .*

Proof. Let $a = a_1 a_2 \cdots a_p$ be a product of homogeneous semistable elements whose slopes are in J . Let σ be a permutation of $\llbracket 1, p \rrbracket$ that reorder the sequence (a_1, a_2, \dots, a_p) by weakly decreasing slopes. Similar arguments as in Lemma 2.4 prove that the upper rim of $\text{Pol}(a)$ is

the polygonal line $[0, |a_{\sigma(1)}|, |a_{\sigma(1)}| + |a_{\sigma(2)}|, \dots, |a|]$. Applying Proposition 2.5, we expand a as a sum of ordered monomials b_i such that $|b_i| = |a|$ and $\text{Pol}^\wedge(b_i) \subset \text{Pol}^\wedge(a)$ for each index i . The slopes of all the semistable factors of b_i are then less than or equal to the slope of $a_{\sigma(1)}$ and greater or equal than the slope of $a_{\sigma(p)}$, so are in J . We conclude that a belongs to $m\left(\bigotimes_{\mu \in J} A_{[\mu]}\right)$. As a particular case, we see that this space is stable by multiplication. \square

Given $\mu \in \mathbb{R}$, we define the subalgebras $A_{[\leq \mu]} = m\left(\bigotimes_{\nu \leq \mu} A_{[\nu]}\right)$ and $A_{[\geq \mu]} = m\left(\bigotimes_{\nu \geq \mu} A_{[\nu]}\right)$.

Proposition 2.8 *The subalgebra $A_{[\leq \mu]}$ (respectively, $A_{[\geq \mu]}$) is a right (respectively, left) coideal of A .*

Proof. Lemma 2.4 and Proposition 2.5 imply that a homogeneous element a belongs to $A_{[\leq \mu]}$ (respectively, $A_{[\geq \mu]}$) if and only if $L(a) \subset \Gamma_{\leq \mu}$ (respectively, $R(a) \subset \Gamma_{\geq \mu}$). The proposition then follows from Lemma 2.1. \square

Likewise we define subalgebras $A_{[< \mu]}$ and $A_{[> \mu]}$.

2.3 The splitting isomorphisms

We will need to look at the multiplication map $A_{[>0]} \otimes A_{[0]} \otimes A_{[<0]} \rightarrow A$ through the prism of filtrations.

Let $\Pi' = \{(r, d) \in \mathbb{R}^2 \mid d < 0 \text{ or } (d = 0 \text{ and } r \geq 0)\}$. For $(\alpha_1, \alpha_2) \in \Gamma^2$, we write $\alpha_1 \leq' \alpha_2$ if $\alpha_1 + \Pi' \subset \alpha_2 + \Pi'$. This relation \leq' is a total order on Γ .

We define an increasing filtration $F'_\bullet A$ of the vector space A by the poset (Γ, \leq') as follows: for $\alpha \in \Gamma$, the subspace $F'_\alpha A$ is the linear span of all homogeneous elements $a \in A$ such that $L(a) \subset \alpha + \Pi'$. This filtration is compatible with the algebra structure of A , and the associated graded algebra $\text{gr}'_\bullet A$ is bigraded by Γ , for the original Γ -grading of A comes through.

As an example, we see that $F'_0 A = A_{[\leq 0]}$ (see the proof of Proposition 2.8) and $F'_\alpha A = 0$ if $\alpha <' 0$ (if $F'_\alpha A$ contains a nonzero homogeneous element a , then $0 \in L(a)$ and $L(a) \subset \alpha + \Pi'$, so $0 \leq' \alpha$), therefore $\text{gr}'_0 A = A_{[\leq 0]}$. Similarly, we see that $A_{[>0]} = \bigoplus_{\gamma \in \Gamma} (F'_\gamma A)^\gamma$.

Proposition 2.9 *There is a unique linear map $\bar{\Delta}' : \text{gr}'_\bullet A \rightarrow A_{[>0]} \otimes A_{[\leq 0]}$ that sends a homogeneous element $a \in \text{gr}'_\alpha A$ to the homogeneous component of degree $(\alpha, |a| - \alpha)$ of $\Delta(a)$. This map $\bar{\Delta}'$ is an isomorphism of algebras. The inverse map is induced by the multiplication in A .*

Proof. Let $\alpha \in \Gamma$. The vector space $F'_\alpha A$ is spanned by the homogeneous elements a that satisfy $\text{Pol}^\wedge(a) \subset \alpha + \Pi'$, so is spanned by the ordered monomials that satisfy this same condition

(Proposition 2.5). Likewise, the space $\sum_{\beta < \alpha} F'_\beta A$ is spanned by the ordered monomials that satisfy $\text{Pol}^\wedge(a) \subset (\alpha + \Pi') \setminus \{\alpha\}$. In addition, if an ordered monomial $a = a_1 a_2 \cdots a_p$ belongs to $F'_\alpha A$, if $\alpha \in L(a)$, and if j is the largest index in $\llbracket 1, p \rrbracket$ such that a_j has positive slope, then $|a_1| + |a_2| + \cdots + |a_j| = \alpha$ and the homogeneous component of degree $(\alpha, |a| - \alpha)$ of $\Delta(a)$ is $a_1 \cdots a_j \otimes a_{j+1} \cdots a_p$. These remarks entail that the map $\overline{\Delta}'$ is well-defined.

In the other direction, if a is a homogeneous element in $A_{[>0]}$ of degree α and if b is in $A_{[\leq 0]}$, then ab belongs to $F'_\alpha A$. Let $\overline{m} : A_{[>0]} \otimes A_{[\leq 0]} \rightarrow \text{gr}'_\bullet A$ be the map that sends $a \otimes b$ to the image of ab in $\text{gr}'_\alpha A$, where a and b are as in the previous sentence. Then $\overline{\Delta}'$ and \overline{m} are mutually inverse linear bijections. Further, $\overline{\Delta}'$ inherits from the coproduct the property of being an algebra morphism. \square

Let $\Pi'' = \{(r, d) \in \mathbb{R}^2 \mid d > 0 \text{ or } (d = 0 \text{ and } r \geq 0)\}$. For $(\beta_1, \beta_2) \in \Gamma^2$, we write $\beta_1 \leq'' \beta_2$ if $\beta_1 + \Pi'' \subset \beta_2 + \Pi''$. This relation \leq'' is a total order on Γ .

We define an increasing filtration $F''_\bullet A$ of the algebra A by the poset (Γ, \leq'') as follows: for $\beta \in \Gamma$, the space $F''_\beta A$ is the linear span of all homogeneous elements $a \in A$ such that $R(a) \subset \beta + \Pi''$. The associated graded algebra $\text{gr}''_\bullet A$ is then bigraded by Γ .

Proposition 2.10 *There is a unique linear map $\overline{\Delta}'' : \text{gr}''_\bullet A \rightarrow A_{[\geq 0]} \otimes A_{[\leq 0]}$ that sends a homogeneous element $a \in \text{gr}''_\beta A$ to the homogeneous component of degree $(|a| - \beta, \beta)$ of $\Delta(a)$. This map $\overline{\Delta}''$ is an isomorphism of algebras. The inverse map is induced by the multiplication in A .*

The filtration F'_\bullet on A induces a filtration on $\text{gr}''_\bullet A$, and likewise the filtration F''_\bullet induces a filtration on $\text{gr}'_\bullet A$. The associated graded algebras $\text{gr}'_\bullet \text{gr}''_\bullet A$ and $\text{gr}''_\bullet \text{gr}'_\bullet A$ are naturally isomorphic and are trigraded by Γ .

Proposition 2.11 *There is a unique linear map $\overline{\Delta}_2 : \text{gr}'_\bullet \text{gr}''_\bullet A \rightarrow A_{[>0]} \otimes A_{[0]} \otimes A_{[\leq 0]}$ that sends a homogeneous element $a \in \text{gr}'_\alpha \text{gr}''_\beta A$ to the homogeneous component of degree $(\alpha, |a| - \alpha - \beta, \beta)$ of $(\Delta \otimes \text{id}) \circ \Delta(a)$. This map $\overline{\Delta}_2$ is an isomorphism of algebras. The inverse map is induced by the multiplication in A .*

We omit the proofs of the last two propositions. For future use, we record that $A_{[\geq 0]} = F''_0 A = \text{gr}''_0 A$, that $A_{[\leq 0]} = \bigoplus_{\gamma \in \Gamma} (F''_\gamma A)^\gamma$, and that $A_{[0]} = F'_0 A \cap F''_0 A = \text{gr}'_0 \text{gr}''_0 A$.

2.4 Duality

In this section we assume that each homogeneous component A^γ is finite-dimensional. The graded dual of A , which we denote by B , is also a Γ -graded connected bialgebra. We use the same notation Δ for the coproduct of B as for A . We can define the bialgebras $B_{[\mu]}$ (Proposition 2.2) and we can look at the basis of ordered monomials in B .

Proposition 2.12 (i) Let $\langle \cdot, \cdot \rangle : A \times B \rightarrow k$ be the duality bracket, and let a and b be ordered monomials in A and B , respectively. Then $\langle a, b \rangle = 0$ unless $\text{Pol}^\wedge(a) = \text{Pol}^\wedge(b)$.

(ii) For each $\mu \in \overline{\mathbb{R}}$, the bialgebra $B_{[\mu]}$ identifies with the graded dual of $A_{[\mu]}$.

Proof. Let $a = a_1 \cdots a_p$ and $b = b_1 \cdots b_q$ be ordered monomials in A and B , respectively. Suppose that $|a| = |b|$, but that $\text{Pol}^\wedge(a) \neq \text{Pol}^\wedge(b)$. Exchanging A and B if necessary, we may assume that one extremal point on the upper rim of $\text{Pol}(a)$, say α , does not belong to $\text{Pol}^\wedge(b)$. By Lemma 2.4, there exists $j \in \llbracket 1, p-1 \rrbracket$ such that $\alpha = |a_1| + \cdots + |a_j|$. As the homogeneous component of degree $(\alpha, |a| - \alpha)$ of $\Delta(b)$ is zero, we have

$$\langle a, b \rangle = \langle (a_1 \cdots a_j)(a_{j+1} \cdots a_p), b \rangle = 0.$$

This shows the first assertion. The second one then follows from Theorem 2.6. \square

3 Polytopes and bases of $\mathcal{O}(N)$

3.1 Notation

For the rest of the paper we consider a split connected reductive group G over a field k of characteristic 0. We fix a Borel subgroup B and a maximal torus T contained in B . Let P be the character lattice of T , let Φ be the root system of (G, T) , let Φ_+ (respectively, Φ_-) be the set of positive (respectively, negative) roots determined by B , and let $\{\alpha_i\}_{i \in I}$ be the set of simple roots. We denote by Q (respectively, Q_\pm) the subgroup (respectively, submonoid) of P generated by Φ (respectively, Φ_\pm).

Let N be the unipotent radical of B and let \mathfrak{n} be the Lie algebra of N . For $i \in I$, we choose a root vector $e_i \in \mathfrak{n}$ of weight α_i . The enveloping algebra $U(\mathfrak{n})$ is generated by the elements e_i . It is graded by Q_+ , with $\deg e_i = \alpha_i$. For $(i, \ell) \in I \times \mathbb{N}$, the divided power $e_i^\ell / \ell!$ in $U(\mathfrak{n})$ is denoted by $e_i^{(\ell)}$.

Since N is an algebraic group, the algebra $\mathcal{O}(N)$ of regular functions on N is a Hopf algebra. It acquires a grading $\bigoplus_{\lambda \in Q_-} \mathcal{O}(N)_\lambda$ by means of the conjugation action of T on N . The group N and its Lie algebra \mathfrak{n} acts on both sides on $\mathcal{O}(N)$; our convention is that $n \cdot f = f(?n)$ and $f \cdot n = f(n?)$ for all $(n, f) \in N \times \mathcal{O}(N)$. Denoting by 1_N the identity element of N , we then have $(x \cdot f)(1_N) = (f \cdot x)(1_N)$ for all $(x, f) \in U(\mathfrak{n}) \times \mathcal{O}(N)$, so it makes sense to define a pairing between $U(\mathfrak{n})$ and $\mathcal{O}(N)$ by $\langle x, f \rangle = (x \cdot f)(1_N)$. This pairing is perfect and identifies $\mathcal{O}(N)$ as a bialgebra to the graded dual of $U(\mathfrak{n})$. For $i \in I$, we denote by $\zeta_i \in \mathcal{O}(N)$ the function of weight $-\alpha_i$ such that $\langle e_i, \zeta_i \rangle = 1$.

We define the height $\text{ht } \lambda$ of an element $\lambda \in Q$ as the sum of the coordinates of λ in the basis $\{\alpha_i\}_{i \in I}$. We define an involution $*$ on the vector space $\mathcal{O}(N)_\lambda$ by setting $f^*(n) = (-1)^{\text{ht } \lambda} f(n^{-1})$ for all $(n, f) \in N \times \mathcal{O}(N)_\lambda$. This involution $*$ is extended linearly to the whole algebra $\mathcal{O}(N)$.

For a function $f \in \mathcal{O}(N)$ of weight λ , we denote by $\underline{L}(f)$ the set of all weights μ such that the coproduct $\Delta(f)$ has a nonzero component along the summand $\mathcal{O}(N)_\mu \otimes \mathcal{O}(N)_{\lambda-\mu}$ of $\mathcal{O}(N)^{\otimes 2}$. In other words, $\underline{L}(f)$ is the set of weights of the left N -submodule of $\mathcal{O}(N)$ generated by f . We set $Q_{\mathbb{R}} = Q \otimes_{\mathbb{Z}} \mathbb{R}$ and we denote the convex hull of $\underline{L}(f)$ in this vector space by $\underline{\text{Pol}}(f)$. This definition mimics the notion of KLR polytope introduced in [26].

In the sequel, we will just write U and \mathcal{O} for respectively $U(\mathfrak{n})$ and $\mathcal{O}(N)$ when this does not cause any confusion.

3.2 Stability in $U(\mathfrak{n})$ and $\mathcal{O}(N)$

In our story the dual $Q_{\mathbb{R}}^*$ of $Q_{\mathbb{R}}$ plays the role of the space of stability parameters.

Let us pick an element $\theta \in Q_{\mathbb{R}}^*$. The map $\lambda \mapsto (\text{ht } \lambda, \theta(\lambda))$ projects Q_+ onto a submonoid Γ_θ of the right half-plane. The Q_+ -grading on U then descends to a Γ_θ -grading, for which the results in section 2 can be applied. Likewise the Q_- -grading on \mathcal{O} descends to a Γ_θ -grading by means of the map $J_\theta : \lambda \mapsto -(\text{ht } \lambda, \theta(\lambda))$.

For each slope $\mu \in \mathbb{R}$, the set of positive roots α such that $\theta(\alpha)/\text{ht } \alpha = \mu$ is closed in Φ . The corresponding root vectors span a subalgebra \mathfrak{n}_μ of the Lie algebra \mathfrak{n} . As homogeneous elements in \mathfrak{n} are primitive hence semistable, the subalgebra of U generated by \mathfrak{n}_μ is contained in $U_{[\mu]}$. Comparing then Theorem 2.6 with the Poincaré–Birkhoff–Witt theorem, we conclude that $U(\mathfrak{n}_\mu) = U_{[\mu]}$.

We set $\mathfrak{n}_{>0} = \sum_{\mu>0} \mathfrak{n}_\mu$ and $\mathfrak{n}_{<0} = \sum_{\mu<0} \mathfrak{n}_\mu$, and we denote by $N_{>0}$, N_0 and $N_{<0}$ the closed subgroups of N with Lie algebras $\mathfrak{n}_{>0}$, \mathfrak{n}_0 and $\mathfrak{n}_{<0}$, respectively. Then N is the Zappa–Szép product* $N_{>0} \bowtie N_0 \bowtie N_{<0}$.

Given a subgroup H of N , we denote by ${}^H\mathcal{O}$ (respectively, \mathcal{O}^H) the subalgebras of invariant elements in \mathcal{O} with respect to the left (respectively, right) action of H . With this notation, we deduce from Proposition 2.12 the following explicit description:

$$\mathcal{O}_{[\geq 0]} = {}^{N_{<0}}\mathcal{O}, \quad \mathcal{O}_{[> 0]} = {}^{N_0 N_{<0}}\mathcal{O}, \quad \mathcal{O}_{[\leq 0]} = \mathcal{O}^{N_{>0}}, \quad \mathcal{O}_{[< 0]} = \mathcal{O}^{N_{>0} N_0}, \quad \mathcal{O}_{[0]} = {}^{N_{<0}}\mathcal{O}^{N_{>0}}.$$

Since the decomposition $N = N_{>0} \bowtie N_0 \bowtie N_{<0}$ is an isomorphism of algebraic varieties, we obtain isomorphisms from $\mathcal{O}_{[> 0]}$, $\mathcal{O}_{[0]}$ and $\mathcal{O}_{[< 0]}$ onto the algebras of regular functions on $N_{>0}$, N_0 and $N_{<0}$, respectively.

In the next sections, we will often decorate the objects with a label θ when we want to stress that they depend on the choice of a stability parameter. As an example, the filtrations defined in section 2.3 will be denoted by ${}^\theta F'_\bullet \mathcal{O}$ and ${}^\theta F''_\bullet \mathcal{O}$. Likewise, given a weight function $f \in \mathcal{O}$, the subsets $L(f)$, $\text{Pol}(f)$ and $\text{Pol}^\wedge(f)$ defined by the Γ_θ -grading of \mathcal{O} will be adorned with an index θ , so for instance $L_\theta(f) = J_\theta(\underline{L}(f))$.

*We recall that a group N is the (internal) Zappa–Szép product of two subgroups N' and N'' (in symbols, $N = N' \bowtie N''$) if the product in N induces a bijection $N' \times N'' \rightarrow N$.

3.3 Polite bases

The following definition highlights properties that the dual canonical basis, the dual semi-canonical basis and Mirković and Vilonen’s basis share.

Definition 3.1. A basis \mathbf{B} of \mathcal{O} is said to be *polite* if it satisfies the following four conditions:

- (i) The elements of \mathbf{B} are weight vectors.
- (ii) For each $(i, n) \in I \times \mathbb{N}$, the function ζ_i^n belongs to \mathbf{B} .
- (iii) For each $\theta \in Q_{\mathbb{R}}^*$, the basis \mathbf{B} is compatible with the filtration ${}^\theta F_{\bullet}' \mathcal{O}$ and with the isomorphism defined in Proposition 2.9.
- (iv) For each $\theta \in Q_{\mathbb{R}}^*$, the basis \mathbf{B} is compatible with the filtration ${}^\theta F_{\bullet}'' \mathcal{O}$ and with the isomorphism defined in Proposition 2.10.

The compatibility required in condition (iii) means the following. First, for each degree $\gamma \in \Gamma_\theta$, the subspace ${}^\theta F_{\gamma}' \mathcal{O}$ should be spanned by $\mathbf{B} \cap {}^\theta F_{\gamma}' \mathcal{O}$. In particular, \mathbf{B} induces bases in $\text{gr}'_{\bullet} \mathcal{O}$, $\mathcal{O}_{[\leq 0]}$ and $\mathcal{O}_{[> 0]}$. Then $\overline{\Delta}'$ should map this basis of $\text{gr}'_{\bullet} \mathcal{O}$ onto the tensor product basis of $\mathcal{O}_{[\leq 0]} \otimes \mathcal{O}_{[> 0]}$.

A similar clarification is in order for condition (iv). However, one can show that conditions (iii) and (iv) are in fact equivalent, so including them both is redundant.

Example 3.2. The quantized enveloping algebra $U_q(\mathfrak{n})$ can be categorified by modules over the Khovanov–Lauda–Rouquier algebras. In this context, Varagnolo and Vasserot showed that for G simply laced, the elements of the canonical basis of $U_q(\mathfrak{n})$ (up to a power of q) correspond to the indecomposable projective graded objects [27]. From Tingley and Webster’s results in [26], we then deduce that the dual canonical basis is a polite basis of \mathcal{O} . Now choose $\theta \in Q_{\mathbb{R}}^*$ and adopt the notation of section 3.2. Then the subspaces ${}^\theta F_0' \mathcal{O} = \mathcal{O}_{[\leq 0]} = \mathcal{O}^{N_{> 0}}$ and ${}^\theta F_0'' \mathcal{O} = \mathcal{O}_{[\geq 0]} = \mathcal{O}^{N_{< 0}}$ are spanned by their intersection with the dual canonical basis. This proves afresh a result of Kimura [18]. (Kimura’s actual result is in truth deeper, as it is valid for quantized symmetrizable Kac–Moody algebra.)

Given a convex polytope K contained in $Q_{\mathbb{R}}$, we define $\mathcal{S}(K)$ as the vector subspace of \mathcal{O} spanned by weight functions f such that $\underline{\text{Pol}}(f) \subset K$.

Proposition 3.3 *Let \mathbf{B} be a polite basis of \mathcal{O} . Then for each convex polytope K in $Q_{\mathbb{R}}$ the subspace $\mathcal{S}(K)$ is spanned by $\mathbf{B} \cap \mathcal{S}(K)$.*

Proof. We may without loss of generality restrict our considerations to a weight subspace \mathcal{O}_λ with $\lambda \in Q_-$. We set $M = Q_- \cap (\lambda - Q_-)$. The Hahn–Banach theorem implies the existence of finitely many elements $\theta_1, \dots, \theta_n$ in $Q_{\mathbb{R}}^*$ and of degrees $\gamma_1 \in \Gamma_{\theta_1}, \dots, \gamma_n \in \Gamma_{\theta_n}$ such that, for each $\mu \in M$,

$$\mu \in K \Leftrightarrow \left(\forall i \in \llbracket 1, n \rrbracket, J_{\theta_i}(\mu) \in \gamma_i + \Pi' \right).$$

As $\underline{L}(f) \subset M$ for any function $f \in \mathcal{O}_\lambda$, we obtain that

$$\mathcal{O}_\lambda \cap \mathcal{S}(K) = \{f \in \mathcal{O}_\lambda \mid \underline{L}(f) \subset K\} = \mathcal{O}_\lambda \cap \bigcap_{i=1}^n \left(\theta_i F_{\gamma_i}' \mathcal{O} \right).$$

The proposition now follows from the compatibility of \mathbf{B} with all the subspaces occurring in the right-hand side. \square

Corollary 3.4 *Let \mathbf{B} be a polite basis of \mathcal{O} , let $f \in \mathbf{B}$, and let $\theta \in Q_{\mathbb{R}}^*$. Expand f as a sum $\sum_{i=1}^{\ell} a_i$ of ordered monomials with respect to the Γ_θ -grading of \mathcal{O} . Then there exists a term a_i such that $\text{Pol}_\theta^\wedge(f) = \text{Pol}^\wedge(a_i)$.*

Proof. We may assume without loss of generality that the functions a_i are homogeneous with respect to the weight grading on \mathcal{O} . From Proposition 2.5, we know that $\text{Pol}_\theta^\wedge(a_i) \subset \text{Pol}_\theta^\wedge(f)$ for each $i \in \llbracket 1, \ell \rrbracket$. On the other hand, a_i belongs to $\mathcal{S}(\underline{\text{Pol}}(a_i))$, which is spanned by its intersection with \mathbf{B} . So each a_i , and therefore f , can be written as a linear combination of elements in $\bigcup_{i=1}^{\ell} (\mathbf{B} \cap \mathcal{S}(\underline{\text{Pol}}(a_i)))$. As \mathbf{B} is a basis, there necessarily exists an index $i \in \llbracket 1, \ell \rrbracket$ such that $f \in \mathcal{S}(\underline{\text{Pol}}(a_i))$. We obtain $\text{Pol}_\theta(f) \subset \text{Pol}_\theta(a_i)$, then $\text{Pol}_\theta^\wedge(f) = \text{Pol}_\theta^\wedge(a_i)$. \square

The root hyperplanes draw a fan in the vector space $Q_{\mathbb{R}}^*$, called the Weyl fan. We say that a convex polytope P contained in $Q_{\mathbb{R}}$ is GGMS if its normal fan is a coarsening of the Weyl fan (GGMS stands for Gelfand, Goresky, MacPherson and Serganova; see [14], section 2.4).

Corollary 3.5 *Let \mathbf{B} be a polite basis of \mathcal{O} and let $f \in \mathbf{B}$. Then the polytope $\underline{\text{Pol}}(f)$ is GGMS.*

Proof. Let $-\theta$ be an element in a face of the normal fan to $\underline{\text{Pol}}(f)$, so that the locus where θ achieves its minimum on $\underline{\text{Pol}}(f)$ is a face of positive dimension. The upper rim of $\text{Pol}_\theta(f)$ has then a nontrivial segment of slope 0. Applying Corollary 3.4, we conclude that $\mathcal{O}_{[0]}$ is not reduced to the base field k . According to the discussion in section 3.2, there exists a positive root α such that $\theta(\alpha) = 0$, so θ belongs to a face of the Weyl fan. To sum up, each chamber of the Weyl fan is contained in a chamber of the normal fan to $\underline{\text{Pol}}(f)$. \square

Proposition 3.6 *Let \mathbf{B} be a polite basis of \mathcal{O} and let $\theta \in Q_{\mathbb{R}}^*$ be such that the slopes $\theta(\alpha)/\text{ht } \alpha$ of the positive roots α are pairwise different. Then the map $f \mapsto \text{Pol}_\theta^\wedge(f)$ is injective on \mathbf{B} .*

Proof. Let f_1 and f_2 be two elements of \mathbf{B} that have the same image by the map Pol_θ^\wedge . By our assumption on θ , there is a unique ordered monomial a relative to the Γ_θ -grading of \mathcal{O} such that $\text{Pol}^\wedge(a) = \text{Pol}_\theta^\wedge(f_1) = \text{Pol}_\theta^\wedge(f_2)$. By Corollary 3.4, a occurs in the expansions of both f_1 and f_2 as sums of ordered monomials, and there exists a scalar t such that $f_1 + tf_2$ is a sum $\sum_{i=1}^\ell b_i$ of ordered monomials, homogeneous with respect to the weight grading on \mathcal{O} , with $\text{Pol}^\wedge(b_i) \subsetneq \text{Pol}^\wedge(a)$ for all $i \in \llbracket 1, \ell \rrbracket$. Certainly then $f_1 + tf_2$ is a linear combination of elements in $\bigcup_{i=1}^\ell (\mathbf{B} \cap \mathcal{S}(\underline{\text{Pol}}(b_i)))$, but as neither f_1 nor f_2 belongs to this union, we necessarily have $f_1 + tf_2 = 0$, in particular $f_1 = f_2$. \square

Corollary 3.7 *Let \mathbf{B} be a polite basis of \mathcal{O} . Then the map $f \mapsto \underline{\text{Pol}}(f)$ is injective on \mathbf{B} .*

The map $K \mapsto \mathcal{S}(K)$ is a filtration of \mathcal{O} indexed by the set of all convex polytopes, endowed with the inclusion order. We denote the associated graded by $\text{gr}_\bullet \mathcal{O}$. By Proposition 3.3, any polite basis \mathbf{B} of \mathcal{O} is compatible with this filtration and thus allows to compute this associated graded. Explicitly, we get that $\text{gr}_K \mathcal{O}$ is one dimensional if K is of the form $\underline{\text{Pol}}(f)$ for $f \in \mathbf{B}$ and is zero otherwise.

Accordingly, the set $\underline{\text{Pol}}(\mathbf{B})$ is the same for all polite bases \mathbf{B} . As mentioned in the introduction, the elements of this set are the Mirković–Vilonen polytopes [1]. Kamnitzer proved that among the GGMS polytopes, these are characterized by the shape of their 2-faces, which is constrained by the tropical Plücker relations [14]. A more general statement is that faces of Mirković–Vilonen polytopes are Mirković–Vilonen polytopes of smaller ranks [7].

Remark 3.8. This last fact can be proved in our set-up in the following way. As in section 3.2, the datum of an element $\theta \in Q_{\mathbb{R}}^*$ defines subgroups $N_{>0}$, N_0 and $N_{<0}$ of N , and the space $\mathcal{O}_{[0]} = {}^{N_{<0}}\mathcal{O}^{N_{>0}}$ is naturally isomorphic to the algebra of regular functions on N_0 . As N_0 can be regarded as the maximal unipotent subgroup of a reductive group with root system $\Phi \cap \ker \theta$, it makes sense to speak of polite bases of $\mathcal{O}_{[0]}$. (The only true matter of concern is the normalization condition (ii), which should be adequately managed.) And as a matter of fact, one easily shows that if \mathbf{B} is a polite basis of \mathcal{O} , then $\mathbf{B} \cap \mathcal{O}_{[0]}$ is a polite basis of $\mathcal{O}(N_0)$. Now consider such a basis \mathbf{B} and pick $f \in \mathbf{B}$. Any face of $\underline{\text{Pol}}(f)$ can be obtained as the minimum locus on $\underline{\text{Pol}}(f)$ of an element $\theta \in Q_{\mathbb{R}}^*$. Using this θ as a stability parameter, the upper ridge of $\text{Pol}_\theta(f)$ contains a segment of slope 0, with endpoints α and β . Denoting by \bar{f} the image of f in $\text{gr}'_\alpha \text{gr}''_\beta \mathcal{O}$ and adopting the notation of Proposition 2.11, we can write $\bar{\Delta}_2(\bar{f}) = f_+ \otimes f_0 \otimes f_-$ with f_+ , f_0 and f_- in \mathbf{B} . Then f_0 belongs to $\mathbf{B} \cap \mathcal{O}_{[0]}$, and the face of $\underline{\text{Pol}}(f)$ that we considered is a translate of the polytope $\underline{\text{Pol}}(f_0)$.

Proposition 3.9 *Let \mathbf{B}_1 and \mathbf{B}_2 be polite bases of \mathcal{O} . We index both of them by the Mirković–Vilonen polytopes by means of the map $\underline{\text{Pol}}$. Then the transition matrix between \mathbf{B}_1 and \mathbf{B}_2 is upper unitriangular with respect to the inclusion order of the polytopes.*

Proof. The triangularity of the transition matrix is a consequence of the compatibility of both bases with the filtration $K \mapsto \mathcal{S}(K)$. The fastest way to prove that there are only ones on

the diagonal is to observe that polite bases are always perfect (see section 3.5 for the specific definition we use in this paper) and to invoke the fact that the transition matrix between two perfect bases is unitriangular (see for instance Proposition 2.6 in [3] for the dual version). \square

3.4 The dual semicanonical basis

In this section we assume that G is simply laced and we show that the dual semicanonical basis of \mathcal{O} is polite. We start with a few recollections to settle the notation and refer to the literature for complete explanations [11, 21].

Let Λ be the proprojective algebra on the Dynkin diagram of G over the field of complex numbers. This is the path algebra of a quiver with relations, with vertex set I . We regard the dimension-vector $\underline{\dim}M$ of a Λ -module M as an element of Q_+ , identified to \mathbb{N}^I by means of the basis $\{\alpha_i\}_{i \in I}$. The Harder–Narasimhan polytope of a Λ -module M is the convex hull in the space $Q_{\mathbb{R}}$ of the set of dimension-vectors of all the submodules of M .

We define the weight of a word $\mathbf{i} = i_1 \cdots i_n$ in the alphabet I as $\alpha_{i_1} + \cdots + \alpha_{i_n}$. To such a word, we associate the monomial $e_{\mathbf{i}} = e_{i_1} \cdots e_{i_n}$ in the Chevalley generators of \mathfrak{n} . If M is a Λ -module of dimension-vector ν and \mathbf{i} is a word of weight ν , we denote by $\mathcal{F}_{\mathbf{i},M}$ the set of all flags $0 = N_0 \subset N_1 \subset \cdots \subset N_n = M$ of Λ -submodules of M such that $\underline{\dim}(N_j/N_{j-1}) = \alpha_{i_j}$ for all $j \in \llbracket 1, n \rrbracket$. With this notation, each Λ -module M of dimension-vector ν determines a function $\varphi_M \in \mathcal{O}_{-\nu}$ such that $\langle e_{\mathbf{i}}, \varphi_M \rangle$ is the Euler characteristic of $\mathcal{F}_{\mathbf{i},M}$ for all words \mathbf{i} of weight ν .

Proposition 3.10 *The Harder–Narasimhan polytope of a Λ -module M is equal to $-\underline{\text{Pol}}(\varphi_M)$.*

Proof. We set $\nu = \underline{\dim}M$ and pick $\mu \in (-\underline{L}(\varphi_M))$. The component of weight $(-\mu, -\nu + \mu)$ of $\Delta(\varphi_M)$ is nonzero, so there exist words \mathbf{i} and \mathbf{j} of weights μ and $\nu - \mu$, respectively, such that $\langle e_{\mathbf{i}} \otimes e_{\mathbf{j}}, \Delta(\varphi_M) \rangle \neq 0$. Denoting by \mathbf{ij} the concatenation of these words, we get $\langle e_{\mathbf{ij}}, \varphi_M \rangle \neq 0$, so $\mathcal{F}_{\mathbf{ij},M} \neq \emptyset$ and M has a submodule of dimension-vector μ . We conclude that $-\underline{L}(\varphi_M)$, and therefore its convex hull $-\underline{\text{Pol}}(\varphi_M)$, is contained in the Harder–Narasimhan polytope of M .

To prove the reverse inclusion, we choose a vertex μ of the Harder–Narasimhan polytope of M . Then M contains a unique submodule N of dimension-vector μ ([7], section 3.2). If \mathbf{i} and \mathbf{j} are words of weights μ and $\nu - \mu$, respectively, then any flag in $\mathcal{F}_{\mathbf{ij},M}$ contains N , whence an isomorphism $\mathcal{F}_{\mathbf{ij},M} \cong \mathcal{F}_{\mathbf{i},N} \times \mathcal{F}_{\mathbf{j},M/N}$. Taking Euler characteristics, we get $\langle e_{\mathbf{ij}}, \varphi_M \rangle = \langle e_{\mathbf{i}}, \varphi_N \rangle \langle e_{\mathbf{j}}, \varphi_{M/N} \rangle$. Since this holds for all such words \mathbf{i} and \mathbf{j} , we deduce that the component of weight $(-\mu, -\nu + \mu)$ of $\Delta(\varphi_M)$ is equal to $\varphi_N \otimes \varphi_{M/N}$, hence is not zero ([12], Lemma 9.5). Therefore $-\mu \in \underline{L}(\varphi_M)$. We conclude that all the vertices of the Harder–Narasimhan polytope of M belong to $-\underline{\text{Pol}}(\varphi_M)$. \square

Given $\nu \in Q_+$, we denote the representation space of Λ -modules (aka Lusztig’s nilpotent variety) in vector-dimension ν by $\text{rep}(\Lambda, \nu)$. If M is the Λ -module defined by the point $m \in$

$\text{rep}(\Lambda, \nu)$, we allow ourselves to write φ_m instead of φ_M . The map $m \mapsto \varphi_m$ from $\text{rep}(\Lambda, \nu)$ to $\mathcal{O}_{-\nu}$ is constructible. To any irreducible component Y of $\text{rep}(\Lambda, \nu)$, one can therefore associate a function $\rho_Y \in \mathcal{O}_{-\nu}$ so that $\{m \in Y \mid \varphi_m = \rho_Y\}$ contains a dense open subset of Y . The dual semicanonical basis of \mathcal{O} is the family of all these functions ρ_Y .

Each $\theta \in Q_{\mathbb{R}}^*$ defines a torsion pair $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta)$ in $\Lambda\text{-mod}$ ([7], section 3.1). For each Λ -module M , denote by M_t and $M_f = M/M_t$ the torsion and locally-free part of M with respect to this torsion pair. The map $M \mapsto (M_t, M_f)$ extends to irreducible components: given a dimension-vector ν and an irreducible component Y of $\text{rep}(\Lambda, \nu)$, there exists a dimension-vector μ and irreducible components Y_t and Y_f of $\text{rep}(\Lambda, \mu)$ and $\text{rep}(\Lambda, \nu - \mu)$, respectively, such that the general point of $Y_t \times Y_f$ is of the form (M_t, M_f) with M a general point in Y . Moreover μ , Y_t and Y_f are uniquely defined and the map $Y \mapsto (Y_t, Y_f)$ is injective (*loc. cit.*, Theorem 4.4). From Proposition 3.10 and its proof, we deduce that $-J_\theta(\mu)$ is the largest element in $L_\theta(\rho_Y)$ with respect to the total order \leq' and that the component of weight $(-\mu, -\nu + \mu)$ of $\Delta(\rho_Y)$ is $\rho_{Y_t} \otimes \rho_{Y_f}$.

Proposition 3.11 *The dual semicanonical basis of \mathcal{O} is polite.*

Proof. By construction, the dual semicanonical basis satisfies the condition (i) in the definition of polite bases. For condition (ii), we choose $(i, n) \in I \times \mathbb{N}$, denote the unique Λ -module of dimension-vector $n\alpha_i$ by nS_i , observe that $\langle e_{i \dots i}, \varphi_{nS_i} \rangle$ is equal to $n!$, the Euler characteristic of the complete flag variety of an n -dimensional complex vector space, and conclude that ζ_i^n is equal to φ_{nS_i} , so belongs to the dual semicanonical basis.

We now show that the dual semicanonical basis satisfies condition (iii). Let $\theta \in Q_{\mathbb{R}}^*$, let $(\alpha, \gamma) \in \Gamma_\theta^2$, and let f be a homogenous element in ${}^\theta F'_\alpha \mathcal{O}$ of degree γ . Let us expand f on the dual semicanonical basis and write $f = \sum_{i=1}^\ell c_i \rho_{Y_i}$, where each scalar c_i is nonzero and where Y_i is an irreducible component of a representation space $\text{rep}(\Lambda, \nu_i)$. We decompose each Y_i according to the torsion pair $(\mathcal{I}_\theta, \overline{\mathcal{P}}_\theta)$, producing a dimension-vector μ_i and irreducible components $(Y_i)_t$ and $(Y_i)_f$ of $\text{rep}(\Lambda, \mu_i)$ and $\text{rep}(\Lambda, \nu_i - \mu_i)$, respectively. Let β be the largest element among the degrees $-J_\theta(\mu_i)$ with respect to the total order \leq' and let I be the set of all indices i such that $-J_\theta(\mu_i) = \beta$. Then the homogeneous component of degree $(\beta, \gamma - \beta)$ of $\Delta(f)$ is $\sum_{i \in I} c_i \rho_{(Y_i)_t} \otimes \rho_{(Y_i)_f}$. Since the pairs $((Y_i)_t, (Y_i)_f)$ are all different, this sum cannot be zero, so $\beta \in L(f)$ and therefore $\beta \leq' \alpha$. It follows that for each $i \in \llbracket 1, \ell \rrbracket$, the set $L_\theta(\rho_{Y_i})$ is contained in $\alpha + \Pi'$ and the function ρ_{Y_i} belongs to ${}^\theta F'_\alpha \mathcal{O}$. We conclude that ${}^\theta F'_\alpha \mathcal{O}$ is spanned by its intersection with the dual semicanonical basis. The same arguments imply the compatibility of the dual semicanonical basis with the isomorphism $\overline{\Delta}'$.

Condition (iv) in the definition of polite bases can be checked similarly, using instead the torsion pair $(\overline{\mathcal{I}}_\theta, \mathcal{P}_\theta)$ in $\Lambda\text{-mod}$. \square

3.5 Comparison with perfect bases

Berenstein and Kazhdan introduced in [8] the notion of perfect bases for locally finite $U(\mathfrak{n})$ -modules. The following is a reformulation of their definition in the case of the $U(\mathfrak{n})$ -module \mathcal{O} , actually a strengthened version of it where the values of some structure constants is prescribed. For $i \in I$ and $f \in \mathcal{O}$, let $\ell_i(f)$ denote the smallest nonnegative integer n such that $e_i^{n+1} \cdot f = 0$. Then a basis \mathbf{B} of \mathcal{O} consisting of weight vectors is said to be perfect if, for each $(i, n) \in I \times \mathbb{N}$, the rule $b \mapsto e_i^{(n)} \cdot b$ defines an injective map $\{b \in \mathbf{B} \mid \ell_i(b) = n\} \rightarrow \{b \in \mathbf{B} \mid e_i \cdot b = 0\}$.

We will say that a basis \mathbf{B} of \mathcal{O} is biperfect if both \mathbf{B} and its image by the involution $*$ are perfect.

Proposition 3.12 *A polite basis of \mathcal{O} is biperfect.*

Proof. We fix $i \in I$ and choose $\theta \in Q_{\mathbb{R}}^*$ so that $\theta(\alpha_i) < 0$ and that θ takes positive values on all the other positive roots. With this stability parameter, the group $N_{<0}$ defined in section 3.2 is the additive group \mathbb{G}_a defined by the root α_i and the algebra $\mathcal{O}_{<0}$ is the polynomial algebra $k[\zeta_i]$. Also, $\mathcal{O}_{\geq 0} = {}^{N_{<0}}\mathcal{O} = \{f \in \mathcal{O} \mid e_i \cdot f = 0\}$.

We set $\gamma = J_{\theta}(-\alpha_i)$; this is the degree of ζ_i relative to the Γ_{θ} -grading on \mathcal{O} . From Proposition 2.10, we deduce that $\text{gr}_{\bullet}'' \mathcal{O}$ is concentrated in degrees $n\gamma$ with $n \in \mathbb{N}$.

If a homogeneous element $f \in \mathcal{O}$ is in $F_{n\gamma}'' \mathcal{O}$, then $(n+1)\gamma \notin R(f)$ and therefore $e_i^{n+1} \cdot f = 0$. Conversely, if f is not in $F_{n\gamma}'' \mathcal{O}$, then there exists $\beta \in R(f)$ such that $\beta \notin n\gamma + \Pi''$. In other words, there exists an element $x \in U(\mathfrak{n})$ of degree β (for the Γ_{θ} -grading of $U(\mathfrak{n})$) such that $x \cdot f \neq 0$, and without loss of generality we can assume that x is a PBW monomial ending with some power of e_i . The fact that $\beta \notin n\gamma + \Pi''$ forces this power to be at least e_i^{n+1} , and we conclude that $e_i^{n+1} \cdot f \neq 0$.

The previous paragraph shows that for a homogeneous element $f \in \mathcal{O}$, the smallest nonnegative integer n such that $f \in F_{n\gamma}'' \mathcal{O}$ is $\ell_i(f)$. For this integer $n = \ell_i(f)$, denoting by \bar{f} the image of f in $\text{gr}_{n\gamma}'' \mathcal{O}$, we then have $\bar{\Delta}''(\bar{f}) = e_i^{(n)} \cdot f \otimes \varphi_i^n$, because $e_i^{(n)}$ spans the homogeneous component of degree $n\gamma$ in $U(\mathfrak{n})$ and φ_i^n is the dual basis element in \mathcal{O} .

Now let \mathbf{B} be a polite basis of \mathcal{O} . For $n \in \mathbb{N}$, set $\mathbf{B}_n = \{b \in \mathbf{B} \mid \ell_i(b) = n\}$. In view of our previous observations, the axioms of a polite basis require that \mathbf{B}_0 is a basis of $F_0'' \mathcal{O} = \mathcal{O}_{\geq 0}$, that $\{\bar{b} \mid b \in \mathbf{B}_n\}$ is a basis of $\text{gr}_{n\gamma}'' \mathcal{O}$, and that $\bar{b} \mapsto e_i^{(n)} \cdot b$ maps bijectively this basis onto \mathbf{B}_0 . Therefore \mathbf{B} is a perfect basis of \mathcal{O} .

Employing $-\theta$ as stability parameter and arguing in similar fashion, we deduce from the compatibility of \mathbf{B} with the filtration F'_{\bullet} and with the isomorphism $\bar{\Delta}'$ that \mathbf{B} enjoys the same property with respect to the right action of e_i on \mathcal{O} . Therefore $\{b^* \mid b \in \mathbf{B}\}$ is also perfect. \square

Perfect bases are automatically endowed with a structure of crystal in the sense of Kashiwara. We will not recall the definition of this notion; let us just say that we will use the standard

notation, with crystal operators denoted by \tilde{e}_i and \tilde{f}_i and with functions ε_i and φ_i for each $i \in I$. One important result from [8] is that the crystals of the perfect bases of \mathcal{O} are all isomorphic. The abstract crystal they share is denoted by $B(\infty)$.

Proposition 3.12 therefore entails that any polite basis \mathbf{B} is indexed by the crystal $B(\infty)$. Also, as we saw in section 3.3, the map $\underline{\text{Pol}}$ defines a bijection from \mathbf{B} onto the set of Mirković–Vilonen polytopes. We thus obtain a bijection from $B(\infty)$ onto the set of Mirković–Vilonen polytopes, and this bijection does not depend on \mathbf{B} ([3], Lemma 2.1). It follows that the set of Mirković–Vilonen polytopes has a natural crystal structure, isomorphic to $B(\infty)$. This result goes back to Kamnitzer, who described this bijection in the most explicit manner [13].

Remark 3.13. The converse of Proposition 3.12 is not true: bipерfect bases are not necessarily polite. Suppose indeed given a pair (b', b'') of elements in $B(\infty)$ which have the same weight and satisfy $\varepsilon_i(b') < \varepsilon_i(b'')$ and $\varepsilon_i^*(b') < \varepsilon_i^*(b'')$ for each $i \in I$, but also $\underline{\text{Pol}}(b') \not\subset \underline{\text{Pol}}(b'')$. By Proposition 3.9, the transition matrix between any two polite bases of \mathcal{O} have a zero entry at position (b'', b') . However, Fei explains in sect. 10.1 of [10] that one can find two bipерfect bases of $\mathcal{O}(N)$ such that the transition matrix between these bases has a nonzero entry at that position. In this case certainly one of these bipерfect bases is not polite.

The main point now is to show that such pairs (b', b'') do exist. The following is Example 2.7 (ii) in [3] (a smaller example is given in [10], Example 10.7). Here we are in type A_4 with the standard numbering of the vertices of the Dynkin diagram. We consider the two elements of $B(\infty)$

$$b' = \left(\tilde{f}_3 \tilde{f}_2 \tilde{f}_1 \tilde{f}_4 \tilde{f}_3 \tilde{f}_2 \right)^4 (1) \quad \text{and} \quad b'' = \left(\tilde{f}_1^2 \tilde{f}_3^5 \tilde{f}_2^5 \tilde{f}_4^2 \right) \left(\tilde{f}_2 \tilde{f}_3 \tilde{f}_4 \tilde{f}_1 \tilde{f}_2 \tilde{f}_3 \right) \left(\tilde{f}_4 \tilde{f}_3 \tilde{f}_2 \tilde{f}_1 \right) (1).$$

They have the same weight ν , and in the preprojective model [17, 19], they correspond to the irreducible components of $\text{rep}(\Lambda, \nu)$ whose general points are the Λ -modules

$$M' = \left(\begin{array}{ccc} & 2 & 3 \\ & 1 & 3 \\ & & 2 \end{array} \begin{array}{ccc} & & 4 \\ & & 4 \\ & & 4 \end{array} \right)^{\oplus 4} \quad \text{and} \quad M'' = \begin{array}{ccc} & 2 & 3 \\ & 1 & 3 \\ & & 2 \end{array} \begin{array}{ccc} & & 4 \\ & & 4 \\ & & 4 \end{array} \oplus \begin{array}{ccc} & 1 & 2 \\ & 2 & 3 \\ & & 4 \end{array} \oplus \left(\begin{array}{ccc} & 1 & 3 \\ & & 3 \\ & & 4 \end{array} \right)^{\oplus 2} \oplus \left(\begin{array}{ccc} & & 3 \\ & & 3 \\ & & 3 \end{array} \right)^{\oplus 3},$$

respectively. Looking at the heads and the socles of these modules, we find that

$$\begin{aligned} (\varepsilon_1(b'), \varepsilon_2(b'), \varepsilon_3(b'), \varepsilon_4(b')) &= (0, 0, 4, 0), & (\varepsilon_1(b''), \varepsilon_2(b''), \varepsilon_3(b''), \varepsilon_4(b'')) &= (2, 1, 5, 1), \\ (\varepsilon_1^*(b'), \varepsilon_2^*(b'), \varepsilon_3^*(b'), \varepsilon_4^*(b')) &= (0, 4, 0, 0), & (\varepsilon_1^*(b''), \varepsilon_2^*(b''), \varepsilon_3^*(b''), \varepsilon_4^*(b'')) &= (1, 5, 1, 2). \end{aligned}$$

On the other hand, adopting the notation of [5], we get

$$N(s_2 s_4 s_3 \omega_3) = \begin{array}{ccc} & 1 & 3 \\ & 2 & 3 \end{array}$$

and we can then compute

$$\dim \text{Hom}_\Lambda(N(s_2 s_4 s_3 \omega_3), M') = 4 > 2 = \dim \text{Hom}_\Lambda(N(s_2 s_4 s_3 \omega_3), M'').$$

Using Theorem 6.3 in *loc. cit.* we conclude that $\underline{\text{Pol}}(b') \not\subset \underline{\text{Pol}}(b'')$.

Computer experiments lead us to believe that in this type A_4 , the algebra \mathcal{O} has several bipерfect bases but only one polite basis.

4 Mirković and Vilonen's basis

4.1 Recollection about the Geometric Satake Equivalence

In this section, we present a very brief summary of the Geometric Satake Equivalence. We direct the reader to [22] for additional details.

We carry on with the notation set up in section 3.1 and denote the Borel subgroup opposite to B with respect to T by B_- . Since the group T is the quotient of B_- by its unipotent radical, any character λ of T can be inflated to a linear character of B_- , still denoted by λ . One can then consider the coinduced G -module (customarily called costandard)

$$\nabla(\lambda) = \{f \in \mathcal{O}(G) \mid \forall(b, g) \in B_- \times G, f(bg) = \lambda(b)f(g)\}.$$

If λ is dominant, then $\nabla(\lambda)$ has λ for highest weight and its character is given by Weyl's formula. The Geometric Satake Equivalence realizes $\nabla(\lambda)$ as the homology of a certain perverse sheaf $\mathcal{I}_*(\lambda, k)$ on the affine Grassmannian of the Langlands dual of G .

Let us set up the relevant notation. The Langlands dual of G will be denoted by G^\vee (for us, G^\vee will be a complex algebraic group). Its maximal torus T^\vee has P for cocharacter lattice and the root system of G^\vee is the coroot system of G . We choose an additive one-parameter subgroup $x_{\alpha^\vee} : \mathbb{G}_a \rightarrow G^\vee$ for each coroot α^\vee . We denote by N_\pm^\vee the subgroups of G^\vee generated by the subgroups x_{α^\vee} for $\alpha \in \Phi_\pm$. We introduce the ring $\mathbb{O} = \mathbb{C}[[z]]$ and its fraction field $\mathbb{K} = \mathbb{C}((z))$.

The affine Grassmannian of G^\vee is the homogeneous space $\mathcal{G}r = G^\vee(\mathbb{K})/G^\vee(\mathbb{O})$. It is the set of \mathbb{C} -points of a reduced projective ind-scheme over \mathbb{C} . Each weight $\lambda \in P$ can be regarded as a point z^λ in $T^\vee(\mathbb{K})$, hence defines a point L_λ in $\mathcal{G}r$. The orbit through L_λ under the action of $N_+^\vee(\mathbb{K})$ (respectively, $N_-^\vee(\mathbb{K})$) is denoted by S_λ (respectively, T_λ). Then

$$\mathcal{G}r = \bigsqcup_{\lambda \in P} S_\lambda = \bigsqcup_{\lambda \in P} T_\lambda, \quad \overline{S}_\lambda = \bigsqcup_{\mu \in Q_+} S_{\lambda-\mu}, \quad \overline{T}_\lambda = \bigsqcup_{\mu \in Q_+} T_{\lambda+\mu},$$

and for each $\mu \in P$, the action of z^μ on $\mathcal{G}r$ sends S_λ onto $S_{\lambda+\mu}$ and T_λ onto $T_{\lambda+\mu}$.

Let λ be a dominant weight. We denote the orbit through L_λ under the action of $G^\vee(\mathbb{O})$ by $\mathcal{G}r^\lambda$ and consider the perverse sheaf

$$\mathcal{I}_*(\lambda, k) = {}^p\tau_{\leq 0} (j_\lambda)_* \underline{k}_{\mathcal{G}r^\lambda}[2\rho(\lambda)]$$

in the derived category $D^b(\mathcal{G}r, k)$ of constructible sheaves on $\mathcal{G}r$. Here $2\rho : P \rightarrow \mathbb{Z}$ is the sum of the positive coroots, $j_\lambda : \mathcal{G}r^\lambda \rightarrow \mathcal{G}r$ is the inclusion map, $(j_\lambda)_*$ is the (derived) direct image, and ${}^p\tau_{\leq 0}$ is the truncation functor for the perverse t -structure. Then, under the Geometric Satake Equivalence, the module $\nabla(\lambda)$ is the hypercohomology $H^\bullet(\mathcal{G}r, \mathcal{I}_*(\lambda, k))$, and for all $\nu \in P$, the weight space $\nabla(\lambda)_\nu$ is $H^{2\rho(\nu)}(T_\nu, t_\nu^! \mathcal{I}_*(\lambda, k))$, where $t_\nu : T_\nu \rightarrow \mathcal{G}r$ is the inclusion map. Rewriting the sheaf as

$$\mathcal{I}_*(\lambda, k) = {}^p\tau_{\leq 0} (j_\lambda)_* \mathbb{D}_{\mathcal{G}r^\lambda}[-2\rho(\lambda)],$$

we get (see [22], Proposition 3.10)

$$\nabla(\lambda)_\nu \cong H^{2\rho(\nu-\lambda)}(\mathcal{G}r^\lambda \cap T_\nu, \mathbb{D}_{\mathcal{G}r^\lambda \cap T_\nu}) \cong H_{2\rho(\lambda-\nu)}^{\text{BM}}(\mathcal{G}r^\lambda \cap T_\nu, k).$$

Furthermore, all the irreducible components of $\mathcal{G}r^\lambda \cap T_\nu$ have dimension $\rho(\lambda - \nu)$, so their fundamental classes form a basis of $\nabla(\lambda)_\nu$.

Let $q : P \rightarrow \mathbb{Q}$ be a positive definite quadratic form on P , invariant under the Weyl group. It determines a central extension $\widetilde{\mathfrak{g}}^\vee$ of the Lie algebra $\mathfrak{g}^\vee \otimes \mathbb{C}[z, z^{-1}]$ by \mathbb{C} . The basic representation[†] V of $\widetilde{\mathfrak{g}}^\vee$ provides an embedding $\Upsilon : \mathcal{G}r \rightarrow \mathbb{P}(V)$ and a $G^\vee(\mathbb{K})$ -equivariant line bundle $\mathcal{L} = \Upsilon^* \mathcal{O}(1)$ on $\mathcal{G}r$. For each dominant weight λ , the cup-product with $c_1(\mathcal{L})$ defines an endomorphism of the vector space $H^\bullet(\mathcal{G}r, \mathcal{I}_*(\lambda, k))$. Then the Geometric Satake Equivalence, suitably normalized, identifies this endomorphism with the action on $\nabla(\lambda)$ of the principal nilpotent element $\sum_{i \in I} q(\alpha_i) e_i$.

4.2 Cutting Mirković–Vilonen cycles

For $(\lambda, \nu) \in P^2$, it is known that $S_\lambda \cap T_\lambda = \{L_\lambda\}$, that the intersection $S_\lambda \cap T_\nu$ is non-empty if and only if $\lambda - \nu \in Q_+$, and that in this case this intersection has pure dimension $\rho(\lambda - \nu)$. The irreducible components of the closure $\overline{S_\lambda \cap T_\nu}$ are called Mirković–Vilonen cycles of weight (λ, ν) . We shall denote the set they form by $\mathcal{X}_{\lambda, \nu}$.

We now fix a regular element $\theta \in Q_{\mathbb{R}}^*$ and define

$$\Phi_\theta = \{\alpha \in \Phi \mid \theta(\alpha) < 0\}.$$

We define $\mathbb{M} = z^{-1}\mathbb{C}[z^{-1}]$, a subspace of \mathbb{K} . For $\mathbb{A} \in \{\mathbb{K}, \mathbb{O}, \mathbb{M}\}$, we denote by $U^\vee(\mathbb{A})$ (respectively, $U_\pm^\vee(\mathbb{A})$) the subgroup of $G^\vee(\mathbb{K})$ generated by the elements $x_{\alpha^\vee}(a)$ with $a \in \mathbb{A}$ and $\alpha \in \Phi_\theta$ (respectively, $\alpha \in \Phi_\theta \cap \Phi_\pm$).

Lemma 4.1 (i) For $\mathbb{A} \in \{\mathbb{K}, \mathbb{O}, \mathbb{M}\}$, we have $U^\vee(\mathbb{A}) = U_+^\vee(\mathbb{A}) \bowtie U_-^\vee(\mathbb{A})$.

(ii) We have the following decompositions:

$$U^\vee(\mathbb{K}) = U^\vee(\mathbb{M}) \bowtie U^\vee(\mathbb{O}) \quad \text{and} \quad U_\pm^\vee(\mathbb{K}) = U_\pm^\vee(\mathbb{M}) \bowtie U_\pm^\vee(\mathbb{O}).$$

Proof. For all coroots α^\vee and β^\vee that are not opposite, there exist commutation relations of the form

$$x_{\alpha^\vee}(a) x_{\beta^\vee}(b) x_{\alpha^\vee}(a)^{-1} x_{\beta^\vee}(b)^{-1} = \prod_{i,j>0} x_{i\alpha^\vee + j\beta^\vee}(C_{i,j} a^i b^j)$$

[†]If G is simple and if one chooses q such that $q(\alpha) = 1$ for each short root α , then $\widetilde{\mathfrak{g}}^\vee$ is an affine untwisted Kac–Moody algebra—up to the derivation d/dz —and the basic representation is the integrable representation with highest weight Λ_0 .

with $C_{i,j} \in \mathbb{C}$; see for instance [9], chapter 5. These relations allow to write any element in $U^\vee(\mathbb{K})$ in a unique way as a product

$$\prod_{\alpha \in \Phi_\theta} x_{\alpha^\vee}(a_\alpha) \quad (*)$$

computed according to any given convex order on Φ_θ . Choosing a convex order for which the elements in $\Phi_\theta \cap \Phi_+$ are smaller than the elements in $\Phi_\theta \cap \Phi_-$, this fact gives the decomposition $U^\vee(\mathbb{A}) = U_+^\vee(\mathbb{A}) \bowtie U_-^\vee(\mathbb{A})$ for $\mathbb{A} = \mathbb{K}$. Noting that, in the commutation relation, the monomials $C_{i,j} a^i b^j$ belong to \mathbb{O} (respectively, \mathbb{M}) as soon as a and b do so, we can use the same process to obtain the decomposition for $\mathbb{A} = \mathbb{O}$ (respectively, $\mathbb{A} = \mathbb{M}$). Along the way, we note that for $\mathbb{A} \in \{\mathbb{O}, \mathbb{M}\}$, the element $(*)$ belongs to $U^\vee(\mathbb{A})$ if and only if all the a_α belong to \mathbb{A} .

We have so far proved the first item. To prove the second, we enumerate the roots $\alpha_1, \dots, \alpha_n$ in Φ_θ according to a convex order. To simplify the notation, we write x_i instead of $x_{\alpha_i^\vee}$. Let $g \in U^\vee(\mathbb{K})$. We claim that for any $k \in \llbracket 0, n \rrbracket$, g can be written as a product

$$x_1(b_1) \cdots x_k(b_k) x_{k+1}(a_{k+1}) \cdots x_n(a_n) x_k(c_k) \cdots x_1(c_1),$$

with a_i in \mathbb{K} , $b_i \in \mathbb{M}$ and $c_i \in \mathbb{O}$. For $k = 0$, this is $(*)$. Let us assume that such a factorization exists for $k \in \llbracket 0, n-1 \rrbracket$ and let us decompose $a_{k+1} = b_{k+1} + c_{k+1}$ with $(b_{k+1}, c_{k+1}) \in \mathbb{M} \times \mathbb{O}$. As the ordering is convex, each coroot of the form $i\alpha_{k+1}^\vee + j\alpha_\ell^\vee$ with i and j positive and $\ell \geq k+2$ belongs to $\{\alpha_{k+2}^\vee, \dots, \alpha_n^\vee\}$, so there exists a'_{k+2}, \dots, a'_n in \mathbb{K} such that

$$x_{k+1}(c_{k+1}) x_{k+2}(a_{k+2}) \cdots x_n(a_n) x_{k+1}(c_{k+1})^{-1} = x_{k+2}(a'_{k+2}) \cdots x_n(a'_n).$$

We then obtain the desired writing

$$g = x_1(b_1) \cdots x_{k+1}(b_{k+1}) x_{k+2}(a'_{k+2}) \cdots x_n(a'_n) x_{k+1}(c_{k+1}) \cdots x_1(c_1)$$

for $k+1$, which establishes our claim by induction. For $k = n$, we get a factorization $g = g'g''$ with $(g', g'') \in U^\vee(\mathbb{M}) \times U^\vee(\mathbb{O})$. Moreover, the uniqueness of the writing $(*)$ implies that $U^\vee(\mathbb{M}) \cap U^\vee(\mathbb{O}) = \{1\}$. We conclude that $U^\vee(\mathbb{K}) = U^\vee(\mathbb{M}) \bowtie U^\vee(\mathbb{O})$. Similar arguments prove the remaining decompositions. \square

For $\mu \in P$, we denote by R_μ the orbit through L_μ under the action of the group $U^\vee(\mathbb{K})$. The stabilizer of L_0 under this action is $U^\vee(\mathbb{O})$, so by Lemma 4.1 the map $g \mapsto g \cdot L_0$ is a bijection from $U^\vee(\mathbb{M})$ onto R_0 .

Lemma 4.2 *For each $\mu \in P$, the intersection $S_\mu \cap R_\mu$ (respectively, $T_\mu \cap R_\mu$) is the orbit through L_μ under the action of the group $U_+^\vee(\mathbb{K})$ (respectively, $U_-^\vee(\mathbb{K})$).*

Proof. The inclusion $U_+^\vee(\mathbb{K}) \cdot L_\mu \subset S_\mu \cap R_\mu$ is banal. For the reverse direction, we choose an element $x \in S_\mu \cap R_\mu$ and write $x = g \cdot L_\mu$ with $g \in U^\vee(\mathbb{K})$. Using Lemma 4.1, we decompose $g = g_+ g_-$ with $g_\pm \in U_\pm^\vee(\mathbb{K})$. Since $x \in S_\mu$, we get $g_+^{-1} \cdot x \in S_\mu$. As $S_\mu \cap T_\mu = \{L_\mu\}$, we obtain $g_- \cdot L_\mu = L_\mu$, and therefore $x = g_+ \cdot L_\mu$ belongs to $U_+^\vee(\mathbb{K}) \cdot L_\mu$. \square

We define a map

$$\Omega_\mu : R_\mu \rightarrow (T_\mu \cap R_\mu) \times (S_\mu \cap R_\mu)$$

as follows. Given $x \in R_\mu$, there is a unique $g \in U^\vee(\mathbb{M})$ such that $z^{-\mu} \cdot x = g \cdot L_0$. By Lemma 4.1, we can write in a unique fashion $g = g_+ g_- = h_- h_+$ with g_+, h_+ in $U_+^\vee(\mathbb{M})$ and g_-, h_- in $U_-^\vee(\mathbb{M})$. We then set $\Omega_\mu(x) = (z^\mu g_- \cdot L_0, z^\mu h_+ \cdot L_0)$.

Proposition 4.3 *Let $(\lambda, \mu, \nu) \in P^3$. Then the map Ω_μ is bijective and restricts to an isomorphism of algebraic varieties from $S_\lambda \cap T_\nu \cap R_\mu$ onto $(S_\lambda \cap T_\mu \cap R_\mu) \times (T_\nu \cap S_\mu \cap R_\mu)$.*

Proof. We readily reduce to the case $\mu = 0$.

Given $(y, z) \in (T_0 \cap R_0) \times (S_0 \cap R_0)$, we write $y = g_- \cdot L_0$ and $z = h_+ \cdot L_0$ with g_- in $U_-^\vee(\mathbb{M})$ and h_+ in $U_+^\vee(\mathbb{M})$, and we write the element $g_- h_+^{-1}$ of $U^\vee(\mathbb{M})$ as $g_+^{-1} h_-$ with $(g_+^{-1}, h_-) \in U_+^\vee(\mathbb{M}) \times U_-^\vee(\mathbb{M})$. We then have $g_+ g_- = h_- h_+$, and we check that $x = g_+ g_- \cdot L_0$ is the unique element in R_0 such that $(y, z) = \Omega_0(x)$. The map Ω_0 is therefore bijective.

Suppose that $x \in R_0$ and set $(y, z) = \Omega_0(x)$. Then x and y belong to the same orbit under the action of $N_+^\vee(\mathbb{K})$, so $x \in S_\lambda$ if and only if $y \in S_\lambda$. Likewise, $x \in T_\nu$ if and only if $z \in T_\nu$. We conclude that Ω_0 induces a bijection from $S_\lambda \cap T_\nu \cap R_0$ onto $(S_\lambda \cap T_0 \cap R_0) \times (T_\nu \cap S_0 \cap R_0)$. This bijection is in fact an isomorphism of algebraic varieties because the definition of Ω_0 and the construction of its inverse depend only on the operations that provide the decomposition $U^\vee(\mathbb{M}) = U_+^\vee(\mathbb{M}) \bowtie U_-^\vee(\mathbb{M})$. \square

The subsets R_μ are locally closed and form a partition of $\mathcal{G}r$, so for any irreducible subvariety Z of $\mathcal{G}r$, there is a unique weight $\mu \in P$ such that $Z \cap R_\mu$ is open and dense in Z . For $(\lambda, \mu, \nu) \in P^3$, we denote by $\mathcal{Z}_{\lambda, \nu}^\mu$ the subset of $\mathcal{Z}_{\lambda, \nu}$ formed by the cycles Z whose general point belongs to R_μ . Therefore $\mathcal{Z}_{\lambda, \nu}^\mu$ is the set of irreducible components of $\overline{S_\lambda \cap T_\nu \cap R_\mu}$ of dimension $\rho(\lambda - \nu)$.

Proposition 4.4 *The map $Z \mapsto (\overline{Z \cap T_\mu \cap R_\mu}, \overline{Z \cap S_\mu \cap R_\mu})$ defines a bijection*

$$\Xi : \mathcal{Z}_{\lambda, \nu}^\mu \xrightarrow{\cong} \mathcal{Z}_{\lambda, \mu}^\mu \times \mathcal{Z}_{\mu, \nu}^\mu.$$

Proof. We denote the set of irreducible components of a topological space X by $\text{Irr}(X)$. The proof of Proposition 4.3 implies that Ω_μ restricts to an isomorphism of algebraic varieties

$$\overline{S_\lambda \cap T_\nu \cap R_\mu} \xrightarrow{\cong} (\overline{S_\lambda \cap T_\mu \cap R_\mu}) \times (\overline{T_\nu \cap S_\mu \cap R_\mu}),$$

so induces a bijection

$$\text{Irr}(\overline{S_\lambda \cap T_\nu \cap R_\mu}) \xrightarrow{\cong} \text{Irr}(\overline{S_\lambda \cap T_\mu \cap R_\mu}) \times \text{Irr}(\overline{T_\nu \cap S_\mu \cap R_\mu}).$$

The dimension of $\overline{S_\lambda \cap T_\nu \cap R_\mu}$ is less than or equal to $\rho(\lambda - \nu)$, and likewise the dimensions of $\overline{S_\lambda \cap T_\mu \cap R_\mu}$ and $\overline{T_\nu \cap S_\mu \cap R_\mu}$ are bounded by $\rho(\lambda - \mu)$ and $\rho(\mu - \nu)$, respectively.

The map $Z \mapsto Z \cap R_\mu$ defines a bijection from $\mathcal{Z}_{\lambda,\nu}^\mu$ onto the set of irreducible components of $\overline{S_\lambda} \cap \overline{T_\nu} \cap R_\mu$ of dimension $\rho(\lambda - \nu)$, and there are similar bijections from $\mathcal{Z}_{\lambda,\mu}^\mu$ and $\mathcal{Z}_{\mu,\nu}^\mu$ onto the sets of irreducible components of $\overline{S_\lambda} \cap \overline{T_\mu} \cap R_\mu$ and $\overline{T_\nu} \cap \overline{S_\mu} \cap R_\mu$ of dimension $\rho(\lambda - \mu)$ and $\rho(\mu - \nu)$, respectively. Therefore the above bijection produces by restriction a bijection $\Xi : \mathcal{Z}_{\lambda,\nu}^\mu \xrightarrow{\cong} \mathcal{Z}_{\lambda,\mu}^\mu \times \mathcal{Z}_{\mu,\nu}^\mu$, and to conclude the proof it remains to check that Ξ is the map described in the statement.

To simplify the notation, we shall take $\mu = 0$. By construction, Ξ relates a Mirković–Vilonen cycle $Z \in \mathcal{Z}_{\lambda,\nu}^0$ and a pair $(Z_t, Z_s) \in \mathcal{Z}_{\lambda,0}^0 \times \mathcal{Z}_{0,\nu}^0$ if and only if

$$Z \cap R_0 = \{g \cdot L_0 \mid g_- \cdot L_0 \in Z_t \cap (T_0 \cap R_0) \text{ and } h_+ \cdot L_0 \in Z_s \cap (S_0 \cap R_0)\},$$

where as before $g \in U^\vee(\mathbb{M})$ is written in the form g_+g_- or h_-h_+ with g_+, h_+ in $U_+^\vee(\mathbb{M})$ and g_-, h_- in $U_-^\vee(\mathbb{M})$.

With these notations, if $g \cdot L_0 \in S_0$, then $g \in U_+^\vee(\mathbb{M})$, therefore $g = h_+$. It follows that $Z \cap (S_0 \cap R_0) \subset Z_s \cap (S_0 \cap R_0)$, whence $\overline{Z \cap S_0 \cap R_0} \subset Z_s$. For the reverse inclusion, we note that $L_0 \in Z_t \cap (T_0 \cap R_0)$, so (taking $g_- = h_- = 1$) $Z_s \cap (S_0 \cap R_0) \subset Z \cap R_0$, and therefore $Z_s = \overline{Z_s \cap S_0 \cap R_0} \subset \overline{Z \cap S_0 \cap R_0}$. We conclude that $Z_s = \overline{Z \cap S_0 \cap R_0}$. The equality $Z_t = \overline{Z \cap T_0 \cap R_0}$ is established in like manner. \square

We have assumed thus far in this section that $\theta \in Q_{\mathbb{R}}^*$ is regular, but this assumption was only made to simplify the exposition. All the constructions presented are indeed valid in general provided we replace Φ_θ by either

$$'\Phi_\theta = \{\alpha \in \Phi \mid \theta(\alpha) < 0\} \cup \{\alpha \in \Phi_+ \mid \theta(\alpha) = 0\}$$

or

$$''\Phi_\theta = \{\alpha \in \Phi \mid \theta(\alpha) < 0\} \cup \{\alpha \in \Phi_- \mid \theta(\alpha) = 0\}.$$

4.3 Politeness of Mirković and Vilonen's basis

For each dominant weight λ , the costandard module $\nabla(\lambda)$, as defined in section 4.1, is a subspace of $\mathcal{O}(G)$. It contains a unique vector v_λ whose restriction to N is the constant function equal to 1, and this vector has weight λ . Given a second dominant weight μ , the multiplication map $f \mapsto fv_\mu$ embeds $\nabla(\lambda)$ as an N -submodule of $\nabla(\lambda + \mu)$. The restriction to N , to wit the map $f \mapsto f|_N$, defines an embedding of N -modules $\Psi_\lambda : \nabla(\lambda) \rightarrow \mathcal{O}$, and the diagram

$$\begin{array}{ccc} \nabla(\lambda) & \xrightarrow{fv_\mu} & \nabla(\lambda + \mu) \\ \Psi_\lambda \searrow & & \swarrow \Psi_{\lambda+\mu} \\ & \mathcal{O} & \end{array}$$

commutes. In this fashion \mathcal{O} is identified with the direct limit of the spaces $\nabla(\lambda)$.

As we saw, the Geometric Satake Equivalence provides an identification

$$\nabla(\lambda) \cong \bigoplus_{\nu \in P} H_{2\rho(\lambda-\nu)}^{\text{BM}}(\mathcal{G}r^\lambda \cap T_\nu, k)$$

and the fundamental classes of the irreducible components of $\mathcal{G}r^\lambda \cap T_\nu$, for all $\nu \in P$, form a basis of $\nabla(\lambda)$. Now let $\nu \in Q_-$ and let $Z \in \mathcal{Z}_{0,\nu}$. If λ is a dominant enough weight, then Z is contained in $z^{-\lambda} \overline{\mathcal{G}r^\lambda}$. In this case, $z^\lambda Z$ is the closure of an irreducible component Y of $\mathcal{G}r^\lambda \cap T_{\lambda+\nu}$ ([1], Proposition 3); furthermore, the image by $\Psi_\lambda : \nabla(\lambda) \rightarrow \mathcal{O}$ of the fundamental class of Y depends only on Z and not on λ ([6], Proposition 6.1). We denote this image by b_Z . For each $\nu \in Q_-$, the elements b_Z with $Z \in \mathcal{Z}_{0,\nu}$ form a basis of \mathcal{O}_ν . The Mirković–Vilonen basis of \mathcal{O} is the family of all these functions b_Z .

Let us resume the discussion in the last paragraph of section 4.1, with the projective embedding $\Upsilon : \mathcal{G}r \rightarrow \mathbb{P}(V)$ derived from the choice of a quadratic form q . Let \mathfrak{H} be the direct sum of all weight subspaces of V save for the highest weight (with respect to the Cartan subalgebra of the semi-direct product $\widetilde{\mathfrak{g}^\vee} \rtimes \mathbb{C} \frac{d}{dz}$); this is a hyperplane of V . We denote by D the divisor on $\mathcal{G}r$ cut by $\mathbb{P}(\mathfrak{H})$, and for $\nu \in P$, we set $D_\nu = z^\nu D$. By [22], Proposition 3.1, we then have

$$D_\nu \cap \overline{T_\nu} = \bigcup_{i \in I} \overline{T_{\nu+\alpha_i}}.$$

Given two Mirković–Vilonen cycles Z and Z' of weights (λ, ν) and $(\lambda, \nu + \alpha_i)$, respectively, we denote by $i(Z', D_\nu \cdot Z)$ the multiplicity of Z' in the intersection product $D_\nu \cdot Z$.

The basis elements b_Z can now be explicitly described in the following way. If Z is a Mirković–Vilonen cycle of weight $(0, \nu)$ and $\mathbf{i} = i_1 \cdots i_n$ is a word of weight $-\nu$, we denote by $\mathcal{C}_{\mathbf{i}, Z}$ the set of all chains $\{L_0\} = Y_0 \subset Y_1 \subset \cdots \subset Y_n = Z$ of Mirković–Vilonen cycles such that $Y_j \in \mathcal{Z}_{0,\nu_j}$ for each $j \in \llbracket 0, n \rrbracket$, where $\nu_j = -(\alpha_{i_1} + \cdots + \alpha_{i_j})$. By [6], Theorem 5.4, we then have

$$\langle e_{\mathbf{i}}, b_Z \rangle = \frac{1}{q(\alpha_{i_1}) \cdots q(\alpha_{i_n})} \sum_{Y_\bullet \in \mathcal{C}_{\mathbf{i}, Z}} i(Y_0, D_{\nu_1} \cdot Y_1) \cdots i(Y_{n-1}, D_{\nu_n} \cdot Y_n).$$

We will need the following technical lemmas. The second one confirms in part Anderson’s description of the coproduct of \mathcal{O} in the Mirković–Vilonen basis ([1], section 10). In both statements, we assume that a stability parameter θ has been fixed in $Q_{\mathbb{R}}^*$ and we adopt the notation of section 4.2, with Φ_θ replaced by either $'\Phi_\theta$ or $''\Phi_\theta$ if θ is not regular.

Lemma 4.5 *Let $(\lambda, \mu, \nu) \in P^3$, let $i \in I$, and let $(Z, Z') \in \mathcal{Z}_{\lambda,\nu}^\mu \times \mathcal{Z}_{\lambda,\nu+\alpha_i}^\mu$ be such that $Z' \subset Z$. Write $\Xi(Z) = (Z_t, Z_s)$ and $\Xi(Z') = (Z'_t, Z'_s)$. Then $Z_t = Z'_t$ and $i(Z'_s, D_\nu \cdot Z_s) = i(Z', D_\nu \cdot Z)$.*

Proof. We observe that $Z'_t = \overline{Z' \cap T_\mu \cap R_\mu} \subset \overline{Z \cap T_\mu \cap R_\mu} = Z_t$. As both Z_t and Z'_t are Mirković–Vilonen cycles of weight (λ, μ) , they have the same dimension, and therefore the inclusion is an equality.

We set $\dot{Z} = Z \cap R_\mu$, $\dot{Z}_t = Z \cap T_\mu \cap R_\mu$, $\dot{Z}_s = Z \cap S_\mu \cap R_\mu$, $\dot{Z}' = Z' \cap R_\mu$ and $\dot{Z}'_s = Z' \cap S_\mu \cap R_\mu$. These are affine varieties. The map $\tilde{\Omega}_\mu$ restricts to an isomorphism of algebraic varieties $\tilde{\Omega}_\mu : \dot{Z} \rightarrow \dot{Z}_t \times \dot{Z}_s$. Let $\text{pr}_2 : \dot{Z}_t \times \dot{Z}_s \rightarrow \dot{Z}_s$ be the second projection. We have $\tilde{\Omega}_\mu(\dot{Z}') = \dot{Z}_t \times \dot{Z}'_s$, that is, $\dot{Z}' = (\text{pr}_2 \circ \tilde{\Omega}_\mu)^{-1}(\dot{Z}'_s)$.

The Lie algebra $\widehat{\mathfrak{g}}^\vee$ integrates to a central extension $E(G^\vee(\mathbb{K}))$ of the loop group $G^\vee(\mathbb{K})$, so that V is an ordinary representation of $E(G^\vee(\mathbb{K}))$. The cocycle on $G^\vee(\mathbb{K})$ that defines this central extension $E(G^\vee(\mathbb{K}))$ is trivial when restricted to the subgroups $N_\pm^\vee(\mathbb{K})$ or $U^\vee(\mathbb{K})$. Therefore these subgroups lift (non-canonically) to the central extension, in other words their action on $\mathcal{G}r$ lifts to an action on V .

Let σ and τ be linear forms on V with kernels $z^\nu \mathfrak{H}$ and $z^\mu \mathfrak{H}$, respectively. Then τ is nonzero on the line $\Upsilon(L_\mu)$ and it vanishes on all the other weight subspaces of V , so it takes a constant nonzero value on the $U^\vee(\mathbb{K})$ -orbit of any nonzero vector v_1 in $\Upsilon(L_\mu)$. In particular τ , regarded as a section of the line bundle \mathcal{L} , does not vanish on R_μ . If we also regard σ as a section of \mathcal{L} , then $f = \sigma/\tau$ is a rational function on $\mathcal{G}r$ which is regular on R_μ . This function f is the equation of the divisor D_ν on R_μ .

We claim that if x is in \dot{Z} and if $y = \text{pr}_2 \circ \tilde{\Omega}_\mu(x)$, then $f(x) = f(y)$. To show this, we write $x = g \cdot L_\mu$ with $g \in z^\mu U^\vee(\mathbb{M}) z^{-\mu}$, we decompose $g = h_- h_+$ with $h_\pm \in z^\mu U_\pm^\vee(\mathbb{M}) z^{-\mu}$, and we note that $y = h_+ \cdot L_\mu$ and $x = h_- \cdot y$. The closure $\overline{T_\nu}$ contains Z and is the disjoint union of the semi-infinite orbit T_ν and its boundary $D \cap \overline{T_\nu}$, which are both $N_-^\vee(\mathbb{K})$ -invariant. The points x and y are therefore both either in $D \cap \overline{T_\nu}$ or in T_ν . In the first case, we have $f(x) = f(y) = 0$. In the second case, we note that the vector $v_3 = h_- \cdot v_1$ belongs to the line $\Upsilon(y)$. As τ assumes a constant value on the orbit of v_1 under the action of $U^\vee(\mathbb{K})$, we have $\tau(h_- \cdot v_3) = \tau(v_3)$. As y is assumed to be in T_ν , there exists $g' \in N_-^\vee(\mathbb{K})$ such that $y = g' \cdot L_\nu$, so $v_2 = g'^{-1} \cdot v_3$ is in $\Upsilon(L_\nu)$. We can now repeat our argument: σ is nonzero on the line $\Upsilon(L_\nu)$ and it vanishes on all the other weight subspaces of V , so it takes a constant nonzero value on the $N_-^\vee(\mathbb{K})$ -orbit of v_2 . We deduce that $\sigma(h_- \cdot v_3) = \sigma(v_3)$, and therefore $f(x) = f(y)$ in this case as well. Our claim is proved.

In conclusion, the equation of the divisor D_ν and the equation of the subvariety \dot{Z}' inside \dot{Z} both factorize through $\text{pr}_2 \circ \tilde{\Omega}_\mu$. This implies that $i(Z', D_\nu \cdot Z) = i(Z'_s, D_\nu \cdot Z_s)$. \square

Lemma 4.6 *Let $(\mu, \nu) \in P \times Q_-$ and let $Z \in \mathcal{Z}_{0,\nu}^\mu$.*

- (i) *We have $L_\mu \in Z$, and if an element L_η belongs to Z , then $\mu - \eta$ is a nonnegative linear combination of roots in Φ_θ .*
- (ii) *Cut Z in two Mirković–Vilonen cycles $Z_t = \overline{Z \cap T_\mu \cap R_\mu}$ and $Z_s = \overline{Z \cap S_\mu \cap R_\mu}$. Then the component of weight $(\mu, \nu - \mu)$ of $\Delta(b_Z)$ is equal to $b_{Z_t} \otimes b_{z^{-\mu} Z_s}$.*

Proof. The first item is standard: L_μ is in Z because Z meets R_μ and is closed and invariant under the action of $T^\vee(\mathbb{C})$, and the second point follows from the inclusion $Z \subset \overline{R_\mu}$.

Since $\mathcal{Z}_{0,\nu}^\mu$ is non-empty, both μ and $\nu - \mu$ are in Q_- . Let $\mathbf{i} = i_1 \cdots i_m$ and $\mathbf{j} = i_{m+1} \cdots i_n$ be words in the alphabet I of weights $-\mu$ and $-\nu + \mu$, respectively. We set $\nu_j = -(\alpha_{i_1} + \cdots + \alpha_{i_j})$ for each $j \in \llbracket 0, n \rrbracket$. Let Y_\bullet be an element in $\mathcal{C}_{\mathbf{ij}, Z}$. For each $j \in \llbracket 0, n \rrbracket$, we have $Y_j \subset Z \subset \overline{R_\mu}$, so $Y_j \cap R_\mu$ is open in Y_j . We observe that $L_\mu \in Y_m$, so $Y_j \cap R_\mu$ is non-empty if $j \geq m$. Therefore for all $j \in \llbracket m, n \rrbracket$, the general point of Y_j belongs to R_μ . Since both $Y_m \cap T_\mu$ and $Y_m \cap R_\mu$ are open dense in Y_m , we have

$$Y_m = \overline{Y_m \cap T_\mu \cap R_\mu} \subset \overline{Z \cap T_\mu \cap R_\mu} = Z_t.$$

This implies that $Y_m = Z_t$, because both Y_m and Z_t are Mirković–Vilonen cycles of weight $(0, \mu)$, and we deduce that $\overline{Y_j \cap T_\mu \cap R_\mu} = Z_t$ for all $j \in \llbracket m, n \rrbracket$. In the notation of Proposition 4.4, we can then write $\Xi(Y_j) = (Z_t, Y'_j)$, where $Y'_j = \overline{Y_j \cap S_\mu \cap R_\mu}$ is in \mathcal{Z}_{μ, ν_j} . Plainly then, $(Y_0, \dots, Y_m) \in \mathcal{C}_{\mathbf{i}, Z_t}$ and $(z^{-\mu} Y'_m, \dots, z^{-\mu} Y'_n) \in \mathcal{C}_{\mathbf{j}, z^{-\mu} Z_s}$. We can thus define a map $\mathcal{C}_{\mathbf{ij}, Z} \rightarrow \mathcal{C}_{\mathbf{i}, Z_t} \times \mathcal{C}_{\mathbf{j}, z^{-\mu} Z_s}$, which is patently bijective.

From Lemma 4.5, we deduce that for all $j \in \llbracket m+1, n \rrbracket$,

$$i(Y_{j-1}, D_{\nu_j} \cdot Y_j) = i(Y'_{j-1}, D_{\nu_j} \cdot Y'_j) = i(z^{-\mu} Y'_{j-1}, D_{\nu_j - \mu} \cdot z^{-\mu} Y'_j).$$

A straightforward calculation then shows that $\langle e_{\mathbf{ij}}, b_Z \rangle = \langle e_{\mathbf{i}}, b_{Z_t} \rangle \langle e_{\mathbf{j}}, b_{z^{-\mu} Z_s} \rangle$. The truthfulness of this equality for all words \mathbf{i} and \mathbf{j} of weights $-\mu$ and $-\nu + \mu$ establishes that the component of weight $(\mu, \nu - \mu)$ of $\Delta(b_Z)$ is equal to $b_{Z_t} \otimes b_{z^{-\mu} Z_s}$. \square

The group $G^\vee(\mathbb{K})$ and its subgroup $T^\vee(\mathbb{C})$ act on the affine Grassmannian $\mathcal{G}r$. We define the moment polytope of a closed irreducible $T^\vee(\mathbb{C})$ -invariant subvariety X of $\mathcal{G}r$ as the convex hull in $P \otimes_{\mathbb{Z}} \mathbb{R}$ of the set $\{\mu \in P \mid L_\mu \in X\}$. We refer to [1], section 6 for an explanation of the terminology ‘moment polytope’. The moment polytope of a closed irreducible $T^\vee(\mathbb{C})$ -invariant subvariety is always a GGMS polytope (see *loc. cit.*; see also Lemma 2.3 in [14]).

Proposition 4.7 *The moment polytope of a Mirković–Vilonen cycle Z is equal to $\underline{\text{Pol}}(b_Z)$.*

Proof. Let ν be the weight of b_Z , so that $Z \in \mathcal{Z}_{0,\nu}$. Let us pick $\mu \in \underline{L}(b_Z)$. The component of weight $(\mu, \nu - \mu)$ of $\Delta(b_Z)$ is nonzero, so there exist words \mathbf{i} and \mathbf{j} of weights $-\mu$ and $-\nu + \mu$, respectively, such that $\langle e_{\mathbf{ij}}, b_Z \rangle \neq 0$. Therefore $\mathcal{C}_{\mathbf{ij}, Z} \neq \emptyset$, so there exists a Mirković–Vilonen cycle $Z' \in \mathcal{Z}_{0,\mu}$ which is contained in Z . Then $L_\mu \in Z$ and μ belongs to the moment polytope of Z . We conclude that $\underline{L}(b_Z)$, and therefore its convex hull $\underline{\text{Pol}}(b_Z)$, is contained in the moment polytope of Z .

To prove the reverse inclusion, we choose a vertex μ of the moment polytope of Z . Let θ be a regular element in $Q_{\mathbb{R}}^*$ such that $-\theta$ lies in the normal cone at μ of this polytope. Adopting the notation of section 4.2, we claim that the general point of Z belongs to the semi-infinite orbit R_μ .

In fact, denote by λ the weight such that $Z \cap R_\lambda$ is open dense in Z . By Lemma 4.6, as L_μ belongs to Z , the difference $\lambda - \mu$ is a nonnegative linear combination of roots α such that

$\theta(\alpha) < 0$. It follows that $\theta(\lambda - \mu) \leq 0$, with equality only if $\lambda = \mu$. On the other hand, λ belongs to the moment polytope of Z , and our choice of θ implies that $\theta(\lambda - \mu) \geq 0$. We then conclude that $\lambda = \mu$, as announced.

Returning to the main proof, we deduce from Lemma 4.6 that the component of weight $(\mu, \nu - \mu)$ of $\Delta(b_Z)$ is equal to $b_{Z_t} \otimes b_{z^{-\mu}Z_s}$ with $(Z_t, Z_s) \in \mathcal{Z}_{0,\mu} \times \mathcal{Z}_{\mu,\nu}$. This is not zero, so $\mu \in \underline{L}(b_Z)$. We conclude that all the vertices of the moment polytope of Z belong to $\underline{\text{Pol}}(b_Z)$. \square

Theorem 4.8 *The Mirković–Vilonen basis of \mathcal{O} is polite.*

Proof. By construction, the Mirković–Vilonen basis satisfies the condition (i) in the definition of polite bases. The condition (ii) follows from Proposition 5.5 in [6].

We now show that the Mirković–Vilonen basis satisfies condition (iii). Let $\theta \in Q_{\mathbb{R}}^*$, let $(\alpha, \gamma) \in \Gamma_{\theta}^2$, and let f be a homogenous element in ${}^{\theta}F'_{\alpha}\mathcal{O}$ of degree γ . Let us expand f on the Mirković–Vilonen basis and write $f = \sum_{i=1}^{\ell} c_i b_{Z_i}$, where each scalar c_i is nonzero and where Z_i is in \mathcal{Z}_{0,ν_i} for a certain weight ν_i . We adopt the notation of section 4.2 with Φ_{θ} replaced by $'\Phi_{\theta}$. We denote by μ_i the weight such that the general point of Z_i belongs to the semi-infinite orbit R_{μ_i} and we cut Z_i at L_{μ_i} , producing cycles $(Z_i)_t$ and $(Z_i)_s$ in \mathcal{Z}_{0,μ_i} and $\mathcal{Z}_{\mu_i,\nu_i}$, respectively.

By the proof of Proposition 4.7, any element $\eta \in \underline{L}(b_{Z_i})$ verifies $L_{\eta} \in Z_i$. It then follows from Lemma 4.6 that $\mu_i - \eta$ is a nonnegative linear combination of roots α such that either $\theta(\alpha) < 0$, or $\theta(\alpha) = 0$ and $\text{ht } \alpha > 0$. Therefore $J_{\theta}(\eta) \leq' J_{\theta}(\mu_i)$, with equality only if $\eta = \mu_i$. We conclude that $J_{\theta}(\mu_i)$ is the largest element of $L_{\theta}(b_{Z_i})$ with respect to the order \leq' .

Let β be the largest element among the degrees $J_{\theta}(\mu_i)$ with respect to the total order \leq' and let I be the set of all indices i such that $J_{\theta}(\mu_i) = \beta$. Then the homogeneous component of degree $(\beta, \gamma - \beta)$ of $\Delta(f)$ is $\sum_{i \in I} c_i b_{(Z_i)_t} \otimes b_{z^{-\mu_i}(Z_i)_s}$. Since the pairs $((Z_i)_t, (Z_i)_s)$ are all different, this sum cannot be zero, so $\beta \in L(f)$ and therefore $\beta \leq' \alpha$. It follows that for each $i \in \llbracket 1, \ell \rrbracket$, the set $L_{\theta}(b_{Z_i})$ is contained in $\alpha + \Pi'$ and the function b_{Z_i} belongs to ${}^{\theta}F'_{\alpha}\mathcal{O}$. We conclude that ${}^{\theta}F'_{\alpha}\mathcal{O}$ is spanned by its intersection with the Mirković–Vilonen basis. The same arguments imply the compatibility of the Mirković–Vilonen basis with the isomorphism $\overline{\Delta}'$.

Condition (iv) in the definition of polite bases can be checked similarly, replacing instead Φ_{θ} by $''\Phi_{\theta}$ in section 4.2. \square

Example 4.9. Let W be the Weyl group of G , endowed with the Bruhat order. Let λ be a dominant weight and let w be an element in W . The costandard module $\nabla(\lambda)$ contains a vector $v_{w\lambda}$ of weight $w\lambda$, unique up to multiplication by a scalar. Let $\nabla(\lambda)_w$ be the N -submodule of $\nabla(\lambda)$ generated by $v_{w\lambda}$ and let $f_{w,\lambda}$ be the image of $v_{w\lambda}$ by the map $\Psi_{\lambda} : \nabla(\lambda) \rightarrow \mathcal{O}$. It is well-known that $f_{w,\lambda}$ belongs to all the perfect bases of \mathcal{O} (provided that $v_{w\lambda}$ is suitably normalized), therefore to all the polite bases. In particular, $\underline{\text{Pol}}(f_{w,\lambda})$ is a Mirković–Vilonen polytope. Now, since Ψ_{λ} is an embedding of N -modules, the polytope $\lambda + \underline{\text{Pol}}(f_{w,\lambda})$ is the convex hull of the set of weights of $\nabla(\lambda)_w$. On the other hand, it follows from the Demazure character formula that this polytope is the convex hull of $\{x\lambda \mid x \in W, x \leq w\}$. We recover in this way a result of Naito and Sagaki ([23], Theorem 4.1.5 and Remark 4.1.6).

References

- [1] J. Anderson, *A polytope calculus for semisimple groups*, Duke Math. J. **116** (2003), 567–588.
- [2] Y. André, *Slope filtrations*, Confluentes Math. **1** (2009), 1–85.
- [3] P. Baumann, *The canonical basis and the quantum Frobenius morphism*, arXiv:1201.0303.
- [4] P. Baumann and S. Gaussent, *On Mirković–Vilonen cycles and crystal combinatorics*, Represent. Theory **12** (2008), 83–130.
- [5] P. Baumann and J. Kamnitzer, *Preprojective algebras and MV polytopes*, Represent. Theory **16** (2012), 152–188.
- [6] P. Baumann, J. Kamnitzer and A. Knutson, *The Mirković–Vilonen basis and Duistermaat–Heckman measures*, with an appendix by A. Dranowski, J. Kamnitzer and C. Morton-Ferguson, Acta Math. **227** (2021), 1–101.
- [7] P. Baumann, J. Kamnitzer and P. Tingley, *Affine Mirković–Vilonen polytopes*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 113–205.
- [8] A. Berenstein and D. Kazhdan, *Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases*. In *Quantum groups*, pp. 13–88, Contemp. Math. vol. 433, Israel Math. Conf. Proc., Amer. Math. Soc., 2007.
- [9] R. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics vol. 28, John Wiley & Sons, London–New York–Sydney, 1972.
- [10] J. Fei, *Crystal structure of upper cluster algebras*, arXiv:2309.08326.
- [11] C. Geiß, B. Leclerc and J. Schröer, *Semicanonical bases and preprojective algebras*, Ann. Sci. Ec. Norm. Super. **38** (2005), 193–253.
- [12] C. Geiß, B. Leclerc and J. Schröer, *Kac–Moody groups and cluster algebras*, Adv. Math. **228** (2011), 329–433.
- [13] J. Kamnitzer, *The crystal structure on the set of Mirković–Vilonen polytopes*, Adv. Math. **215** (2007), 66–93.
- [14] J. Kamnitzer, *Mirković–Vilonen cycles and polytopes*, Ann. of Math. **171** (2010), 245–294.
- [15] S.-J. Kang, S.-J. Oh and E. Park, *Perfect bases for integrable modules over generalized Kac–Moody algebras*, Algebr. Represent. Theory **14** (2011), 571–587.
- [16] M. Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), 455–485.
- [17] M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, Duke Math. J. **89** (1997), 9–36.

- [18] Y. Kimura, *Quantum unipotent subgroup and dual canonical basis*, Kyoto J. Math. **52** (2012), 277–331.
- [19] G. Lusztig, *Canonical bases arising from quantized enveloping algebras. II*, Progr. Theoret. Phys. Suppl. **102** (1990), 175–201.
- [20] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics vol. 110, Birkhäuser Boston, Boston, 1993.
- [21] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129–139.
- [22] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. **166** (2007), 95–143.
- [23] S. Naito and D. Sagaki, *Mirković–Vilonen polytopes lying in a Demazure crystal and an opposite Demazure crystal*, Adv. Math. **221** (2009), 1804–1842.
- [24] Y. Saito, *PBW basis of quantized universal enveloping algebras*, Publ. Res. Inst. Math. Sci. **30** (1994), 209–232.
- [25] O. Schiffmann, *Spherical Hall algebras of curves and Harder–Narasimhan stratas*, J. Korean Math. Soc. **48** (2011), 953–967.
- [26] P. Tingley and B. Webster, *Mirković–Vilonen polytopes and Khovanov–Lauda–Rouquier algebras*, Compos. Math. **152** (2016), 1648–1696.
- [27] M. Varagnolo and E. Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math. **659** (2011), 67–100.

Pierre Baumann, Institut de Recherche Mathématique Avancée, Université de Strasbourg et CNRS UMR 7501, 7 rue René Descartes, 67000 Strasbourg, France.

baumann@math.unistra.fr