Avril 2012
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rev. Juin 2015

Construction and combinatorics of perfect bases

I Perfect bases (mainly after Berenstein, Kazhdan, Zelevinsky)

1) Definition

$A = (a_{ij})_{i,j \in I}$ symmetrizable Cartan matrix

$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ Kac-Moody algebra / $\mathbb{C}$

$U = U(\mathfrak{n}_+)$ presented by generators e_i $(i \in I)$

relations $\displaystyle\sum_{p+q=1-a_{ij}} (-1)^p \, e_i^{(p)} e_j e_i^{(q)} = 0 \qquad (i \neq j)$

$\left(e_i^{(n)} = \dfrac{e_i^n}{n!}\right.$ divided power$\Big)$

graded by $Q_+ = \left\{ \sum n_i \alpha_i \;\middle|\; n_i \in \mathbb{N} \right\} \subset \mathfrak{h}^*$ with $\deg e_i = \alpha_i$

(write $U = \bigoplus_{\nu \in Q_+} U_\nu$)

endowed with an involutive antiautomorphism $x \mapsto x^\dagger$ which fixes the e_i.

Main problem: construct bases of U related to this presentation. Already a lot of work on this problem: Gelfand, Zelevinsky, Retakh, Berenstein, Mathieu, Lusztig, Kashiwara, Kazhdan, ...

$N = \exp \mathfrak{n}_+$ unipotent group with Lie algebra $\mathfrak{n}_+$.

$R = \mathbb{C}[N]$ algebra of regular functions on N.

$\cong U^*$ graded dual $\qquad R = \bigoplus_{\nu \in Q_+} R_{-\nu}, \quad R_{-\nu} = (U_\nu)^*$.

$\mathfrak{n}_+$ acts by derivations on R (left invariant vector fields on N)

$\eta \mapsto \eta^\dagger$ the transpose of $x \mapsto x^\dagger$.

For $i \in I$ and $\eta \in R \setminus \{0\}$, set $\varepsilon_i(\eta) = \max\{ n \in \mathbb{N} \mid e_i^n \eta \neq 0 \}$

$\tilde{e}_i^{\max} \eta = e_i^{(\varepsilon_i(\eta))} \cdot \eta$.

Definition: A linear basis B of R is perfect if:

(P0) $1 \in B$

(P1) B is graded w.r.t the Q_+-gradation

(P2) $\forall b \in B$, $\forall i \in I$, $\tilde{e}_i^{\max} b \in B$.

Further, $\forall n \in \mathbb{N}$, $\tilde{e}_i^{\max}$ is injective on $\{b \in B \mid \varepsilon_i(b) = n\}$.

(P3) B is stable under $\dagger$.

Observation: For $i \in I$ and $n \in \mathbb{N}$, set $K_{i,n} = \ker e_i^{n+1} \subset R$.

- B perfect basis $\Rightarrow$ $\{b \in B \mid \varepsilon_i(b) \leq n\}$ basis of $K_{i,n}$.

  (proof: let $\eta \in K_{i,n}$. Write $\eta = \sum_{b \in B} a_b b$, set $m = \max\{\varepsilon_i(b) \mid a_b \neq 0\}$.
  If we had $m > n$, then we would have $0 = e_i^{(m)} \eta = \sum_{\substack{b \in B \\ \varepsilon_i(b) = m}} a_b \underbrace{\left(\tilde{e}_i^{\max} b\right)}_{\in B \text{ and pairwise } \neq}$
  which is impossible.)

- $e_i^{(n)}$ induces a linear bijection $K_{i,n} / K_{i,n-1} \longrightarrow K_{i,0}$

  (proof of surjectivity: e_i is not a zero divisor in U)

  so $\tilde{e}_i^{\max}$ induces a bijection $\{b \in B \mid \varepsilon_i(b) = n\} \longrightarrow \{b \in B \mid \varepsilon_i(b) = 0\}$.

Conclusion: Each perfect basis carries a combinatorial structure:

- a map $\mathrm{wt} : B \to Q_-$
- maps $\varepsilon_i : B \to \mathbb{N}$
- bijections

$$\begin{tikzcd} \{b \in B \mid \varepsilon_i(b) = n\} \arrow[rr, shift left, "\tilde{f}_i"] \arrow[dr, "\tilde{e}_i^{\max}"'] & & \{b \in B \mid \varepsilon_i(b) = n+1\} \arrow[ll, shift left, "\tilde{e}_i"] \arrow[dl, "\tilde{e}_i^{\max}"] \\ & \{b \in B \mid \varepsilon_i(b) = 0\} & \end{tikzcd}$$

- the involution $b \mapsto b^\dagger$ (Kashiwara's involution $*$ / Lusztig's σ)
- for convenience, define $\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}\, b \rangle$

  and set $\tilde{e}_i b = 0$ if $\varepsilon_i(b) = 0$

2) Usefulness

$P_+ = \{$ dominant integral weights $\}$

$\lambda \in P_+ \longmapsto L(\lambda)$ irreducible integrable module with hw λ

$ \ \ v_\lambda \in L(\lambda)$ hw vector

$\Psi_\lambda : L(\lambda) \hookrightarrow R$ the unique morphism of n_+-modules that maps $v_\lambda \mapsto 1$

$W = \langle s_i, i \in I \rangle$ Weyl group

For $w \in W$, $\lambda \in P_+$ $\quad v_{w\lambda} \in L(\lambda)$ extremal weight vector (suitably normalized)

Flag minor: any element in R of the form $\Psi_\lambda(v_{w\lambda})$.

Proposition: Let B be a perfect basis of R. Then

1) B is compatible with all subspaces $\operatorname{im} \Psi_\lambda$ and contains all flag minors.

2) for each $\lambda \in P_+$, the basis $\Psi_\lambda^{-1}(B)$ of $L(\lambda)$ consists of weight vectors and is compatible with all subspaces $\ker e_i^{n+1}$ and $\ker f_i^{n+1}$.

Proof: Let $\lambda \in P_+$, set $n_i = \langle \alpha_i^\vee, \lambda \rangle$. Then $\operatorname{im} \Psi_\lambda = \{ \eta \in R \mid \eta^\dagger \in \bigcap_{i \in I} K_{i, n_i} \}$.

Application: Let B be a perfect basis of R, let $\lambda, \mu, \nu \in P_+$. Then

$$\dim \operatorname{Hom}_{\mathfrak{g}}(L(\lambda), L(\mu) \otimes L(\nu)) = \operatorname{card} \{ b \in B \mid \operatorname{wt} b = \mu + \nu - \lambda,\ \varepsilon_i(b) \leq \langle \alpha_i^\vee, \mu \rangle,\ \varepsilon_i(b^\dagger) \leq \langle \alpha_i^\vee, \nu \rangle \}$$

Proof: $f \mapsto \Psi_\nu(f(v_\lambda \otimes v_\mu^*))$ envoie bijectivement $\operatorname{Hom}_{\mathfrak{g}}(L(\lambda) \otimes L(\mu)^*, L(\nu))$ sur $\{ \eta \in R \mid \operatorname{wt}(\eta) = \mu + \nu - \lambda,\ \eta \in \bigcap_{i \in I} K_{i, \langle \alpha_i^\vee, \mu \rangle},\ \eta^\dagger \in \bigcap_{i \in I} K_{i, \langle \alpha_i^\vee, \nu \rangle} \}$.

Theorem (Kashiwara): B perfect basis, $\lambda \in P_+$. The basis of $L(\lambda)$ dual to $\Psi_\lambda^{-1}(B)$ (w.r.t. a contravariant form) is compatible with the Demazure submodules.

(Reference: M. Kashiwara, The crystal base and Littelmann's refined Demazure character formula, §§ 3.1–3.2. Kashiwara proves this for the upper crystal basis, but the argument is general.)

3) Examples

Type A_1: $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{C} \right\}$ $\mathbb{C}[N] = \mathbb{C}[x]$

$e = \frac{d}{dx}$ $B = \{ x^n \mid n \in \mathbb{N} \}$ (exists and is unique)

Type A_2: $N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{C} \right\}$ $\mathbb{C}[N] = \mathbb{C}[x, y, z]$.

$e_1 = \frac{\partial}{\partial x}$ $e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$

Flag minors of $L(\omega_1)$: x, z

Flag minors of $L(\omega_2)$: y, $xy - z$

$B = \{ x^a z^b (xy-z)^c \mid a, b, c \in \mathbb{N} \} \cup \{ y^a z^b (xy-z)^c \mid a, b, c \in \mathbb{N} \}$

(exists and is unique)

(One can check here that in this basis, e_1 and e_2 act with coefficients in $\mathbb{N}$, and that the structure constants of the multiplication belong to $\mathbb{N}$.)

Type A_3: Still have existence, uniqueness, and explicit formulas

(Reference: A. Berenstein, A. Zelevinsky, String bases for quantum groups of type A_n. This paper is the starting point of the theory of cluster algebras.)

In general: no uniqueness; existence ensured by several constructions:

- Lusztig's dual canonical basis = Kashiwara's upper crystal basis (specialized at $q = 1$).
- basis arising from KLR algebras: $R_{-\nu} \cong G_0(R(\nu)) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}$ (at $q = 1$)

  (simple graded $R(\nu)$-modules up to isomorphism and up to a shift in the graduation give a perfect basis)
- Lusztig's dual semicanonical basis (A symmetric)
- the MV basis, arising from geometric Satake equivalence (A of finite type)

  (proof: see II)

4) Uniqueness of crystal

Theorem (Berenstein-Kazhdan): Let B', B'' be two perfect bases of R.
Then $\exists !$ bijection $B' \to B''$ that preserves the combinatorial data $(\mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i)$. In addition, it commutes with $\dagger$.

Notation: $(B(\infty), \mathrm{wt}, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \dagger)$ the abstract set with combinatorial data common to all perfect bases of R (Kashiwara's crystal).

5) Saito's reflections

For $i \in I$ and $b \in B$, define $\tilde{e}_i^{\dagger} b = (\tilde{e}_i b^{\dagger})^{\dagger}$ (right action of U on R)

Theorem-Definition (Saito): Let $i \in I$.
$\exists$ inverse bijections $\{ b \in B(\infty) \mid \varepsilon_i(b) = 0 \} \underset{\sigma_i^{\dagger}}{\overset{\sigma_i}{\rightleftarrows}} \{ b \in B(\infty) \mid \varepsilon_i(b^{\dagger}) = 0 \}$

given by
$$\left.\begin{aligned} \sigma_i(b) &= \tilde{f}_i^{\,\varphi_i(b^{\dagger})} (\tilde{e}_i^{\dagger})^{\varepsilon_i(b^{\dagger})} b \\ \sigma_i^{\dagger}(b^{\dagger}) &= (\sigma_i(b))^{\dagger} \end{aligned}\right\} \text{ if } \varepsilon_i(b) = 0.$$

Note that $\mathrm{wt}\, \sigma_i(b) = \mathrm{wt}\, b - (\varphi_i(b^{\dagger}) - \varepsilon_i(b^{\dagger}))\alpha_i = \mathrm{wt}\, b - \langle \alpha_i^{\vee}, \mathrm{wt}\, b^{\dagger} \rangle \alpha_i = s_i(\mathrm{wt}\, b)$.

For convenience, set $\hat{\sigma}_i(b) = \sigma_i(\tilde{e}_i^{\max} b)$ for all $b \in B(\infty)$, $i \in I$.

Remark (Tingley): If $n \geq \underbrace{\varepsilon_i(b) + \varepsilon_i(b^{\dagger}) + \langle \alpha_i^{\vee}, \mathrm{wt}\, b \rangle}_{\text{(always a } \geq 0 \text{ number)}}$, then $\sigma_i(b) = (\tilde{e}_i^{\dagger})^{\max} \tilde{f}_i^{\,n} b$.

Proposition: The $\hat{\sigma}_i : B(\infty) \to B(\infty)$ satisfy the braid relations.

Proof: see III

Notation: For $w = s_{i_1} \cdots s_{i_k}$ reduced, set $\hat{\sigma}_w = \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_k}$.

6) Mirković-Vilonen polytopes

$\boxed{A \text{ of finite type}}$

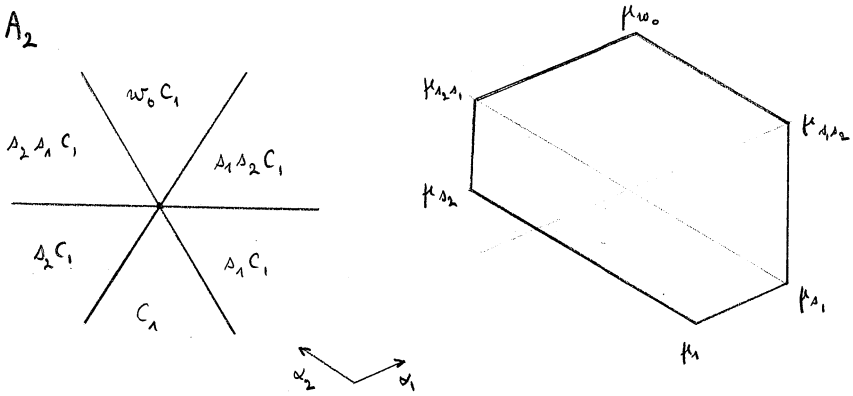
Notations: Φ root system of $\mathfrak{g}$

$\mathcal{W}$ Weyl fan in $\mathfrak{h}_{\mathbb{R}}$ (described by the root hyperplanes)

$\overline{Q}_+ = \{ \sum a_i \alpha_i \mid a_i \in \mathbb{R}_+ \} \subset \mathfrak{h}^*_{\mathbb{R}}$

Definition (Kamnitzer): A Gelfand-Goresky-McPherson-Serganova polytope is a convex polytope $P \subset \mathfrak{h}^*_{\mathbb{R}}$ whose dual fan is a coarsening of $\mathcal{W}$.

Picture in type A_2



To a chamber wC_1 corresponds a vertex μ_w of P. (Vertices are allowed to be non-distinct.)

Lemma (Kamnitzer): $\{\text{GGMS polytopes}\} \xleftrightarrow{1:1} \{ \text{collections } (\mu_w) \in (\mathfrak{h}^*_{\mathbb{R}})^W \mid \forall x, w,\ \mu_x \in \mu_w + w\overline{Q}_+ \}$

$\xleftrightarrow{1:1} \{ \text{collections } (\mu_w) \mid \forall w, \forall i,\ \mu_{ws_i} - \mu_w \in \mathbb{R}_+ \cdot w\alpha_i \}$.

Back to $B(\infty)$.

Definition: For $b \in B(\infty)$ and $w \in W$, set $\mu_w(b) = w \cdot \mathrm{wt}(\hat{\sigma}_{w^{-1}} b)$.

Observation: $ws_i > w \Rightarrow \mu_{ws_i}(b) - \mu_w(b) = w\left[s_i \,\mathrm{wt}(\hat{\sigma}_i b') - \mathrm{wt}(b') \right]$ $\quad b' = \hat{\sigma}_{w^{-1}} b$

$= w \left[\cancel{s_i} \,\mathrm{wt}(\cancel{\sigma_i}\, \tilde{e}_i^{\varepsilon_i(b')} b') - \mathrm{wt}(b') \right]$

$= \underbrace{\varepsilon_i(b')}_{\geq 0} \cdot w\alpha_i$

So $\underbrace{\mathrm{Conv}\{ \mu_w(b) \mid w \in W \}}_{\mathrm{Pol}(b), \text{ the MV polytope of } b.}$ is GGMS

7) Lusztig data

$\boxed{\text{A still of finite type}}$

$U_q(\mathfrak{g})$ the quantum group $/\mathbb{C}(q)$; generators $E_i, F_i, K_i^{\pm 1}$

$T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ Lusztig's automorphism
a quantum analogue of $\mathrm{Ad}(\bar{s}_i)$.

$\bar{\cdot} : U_q(n_+) \to U_q(n_+)$ the bar involution: $\mathbb{C}$-algebra automorphism, $\bar{q} = q^{-1}$, $\bar{E_i} = E_i$

Given $\underline{i} = (i_1, \dots, i_r)$ such that $s_{i_1} \cdots s_{i_r}$ reduced decomposition of w_0:

- enumeration of the positive roots $\beta_1, \dots, \beta_r$ $\qquad \beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$
- PBW basis of $U_q(n_+)$ $\qquad \{ E_{\underline{i}}^{(\underline{n})} \mid \underline{n} = (n_1, \dots, n_r) \in \mathbb{N}^r \}$

$$E_{\underline{i}}^{(\underline{n})} = E_{\beta_r}^{(n_r)} \cdots E_{\beta_2}^{(n_2)} E_{\beta_1}^{(n_1)}, \qquad E_{\beta_k}^{(n_k)} = T_{i_1} \cdots T_{i_{k-1}} \left(\frac{E_{i_k}^{n_k}}{[n_k]_{i_k}^!} \right)$$

Theorem (Lusztig): $\forall\, \underline{n} \in \mathbb{N}^r$, $\exists !$ bar-invariant element in $U_q(n_+)$

$$\mathcal{E}_{\underline{i}}(\underline{n}) = \sum_{\underline{m} \in \mathbb{N}^r} \zeta_{\underline{m}}^{\underline{n}} \, E_{\underline{i}}^{(\underline{m})}$$

such that $\zeta_{\underline{n}}^{\underline{n}} = 1$ and $\zeta_{\underline{m}}^{\underline{n}} \in q\mathbb{Z}[q]$ for all $\underline{m} \neq \underline{n}$.

$\{ \mathcal{E}_{\underline{i}}(\underline{n}) \mid \underline{n} \in \mathbb{N}^r \}$ basis of $U_q(n_+)$, independent of $\underline{i}$: canonical basis.

Specialization at $q=1$ gives a basis of U, whose dual is perfect.

Notation: $B(\infty)$ indexes the dual canonical basis, whence a bijection $\quad \begin{array}{l} B(\infty) \to \mathbb{N}^r \\ b \mapsto \underline{n} = N(\underline{i}, b) \end{array}$

$N(\underline{i}, b) =$ Lusztig data of b in direction $\underline{i}$.

Theorem (Saito): $n_1 = \varepsilon_{i_1}(b)$; $n_2 = \varepsilon_{i_2}(\hat{\sigma}_{i_1}(b))$; $\dots$ $n_k = \varepsilon_{i_k}(\hat{\sigma}_{i_{k-1}} \cdots \hat{\sigma}_{i_1}(b))$; ....

(So $\hat{\sigma}_{i_1}$ mimicks on $B(\infty)$ the action of T_i^{-1} on PBW mononials.)

Corollaries: 1) $\mu_{s_{i_1} \cdots s_{i_k}}(b) - \mu_{s_{i_1} \cdots s_{i_{k-1}}}(b) = n_k \beta_k$

(The length of the edges of $\mathrm{Pol}(b)$ are Lusztig data of b.)

2) $b \mapsto \mathrm{Pol}(b)$ is injective.

II The MV basis

A of finite type

1) Background on geometric Satake equivalence

G connected alg. gp. s.t. Lie $G = \mathfrak{g}$

$\cup$

$B_\pm$

$\cup$

T

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

$X = \mathrm{Hom}(T, \mathbb{C}^*)$ ($X = P$ if G simply connected)

$T^\vee = X \otimes_{\mathbb{Z}} \mathbb{C}^*$ dual torus

$\mathrm{Hom}(T^\vee, \mathbb{C}^*) = X^\vee$ dual lattice

$\cup$

$\Phi^\vee$

$G^\vee$ Langlands dual

$\mathcal{O} = \mathbb{C}[[t]]$, $\mathcal{K} = \mathbb{C}((t))$

$\mathcal{G}r = G^\vee(\mathcal{K}) / G^\vee(\mathcal{O})$ affine grassmannian of $G^\vee$

(like a G/P with P parabolic maximal, but for a Kac-Moody group, so infinite dimensional. However, this is the limit of a direct system of projective varieties and closed embeddings, namely the Schubert varieties)

$\mathrm{Perv} = \{ G^\vee(\mathcal{O})$ - equiv. perverse sheaves on $\mathcal{G}r$ with coef in $\mathbb{C}$ and fin. dim. support$\}$

abelian rigid monoidal category

$H : \mathrm{Perv} \longrightarrow \mathrm{Vect}$ exact, faithful, monoidal

$\Longrightarrow \mathrm{Perv} \cong \mathrm{Rep}\, \overline{G}$ $\quad \overline{G}$ pro-algebraic gp (Saavedra Rivano's theorem)

Beilinson - Drinfeld, Ginzburg, Mirković-Vilonen (+ Lusztig) : $\overline{G} \cong G$

$X = \mathrm{Hom}(\mathbb{C}^*, T^\vee)$
Ψ
$\lambda \longmapsto t^\lambda$ = image of $t \in \mathcal{K}^*$ in $T^\vee(\mathcal{K})$ or in $\mathcal{G}r$.

$\mathcal{G}r_\lambda \subset \mathcal{G}r$ the $G^\vee(\mathcal{O})$-orbit of t^λ

$$\mathcal{G}r = \coprod_{\substack{\lambda \in X_+ \\ \uparrow \\ X \cap P_+}} \mathcal{G}r_\lambda$$

Simple objects in Perv: $\mathcal{I}_\lambda = IC(\overline{\mathcal{G}r_\lambda}, 1)$

$$\mathcal{I}_\lambda \overset{H}{\rightsquigarrow} L(\lambda)$$

2) The MV basis

$G^\vee$
$\cup$
$B^\vee_- \supset N^\vee_-$
$\cup$
$T^\vee$

For $\nu \in X$, let $T_\nu \subset \mathcal{G}r$ the $N^\vee_-(\mathcal{K})$-orbit of t^ν.

Define $\rho : \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R}$, $\alpha_i \mapsto 1$.

Mirković-Vilonen:

$\forall A \in \mathrm{Perv}$, $\forall k \in \mathbb{Z}$,
$$\bigoplus_{\substack{\nu \in X \\ 2\rho(\nu) = k}} H^k_{\overline{T_\nu}}(\mathcal{G}r, A) \longrightarrow H^k(\mathcal{G}r, A) \text{ isomorphism} \quad (\diamondsuit)$$

$\forall \lambda \in X_+$, $\forall \nu \in X$,
$$H^{2\rho(\nu)}_{\overline{T_\nu}}(\mathcal{G}r, \mathcal{I}_\lambda) \cong H^{BM}_{2\rho(\lambda-\nu)}(\overline{\mathcal{G}r_\lambda} \cap \overline{T_\nu}) \text{ with coefs in } \mathbb{C}$$
of pure dim $\rho(\lambda-\nu)$

$$\mathcal{Z}(\lambda)_\nu = \mathrm{Irr}(\overline{\mathcal{G}r_\lambda} \cap \overline{T_\nu}) \ni Z \longmapsto [Z] \in H^{2\rho(\nu)}_{\overline{T_\nu}}(\mathcal{G}r, \mathcal{I}_\lambda) \text{ fundamental class}$$

$$\coprod_\nu \{[Z] \mid Z \in \mathcal{Z}(\lambda)_\nu\} \text{ basis of } H(\mathcal{G}r, \mathcal{I}_\lambda) \cong L(\lambda)$$

Theorem (B.-Kamnitzer): Via $\Psi_\lambda : L(\lambda) \hookrightarrow R$, these bases glue together and give a perfect basis of R. In this basis, the structure constants of the multiplication $\in \mathbb{N}$.

3) Action of G (Ginzburg, Vasserot)

? fix the isomorphism $G \cong \overline{G}$.

action of T defined by $(\Diamond)$: $T \xrightarrow{\simeq} \overline{T}$ maximal torus of $\overline{G}$

$$\begin{array}{ccc} h = 2\rho & \longmapsto & \overline{h} \\ \cap & & \\ \mathfrak{h} & \xrightarrow{\simeq} & \overline{\mathfrak{h}} \\ \cap & & \cap \\ \mathfrak{g} & & \overline{\mathfrak{g}} \end{array}$$

Consider the Plücker embedding: $j : Gr \hookrightarrow \mathbb{P}(L(\Lambda_0))$

basic representation of $\widehat{\mathfrak{g}^\vee}$, the affine KM algebra corresponding to $\mathfrak{g}^\vee$.

(to simplify, assume here G simple of adjoint type)

$\mathcal{L} = j^* \mathcal{O}(1)$ ample line bundle. Set $\overline{e} = (c_1(\mathcal{L}) \cup ?) \in \overline{\mathfrak{g}}$.

Since $[\overline{h}, \overline{e}] = 2\overline{e}$ ($\overline{h}$ acts on H^k by multiplication by k)

we can write $\overline{e} = \sum_i Q(\alpha_i)\,\overline{e_i}$ with $\overline{e_i} \in \overline{\mathfrak{g}}^{\alpha_i}$ ($Q(\alpha_i)$ = square of length of α_i, 1 if α_i short root)

Hard Lefschetz $\Rightarrow \exists$ sl_2-triple $(\overline{e}, \overline{h}, \overline{f}) \Rightarrow$ each $\overline{e_i} \neq 0$

Define $G \cong \overline{G}$ by integrating the isomorphism $\mathfrak{g} \cong \overline{\mathfrak{g}}$ s.t. $e_i \mapsto \overline{e_i}$

Geometric translation:

Choose $\nu \in X$. Mirković-Vilonen $\Rightarrow$ $\overline{T_\nu} = \bigcup_{\mu \in Q_+} T_{\nu+\mu}$ and $D \cap \overline{T_\nu} = \overline{T_\nu} \setminus T_\nu = \bigcup_{i \in I} \overline{T_{\nu+\alpha_i}}$

for a well-chosen hyperplane $D \subset \mathbb{P}(L(\Lambda_0))$.

For $k = 2\rho(\nu)$ and $\lambda \in X_+$, $d = 2\rho(\lambda)$:

$$\begin{tikzcd} H^k(Gr, \mathcal{J}_\lambda) \arrow[d, "\cup c_1(\mathcal{L})"'] & H^k_{\overline{T_\nu}}(Gr, \mathcal{J}_\lambda) \arrow[l] \arrow[r, "MV"] & H^{BM}_{d-k}(\overline{T_\nu} \cap \overline{Gr_\lambda}) \arrow[d, "\cdot [D]"] \\ H^{k+2}(Gr, \mathcal{J}_\lambda) & \bigoplus_{i \in I} H^{k+2}_{\overline{T_{\nu+\alpha_i}}}(Gr, \mathcal{J}_\lambda) \arrow[l] \arrow[r] & H^{BM}_{d-k-2}\left(\bigcup_{i \in I} \overline{T_{\nu+\alpha_i}} \cap \overline{Gr_\lambda}\right) \end{tikzcd}$$

4) Polytopes

Let $Y \subset \mathcal{G}r$ be closed, $T^\vee$-invariant, finite dim.

For $\nu \in X$, $t^\nu \in Y \Leftrightarrow Y$ meets T^ν, and this holds for finitely many ν.

If Y irreducible, then $\exists\, \nu \in X$ s.t. $Y \cap T_\nu$ open dense in Y;

concretely, $t^\nu \in Y$ and any $\mu \in X$ s.t. $t^\mu \in Y$ belongs to $\nu + Q_+$.

Denote this ν by $\mu_1(Y)$.

For $w \in W$, set $\mu_w(Y) = w\, \mu_1(\dot{w}^{-1} Y)$, where $\dot{w} \in G^\vee$ lift of w.

Fact: $\forall x, w$, $\mu_x(Y) \in \mu_w(Y) + w\, Q_+$

$\longrightarrow$ GGMS polytope $\mathrm{Pol}(Y)$.

Remark: $Y \subset Z \Rightarrow t^{\mu_1(Y)} \in Z \Rightarrow \mu_1(Y) \in \mu_1(Z) + Q_+$

$\Rightarrow$ plus généralement, $\forall w$, $\mu_w(Y) \in \mu_w(Z) + w\, Q_+$

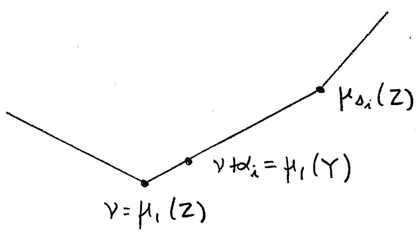
$\Rightarrow \mathrm{Pol}(Y) \subset \mathrm{Pol}(Z)$

Notation: $\mu_{s_i}(Y) - \mu_1(Y) = \underbrace{\varepsilon_i(Y)}_{\in \mathbb{N}}\, \alpha_i$

Facts: 1) $Z \in \mathcal{Z}(\lambda)_\nu \Rightarrow \mu_1(Z) = \nu$ and $\mu_{w_0}(Z) = \lambda$

2) Let $Z \in \mathcal{Z}(\lambda)_\nu$, let $i \in I$.

Any $Y \in \mathcal{Z}(\lambda)_{\nu+\alpha_i}$ contained in Z satisfies $\varepsilon_i(Y) < \varepsilon_i(Z)$



Assume $\varepsilon_i(Z) \geq 1$. Then $\exists!\, Y \in \mathcal{Z}(\lambda)_{\nu+\alpha_i}$ such that $Y \subset Z$ and $\varepsilon_i(Y) = \varepsilon_i(Z) - 1$ (Braverman-Gaitsgory), and with D as in §3, the multiplicity of $[Y]$ in $[Z]\cdot[D]$ is $Q(\alpha_i)\, \varepsilon_i(Z)$.

5) End of the proof

a) Look first at $L(\lambda)$ for $\lambda \in X_+$

Take $Z \in \mathcal{Z}(\lambda)_\nu$, choose D as in §3.

Then $\bar{e}[Z] = [Z]\cdot[D]$ so $\bar{e}_i[Z] = \sum_{\substack{Y \in \mathcal{Z}(\lambda)_{\nu+\alpha_i} \\ Y \subset Z}} \frac{\text{multiplicity}}{Q(\alpha_i)} [Y]$

b) For all Y here, $\varepsilon_i(Y) \leq \varepsilon_i(Z) - 1$; exactly one Y has $\varepsilon_i(Y) = \varepsilon_i(Z) - 1$, and it appears with coefficient $\varepsilon_i(Z)$.

$$\Rightarrow \quad \bar{e}_i^{(k)}[Z] = \begin{cases} 0 & \text{if } k > \varepsilon_i(Z) \\ [V] & \text{for a } V \in \mathcal{Z}(\lambda)_{\nu+k\alpha_i} \text{ if } k = \varepsilon_i(Z) \end{cases}$$

c) For any $\lambda, \rho \in X_+$ and $\nu \in X$, $\exists$ injection $\mathcal{Z}(\lambda)_\nu \longrightarrow \mathcal{Z}(\lambda+\rho)_{\nu+\rho}$, $Z \longmapsto t^\rho Z$ (J. Anderson)

whence a linear injection $i : L(\lambda) \hookrightarrow L(\lambda+\rho)$.

Step a) $\Rightarrow$ i is a map of n_+-module

$$\Rightarrow \quad \begin{tikzcd} L(\lambda) \arrow[d, hook, "i"'] \arrow[dr, hook, "\Psi_\lambda"] & \\ L(\lambda+\rho) \arrow[r, hook, "\Psi_{\lambda+\rho}"'] & R \end{tikzcd} \quad \text{commutes}$$

so the bases of the $L(\lambda)$ glue correctly and give a basis of R.

d) Step b) $\Rightarrow$ this basis satisfies (P2)

(I omit the proof of (P3) and of the last assertion of the theorem.)

6) Relation to MV polytopes

Theorem (Kamnitzer's thesis): Let $\lambda \in X_+$, let $Z \in \mathcal{Z}(\lambda)_\nu$. Suppose $\Psi_\lambda([Z])$ is indexed by $b \in B(\infty)_{\nu-\lambda}$. Then $\mathrm{Pol}(Z) = \lambda + \mathrm{Pol}(b)$.

Proof: requires finer description of MV cycles + Berenstein, Fomin and Zelevinsky's chamber Ansatz, whose tropicalization describe Lusztig data.

Proposition: The transition matrix between the MV basis and the dual canonical basis is upper unitriangular wrt the order on $B(\infty)$ given by inclusion of MV polytopes.

III Preprojective algebras and the semicanonical basis

A symmetric

1) Preprojective algebras

Oriented graph without loops: $\{\text{vertices}\} = I$

$\{\text{edges}\} = H$: between i and j, $-a_{ij}$ edges in each direction

$h \mapsto \bar{h}$ orientation reversing involution

$\varepsilon : H \to \{\pm 1\}$ such that $\varepsilon(h) + \varepsilon(\bar{h}) = 0$

K field

$\Lambda = K$ - path algebra of the quiver $(I, H) \Big/ \Big\langle \sum\limits_{h \in H} \varepsilon(h) h\bar{h} \Big\rangle$ completed wrt the ideal generated by the arrows $\Bigg\}$ completed preprojective algebra

Example: type A_3 $\quad (I, H) = \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "h"] & \overset{2}{\bullet} \arrow[l, shift left, "\bar{h}"] \arrow[r, shift left, "k"] & \overset{3}{\bullet} \arrow[l, shift left, "\bar{k}"] \end{tikzcd}$ $\quad \varepsilon(h) = \varepsilon(k) = 1$

Λ-module $=$ $\begin{tikzcd} M_1 \arrow[r, shift left, "M_h"] & M_2 \arrow[l, shift left, "M_{\bar{h}}"] \arrow[r, shift left, "M_k"] & M_3 \arrow[l, shift left, "M_{\bar{k}}"] \end{tikzcd}$ s.t. $M_{\bar{h}} M_h = M_k M_{\bar{k}} = M_h M_{\bar{h}} - M_{\bar{k}} M_k = 0$

M a Λ-module $\rightsquigarrow$ $\underline{\dim}\, M = \sum\limits_{i \in I} (\dim M_i)\, \alpha_i \in Q_+$ dimension-vector

Simple Λ-modules: S_i ($i \in I$), 1-dimensional, concentrated on vertex i; $\underline{\dim}\, S_i = \alpha_i$

$M \mapsto \underline{\dim}\, M$ induces $K(\Lambda\text{-mod}) \underline{\simeq} \bigoplus\limits_{i \in I} \mathbb{Z} \alpha_i$

Duality operation on Λ-mod: $M = \Big(\bigoplus\limits_{i \in I} M_i, (M_h)\Big) \rightsquigarrow M^{\dagger} = \Big(\bigoplus\limits_{i \in I} M_i^*, (M_{\bar{h}}^*)\Big)$

Representation spaces:

For $\nu = \sum\limits_i \nu_i \alpha_i \in Q_+$, let $\Lambda(\nu) \subset \prod\limits_{h \in H} \mathrm{Hom}_K\big(K^{\nu_{s(h)}}, K^{\nu_{t(h)}}\big)$ $\quad G(\nu) = \prod\limits_i GL_{\nu_i}(K)$

source of h $\quad$ target of h

affine variety of Λ-module structures on $\bigoplus\limits_{i \in I} K^{\nu_i}$

"Lusztig's nilpotent varieties"

2) Lusztig's semicanonical basis

Take here $K = \mathbb{C}$.

Let $\nu \in Q_+$. For a Λ-module M of dim. vect. ν, define $\delta_M : U_\nu \longrightarrow \mathbb{C}$ by:
if $\alpha_{i_1} + \dots + \alpha_{i_k} = \nu$, then (Lusztig; Geiß-Leclerc-Schröer)

$$\delta_M(e_{i_1} \cdots e_{i_k}) = \chi\left(\left\{ 0 = M_0 \subset M_1 \subset \dots \subset M_k = M \;\middle|\; \begin{array}{l} M_p \text{ submodule of } M \\ \underline{\dim}\, M_p / M_{p-1} = \alpha_{i_p} \end{array} \right\} \right)$$

closed subset of the product of flag manifolds

Example: type A_2 $\quad 1 \bullet \underset{\bar{h}}{\overset{h}{\rightleftarrows}} \bullet 2 \quad$ 4 indecomposables $\quad S_1 : \mathbb{C} \rightleftarrows 0 \quad S_2 : 0 \rightleftarrows \mathbb{C}$

$T_1 : \mathbb{C} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{C} \quad T_2 : \mathbb{C} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{C}$

in dimension-vector $\nu = 2\alpha_1 + \alpha_2$

$\mathbb{C}^2 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}$

$\mathbb{C}^2 \underset{(0\ 1)}{\overset{0}{\rightleftarrows}} \mathbb{C}$

M	$\delta_M(e_1^2 e_2)$	$\delta_M(e_1 e_2 e_1)$	$\delta_M(e_2 e_1^2)$
$2S_1 \oplus S_2$	$\chi(\mathbb{P}^1) = 2$	$\chi(\mathbb{P}^1) = 2$	$\chi(\mathbb{P}^1) = 2$
$S_1 \oplus T_1$	$\chi(\mathbb{P}^1) = 2$	$\chi(\bullet) = 1$	$\chi(\emptyset) = 0$
$S_1 \oplus T_2$	0	1	2

the module has no submodule of $\underline{\dim} = \alpha_2$

unique submodule of $\underline{\dim} = \alpha_1 + \alpha_2$

As one can see, the Serre relation $\delta_M(e_1^2 e_2) - 2\,\delta_M(e_1 e_2 e_1) + \delta_M(e_2 e_1^2) = 0$ is always satisfied, so δ_M is well defined.

Observation: The δ_M are not linearly independent. But they span $R_{-\nu} = (U_\nu)^*$.

Problem: Extract a basis

$(\Lambda(\nu) \longrightarrow R_{-\nu},\ M \longmapsto \delta_M)$ is constructible; for each $Z \in \operatorname{Irr} \Lambda(\nu)$, define δ_Z as δ_M for M general in Z.

Theorem (Lusztig): $\displaystyle\coprod_{\nu \in Q_+} \{ \delta_Z \mid Z \in \operatorname{Irr} \Lambda(\nu) \}$ is a basis of R "dual semicanonical basis".

Definition: M a Λ-module, $i \in I$

i-head of M: $\mathrm{hd}_i M$ = largest quotient of M isomorphic to $S_i^{\oplus \cdots}$

(i-th part of the head of M)

Observations: * $\delta_M(\cdots e_i^n) \neq 0 \implies \exists\, M \twoheadrightarrow S_i^{\oplus n} \implies n \leq \dim \mathrm{hd}_i M$

and for $n = \dim \mathrm{hd}_i M$, $e_i^{(n)} \delta_M = \delta_N \neq 0$, where $N = \ker(M \to \mathrm{hd}_i M)$.

(Note here that the divided power $n!$ is the Euler characteristic of the flag variety of $\mathrm{hd}_i M$.)

Moreover, if M is generic, then N is generic.

} the dual semicanonical basis satisfies (P2)

* $(\delta_M)^\dagger = \delta_{M^\dagger}$.

Conclusion: 1) The dual semicanonical basis is perfect.

2) $\coprod_{\nu \in Q_+} \mathrm{Irr}\, \Lambda(\nu)$ is canonically indexed by $B(\infty)$ (Kashiwara-Saito)

$$\begin{array}{ccc} \Psi & & \Psi \\ \Lambda_b & \longleftarrow & b \end{array}$$

an ingredient of the proof of Lusztig's theorem.

3) Reflection functors

Let $i \in I$.

Local description around i of a Λ-module M:

$$\bigoplus_{\substack{h \in H \\ s(h) = i}} M_{t(h)} \xrightarrow{(M_{\bar{h}})} M_i \xrightarrow{(\varepsilon(h) M_h)} \bigoplus_{\substack{h \in H \\ s(h) = i}} M_{t(h)}$$

for brevity: $\quad \tilde{M}_i \xrightarrow{M_{in(i)}} M_i \xrightarrow{M_{out(i)}} \tilde{M}_i \qquad (*)$

Note: $\mathrm{hd}_i M = \mathrm{coker}\, M_{in(i)}$; set $\mathrm{soc}_i M = \ker M_{out(i)}$ i-socle of M

Define $\Sigma_i M$ by replacing in M the part $(*)$ by $\tilde{M}_i \xrightarrow{M_{out(i)} M_{in(i)}} \ker M_{in(i)} \hookrightarrow \tilde{M}_i$

$\Sigma_i^\dagger M$ ———————— $\tilde{M}_i \twoheadrightarrow \mathrm{coker}\, M_{out(i)} \xrightarrow{M_{out(i)} M_{in(i)}} \tilde{M}_i$

Still get Λ-modules, because $\tilde{M}_i \to \tilde{M}_i$ hasn't changed, and at vertex i, the composed of the two maps is zero.

Fact: These functors induce equivalences of categories

$$\{M \in \Lambda\text{-mod} \mid \mathrm{hd}_i M = 0\} \underset{\Sigma_i^\dagger}{\overset{\Sigma_i}{\rightleftarrows}} \{M \in \Lambda\text{-mod} \mid \mathrm{soc}_i M = 0\}$$

Moreover $\mathrm{hd}_i M = 0 \Rightarrow \underline{\dim}\, \Sigma_i M = s_i(\underline{\dim}\, M)$.

Theorem (B.): Let $i \in I$, set $T_i = \mathrm{Ad}(\bar{s}_i) \in \mathrm{Aut}(U(\mathfrak{g}))$ $\quad (\bar{s}_i = \exp(e_i)\exp(-f_i)\exp(e_i))$.
Let $\nu \in Q_+$, let $M \in \Lambda\text{-mod}$ s.t. $\underline{\dim}\, M = \nu$ and $\mathrm{hd}_i M = 0$. Let $x \in U_\nu$ such that $T_i(x) \in U$. Then $\langle \delta_M, x \rangle = \langle \delta_{\Sigma_i M}, T_i(x) \rangle$.

Theorem (B.-Kamnitzer): Let $b \in B(\infty)$. M general in $\Lambda_b \Rightarrow \Sigma_i M$ general in $\Lambda_{\hat{\sigma}_i(b)}$ (loosely stated).

(Another interpretation of the Saito reflection: they now act on irreducible components of nilpotent varieties.)

4) Tilting theory in Λ-mod

$I_i = \mathrm{ann}_\Lambda S_i$. Then $\Sigma_i = \mathrm{Hom}_\Lambda(I_i, ?)$ and $\Sigma_i^\dagger = I_i \otimes_\Lambda ?$

Theorem (Buan-Iyama-Reiten-Scott):

i) The Λ-bimodules I_i satisfy the braid relations: set $I_w = I_{i_1} \otimes_\Lambda \cdots \otimes_\Lambda I_{i_k}$ for $w = s_{i_1} \cdots s_{i_k}$ reduced.

(Consequence: the Σ_i satisfy the braid relations, hence the $\hat{\sigma}_i$ also do.)

ii) I_w tilting Λ-bimodule, $\mathrm{End}_\Lambda(I_w) = \Lambda$.

Brenner-Butler theory $\Rightarrow$

- I_w defines a (in fact, two) torsion pair in Λ-mod.
- Each $M \in \Lambda\text{-mod}$ has a largest quotient M/N such that $\mathrm{Hom}_\Lambda(I_w, M/N) = 0$, namely $N = \mathrm{im}\left(I_w \otimes_\Lambda \mathrm{Hom}_\Lambda(I_w, M) \xrightarrow{\mathrm{eval}} M\right)$.

Write M^w for N.

Examples: $M^1 = M$, $M^{s_i} = \ker(M \to \mathrm{hd}_i M)$, $M^{uv} \subset M^u$ if $\ell(uv) = \ell(u) + \ell(v)$.

Prop (B.-Kamnitzer-Tingley): Let $b \in B(\infty)$, M general in Λ_b, $w \in W$. Then $\mu_w(b) = -\underline{\dim}\, M^w$.

they were defined only for A of finite type, but the definition is general

Proof: $w = s_{i_1} \cdots s_{i_k}$ reduced.

$\mathrm{Hom}_\Lambda(I_w, M) = \Sigma_{i_k} \cdots \Sigma_{i_1} M$ is general in $\Lambda_{b'}$, where $b' = \hat{\sigma}_{i_k} \cdots \hat{\sigma}_{i_1} b = \hat{\sigma}_{w^{-1}} b$.

$\underline{\dim}\, M^w = w\, \underline{\dim}\, \mathrm{Hom}_\Lambda(I_w, M) = -w \,\mathrm{wt}\, b' = -\mu_w(b)$.

5) Harder-Narasimhan polytopes

$\mathcal{A}$ finite length category

$T \in \mathcal{A} \longmapsto [T] \in K(\mathcal{A})$

Definition: $P(T) =$ convex hull in $K(\mathcal{A})_{\mathbb{R}} = K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ of $[X]$ for $X \subset T$.

HN polytope of T

(convex hull of a finite number of points).

Faces of $P(T)$: each $\theta \in K(\mathcal{A})^*_{\mathbb{R}}$ defines $P_\theta(T) = \{ x \in P(T) \mid \langle \theta, x \rangle = \sup_{P(T)} \theta \}$

Fact: $\{ X \subset T \mid [X] \in P_\theta(T) \}$ has a smallest element, $T_\theta^{\min}$, and a largest one, $T_\theta^{\max}$.

Exercise: Let $\mathcal{R}_\theta = \{ T \in \mathcal{A} \mid \langle \theta, [T] \rangle = 0 \text{ and } \forall X \subset T,\ \langle \theta, [X] \rangle \leq 0 \}$

(θ-semistable objects). Then $\mathcal{R}_\theta$ abelian subcategory, $T_\theta^{\max} / T_\theta^{\min} \in \mathcal{R}_\theta$,

and for $i : \mathcal{R}_\theta \hookrightarrow \mathcal{A}$,

$$P_\theta(T) = [T_\theta^{\min}] + K(i)_{\mathbb{R}} \left(P\left(T_\theta^{\max} / T_\theta^{\min} \right) \right)$$

HN polytope relative to $\mathcal{R}_\theta$

(Heredity property: a face of a HN polytope is a HN polytope, flattened and shifted).

Case $\mathcal{A} = \Lambda$-mod

$K(\Lambda\text{-mod}) = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, $\quad K(\Lambda\text{-mod})_{\mathbb{R}} = \mathfrak{h}^*_{\mathbb{R}}$. Denote $C_0 \subset \mathfrak{h}_{\mathbb{R}}$ dominant chamber.

$$[M] = \underline{\dim}\, M$$

Theorem (B.-Kamnitzer-Tingley): Assume A of finite type.

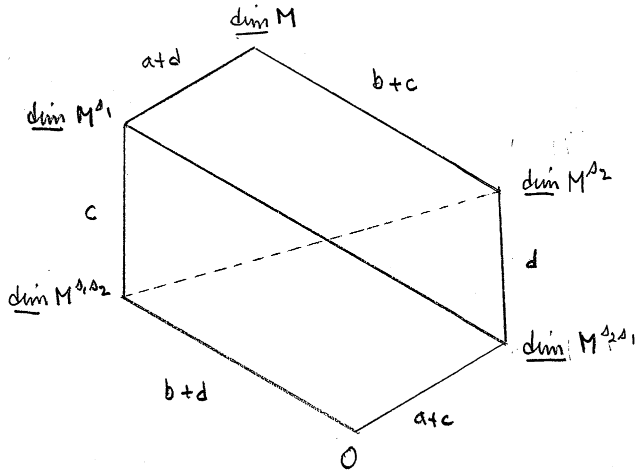
Let $\theta \in \mathfrak{h}_{\mathbb{R}}$. Then $\{ w \in W \mid w^{-1}\theta \in \overline{C_0} \}$ has a shortest element, w_1 / longest element, w_2

Then for each Λ-module M, $M^{\min}_{\theta} = M^{w_2}$ and $M^{\max}_{\theta} = M^{w_1}$.

In particular, $P(M)$ is GGMS and $=$ Convex hull $\left(\{ \underline{\dim}\, M^w \mid w \in W \} \right)$.

Corollary: $b \in B(\infty)$, M general in Λ_b $\Rightarrow$ $\mathrm{Pol}(b) = -P(M)$

Example: Type A_2, $\quad M = S_1^{\oplus a} \oplus S_2^{\oplus b} \oplus T_1^{\oplus c} \oplus T_2^{\oplus d}$



M general $\Rightarrow$ a or $b = 0$ $\Rightarrow$ one of the two diagonals is // to the opposite sides

All 2-faces of type A_2 of an MV polytope have this property ("Tropical Plücker Relations"). This characterizes MV polytopes among all lattice GGMS polytopes ($\exists$ also TPR for 2-faces of type B_2 and G_2.)

In view of the last Corollary in Part I, this condition translates to relations between Lusztig data $N(\underline{i}, b)$ for b fixed, $\underline{i}$ variable. These relations are equivalent to Lusztig's piecewise linear bijections.