The q-Weyl group of a q-Schur algebra

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Abstract

The q-Schur algebras of Dipper and James are quotients of the quantized enveloping algebras $U_q(\mathfrak{gl}_m)$ of Drinfeld and Jimbo. The q-Weyl group of $U_q(\mathfrak{gl}_m)$ (also known as Lusztig's automorphisms braid group) induces a group of inner automorphisms of the q-Schur algebras. We describe precisely elements in the q-Schur algebras that define these inner automorphisms. This description allows us to recover certain known properties of the q-Weyl group.

Introduction

Let G be a complex reductive Lie group, $\mathfrak{g}$ its Lie algebra, and W the Weyl group (relative to some torus and some Borel subgroup B), with its standard generators $\{s_1, \dots, s_\ell\}$. Let $\widetilde{W}$ be the braid group of W, that is the group with generators $\{\bar{s}_1, \dots, \bar{s}_\ell\}$ and relations $\bar{s}_i \bar{s}_j \bar{s}_i \cdots = \bar{s}_j \bar{s}_i \bar{s}_j \cdots$, with m_{ij} factors in each side, (m_{ij}) being the Coxeter matrix. There are several morphisms of $\widetilde{W}$ into G (see [25]), through which $\widetilde{W}$ acts on the integrable $\mathfrak{g}$-modules, and hence (by the adjoint action) on the enveloping algebra $U(\mathfrak{g})$. One can imbed $U(\mathfrak{g})$ and the group algebra $\mathbf{C}G$ into a bigger algebra $\widehat{U(\mathfrak{g})}$, so that these actions of $\widetilde{W}$ on $U(\mathfrak{g})$ become the restrictions of inner automorphisms of $\widehat{U(\mathfrak{g})}$.

Let $U_q(\mathfrak{g})$ be the Drinfeld–Jimbo quantization of $U(\mathfrak{g})$: this is a Hopf algebra over the field $\mathbf{Q}(v)$ of rational functions (with $q = v^2$). Lusztig, Levendorskii and Soibelman have defined invertible elements $\bar{s}_1, \dots, \bar{s}_\ell$ in some completion $\widehat{U_q(\mathfrak{g})}$ of $U_q(\mathfrak{g})$. These elements define a morphism $\widetilde{W} \to \left(\widehat{U_q(\mathfrak{g})}\right)^\times$, so that $\widetilde{W}$ acts on the integrable $U_q(\mathfrak{g})$-modules, this construction being a deformation of the classical case.

One of the results of Lusztig, Levendorskii and Soibelman is that if $\mathfrak{g}$ is of type A, D or E, and if M is the q-deformation of the adjoint $\mathfrak{g}$-module, then the action of $\widetilde{W}$ on the zero-weight subspace of M satisfies quadratic relations and factorizes through the Hecke algebra of W (at the value q of the parameter). This result may be proved by a simple computation [22, § 1][20, Sect. 4].

Another way to see that is the following. The Hecke algebra admits a geometric realization in terms of double B-cosets in G, hence in terms of G-orbits of pairs of flags

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$(f, f') \in (G/B)^2$. In the same vein, Beilinson, Lusztig and MacPherson [1] have constructed certain finite-dimensional algebras $\mathcal{S}_q^d(m)$ in terms of the geometry of relative positions of two m-step filtrations in a d-dimensional vector space, and have shown that these algebras were epimorphic images of $U_q(\mathfrak{gl}_m)$. (Du [8] has observed that the algebra $\mathcal{S}_q^d(m)$ was isomorphic to the q-Schur algebra of Dipper and James, the construction of Beilinson, Lusztig and MacPherson being the translation in a geometric framework of the original algebraic definition.) It is easy to see that $\mathcal{S}_q^d(m)$ is a quotient not only of $U_q(\mathfrak{gl}_m)$ but even of $\widehat{U_q(\mathfrak{gl}_m)}$. It therefore makes sense to consider the images in $\mathcal{S}_q^d(m)$ of the generators $\bar{s}_1, \dots, \bar{s}_{m-1}$ of the q-Weyl group of $U_q(\mathfrak{gl}_m)$. Despite the complicated original definition of the $\bar{s}_i$, these images have a very simple description, that enables us to give a perhaps more conceptual proof (in case A) of the result of Lusztig, Levendorskii and Soibelman recalled above.

Section 1 of this paper is aimed at giving a unified treatment of the basic facts concerning the quantized enveloping algebras $U_q(\mathfrak{gl}_m)$, the q-Schur algebras $\mathcal{S}_q^d(m)$, and the spaces $S_q^d(\mathcal{M}_{m,n}^*)$ of homogeneous functions of degree d on the quantum spaces of matrices of size $m \times n$. Section 1 is supplemented by an appendix explaining how to deduce from a simple argument (given in the non-quantum case by J. A. Green [11]) the standard basis theorems for $S_q^d(\mathcal{M}_{m,n}^*)$ and $\mathcal{S}_q^d(m)$ (proved algorithmically in [15] and [12]). In Section 2, we recall the definition of the q-Weyl group of $U_q(\mathfrak{gl}_m)$ and explain how to compute the images of the generators $\bar{s}_1, \dots, \bar{s}_{m-1}$ under the epimorphism $U_q(\mathfrak{gl}_m) \twoheadrightarrow \mathcal{S}_q^d(m)$. Our main result there is Theorem 3. Section 3 deals with applications of this result. We explain how to recover the quantum Schur-Weyl duality from a quantum $(\mathfrak{gl}_m, \mathfrak{gl}_n)$-duality and how all this is related to the aforementioned result of Lusztig, Levendorskii and Soibelman.

1 q-Deformations

1.1 Basic notations

Let m be a positive integer. We denote by P_m the free $\mathbf{Z}$-module with basis $(\varepsilon_1, \dots, \varepsilon_m)$. Elements of P_m are called weights. A weight $\lambda = \lambda_1\varepsilon_1 + \dots + \lambda_m\varepsilon_m$ is said to be polynomial (respectively dominant) if all the λ_i are non-negative (respectively if $\lambda_1 \geq \dots \geq \lambda_m$). The degree of a weight $\lambda = \lambda_1\varepsilon_1 + \dots + \lambda_m\varepsilon_m$ is the integer $|\lambda| = \lambda_1 + \dots + \lambda_m$. The simple roots are the elements $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ (for $1 \leq i \leq m-1$). We endow P_m with the bi-additive form $P_m \times P_m \to \mathbf{Z}$ defined by $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$ (Kronecker's symbol). The symmetric group on m letters $\mathfrak{S}_m$ acts on the set $\{\varepsilon_1, \dots, \varepsilon_m\}$, hence acts $\mathbf{Z}$-linearly on P_m. The image W of $\mathfrak{S}_m$ in $\mathrm{Aut}(P_m)$ is generated by the simple reflections $s_i : \big(P_m \to P_m, \lambda \mapsto \lambda - (\alpha_i|\lambda)\alpha_i\big)$ (for $1 \leq i \leq m-1$).

Our ground ring will be the field $\mathbf{Q}(v)$ of rational functions in v over the rational numbers, although most of our constructions are valid over an arbitrary commutative base ring. We put $q = v^2$. The q-numbers, q-factorials and q-binomial coefficients are defined as in [23]:

$$[n] = \tfrac{v^n - v^{-n}}{v - v^{-1}}, \qquad [n]! = \textstyle\prod_{i=1}^n [i], \qquad \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \tfrac{[n]!}{[k]![n-k]!}.$$

1.2 Quantized enveloping algebra

Let m be a positive integer and let $(a_{ij})_{1\leq i,j\leq m-1}$ be the Cartan matrix: $a_{ij} = (\alpha_i|\alpha_j)$.

Definition 1 [17] *The quantized enveloping algebra $U_q(\mathfrak{gl}_m)$ is the $\mathbf{Q}(v)$-algebra presented by the generators K_λ (for $\lambda \in P_m$), $E_1, \dots, E_{m-1}$ and $F_1, \dots, F_{m-1}$, with the relations:*

$$
\begin{aligned}
&K_\lambda\, K_\mu = K_{\lambda+\mu} && \textit{(for } \lambda,\mu \in P_m\textit{)},\\
&K_\lambda\, E_i = v^{(\lambda|\alpha_i)} E_i\, K_\lambda \quad \textit{and} \quad K_\lambda\, F_i = v^{-(\lambda|\alpha_i)} F_i\, K_\lambda && \textit{(for } \lambda \in P_m \textit{ and } 1 \leq i \leq m-1\textit{)},\\
&\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix} E_i^r E_j E_i^{1-a_{ij}-r} = 0 && \textit{(for } 1 \leq i \neq j \leq m-1\textit{)},\\
&\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix} F_i^r F_j F_i^{1-a_{ij}-r} = 0 && \textit{(for } 1 \leq i \neq j \leq m-1\textit{)},\\
&[E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{v - v^{-1}} && \textit{(for } 1 \leq i,j \leq m-1\textit{)}.
\end{aligned}
$$

It is a Hopf algebra when endowed with the coproduct Δ and the counit ε given by:

$$
\begin{aligned}
\Delta(K_\lambda) &= K_\lambda \otimes K_\lambda, & \varepsilon(K_\lambda) &= 1 && \textit{(for } \lambda \in P_m\textit{)},\\
\Delta(E_i) &= E_i \otimes K_{\alpha_i} + 1 \otimes E_i, & \varepsilon(E_i) &= 0 && \textit{(for } 1 \leq i \leq m-1\textit{)},\\
\Delta(F_i) &= F_i \otimes 1 + K_{-\alpha_i} \otimes F_i, & \varepsilon(F_i) &= 0 && \textit{(for } 1 \leq i \leq m-1\textit{)}.
\end{aligned}
$$

The natural $U_q(\mathfrak{gl}_m)$-module is the vector space V_m with basis $(e_1, \dots, e_m)$, and the action of $U_q(\mathfrak{gl}_m)$ is given by:

$$
\begin{aligned}
&K_\lambda \cdot e_k = v^{(\lambda|\varepsilon_k)} e_k && \text{(for } \lambda \in P_m \text{ and } 1 \leq k \leq m\text{)},\\
&E_i \cdot e_k = \delta_{i+1,k}\, e_i \quad \text{and} \quad F_i \cdot e_k = \delta_{i,k}\, e_{i+1} && \text{(for } 1 \leq i \leq m-1 \text{ and } 1 \leq k \leq m\text{)}.
\end{aligned}
$$

The dual of V_m is a right $U_q(\mathfrak{gl}_m)$-module. We will denote it by V_m^* and we will denote by $(f_1, \dots, f_m)$ the basis dual to $(e_1, \dots, e_m)$.

The comultiplication of $U_q(\mathfrak{gl}_m)$ allows to endow the tensor power $(V_m)^{\otimes d}$ with the structure of a $U_q(\mathfrak{gl}_m)$-module. We will denote by $e_{j_1,\dots,j_d}$ the element $e_{j_1} \otimes \cdots \otimes e_{j_d}$ in $(V_m)^{\otimes d}$. Analogously, $(V_m^*)^{\otimes d}$ is a right $U_q(\mathfrak{gl}_m)$-module, with basis $f_{i_1,\dots,i_d}$.

The subalgebra U^0 generated in $U_q(\mathfrak{gl}_m)$ by the elements K_λ is commutative. A vector e in a $U_q(\mathfrak{gl}_m)$-module is said to be of weight $\lambda \in P_m$ if the action of U^0 on e is scalar and given by the character $K_\mu \mapsto v^{(\lambda|\mu)}$. A left $U_q(\mathfrak{gl}_m)$-module M will be called a weight module if it is the sum of its weight subspaces:

$$M = \bigoplus\nolimits_{\lambda \in P_m} \{e \in M \mid \forall \mu \in P_m,\ K_\mu \cdot e = v^{(\lambda|\mu)} e\}.$$

The module M is said to be polynomial (respectively polynomial of degree d) if all the weights occurring in the sum are polynomial (respectively polynomial of degree d).

Finally, there is an involutive antiautomorphism of algebras $\Phi : U_q(\mathfrak{gl}_m) \to U_q(\mathfrak{gl}_m)$ given on the generators by $K_\lambda \mapsto K_\lambda,\ E_i \mapsto F_i,\ F_i \mapsto E_i$. With it, one has the notion of contravariant duality.

1.3 Hecke algebras of type A and the quantum Schur–Weyl duality

Let d be a positive integer, $\mathfrak{S}_d$ be the symmetric group on d letters, and $S = \{s_1, \ldots, s_{d-1}\}$ be the set of usual generators of $\mathfrak{S}_d$.

Definition 2 *The Hecke algebra $H_q(\mathfrak{S}_d)$ of the Coxeter system $(\mathfrak{S}_d, S)$ is the $\mathbf{Q}(v)$-algebra presented by the generators $(T_{s_i})_{1\le i\le d-1}$, with the relations:*

$$\begin{aligned}
&T_{s_i}T_{s_j} = T_{s_j}T_{s_i} && \text{(for } |i-j|\ge 2\text{)},\\
&T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}} && \text{(for } 1\le i\le d-2\text{)},\\
&(T_{s_i}-q)(T_{s_i}+1) = 0 && \text{(for } 1\le i\le d-1\text{)}.
\end{aligned}$$

For each $w \in \mathfrak{S}_d$, one can define T_w as the product $T_{s_{i_1}}\cdots T_{s_{i_k}}$, where $s_{i_1}\cdots s_{i_k}$ is a reduced decomposition of w. The $(T_w)_{w\in\mathfrak{S}_d}$ form a basis of $H_q(\mathfrak{S}_d)$.

The algebra $H_q(\mathfrak{S}_d)$ acts on the spaces $(V_m)^{\otimes d}$ and $(V_m^*)^{\otimes d}$, by means of the following formulas:

$$e_{j_1,\ldots,j_d}\cdot T_{s_i} = \begin{cases} v\, e_{j_1,\ldots,j_{i+1},j_i,\ldots,j_d} & \text{if } j_i < j_{i+1},\\ q\, e_{j_1,\ldots,j_d} & \text{if } j_i = j_{i+1},\\ v\, e_{j_1,\ldots,j_{i+1},j_i,\ldots,j_d} + (q-1)e_{j_1,\ldots,j_d} & \text{if } j_i > j_{i+1},\end{cases}$$

$$T_{s_i}\cdot f_{i_1,\ldots,i_d} = \begin{cases} v\, f_{j_1,\ldots,j_{i+1},j_i,\ldots,j_d} & \text{if } j_i < j_{i+1},\\ q\, f_{j_1,\ldots,j_d} & \text{if } j_i = j_{i+1},\\ v\, f_{j_1,\ldots,j_{i+1},j_i,\ldots,j_d} + (q-1)f_{j_1,\ldots,j_d} & \text{if } j_i > j_{i+1}.\end{cases}$$

In this way, $(V_m)^{\otimes d}$ becomes a $U_q(\mathfrak{gl}_m)$-$H_q(\mathfrak{S}_d)$-bimodule and $(V_m^*)^{\otimes d}$ is its dual $H_q(\mathfrak{S}_d)$-$U_q(\mathfrak{gl}_m)$-bimodule. These assertions are part of the theory of quantized Schur–Weyl duality, due to Jimbo (see Section 3.2).

1.4 q-Schur algebras and the BLM construction

In this section, I present a construction of the q-Schur algebras due to Beilinson, Lusztig and MacPherson. One can find detailed proofs for the assertions below in the article [1].

Let m, n and d be three positive integers, and denote by $\Theta^d_{m,n}$ the set of all matrices of size $m\times n$, whose coefficients are non-negative integers of sum d.

Let V be a vector space of finite dimension d over a field $\mathbf{F}$. Let $f = (0 = V_0 \subseteq \cdots \subseteq V_m = V)$ and $f' = (0 = V'_0 \subseteq \cdots \subseteq V'_n = V)$ be two filtrations of V. To the pair (f, f'), one associates the matrix $A = c(f, f')$ of size $m\times n$ with coefficients $a_{ij} = \dim(V_i \cap V'_j) - \dim(V_i\cap V'_{j-1}) - \dim(V_{i-1}\cap V'_j) + \dim(V_{i-1}\cap V'_{j-1})$, so that $\dim(V_i\cap V'_j) = \sum_{r\le i, s\le j} a_{rs}$. Let X_m be the set of all m-step filtrations in V. The group $\mathrm{GL}(V)$ acts on all the X_m, hence acts also (diagonally) on $X_m\times X_n$. The map $(f, f')\mapsto c(f, f')$ defines a bijection between the set of $\mathrm{GL}(V)$-orbits in $X_m\times X_n$ and the set of matrices $\Theta^d_{m,n}$.

If $A\in\Theta^d_{m,n}$, we denote by $\mathcal{O}_A$ the corresponding $\mathrm{GL}(V)$-orbit in $X_m\times X_n$. To a matrix $A\in\Theta^d_{m,n}$, we associate the weights $\mathrm{ro}(A) = \sum_{i,j} a_{ij}\varepsilon_i$ and $\mathrm{co}(A) = \sum_{i,j} a_{ij}\varepsilon_j$ belonging to

P_m and P_n respectively. If $(f, f') \in \mathcal{O}_A$, with $f = (0 = V_0 \subseteq \cdots \subseteq V_m = V)$ and $f' = (0 = V'_0 \subseteq \cdots \subseteq V'_n = V)$, then $\mathrm{ro}(A) = \sum_{i=1}^m \dim(V_i/V_{i-1})\varepsilon_i$ and $\mathrm{co}(A) = \sum_{j=1}^n \dim(V'_j/V'_{j-1})\varepsilon_j$.

Let m, n, p be three positive integers and let $A \in \Theta^d_{m,p}$, $B \in \Theta^d_{p,n}$, $C \in \Theta^d_{m,n}$. There exists a polynomial $g_{A,B,C} \in \mathbf{Z}[v^2]$ satisfying the following property: if $\mathbf{F}$ is a finite field with q elements, if one chooses filtrations $f \in X_m$ and $f' \in X_n$ such that (f, f') belongs to the orbit $\mathcal{O}_C$, then the value at $v^2 = q$ of polynomial $g_{A,B,C}$ is equal to the number of filtrations $f'' \in X_p$ such that $(f, f'') \in \mathcal{O}_A$ and $(f'', f') \in \mathcal{O}_B$. One has $g_{A,B,C} = 0$ except if $\mathrm{ro}(A) = \mathrm{ro}(C)$, $\mathrm{co}(A) = \mathrm{ro}(B)$, and $\mathrm{co}(B) = \mathrm{co}(C)$. If $m = p$ and $A \in \Theta^d_{m,m}$ is a diagonal matrix, then $g_{A,B,C}$ is $\delta_{B,C}$ or 0 according as $\mathrm{ro}(B)$ is equal or different from $\mathrm{co}(A)$. If $p = n$ and $B \in \Theta^d_{n,n}$ is a diagonal matrix, then $g_{A,B,C}$ is $\delta_{A,C}$ or 0 according as $\mathrm{co}(A)$ is equal or different from $\mathrm{ro}(B)$.

We denote by $\mathcal{S}^d_q(m, n)$ the $\mathbf{Q}(v)$-vector space with basis the family of symbols $(e_A)_{A\in\Theta^d_{m,n}}$. The coefficients $g_{A,B,C}$ afford a bilinear map $\big(\mathcal{S}^d_q(m, p) \times \mathcal{S}^d_q(p, n) \to \mathcal{S}^d_q(m, n), (e_A, e_B) \mapsto \sum_{c\in\Theta^d_{m,n}} g_{A,B,C}\ e_C\big)$, and we have an associativity property for these "products". Letting $m = n = p$, we see that $\mathcal{S}^d_q(m, m)$ is an algebra; the unit is the element $\sum_A e_A$, where the sum runs over the set of diagonal matrices in $\Theta^d_{m,m}$. Similarly, $\mathcal{S}^d_q(m, n)$ is a $\mathcal{S}^d_q(m, m)$-$\mathcal{S}^d_q(n, n)$-bimodule. We will simplify the notation and denote $\mathcal{S}^d_q(m, m)$ by $\mathcal{S}^d_q(m)$.

Finally if $A \in \Theta^d_{m,n}$, one puts $[A] = v^{-\sum_{i\geq k, j<l} a_{ij}a_{kl}} e_A$ in $\mathcal{S}^d_q(m, n)$. From the above, it follows that if $A \in \Theta^d_{m,m}$ is a diagonal matrix, then $[A] \in \mathcal{S}^d_q(m)$ is an idempotent and the left multiplication by $[A]$ in $\mathcal{S}^d_q(m, n)$ is the projection on the subspace spanned by the set $\{[B] \mid B \in \Theta^d_{m,n},\ \mathrm{ro}(B) = \mathrm{co}(A)\}$. An analogous property holds on the right. The unit in $\mathcal{S}^d_q(m)$ is the element $\sum_A [A]$, where the sum runs over the set of diagonal matrices in $\Theta^d_{m,m}$.

Remarks. (a) The original definition of the q-Schur algebra goes back to Dipper and James. The fact that both constructions define the same object has been noticed by Du [8, § 1.4]. Let me explain here briefly and without proofs this correspondence.

To a polynomial weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m$ in P_m of degree d, one can associate the parabolic subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_m}$ of $\mathfrak{S}_d$ and the element $x_\lambda = \sum_{w\in\mathfrak{S}_\lambda} T_w$ in $H_q(\mathfrak{S}_d)$. In the $U_q(\mathfrak{gl}_m)$-$H_q(\mathfrak{S}_d)$-bimodule $(V_m)^{\otimes d}$, the weight space (for the action of $U_q(\mathfrak{gl}_m)$) for the weight λ is a right $H_q(\mathfrak{S}_d)$-submodule isomorphic to the induced module $x_\lambda H_q(\mathfrak{S}_d)$. Hence the commutant of the image of the algebra $H_q(\mathfrak{S}_d)$ inside $\mathrm{End}_{\mathbf{Q}(v)}\big((V_m)^{\otimes d}\big)$ is isomorphic to the algebra $\mathrm{End}_{H_q(\mathfrak{S}_d)}\big(\bigoplus_\lambda x_\lambda H_q(\mathfrak{S}_d)\big)$, where the sum runs over all polynomial weights $\lambda \in P_m$ of degree d. This latter object is the q-Schur algebra of Dipper and James (see [5, § 1.2]). The Schur–Weyl duality (see the end of Section 1.3) implies the existence of an algebra morphism from $U_q(\mathfrak{gl}_m)$ to $\mathcal{S}^d_q(m)$.

Dipper and James' definition for the q-Schur algebra is valid over an arbitrary base ring R. Let q be a power of a prime number and take R so that the image of q in the prime subring of R is invertible and has a square root v. Let $\mathbf{F}_q$ be the field with q elements, let G be the group $\mathrm{GL}_d(\mathbf{F}_q)$, let B be its standard Borel subgroup, and identify the symmetric group $\mathfrak{S}_d$ with the Weyl group of G in the usual way. To each weight $\lambda \in P_m$ of degree d is associated the standard parabolic subgroup $P_\lambda = B\mathfrak{S}_\lambda B$ of G. Let M (respectively N_λ) be the right RG-module obtained by inducing the trivial representation of B (respectively P_λ) to G. The module M (respectively N_λ) can be identified with the set of functions from G/B (respectively G/P_λ) to R, where G acts by left translation.

The endomorphism algebra $\mathrm{End}_G(M)$ is classically identified with the set of functions from $B\backslash G/B$ to R, endowed with the convolution product (see [2, Chap. IV, § 2, exerc. 22]), that is, with the "specialized" Hecke algebra $H_q(\mathfrak{S}_d)_R$. The subspace $x_\lambda H_q(\mathfrak{S}_d)_R$ may be identified with the set of functions from $B\backslash G/P_\lambda$ to R.

Now M is an $H_q(\mathfrak{S}_d)_R$-RG-bimodule, so we have the functor $\mathrm{Hom}_G(M,-)$ from the category of right RG-modules to the category of right $H_q(\mathfrak{S}_d)_R$-modules. It sends the object N_λ to the object $x_\lambda H_q(\mathfrak{S}_d)_R$, and if $\mu \in P_m$ is another weight of degree d, it defines a bijection between the spaces of homomorphisms $\mathrm{Hom}_G(N_\lambda, N_\mu)$ and $\mathrm{Hom}_{H_q(\mathfrak{S}_d)_R}\big(x_\lambda H_q(\mathfrak{S}_d)_R, x_\mu H_q(\mathfrak{S}_d)_R\big)$. (This is Theorem 2.24 of [4]. Both spaces are naturally in bijection with the space of functions from $P_\lambda\backslash G/P_\mu$ to R, which can be identified with the subspace $H_q(\mathfrak{S}_d)_R\, x_\lambda \cap x_\mu H_q(\mathfrak{S}_d)_R$; see [7, Lemma 1.1 (e)].)

Thus the "specialized" q-Schur algebra in the definition of Dipper and James identifies with the endomorphism ring $\mathrm{End}_G(\bigoplus_\lambda N_\lambda)$. This correspondence, traced at the level of "generic" algebras, explains the aforementioned result of [8].

(b) There is a stabilization procedure in this framework. Let N be an integer greater that two given positive integers m and n. There is a canonical injection $\Theta^d_{m,n} \hookrightarrow \Theta^d_{N,N}$, obtained by putting a matrix A of size $m \times n$ in the top left corner of an $N \times N$ matrix and by padding with zeros elsewhere. This gives rise to a map $\mathcal{S}^d_q(m,n) \hookrightarrow \mathcal{S}^d_q(N)$, defined on the basic vectors. If N is also greater than p and if $A \in \Theta^d_{m,p}$, $B \in \Theta^d_{p,n}$, then the product $[A][B]$ is the same computed in $\mathcal{S}^d_q(N)$ or by the map $\mathcal{S}^d_q(m,p) \times \mathcal{S}^d_q(p,n) \to \mathcal{S}^d_q(m,n)$. In particular, we have a map $\mathcal{S}^d_q(m) \hookrightarrow \mathcal{S}^d_q(N)$, and we denote by η_m the image in $\mathcal{S}^d_q(N)$ of the unit element in $\mathcal{S}^d_q(m)$. Thus η_m is the sum of elements $[A]$, for $A \in \Theta^d_{N,N}$ running over the set of diagonal matrices with $\mathrm{co}(A) \in P_m$. Then η_m is an idempotent in $\mathcal{S}^d_q(N)$, the algebra $\mathcal{S}^d_q(m)$ may be identified with the algebra $\eta_m\ \mathcal{S}^d_q(N)\ \eta_m$, and the $\mathcal{S}^d_q(m)$-$\mathcal{S}^d_q(n)$-bimodule $\mathcal{S}^d_q(m,n)$ may be identified with the subspace $\eta_m\ \mathcal{S}^d_q(N)\ \eta_n$.

There is an algebra homomorphism from $U_q(\mathfrak{gl}_m)$ to $\mathcal{S}^d_q(m)$. The main result obtained by Beilinson, Lusztig and MacPherson asserts its surjectivity and provides formulas for it:

Theorem 1 [1, § 5.7] *There exists a (unique) surjective homomorphism ζ of $\mathbf{Q}(v)$-algebras from $U_q(\mathfrak{gl}_m)$ to $\mathcal{S}^d_q(m)$ which:*

- *sends K_λ to $\sum v^{\lambda_1 j_1+\cdots+\lambda_m j_m}[j_1E_{11}+\cdots+j_mE_{mm}]$, the sum running over all the m-uples of non-negative integers $(j_1,\ldots,j_m)$ of sum d;*
- *sends E_i to $\sum[j_1E_{11}+\cdots+j_mE_{mm}+E_{i,i+1}]$, the sum running over all the m-uples of non-negative integers $(j_1,\ldots,j_m)$ of sum $d-1$;*
- *sends F_i to $\sum[j_1E_{11}+\cdots+j_mE_{mm}+E_{i+1,i}]$, the sum running over all the m-uples of non-negative integers $(j_1,\ldots,j_m)$ of sum $d-1$.*

So $\mathcal{S}_q^d(m)$ is a quotient of $U_q(\mathfrak{gl}_m)$ and $\mathcal{S}_q^d(m,n)$ is a $U_q(\mathfrak{gl}_m)$-$U_q(\mathfrak{gl}_n)$-bimodule. The laws for this latter structure are given by the formulas in [1, Lemmas 3.2 and 3.4]:

Lemma 1 *For the $U_q(\mathfrak{gl}_m)$-$U_q(\mathfrak{gl}_n)$-bimodule structure on $\mathcal{S}_q^d(m,n)$, one has for all $A \in \Theta_{m,n}^d$:*

$$\begin{aligned}
K_\lambda \cdot [A] &= v^{(\lambda|\,\mathrm{ro}(A))}\,[A],\\
E_h \cdot [A] &= \sum_{\substack{p=1\\ a_{h+1,p}\geq 1}}^{n} [a_{hp}+1]\; v^{\sum_{j>p}(a_{hj}-a_{h+1,j})}\,[A+E_{hp}-E_{h+1,p}],\\
F_h \cdot [A] &= \sum_{\substack{p=1\\ a_{hp}\geq 1}}^{n} [a_{h+1,p}+1]\; v^{\sum_{j<p}(a_{h+1,j}-a_{hj})}\,[A-E_{hp}+E_{h+1,p}],\\
[A]\cdot K_\lambda &= v^{(\lambda|\,\mathrm{co}(A))}\,[A],\\
[A]\cdot E_h &= \sum_{\substack{p=1\\ a_{ph}\geq 1}}^{m} [a_{p,h+1}+1]\; v^{\sum_{i<p}(a_{i,h+1}-a_{ih})}\,[A-E_{ph}+E_{p,h+1}],\\
[A]\cdot F_h &= \sum_{\substack{p=1\\ a_{p,h+1}\geq 1}}^{m} [a_{ph}+1]\; v^{\sum_{i>p}(a_{ih}-a_{i,h+1})}\,[A+E_{ph}-E_{p,h+1}].
\end{aligned}$$

As a consequence, the left $U_q(\mathfrak{gl}_m)$-module $\mathcal{S}_q^d(m)$ is a finite-dimensional polynomial module of degree d.

1.5 The algebra of functions on quantum matrix spaces

Let m and n be two positive integers. The following definition can be traced back at least to Manin in the case $m=n$, and the immediate generalization to the case $m\neq n$ also appears in numerous places.

Definition 3 *The $\mathbf{Q}(v)$-algebra presented by generators X_{ij} ($1\leq i\leq m$ and $1\leq j\leq n$) and relations:*

$$X_{rt}X_{su} = \begin{cases} X_{su}X_{rt} & \text{if } r>s \text{ and } t<u \text{ or if } r<s \text{ and } t>u,\\ v\,X_{su}X_{rt} & \text{if } r>s \text{ and } t=u \text{ or if } r=s \text{ and } t>u,\\ X_{su}X_{rt}+(v-v^{-1})X_{ru}X_{st} & \text{if } r>s \text{ and } t>u. \end{cases}$$

is called the algebra of functions on the quantum matrix space and is denoted by $S_q(\mathcal{M}_{m,n}^*)$.

$S_q(\mathcal{M}_{m,n}^*)$ is in a natural way a $\mathbf{N}$-graded algebra, whose d-th homogeneous components will be denoted by $S_q^d(\mathcal{M}_{m,n}^*)$.

Fix the natural integer d, and let $I_{m,n}^d$ and $J_{m,n}^d$ be the sets of all the d-tuples of pairs of integers $((i_1,j_1),\dots,(i_d,j_d))\in(\{1,\dots,m\}\times\{1,\dots,n\})^d$ defined by the conditions of being lexicographically non-decreasing:

$$\begin{aligned}
I_{m,n}^d &= \{((i_1,j_1),\dots,(i_d,j_d)) \mid i_a\leq i_{a+1}, i_a=i_{a+1}\Rightarrow j_a\leq j_{a+1}\},\\
J_{m,n}^d &= \{((i_1,j_1),\dots,(i_d,j_d)) \mid j_a\leq j_{a+1}, j_a=j_{a+1}\Rightarrow i_a\leq i_{a+1}\}.
\end{aligned}$$

Theorem 2 [24, § 3.5]

(a) The set consisting of the vectors $X_{i_1,j_1}\cdots X_{i_d,j_d}$ for $((i_1,j_1),\dots,(i_d,j_d))$ running over $I^d_{m,n}$ (or running over $J^d_{m,n}$) is a $\mathbf{Q}(v)$-basis in $S^d_q(\mathcal{M}^*_{m,n})$, which is thus of dimension $\binom{mn+d-1}{d}$.

(b) $S_q(\mathcal{M}^*_{m,n})$ is an algebra without zero-divisor.

We will need later another parametrization of the basis given by the above theorem. Let $\mathcal{E}^d_{m,n}$ be the set of all d-uples of pairs of integers $((i_1,j_1),\dots,(i_d,j_d))\in(\{1,\dots,m\}\times\{1,\dots,n\})^d$ such that for all $1\le a<b\le d$, one has $i_a<i_b$, $j_a<j_b$ or $(i_a,j_a)=(i_b,j_b)$.

Recall that $\Theta^d_{m,n}$ is the set of all matrices of size $m\times n$, whose coefficients are non-negative integers of sum d. Let $E_{ij}\in\Theta^1_{m,n}$ denotes the elementary matrix, with a 1 in position (i,j) and with 0 elsewhere. To the element $((i_1,j_1),\dots,(i_d,j_d))$ of $\mathcal{E}^d_{m,n}$, one associates the matrix $\varphi((i_1,j_1),\dots,(i_d,j_d))=\sum_{a=1}^d E_{i_a,j_a}\in\Theta^d_{m,n}$.

On $\mathcal{E}^d_{m,n}$, we put the equivalence relation $\approx$ generated by the following elementary moves: $((i_1,j_1),\dots,(i_d,j_d))\sim((i'_1,j'_1),\dots,(i'_d,j'_d))$ if and only if there exists $1\le a<d$ such that $(i_a-i_{a+1})(j_a-j_{a+1})<0$ and $((i'_1,j'_1),\dots,(i'_d,j'_d))$ is obtained from $((i_1,j_1),\dots,(i_d,j_d))$ by exchanging (i_a,j_a) with (i_{a+1},j_{a+1}).

Proposition 1 *The equivalence classes in $\mathcal{E}^d_{m,n}$ are the fibers of $\varphi:\mathcal{E}^d_{m,n}\to\Theta^d_{m,n}$. The subsets $I^d_{m,n}$ and $J^d_{m,n}$ of $\mathcal{E}^d_{m,n}$ contain each one exactly one element of each equivalence class for $\approx$. If $((i_1,j_1),\dots,(i_d,j_d))$ and $((i'_1,j'_1),\dots,(i'_d,j'_d))$ are two $\approx$-equivalent elements in $\mathcal{E}^d_{m,n}$, then $X_{i_1,j_1}\cdots X_{i_d,j_d}=X_{i'_1,j'_1}\cdots X_{i'_d,j'_d}$.*

Proof. One first remark that φ induces bijections between $I^d_{m,n}$, $J^d_{m,n}$ and $\Theta^d_{m,n}$. If $(i_1,\dots,j_d)\in\mathcal{E}^d_{m,n}$, then $(i_a>i_{a+1})\Rightarrow(j_a<j_{a+1})$. By induction on the number of inversions in the sequence $(i_1,\dots,i_d)$, one then shows that $((i_1,j_1),\dots,(i_d,j_d))$ is equivalent for the relation $\approx$ to at least one element of $I^d_{m,n}$. Since $\varphi:\mathcal{E}^d_{m,n}\to\Theta^d_{m,n}$ is surjective and constant on the equivalence classes, the relation $\approx$ has at least $\operatorname{Card}\Theta^d_{m,n}=\operatorname{Card}I^d_{m,n}$ equivalence classes. One deduces from this that $I^d_{m,n}$ is a set of representatives for the equivalence classes of $\approx$ in $\mathcal{E}^d_{m,n}$. The same is true for $J^d_{m,n}$. From the fact that $\varphi:I^d_{m,n}\to\Theta^d_{m,n}$ is a bijection, one deduces that φ induces a bijection between $\mathcal{E}^d_{m,n}/\approx$ and $\Theta^d_{m,n}$. The last assertion of the lemma follows from the first relation in Definition 3.

If $B\in\Theta^d_{m,n}$, we will denote by $X(B)$ the element $X_{i_1,j_1}\cdots X_{i_d,j_d}\in S^d_q(\mathcal{M}^*_{m,n})$ with $((i_1,j_1),\dots,(i_d,j_d))\in\mathcal{E}^d_{m,n}\cap\varphi^{-1}(B)$. The interesting structure on $S_q(\mathcal{M}^*_{m,n})$ is given in the following proposition.

Proposition 2 [15, p. 210] *Let m,n,p be three natural numbers. There is a unique homomorphism of algebras from $S_q(\mathcal{M}^*_{m,n})$ to $S_q(\mathcal{M}^*_{m,p})\otimes S_q(\mathcal{M}^*_{p,n})$ which sends X_{ij} to $\sum_{k=1}^p X_{ik}\otimes X_{kj}$.*

Letting $m=n=p$ in this proposition, we see that the algebra $S_q(\mathcal{M}^*_{m,m})$ may be endowed with the structure of a bialgebra. The counit is given by $\varepsilon(X_{ij})=\delta_{ij}$ for $1\le i,j\le m$. Letting

$p = m$ (respectively $p = n$) in the proposition, we see that $S_q(\mathcal{M}^*_{m,n})$ is a left $S_q(\mathcal{M}^*_{m,m})$-comodule (respectively a right $S_q(\mathcal{M}^*_{n,n})$-comodule). It is in fact a $S_q(\mathcal{M}^*_{m,m})$-$S_q(\mathcal{M}^*_{n,n})$-bicomodule. Each homogeneous component $S^d_q(\mathcal{M}^*_{m,m})$ is a subcoalgebra of $S_q(\mathcal{M}^*_{m,m})$ and $S^d_q(\mathcal{M}^*_{m,n})$ is a $S^d_q(\mathcal{M}^*_{m,m})$-$S^d_q(\mathcal{M}^*_{n,n})$-bicomodule.

Remark. There is a stabilization procedure here. Let N be an integer greater than m, n and p. Let us denote by $\mathcal{K}_{m,n}$ the two-sided ideal generated in the algebra $S_q(\mathcal{M}^*_{N,N})$ by the elements X_{ij} with $i > m$ or $j > n$. The quotient algebra $S_q(\mathcal{M}^*_{N,N})/\mathcal{K}_{m,n}$ identifies naturally with the algebra $S_q(\mathcal{M}^*_{m,n})$. If N is also greater than p, then the coproduct given in Proposition 2 is compatible with this construction since it sends $\mathcal{K}_{m,n}$ into $\big(\mathcal{K}_{m,p} \otimes S_q(\mathcal{M}^*_{p,n})\big) + \big(S_q(\mathcal{M}^*_{m,p}) \otimes \mathcal{K}_{p,n}\big)$.

Let us denote by $t_{ij} \in (U_q(\mathfrak{gl}_m))^*$ the matrix coefficients of the natural $U_q(\mathfrak{gl}_m)$-module V_m: $t_{ij} : x \mapsto \langle f_i, x \cdot e_j\rangle$. They belong to the restricted (Hopf) dual $(U_q(\mathfrak{gl}_m))^{*\mathrm{res}}$ of $U_q(\mathfrak{gl}_m)$.

Proposition 3 *There is a bialgebra morphism κ from $S_q(\mathcal{M}^*_{m,m})$ to $(U_q(\mathfrak{gl}_m))^{*res}$ which sends X_{ij} to t_{ij}.*

Proof. The R-matrix of $U_q(\mathfrak{gl}_m)$ defines the $U_q(\mathfrak{gl}_m)$-linear map:

$$\left((V_m)^{\otimes 2} \to (V_m)^{\otimes 2}, e_{j_1,j_2} \mapsto \begin{cases} e_{j_2,j_1} & \text{if } j_1 < j_2, \\ v\, e_{j_1,j_2} & \text{if } j_1 = j_2, \\ e_{j_2,j_1} + (v - v^{-1}) e_{j_1,j_2} & \text{if } j_1 > j_2 \end{cases} \right).$$

The $U_q(\mathfrak{gl}_m)$-linearity of this map implies that the elements t_{ij} satisfy the relations of Definition 3. We thus have an algebra homomorphism from $S_q(\mathcal{M}^*_{m,m})$ to $(U_q(\mathfrak{gl}_m))^{*\mathrm{res}}$, that clearly respects the coproduct and the counit.

The $S^d_q(\mathcal{M}^*_{m,m})$-$S^d_q(\mathcal{M}^*_{n,n})$-bicomodule structure on $S^d_q(\mathcal{M}^*_{m,n})$ gives rise to a $(U_q(\mathfrak{gl}_m))^{*\mathrm{res}}$-$(U_q(\mathfrak{gl}_n))^{*\mathrm{res}}$-bicomodule structure on $S^d_q(\mathcal{M}^*_{m,n})$, and thus to a $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_m)$-bimodule structure on $S^d_q(\mathcal{M}^*_{m,n})$. Another way to describe that is to look at the $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_m)$-bimodule $(V_n)^{\otimes d} \otimes (V^*_m)^{\otimes d}$, with basis $e_{j_1,\ldots,j_d} \otimes f_{i_1,\ldots,i_d} = (e_{j_1} \otimes \cdots \otimes e_{j_d}) \otimes (f_{i_1} \otimes \cdots \otimes f_{i_d})$.

Proposition 4 *The linear map*

$$\pi : \big((V_n)^{\otimes d} \otimes (V^*_m)^{\otimes d} \to S^d_q(\mathcal{M}^*_{m,n}), e_{j_1,\ldots,j_d} \otimes f_{i_1,\ldots,i_d} \mapsto X_{i_1,j_1} \cdots X_{i_d,j_d}\big)$$

factorizes through $(V_n)^{\otimes d} \otimes_{H_q(\mathfrak{S}_d)} (V^*_m)^{\otimes d}$ and defines an isomorphism of $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_m)$-bimodules between this latter space and $S^d_q(\mathcal{M}^*_{m,n})$.

Proof. The relations defining the algebra $S_q(\mathcal{M}^*_{m,n})$ imply that the kernel of π is exactly the subspace spanned by the vectors $e \cdot T_{s_k} \otimes f - e \otimes T_{s_k} \cdot f$ for $e \in (V_n)^{\otimes d}$, $f \in (V^*_m)^{\otimes d}$ and $1 \leq k \leq d-1$. Hence π defines an injection from $(V_n)^{\otimes d} \otimes_{H_q(\mathfrak{S}_d)} (V^*_m)^{\otimes d}$ to $S^d_q(\mathcal{M}^*_{m,n})$. Further, π is surjective.

We now fix $(j_1, \ldots, j_d) \in \{1, \ldots, n\}^d$ and consider the map

$$u : \big((V^*_m)^{\otimes d} \to S^d_q(\mathcal{M}^*_{m,n}), f_{i_1,\ldots,i_d} \mapsto X_{i_1,j_1} \cdots X_{i_d,j_d}\big).$$

Let $x \in U_q(\mathfrak{gl}_m)$ and denote its iterated coproduct by the Sweedler notation $\sum x_{(1)} \otimes \cdots \otimes x_{(d)}$. Then

$$\begin{aligned} u(e_{j_1,\ldots,j_d}) \cdot x &= (X_{i_1,j_1} \cdots X_{i_d,j_d}) \cdot x \\ &= \textstyle\sum_{k_1,\ldots,k_d=1}^{m} \langle x, \kappa(X_{i_1,k_1} \cdots X_{i_d,k_d}) \rangle X_{k_1,j_1} \cdots X_{k_d,j_d} \\ &= \textstyle\sum_{k_1,\ldots,k_d=1}^{m} \langle x, t_{i_1,k_1} \cdots t_{i_d,k_d} \rangle X_{k_1,j_1} \cdots X_{k_d,j_d} \\ &= \textstyle\sum_{k_1,\ldots,k_d=1}^{m} \langle x_{(1)}, t_{i_1,k_1} \rangle \cdots \langle x_{(d)}, t_{i_d,k_d} \rangle X_{k_1,j_1} \cdots X_{k_d,j_d} \\ &= \textstyle\sum_{k_1,\ldots,k_d=1}^{m} \langle f_{i_1}, x_{(1)} \cdot e_{k_1} \rangle \cdots \langle f_{i_d}, x_{(d)} \cdot e_{k_d} \rangle X_{k_1,j_1} \cdots X_{k_d,j_d} \\ &= \textstyle\sum_{k_1,\ldots,k_d=1}^{m} \langle f_{i_1,\ldots,i_d}, x \cdot e_{k_1,\ldots,k_d} \rangle u(f_{k_1,\ldots,k_d}) \\ &= u(f_{i_1,\ldots,i_d} \cdot x). \end{aligned}$$

This calculation shows that the linear map in the statement of the proposition is $U_q(\mathfrak{gl}_m)$-linear. An analogous reasoning shows that it respects the left action of $U_q(\mathfrak{gl}_n)$.

In the case $m = n$, under the identification (through κ) of $S_q^d(\mathcal{M}_{m,m}^*)$ with the subcoalgebra of $(U_q(\mathfrak{gl}_m))^{*\mathrm{res}}$ spanned by monomials of degree d in the elements t_{ij}, the $U_q(\mathfrak{gl}_m)$-$U_q(\mathfrak{gl}_m)$-linearity of the map π in Proposition 4 is of course a standard fact (see [6, Proposition 2.7.11] for instance).

Remark. In [3], Dipper and Donkin have introduced another version of the algebra of functions on the quantum matrix space, which has the advantage of being defined over the subring $\mathbf{Z}[q] \subset \mathbf{Q}(v)$. Dipper and Donkin have pointed out that their bialgebra was not isomorphic to $S_q^d(\mathcal{M}_{mm}^*)$. Several techniques of [loc. cit.] may however easily be adapted to our purposes. For instance, Theorem 2 corresponds to [loc. cit., Theorem 1.1.8 and Lemma 1.2.1], Proposition 1 is reminiscent of [loc. cit., Definition 4.3.1], and Proposition 2 corresponds to [loc. cit., Theorem 1.4.2]. The forthcoming Proposition 5 is an analogue of [loc. cit., Theorem 3.2.5] (taking into account [8, § 1.4]) and Lemma 3 has to do with [loc. cit., Lemma 4.3.2].

1.6 Duality with the function algebras

We already asserted the existence of a pairing between $U_q(\mathfrak{gl}_m)$ and $S_q(\mathcal{M}_{m,m}^*)$. By the definition of κ, the orthogonal in $U_q(\mathfrak{gl}_m)$ of $S_q^d(\mathcal{M}_{m,m}^*)$ for this pairing is the annihilator of the left $U_q(\mathfrak{gl}_m)$-module $(V_m)^{\otimes d}$. And by [10, § 1], this latter is also the kernel of the homomorphism ζ defined in Theorem 1. Thus there is a pairing between $\mathcal{S}_q^d(m)$ and $S_q^d(\mathcal{M}_{m,m}^*)$.

The explicit formula giving this pairing does not seem to be widely known. More generally, let us define a pairing between $\mathcal{S}_q^d(m,n)$ and $S_q^d(\mathcal{M}_{m,n}^*)$ by, for A and B in $\Theta_{m,n}^d$:

$$\langle [A], X(B) \rangle = \begin{cases} v^{-\sum_{i<j,k} a_{ik} a_{jk}} & \text{if } A = B, \\ 0 & \text{if } A \neq B. \end{cases}$$

This is a non-degenerate bilinear form.

Proposition 5 *(a) For the structures of $U_q(\mathfrak{gl}_m)$-$U_q(\mathfrak{gl}_n)$-bimodule on $\mathcal{S}_q^d(m,n)$ and of $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_m)$-bimodule on $S_q^d(\mathcal{M}_{m,n}^*)$, one has for all $a \in \mathcal{S}_q^d(m,n)$, $b \in S_q^d(\mathcal{M}_{m,n}^*)$, $x \in U_q(\mathfrak{gl}_m)$ and $y \in U_q(\mathfrak{gl}_n)$:*

$$\langle x \cdot a, b\rangle = \langle a, b \cdot x\rangle \quad \text{and} \quad \langle a \cdot y, b\rangle = \langle a, y \cdot b\rangle.$$

(b) If m, n, p, d are positive integers, if $a \in \mathcal{S}_q^d(m,p)$, $b \in \mathcal{S}_q^d(p,n)$, $c \in S_q^d(\mathcal{M}_{m,n}^*)$, and if we denote by $ab \in \mathcal{S}_q^d(m,n)$ the product of a and b and by $\sum c_{(1)} \otimes c_{(2)} \in S_q^d(\mathcal{M}_{m,p}^*) \otimes S_q^d(\mathcal{M}_{p,n}^*)$ the coproduct of c (as in Proposition 2), then

$$\langle ab, c\rangle \;= \sum \langle a, c_{(1)}\rangle \, \langle b, c_{(2)}\rangle.$$

(c) For $x \in U_q(\mathfrak{gl}_m)$ and $a \in S_q^d(\mathcal{M}_{m,m}^*)$, one has $\langle \zeta(x), a\rangle = \langle x, \kappa(a)\rangle$.

Proof. If we fix $(i_1, \ldots, i_d) \in \{1, \ldots, m\}^d$, the map

$$\big((V_n)^{\otimes d} \to S_q^d(\mathcal{M}_{m,n}^*), e_{j_1,\ldots,j_d} \mapsto X_{i_1,j_1} \cdots X_{i_d,j_d}\big)$$

is a morphism of $U_q(\mathfrak{gl}_n)$-modules (Proposition 4). This allows to compute the action of $U_q(\mathfrak{gl}_n)$ on the elements $X(B) \in S_q^d(\mathcal{M}_{m,n}^*)$, for $B \in \Theta_{m,n}^d$, by taking $((i_1,j_1), \ldots, (i_d,j_d)) \in I_{m,n}^d \cap \varphi^{-1}(B)$ (notations from Section 1.5). Using the relations in Definition 3, one finds after some simple calculations

$$\begin{aligned}
K_\lambda \cdot X(B) &= v^{(\lambda \,|\, \mathrm{co}(B))}\, X(B),\\
E_h \cdot X(B) &= \sum_{p=1}^m [b_{p,h+1}]\, v^{\sum_{i>p} b_{ih} - b_{i,h+1}}\, X(B + E_{ph} - E_{p,h+1}),\\
F_h \cdot X(B) &= \sum_{p=1}^m [b_{ph}]\, v^{\sum_{i<p} b_{i,h+1} - b_{ih}}\, X(B - E_{ph} + E_{p,h+1}).
\end{aligned}$$

In the right hand side of the second (respectively the third) formula, it is understood that a summand corresponding to an index p such that $b_{p,h+1} = 0$ (respectively $b_{ph} = 0$) vanishes. Comparing with the formulas in Lemma 1, this proves the second equality in statement (a).

The first equality in statement (a) follows as for it from the following relations describing the action of $U_q(\mathfrak{gl}_m)$ on $S_q^d(\mathcal{M}_{m,n}^*)$:

$$\begin{aligned}
X(B) \cdot K_\lambda &= v^{(\lambda \,|\, \mathrm{ro}(B))}\, X(B),\\
X(B) \cdot E_h &= \sum_{p=1}^n [b_{hp}]\, v^{\left(\sum_{j\geq p} b_{hj} - b_{h+1,j}\right)-1}\, X(B - E_{hp} + E_{h+1,p}),\\
X(B) \cdot F_h &= \sum_{p=1}^n [b_{h+1,p}]\, v^{\left(\sum_{j\leq p} b_{h+1,j} - b_{hj}\right)-1}\, X(B + E_{hp} - E_{h+1,p}).
\end{aligned}$$

Here again, in the right hand side of the second (respectively the third) formula, it is understood that a summand corresponding to an index p such that $b_{hp} = 0$ (respectively $b_{h+1,p} = 0$) vanishes.

The formula in statement (c) is true for $x = 1$ and any a. Since ζ (respectively κ) is a morphism of left (respectively right) $U_q(\mathfrak{gl}_m)$-modules and since 1 generates the left regular $U_q(\mathfrak{gl}_m)$-module, statement (a) shows that the formula is true in the general case.

Statement (c), the fact that ζ is a surjective morphism of algebras, and the fact that κ is a morphism of coalgebras imply that statement (b) holds in the case $m = n = p$. For the general case, we choose an integer N greater than m, n and p, and use a stabilization procedure, as explained in previous remarks.

One can understand the statement (b) by saying that the algebra $\mathcal{S}_q^d(m)$ is the dual of the coalgebra $S_q^d(\mathcal{M}_{m,m}^*)$ (see [24, § 11.2 and Remark 11.3.3] and [3, Theorem 4.4.8]). Statement (c) and the surjectivity of ζ imply that, in restriction to $S_q^d(\mathcal{M}_{m,m}^*)$, the map κ is injective. (In fact, κ itself is injective.)

1.7 Appendix: Standard basis theorems for $S_q^d(\mathcal{M}_{m,n}^*)$ and $\mathcal{S}_q^d(m,n)$

To study combinatorially the theory of invariants in a way independent of the characteristic of the ground field, Doubilet, Rota and Stein have introduced some objects called bitableaux. These bitableaux give a "standard basis" of the space of polynomial functions on a space of matrices, and J. A. Green has defined in a dual way a "standard basis" for the classical Schur algebra. In turn, these bases allow one to define filtrations on both these spaces, that are linked with the recent notions of tilting theory and quasi-hereditary algebra. (See [11] and references therein for all that.)

In [15] and [12], Huang, Zhang and R. M. Green defined quantum analogues of these constructions and devised straightening algorithms in order to prove standard basis theorems for $S_q^d(\mathcal{M}_{m,n}^*)$ (in a supersymmetric case) and for $\mathcal{S}_q^d(m)$ respectively. Using (and abusing of) an idea of J. A. Green [11], we will show in this appendix how to deduce both standard basis theorems from the explicit formula given in Section 1.6 for the pairing between $\mathcal{S}_q^d(m,n)$ and $S_q^d(\mathcal{M}_{m,n}^*)$. Besides being simpler, this approach has the advantage of giving a more precise information than the straightening algorithms can do.

The set of polynomial dominant weights of degree d in P_m is in bijection with the set of Young diagrams with d boxes and at most m rows. Such a Young diagram, filled with integers between 1 and m, is called semistandard if the entries in each row, from the left to the right, are non-decreasing and if the entries in each column, from the top to the bottom, are increasing.

Let $\mathcal{T}_{m,n}^d$ be the set of pairs (P,Q) consisting of Young semistandard tableaux of the same shape, with d boxes, such that P and Q are filled with positive integers less or equal than m and n respectively. To each $(P,Q) \in \mathcal{T}_{m,n}^d$, one associates a "quantum semistandard bitableau" (or "bideterminant") $X_{P,Q} \in S_q^d(\mathcal{M}_{m,n}^*)$, as in [21, § 1], and a "quantized semistandard codeterminant" $Y_{P,Q} \in \mathcal{S}_q^d(m,n)$, as in [12] (the precise definitions are recalled below). We aim to show that these elements afford bases for the spaces $S_q^d(\mathcal{M}_{m,n}^*)$ and $\mathcal{S}_q^d(m,n)$.

First, I introduce some notations:

- If λ and $\mu \in P_m$ are weights, we write $\lambda \trianglelefteq \mu$ if one can express $\mu - \lambda$ as a linear combination with non-negative coefficients of the simple roots α_i. To a Young tableau P with at most m lines and filled with positive numbers less or equal than m, we associate two polynomial weights in P_m: the shape of P is $\mathrm{sh}(P) = \lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m$, where $\lambda_1 \geq \cdots \geq \lambda_m$ are the lengths of the rows of P, and the content of P is $\mathrm{cont}(P) = n_1\varepsilon_1 + \cdots + n_m\varepsilon_m$, where n_i is the number of boxes in P filled by i. We say that P is column-standard if the entries is each column of P increase from top to bottom. If P is column-standard, then $\mathrm{cont}(P) \trianglelefteq \mathrm{sh}(P)$.

- If d is a positive integer, we let $A_d = \sum_{w\in\mathfrak{S}_d}(-q)^{-\ell(w)}T_w$ be the antisymmetrizer in $H_q(\mathfrak{S}_d)$, where $\ell(w)$ denotes the usual length of the permutation w. If $1 \leq i_1 < \cdots < i_d \leq m$ and $1 \leq j_1 < \cdots < j_d \leq n$, one defines the quantum determinant as the element of $S_q^d(\mathcal{M}_{m,n}^*)$ given by

$$\begin{aligned}\begin{vmatrix} X_{i_1,j_1} & \dots & X_{i_1,j_d} \\ \vdots & & \vdots \\ X_{i_d,j_1} & \dots & X_{i_d,j_d} \end{vmatrix}_q &= \pi(e_{j_1,\dots,j_d}\cdot A_d \otimes f_{i_1,\dots,i_d}) \\ &= \sum_{w\in\mathfrak{S}_d} (-v)^{-\ell(w)}\ \pi(e_{j_{w(1)},\dots,j_{w(d)}} \otimes f_{i_1,\dots,i_d}) \\ &= \sum_{w\in\mathfrak{S}_d} (-v)^{-\ell(w)}\ X_{i_1,j_{w(1)}}\cdots X_{i_d,j_{w(d)}}.\end{aligned}$$

- Let $(P,Q) \in \mathcal{T}_{m,n}^d$. The common shape of P and Q is a Young diagram with say t columns of lengths $d_1,\dots,d_t$. Let $1 \leq i_{s,1} < \cdots < i_{s,d_s} \leq m$ and $1 \leq j_{s,1} < \cdots < j_{s,d_s} \leq n$ be the entries in the s-th columns of P and Q respectively, from top to bottom. One then puts

$$X_{P,Q} = \prod_{s=1}^t \begin{vmatrix} X_{i_{s,1},\, j_{s,1}} & \dots & X_{i_{s,1},\, j_{s,d_s}} \\ \vdots & & \vdots \\ X_{i_{s,d_s},\, j_{s,1}} & \dots & X_{i_{s,d_s},\, j_{s,d_s}} \end{vmatrix}_q .$$

  This definition extends to the case of a pair of Young tableaux of the same shape, with P and Q filled with positive integers less or equal than m and n respectively, but that are only column-standard. In this latter case, we refer to $X_{P,Q}$ as a "quantum column-standard bitableau" (this is a less general notion than the one of "quantum bitabloid" defined in [21, § 1]).

- Let $(P,Q) \in \mathcal{T}_{m,n}^d$. The common shape of P and Q is a Young diagram with at most $p = \min(m,n)$ rows. Let $A_P \in \Theta_{m,p}^d$ be the lower triangular matrix whose element in position (i,j) is the number of times i appears in the j-th line of P. Let $B_Q \in \Theta_{p,n}^d$ be the upper triangular matrix whose element in position (i,j) is the number of times j appears in the i-th line of Q. Then $\mathrm{co}(A) = \mathrm{ro}(B) = \mathrm{sh}(P) = \mathrm{sh}(Q)$, and, following R. M. Green, we define $Y_{P,Q} \in \mathcal{S}_q^d(m,n)$ to be the product $[A_P][B_Q]$.

- Let m, n and d be positive integers. We say that a matrix $A \in \Theta_{m,n}^d$ is less than or equal to a matrix $B \in \Theta_{m,n}^d$ if $\mathrm{ro}(A) = \mathrm{ro}(B)$, $\mathrm{co}(A) = \mathrm{co}(B)$, and if for all $1 \leq \mu \leq m$ and $1 \leq \nu \leq n$, there holds

$$\sum_{(i,j)\in\{1,\dots,\mu\}\times\{1,\dots,\nu\}} a_{ij} \ \leq\ \sum_{(i,j)\in\{1,\dots,\mu\}\times\{1,\dots,\nu\}} b_{ij}.$$

This defines a partial order $\leq$ on $\Theta^d_{m,n}$.

We will also need two lemmas.

Lemma 2 [24, Lemma 4.4.2][15, Proposition 6] *Let m, n, p, d be positive integers, with $d \leq \min(m, n, p)$, and take $1 \leq i_1 < \cdots < i_d \leq m$ and $1 \leq j_1 < \cdots < j_d \leq n$. Then the image of the quantum determinant*

$$\begin{vmatrix} X_{i_1,j_1} & \dots & X_{i_1,j_d} \\ \vdots & & \vdots \\ X_{i_d,j_1} & \dots & X_{i_d,j_d} \end{vmatrix}_q \in S^d_q(\mathcal{M}^*_{m,n})$$

under the coproduct defined in Proposition 2 is equal to

$$\sum_{1\leq k_1<\cdots<k_d\leq p} \begin{vmatrix} X_{i_1,k_1} & \dots & X_{i_1,k_d} \\ \vdots & & \vdots \\ X_{i_d,k_1} & \dots & X_{i_d,k_d} \end{vmatrix}_q \otimes \begin{vmatrix} X_{k_1,j_1} & \dots & X_{k_1,j_d} \\ \vdots & & \vdots \\ X_{k_d,j_1} & \dots & X_{k_d,j_d} \end{vmatrix}_q \in S^d_q(\mathcal{M}^*_{m,p}) \otimes S^d_q(\mathcal{M}^*_{p,n}).$$

Lemma 3 *Let m, n, d be positive integers.*

(a) Take $((i_1, j_1), \dots, (i_d, j_d)) \in (\{1, \dots, m\} \times \{1, \dots, n\})^d$, put $A = \sum_{a=1}^d E_{i_a,j_a} \in \Theta^d_{m,n}$, and expand $X_{i_1,j_1} \cdots X_{i_d,j_d} = \sum_{B\in\Theta^d_{m,n}} c_B\, X(B)$ in terms of our usual basis (see conventions after Proposition 1). Then the coefficients c_B belong to $\mathbf{Z}[v, v^{-1}]$, $c_B \neq 0$ only if $B \leq A$, and c_A is a power of v.

(b) Take $1 \leq i_1 < \cdots < i_d \leq m$ and $1 \leq j_1 < \cdots < j_d \leq n$, put $A = \sum_{a=1}^d E_{i_a,j_a} \in \Theta^d_{m,n}$, and expand

$$\begin{vmatrix} X_{i_1,j_1} & \dots & X_{i_1,j_d} \\ \vdots & & \vdots \\ X_{i_d,j_1} & \dots & X_{i_d,j_d} \end{vmatrix}_q = \sum_{B\in\Theta^d_{m,n}} c_B\, X(B)$$

in terms of our usual basis. Then the coefficients c_B belong to $\mathbf{Z}[v, v^{-1}]$, $c_B \neq 0$ only if $B \leq A$, and $c_A = 1$.

(c) Let (P, Q) be a pair of column-standard Young tableaux of the same shape, such that P and Q are filled with positive integers less or equal than m and n respectively. (For instance, take $(P, Q) \in \mathcal{T}^d_{m,n}$.) Enumerate the entries in the boxes of P and Q by $(i_1, \dots, i_d)$ and $(j_1, \dots, j_d)$ (in the same order), put $A = \sum_{a=1}^d E_{i_a,j_a} \in \Theta^d_{m,n}$, and expand $X_{P,Q} = \sum_{B\in\Theta^d_{m,n}} c_B\, X(B)$ in terms of our usual basis. Then the coefficients c_B belong to $\mathbf{Z}[v, v^{-1}]$, $c_B \neq 0$ only if $B \leq A$, and c_A is a power of v.

Proof. (a) Order the set $\{1, \dots, m\} \times \{1, \dots, n\}$ by the lexicographic order: $(i, j) \leq (i', j')$ if $i < i'$ or if $i = i'$ and $j \leq j'$, and denote by N the number of inversions in the sequence $((i_1, j_1), \dots, (i_d, j_d))$. There is nothing to prove if $N = 0$, since in that case $((i_1, j_1), \dots, (i_d, j_d)) \in I^d_{m,n}$ (see Section 1.5). So assume that $N \geq 1$. There is then an $a \in \{1, \ldots, d-1\}$ such that $(i_a, j_a) > (i_{a+1}, j_{a+1})$. Thanks to the relations in Definition 3, we have

$$X_{i_1,j_1} \cdots X_{i_d,j_d} = \begin{cases} X_{i_1,j_1} \cdots X_{i_{a+1},j_{a+1}} X_{i_a,j_a} \cdots X_{i_d,j_d} & \text{if } i_a > i_{a+1} \text{ and } j_a < j_{a+1}, \\ v X_{i_1,j_1} \cdots X_{i_{a+1},j_{a+1}} X_{i_a,j_a} \cdots X_{i_d,j_d} & \text{if } i_a > i_{a+1} \text{ and } j_a = j_{a+1} \\ & \text{or if } i_a = i_{a+1} \text{ and } j_a > j_{a+1}, \\ X_{i_1,j_1} \cdots X_{i_{a+1},j_{a+1}} X_{i_a,j_a} \cdots X_{i_d,j_d} + & \\ \qquad (v - v^{-1})\, X_{i_1,j_1} \cdots X_{i_{a+1},j_a} X_{i_a,j_{a+1}} \cdots X_{i_d,j_d} & \text{if } i_a > i_{a+1} \text{ and } j_a > j_{a+1}. \end{cases}$$

The sequences arising from each terms above have fewer than N inversions. The matrix associated to the sequence $((i_1, j_1), \ldots, (i_{a+1}, j_{a+1}), (i_a, j_a), \ldots, (i_d, j_d))$ is the matrix A and, in the third case, the matrix associated to the sequence $((i_1, j_1), \ldots, (i_{a+1}, j_a), (i_a, j_{a+1}), \ldots, (i_d, j_d))$ is less than A. An induction concludes the proof.

(b) Choose $w \in \mathfrak{S}_d$ and consider in the quantum determinant the term $X_{i_1,j_{w(1)}} \cdots X_{i_d,j_{w(d)}}$. By statement (a), the expansion of this term involve monomials $X(B)$ such that the matrix B is less or equal than $\sum_{a=1}^d E_{i_a,j_{w(a)}}$. This latter matrix is equal to A if $w = 1$ and is less than A otherwise. Thus after having gathered the expansions of the different terms, only monomials $X(B)$ with $B \leq A$ can occur, and $X(A)$ occurs only for the term $X_{i_1,j_1} \cdots X_{i_d,j_d} = X(A)$.

(c) The assertion (c) is a direct consequence of the statements (a) and (b).

Now we can prove:

Proposition 6 *The matrix* $\big(\langle Y_{P,Q}, X_{P',Q'} \rangle\big)_{(P,Q),(P',Q') \in \mathcal{T}^d_{m,n}}$ *has entries in* $\mathbf{Z}[v, v^{-1}]$ *and is invertible over this ring.*

Proof. Since the structure constants of the q-Schur algebras belong to $\mathbf{Z}[v, v^{-1}]$, Lemma 3 (c) implies that our matrix is $\mathbf{Z}[v, v^{-1}]$-valued. If (P, Q) and $(P', Q') \in \mathcal{T}^d_{m,n}$, we say that (P, Q) is less or equal than (P', Q') if either

- the common shape of P and Q is less than the common shape of P' and Q' : $\operatorname{sh}(P) \lhd \operatorname{sh}(P')$;
- P, Q, P', Q' have the same shape, $A_P \leq A_{P'}$ and $B_Q \leq B_{Q'}$.

This definition endows $\mathcal{T}^d_{m,n}$ with a partial order $\preccurlyeq$ and it is enough to prove that the matrix $\big(\langle Y_{P,Q}, X_{P',Q'} \rangle\big)_{(P,Q),(P',Q') \in \mathcal{T}^d_{m,n}}$ is upper triangular w.r.t. $\preccurlyeq$ with invertible diagonal elements.

The image of $X_{P',Q'}$ under the coproduct (defined in Proposition 2) can be computed using Lemma 2. The result is $\sum_R X_{P',R} \otimes X_{R,Q'}$ where the sum runs over the set of column-standard Young tableaux with the same shape than P' and Q' that are filled with positive integers less or equal than p. Using Proposition 5 (b), we can write

$$\langle Y_{P,Q}, X_{P',Q'} \rangle = \textstyle\sum_R \langle [A_P], X_{P',R} \rangle \, \langle [B_Q], X_{R,Q'} \rangle.$$

If $\langle Y_{P,Q}, X_{P',Q'}\rangle$ is non-zero, then there is an R such that $\langle [A_P], X_{P',R}\rangle$ and $\langle [B_Q], X_{R,Q'}\rangle$ are non-zero. These conditions imply that $\mathrm{cont}(R)$ is equal to $\mathrm{co}(A_P) = \mathrm{ro}(B_Q) = \mathrm{sh}(P)$, and this is possible only if $\mathrm{sh}(P) = \mathrm{cont}(R) \trianglelefteq \mathrm{sh}(R) = \mathrm{sh}(P')$. Suppose further that P, Q, P', Q' have the same shape. In the sum $\sum_R \langle [A_P], X_{P',R}\rangle \, \langle [B_Q], X_{R,Q'}\rangle$, the only possible non-zero term occurs for the so-called Yamanouchi tableau R, filled with 1 in the first line, 2 in the second, and so on. According to Lemma 3 (c), the bracket $\langle [A_P], X_{P',R}\rangle$ (respectively $\langle [B_Q], X_{R,Q'}\rangle$) can be non-zero only if $A_P \leq A_{P'}$ (respectively $B_Q \leq B_{Q'}$). All this proves that $(P,Q) \preccurlyeq (P',Q')$ is a necessary condition to have $\langle Y_{P,Q}, X_{P',Q'}\rangle \neq 0$. Therefore the matrix $\big(\langle Y_{P,Q}, X_{P',Q'}\rangle\big)$ is upper triangular.

The same reasoning proves that, up to a power of v, $\langle Y_{P,Q}, X_{P,Q}\rangle$ equals $\langle [A_P], X(A_P)\rangle \times \langle [B_Q], X(B_Q)\rangle$. Hence $\langle Y_{P,Q}, X_{P,Q}\rangle$ is a power of v. This proves our assertion about the diagonal entries of the matrix.

A direct consequence of this proposition is that the sets $(X_{P,Q})_{(P,Q)\in\mathcal{T}^d_{m,n}}$ and $(Y_{P,Q})_{(P,Q)\in\mathcal{T}^d_{m,n}}$ are bases of the spaces $S^d_q(\mathcal{M}^*_{m,n})$ and $\mathcal{S}^d_q(m,n)$ respectively. More precisely, they are bases of the $\mathbf{Z}[v,v^{-1}]$-lattices spanned by the sets $\{X(A) \mid A \in \Theta^d_{m,n}\}$ and $\{[A] \mid A \in \Theta^d_{m,n}\}$ respectively (compare with [15, Theorem 9] and [12, Corollary 3.7]).

All the other consequences that J. A. Green has obtained from its technique quantify easily. For instance, one can show that the straightening of a quantum column-standard bitableau only affords quantum semistandard bitableaux whose shapes are less or equal than the shape of the original column-standard bitableau w.r.t. the dominance order $\trianglelefteq$ (this fact does not seem to be a direct consequence of the algorithm described by Huang and Zhang [15] and Galdi [9]). One can therefore use the quantum bitableaux and the quantized codeterminants to exhibit bimodule filtrations for $S^d_q(\mathcal{M}^*_{m,n})$ and $\mathcal{S}^d_q(m,n)$. I must point out further that, despite J. A. Green's claim, details of the straightening algorithm are not needed to get the precise isomorphism in [11, Theorem 7.3 (iii)].

2 The q-Weyl group

2.1 The completion and the q-Weyl group of $U_q(\mathfrak{gl}_m)$

In the above presentation of the q-Weyl group, I will use Levendorskii and Soibelman's idea of seeing the q-Weyl group inside a completion of $U_q(\mathfrak{gl}_m)$ and not as a subgroup of $\mathrm{Aut}(U_q(\mathfrak{gl}_m))$.

Let $\mathcal{J}$ be the set of the annihilators $I \subseteq U_q(\mathfrak{gl}_m)$ of the finite-dimensional weight $U_q(\mathfrak{gl}_m)$-modules. Endowing $\mathcal{J}$ with the order relation given by the inclusion, one may define the inverse limit $\widehat{U_q(\mathfrak{gl}_m)} = \varprojlim (U_q(\mathfrak{gl}_m)/I)$. The canonical map $U_q(\mathfrak{gl}_m) \to \widehat{U_q(\mathfrak{gl}_m)}$ is injective (see [18, Lemma 8.3]) and we identify $U_q(\mathfrak{gl}_m)$ with its image.

Here is an instance where this construction arises naturally. Let $\lambda \in P_m$ be a weight. For each $I \in \mathcal{J}$, one may choose a finite-dimensional weight $U_q(\mathfrak{gl}_m)$-module M whose annihilator is I. This module M is the direct sum of its weight subspaces: $M = \bigoplus_{\lambda\in P_m} M_\lambda$; in particular, it is completely reducible as a U^0-module. The projection on a summand M_λ with respect to this decomposition belongs to the centralizer of $\mathrm{End}_{U^0}(M)$ in $\mathrm{End}_{\mathbf{Q}(v)}(M)$, hence is the image of a (unique) element $(1_\lambda)_I \in U_q(\mathfrak{gl}_m)/I$, by the density theorem. Forming direct sums, one can easily see that $(1_\lambda)_I$ does not depend on the choice of M and that the family $((1_\lambda)_I)_{I\in\mathcal{J}}$ defines an element $1_\lambda \in \widehat{U_q(\mathfrak{gl}_m)}$. (Here, we used the same notation as in [23, § 23.1].)

More generally, an element $x \in \widehat{U_q(\mathfrak{gl}_m)}$ gives rise, for every finite-dimensional weight module M, to an operator $x_M \in \operatorname{End}_{\mathbf{Q}(v)}(M)$, with the property that for all morphism $f : M \to N$, there holds $x_N \circ f = f \circ x_M$.

Now the symmetric group $\mathfrak{S}_m$ identifies with the Weyl group of GL_m. This is a Coxeter system on the set of simple reflections $\{s_1, \ldots, s_{m-1}\}$ and it acts on the weight lattice P_m. Lusztig [23, Chap. 5], and independently Levendorskii and Soibelman [20], have introduced elements $\bar{s}_1, \ldots, \bar{s}_{m-1}$ of the algebra $\widehat{U_q(\mathfrak{gl}_m)}$ such that:

- for each weight $\lambda \in P_m$, one has $1_{s_i(\lambda)}\ \bar{s}_i = \bar{s}_i\ 1_\lambda$ (see [23, Proposition 5.2.7]);
- the $\bar{s}_i$ satisfy the braid relations: $\bar{s}_i\bar{s}_j = \bar{s}_j\bar{s}_i$ if $|i-j| \geq 2$, and $\bar{s}_i\bar{s}_{i+1}\bar{s}_i = \bar{s}_{i+1}\bar{s}_i\bar{s}_{i+1}$ (see [23, Theorem 39.4.3]);
- the $\bar{s}_i$ are invertible in $\widehat{U_q(\mathfrak{gl}_m)}$ and the inner automorphisms they define in $\widehat{U_q(\mathfrak{gl}_m)}$ stabilize the subalgebra $U_q(\mathfrak{gl}_m)$ (see [23, Proposition 37.1.2]).

By the definition of $\widehat{U_q(\mathfrak{gl}_m)}$, the elements $\bar{s}_i$ give rise, for each finite-dimensional weight $U_q(\mathfrak{gl}_m)$-module M, to an operator $T_i = (\bar{s}_i)_M : M \to M$. In fact, Lusztig has found several possible choices for the T_i, which he denotes by $T'_{i,\pm1}$ and $T''_{i,\pm1}$. These operators are defined by the following formulas, in which e is a vector of weight λ, and $E_i^{(a)} = \frac{E_i^a}{[a]!}$ and $F_i^{(b)} = \frac{F_i^b}{[b]!}$ are the q-divided powers (see [23, § 5.2.1]):

$$T'_{i,\pm1}(e) = \textstyle\sum_{a-b+c=(\alpha_i|\lambda)} (-1)^b\ v^{\pm(b-ac)}\ F_i^{(a)}\ E_i^{(b)}\ F_i^{(c)} \cdot e\ ,$$
$$T''_{i,\pm1}(e) = \textstyle\sum_{-a+b-c=(\alpha_i|\lambda)} (-1)^b\ v^{\pm(b-ac)}\ E_i^{(a)}\ F_i^{(b)}\ E_i^{(c)} \cdot e\ .$$

It is known that $T''_{i,\pm1} = (T'_{i,\mp1})^{-1}$ [23, Proposition 5.2.3].

The operators $T'_{i,+1}$ and $T''_{i,+1}$ are induced by elements $\bar{s}'_i$ and $\bar{s}''_i$ in $\widehat{U_q(\mathfrak{gl}_m)}$, $T'_{i,-1}$ and $T''_{i,-1}$ being then induced by $(\bar{s}''_i)^{-1}$ and $(\bar{s}'_i)^{-1}$. The formulas above define the elements $(\bar{s}'_i)^{\pm1}1_\lambda$ and $(\bar{s}''_i)^{\pm1}1_\lambda$ as the sum of series which converge in $\widehat{U_q(\mathfrak{gl}_m)}$.

Finally, recall the involutive antiautomorphism Φ of the algebra $U_q(\mathfrak{gl}_m)$ defined in Section 1.2. It preserves the set $\mathcal{J}$ of ideals, since the contravariant dual of a finite-dimensional weight module is a finite-dimensional weight module. Thus Φ extends to an involutive antiautomorphism of the algebra $\widehat{U_q(\mathfrak{gl}_m)}$, which we denote by $\widehat{\Phi}$. If $\lambda \in P_m$, then $\widehat{\Phi}(1_\lambda) = 1_\lambda$ in

$\widehat{U_q(\mathfrak{gl}_m)}$. One also verifies by the formulas that $\widehat{\Phi}(\bar{s}'_i) = \bar{s}''_i$ in $\widehat{U_q(\mathfrak{gl}_m)}$:

$$\begin{aligned}
\widehat{\Phi}(\bar{s}'_i\, 1_\lambda) &= \widehat{\Phi}\big(\textstyle\sum_{a-b+c=(\alpha_i|\lambda)} (-1)^b\, v^{b-ac}\, F_i^{(a)}\, E_i^{(b)}\, F_i^{(c)}\big)\, \widehat{\Phi}(1_{s_i(\lambda)}) \\
&= \textstyle\sum_{a-b+c=(\alpha_i|\lambda)} (-1)^b\, v^{b-ac}\, E_i^{(c)}\, F_i^{(b)}\, E_i^{(a)}\, 1_{s_i(\lambda)} \\
&= \textstyle\sum_{-a+b-c=(\alpha_i|s_i(\lambda))} (-1)^b\, v^{b-ac}\, E_i^{(c)}\, F_i^{(b)}\, E_i^{(a)}\, 1_{s_i(\lambda)} \\
&= \bar{s}''_i\, 1_{s_i(\lambda)} \\
&= \widehat{\Phi}(1_\lambda)\, \bar{s}''_i\,.
\end{aligned}$$

2.2 The q-Weyl group of the q-Schur algebra

As noticed at the end of Section 1.4, the left $U_q(\mathfrak{gl}_m)$-module $\mathcal{S}_q^d(m)$ is finite-dimensional and is a weight module. Thus the kernel of ζ belongs to $\mathcal{J}$, and so ζ factorizes canonically through an epimorphism $\widehat{\zeta} : \widehat{U_q(\mathfrak{gl}_m)} \to \mathcal{S}_q^d(m)$.

Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m \in P_m$ be a weight. Since the left $U_q(\mathfrak{gl}_m)$-module $\mathcal{S}_q^d(m)$ is polynomial of degree d, the image under $\widehat{\zeta}$ of 1_λ in $\mathcal{S}_q^d(m)$ is zero except if λ is polynomial of degree d, in which case it is equal to $[\sum_{j=1}^m \lambda_j E_{jj}]$. Our aim is to describe similarly the images of $\bar{s}'_i$ and $\bar{s}''_i$ in $\mathcal{S}_q^d(m)$.

We first begin with a proposition giving a pairing between $S_q^d(\mathcal{M}_{n,m}^*)$ and $S_q^d(\mathcal{M}_{m,n}^*)$. (This pairing is the quantum version of the non-supersymmetric case of [13, Proposition 11].) Let us denote by A^T the transposed of a matrix A.

Proposition 7 *The non-degenerate bilinear form between $S_q^d(\mathcal{M}_{n,m}^*)$ and $S_q^d(\mathcal{M}_{m,n}^*)$ defined by, for $A \in \Theta_{n,m}^d$ and $B \in \Theta_{m,n}^d$*

$$(X(A), X(B)) = \begin{cases} \prod_{i,j} \big(v^{a_{ij}(a_{ij}-1)/2}[a_{ij}]!\big) & \text{if } B = A^T, \\ 0 & \text{else,} \end{cases}$$

satisfies, for all $a \in S_q^d(\mathcal{M}_{n,m}^*)$, $b \in S_q^d(\mathcal{M}_{m,n}^*)$, $x \in U_q(\mathfrak{gl}_m)$ and $y \in U_q(\mathfrak{gl}_n)$:

$$(x \cdot a, b) = (a, b \cdot x) \quad \text{and} \quad (a \cdot y, b) = (a, y \cdot b).$$

Proof. One checks the relations directly with the help of the formulas given in the proof of Proposition 5.

An immediate corollary of Propositions 5 and 7 is the following statement.

Proposition 8 *The linear map defined in the standard bases by*

$$\Big(S_q^d(\mathcal{M}_{n,m}^*) \to \mathcal{S}_q^d(m,n), X(A^T) \mapsto \textstyle\prod_{i,j} \big(v^{a_{ij}(a_{ij}-1)/2}[a_{ij}]!\big)\, v^{\sum_{i<j,k} a_{ik}a_{jk}}[A]\Big)$$

is an homomorphism of $U_q(\mathfrak{gl}_m)$-$U_q(\mathfrak{gl}_n)$-bimodules.

One may notice that the expression $\prod_{i,j}\left(v^{a_{ij}(a_{ij}-1)/2}[a_{ij}]!\right)v^{\sum_{i<j,k}a_{ik}a_{jk}}$ also appears, in a slightly different context, in [12, p. 2896].
The proof of the following technical lemma relies on a result due to Levendorskii and Soibelman.

Lemma 4 *Let $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_m\varepsilon_m \in P_m$ be a polynomial weight of degree d. In the $U_q(\mathfrak{gl}_m)$-module $(V_m)^{\otimes d}$, the action of $T''_{i,+1}$ satisfies the relation:*

$$T''_{i,+1}(e_{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{m,\dots,m}_{\lambda_m}}) = (-v)^{\lambda_i} e_{\underbrace{1,\dots,1}_{\lambda_1},\dots,i-1,\underbrace{i+1,\dots,i+1}_{\lambda_i},\underbrace{i,\dots,i}_{\lambda_{i+1}},i+2,\dots,\underbrace{m,\dots,m}_{\lambda_m}}.$$

Proof. It is easy to compute in $(V_m)^{\otimes\lambda_i}$, from the definition of $T''_{i,+1}$, that $T''_{i,+1}\cdot e_{i,\dots,i} = (-v)^{\lambda_i}F_i^{(\lambda_i)}e_{i,\dots,i} = (-v)^{\lambda_i}e_{i+1,\dots,i+1}$. With the help of [23, Proposition 5.2.2], one can also compute, in $(V_m)^{\otimes\lambda_{i+1}}$, that $T''_{i,+1}\cdot e_{i+1,\dots,i+1} = E_i^{(\lambda_{i+1})}e_{i+1,\dots,i+1} = e_{i,\dots,i}$.

One may compute the action of $T''_{i,+1}$ on a tensor product of modules thanks to the link, discovered by Levendorskii and Soibelman, between the action of the q-Weyl group and the action of the "partial R-matrices". Remembering that we are using from the beginning the opposite coproduct as the one in [23], the lemma and the proposition in [23, § 5.3] imply the following equality in $(V_m)^{\otimes\lambda_i}\otimes(V_m)^{\otimes\lambda_{i+1}}$:

$$T''_{i,+1}\cdot(e_{i,\dots,i}\otimes e_{i+1,\dots,i+1}) = (-v)^{\lambda_i}e_{i+1,\dots,i+1}\otimes e_{i,\dots,i}.$$

Our formula can be deduced immediately from that.

Now we can prove our result:

Theorem 3 *The images of $\bar{s}'_i$ and $\bar{s}''_i$ in $\mathcal{S}_q^d(m)$ are given by the following sums, running over all the m-uples $\lambda = (\lambda_1,\dots,\lambda_m)$ of non-negative integers of sum d:*

$$\begin{aligned}\widehat{\zeta}(\bar{s}'_i) &= \textstyle\sum_\lambda (-v)^{\lambda_i}\left[\sum_{j\in\{1,\dots,m\}\setminus\{i,i+1\}}\lambda_jE_{jj} + \lambda_iE_{i,i+1} + \lambda_{i+1}E_{i+1,i}\right],\\ \widehat{\zeta}(\bar{s}''_i) &= \textstyle\sum_\lambda (-v)^{\lambda_i}\left[\sum_{j\in\{1,\dots,m\}\setminus\{i,i+1\}}\lambda_jE_{jj} + \lambda_iE_{i+1,i} + \lambda_{i+1}E_{i,i+1}\right].\end{aligned}$$

Proof. Proposition 4 and Lemma 4 imply that

$$\begin{aligned}&\bar{s}''_i\cdot\pi(e_{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{m,\dots,m}_{\lambda_m}}\otimes f_{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{m,\dots,m}_{\lambda_m}})\\ &\qquad = \pi\big(T''_{i,+1}(e_{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{m,\dots,m}_{\lambda_m}})\otimes f_{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{m,\dots,m}_{\lambda_m}}\big)\\ &\qquad = (-v)^{\lambda_i}\pi(e_{\underbrace{1,\dots,1}_{\lambda_1},\dots,i-1,\underbrace{i+1,\dots,i+1}_{\lambda_i},\underbrace{i,\dots,i}_{\lambda_{i+1}},i+2,\dots,\underbrace{m,\dots,m}_{\lambda_m}}\otimes f_{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{m,\dots,m}_{\lambda_m}}).\end{aligned}$$

Thus in the left $U_q(\mathfrak{gl}_m)$-module $S_q^d(\mathcal{M}^*_{n,m})$, by Proposition 4:

$$\bar{s}''_i\cdot X(\textstyle\sum_j\lambda_jE_{jj}) = (-v)^{\lambda_i}X(\sum_{j\in\{1,\dots,m\}\setminus\{i,i+1\}}\lambda_jE_{jj} + \lambda_iE_{i,i+1} + \lambda_{i+1}E_{i+1,i})$$

and so in the left $U_q(\mathfrak{gl}_m)$-module $\mathcal{S}_q^d(m,n)$, using Proposition 8:

$$\bar{s}_i'' \cdot \left[\sum_j \lambda_j E_{jj}\right] = (-v)^{\lambda_i} \left[\sum_{j\in\{1,\dots,m\}\setminus\{i,i+1\}} \lambda_j E_{jj} + \lambda_i E_{i+1,i} + \lambda_{i+1} E_{i,i+1}\right].$$

Using the fact that the unit in the algebra $\mathcal{S}_q^d(m)$ is the sum of the $[\sum_j \lambda_j E_{jj}]$ for $(\lambda_1,\dots,\lambda_m)$ running over the m-uples of non-negative integers of sum d, we find the second formula.

The $\mathbf{Q}(v)$-linear map from $\mathcal{S}_q^d(m)$ to itself defined by $[A]\mapsto[A^T]$ is an algebra antiautomorphism (see [1, Lemma 3.10]). Comparing with Theorem 1, we see that this automorphism is induced by $\widehat{\Phi}$. Applying this automorphism to our formula for $\bar{s}_i''$, we obtain the first formula.

It would be now easy to describe the elements $\bar{s}_i'$ and $\bar{s}_i''$ with the help of the basis $\phi_{\lambda\mu}^w$ introduced by Dipper and James [5, p. 253]. We leave the translation to the reader, who will use the explicit isomorphism of Du [8, § 1.4]. The simplicity of the result hide the fact that the present approach gives for free the invertibility of $\widehat{\zeta}(\bar{s}_i')$ and $\widehat{\zeta}(\bar{s}_i'')$. This can for instance be used to simplify to proof of [12, Lemma 2.4 (ii)], rendering the analysis of [loc. cit., § 2] unnecessary.

3 Applications

3.1 A result of Lusztig, Levendorskii and Soibelman

Let $d\le m$ be two positive integers. It is well-known that the Hecke algebra $H_q(\mathfrak{S}_d)$ can be identified with a subalgebra of the q-Schur algebra (see [5, p. 256]). Maybe the simplest way to see that is the following. Consider the weight $\omega=\varepsilon_1+\cdots+\varepsilon_d\in P_m$ and the matrix $I_d=E_{11}+\cdots+E_{dd}\in\Theta_{m,m}^d$. Then $\widehat{\zeta}(1_\omega)=[I_d]\in\mathcal{S}_q^d(m)$ is an idempotent in the algebra $\mathcal{S}_q^d(m)$, and the algebra $[I_d]\mathcal{S}_q^d(m)[I_d]$ admits as basis the set of elements $[A]$, for the matrices A such that $\mathrm{ro}(A)=\mathrm{co}(A)=\omega$, that is the matrices A consisting of a permutation matrix of order d in the upper left corner, padded with zeros elsewhere.

Proposition 9 *The map*

$$\left(H_q(\mathfrak{S}_d)\to[I_d]\mathcal{S}_q^d(m)[I_d],\, T_{s_i}\mapsto v\left[\sum_{j\in\{1,\dots,d\}\setminus\{i,i+1\}} E_{jj}+E_{i,i+1}+E_{i+1,i}\right]\right)$$

is an isomorphism of algebras.

Proof. A matrix $A\in\Theta_{m,m}^d$ satisfies $\mathrm{ro}(A)=\mathrm{co}(A)=\omega$ if and only if for $(f,f')\in\mathcal{O}_A$, writing $f=(0=V_0\subseteq\cdots\subseteq V_m=V)$ and $f'=(0=V_0'\subseteq\cdots\subseteq V_m'=V)$, one has that $V_d=V_d'=V$ and that $(V_0\subseteq\cdots\subseteq V_d)$ and $(V_0'\subseteq\cdots\subseteq V_d')$ are complete flags in V. Hence the above isomorphism is the usual geometric realization of the Hecke algebra of type A in terms of the geometry of flag varieties (see [2, Chap. IV, § 2, exerc. 22]).

We can now relate this fact with the quantum Weyl group of $U_q(\mathfrak{gl}_m)$: for $i\le d-1$, the images under $\widehat{\zeta}$ of $-\bar{s}_i'1_\omega=-1_\omega\bar{s}_i'1_\omega$ and of $-\bar{s}_i''1_\omega=-1_\omega\bar{s}_i''1_\omega$ belong to $[I_d]\mathcal{S}_q^d(m)[I_d]$ and are both equal to the image of T_{s_i} under the map of Proposition 9. Taking $m=d$, we can state the following variant of the result of Lusztig, Levendorskii and Soibelman:

Proposition 10 *Let M be a finite-dimensional weight $U_q(\mathfrak{gl}_m)$-module. The operators $T'_{i,+1}$ and $T''_{i,+1}$ of the q-Weyl group of $U_q(\mathfrak{gl}_m)$ act on M. They stabilize the weight subspace of M of weight $\varepsilon_1 + \cdots + \varepsilon_m$ and satisfy on it the relations*

$$(-T_{i,+1} - q)(-T_{i,+1} + 1) = 0.$$

This single result would not justify the development of all our machinery. In fact, we could have followed Levendorskii and Soibelman's method, without any further computation.

3.2 Quantum Schur–Weyl duality and quantum $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality

The space $(V_n)^{\otimes d}$ is a $U_q(\mathfrak{gl}_n)$-$H_q(\mathfrak{S}_d)$-bimodule (see Section 1.3). Let us recall the following result of Jimbo.

Theorem 4 [17] *The images of the algebras $U_q(\mathfrak{gl}_n)$ and $H_q(\mathfrak{S}_d)$ in $\mathrm{End}_{\mathbf{Q}(v)}((V_n)^{\otimes d})$ are each one the full commutant of the other.*

The following result is known to the experts (see [9]), but apparently has not yet appeared in printed form. It may be called quantum $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality.

Proposition 11 *(a) The images in $\mathrm{End}_{\mathbf{Q}(v)}(S_q^d(\mathcal{M}^*_{m,n}))$ of $(U_q(\mathfrak{gl}_m))^{\mathrm{op}}$ and $U_q(\mathfrak{gl}_n)$ are each one the full commutant of the other.*

(b) The images in $\mathrm{End}_{\mathbf{Q}(v)}(\mathcal{S}_q^d(m,n))$ of $U_q(\mathfrak{gl}_m)$ and $(U_q(\mathfrak{gl}_n))^{\mathrm{op}}$ are each one the full commutant of the other.

Proof. We first prove statement (b). Suppose that $m \leq n$ and denote by $\eta \in \mathcal{S}_q^d(n)$ the sum $\sum_A [A]$ over all diagonal matrices $A \in \Theta^d_{n,n}$ such that $\mathrm{ro}(A) \in P_m$. Being the sum of a family of orthogonal idempotents, η is an idempotent. There are canonical isomorphisms $\mathcal{S}_q^d(m) \simeq \eta\ \mathcal{S}_q^d(n)\ \eta$ and $\mathcal{S}_q^d(m,n) \simeq \eta\ \mathcal{S}_q^d(n)$, compatible with the structure of $\mathcal{S}_q^d(m)$-$\mathcal{S}_q^d(n)$-bimodule over $\mathcal{S}_q^d(m,n)$ (see Remark (b) in Section 1.4). Thus we have isomorphisms

$$\mathcal{S}_q^d(m) \simeq \eta\ \mathcal{S}_q^d(n)\ \eta \simeq \mathrm{End}_{\mathcal{S}_q^d(n)}(\eta\ \mathcal{S}_q^d(n)) \simeq \mathrm{End}_{\mathcal{S}_q^d(n)}(\mathcal{S}_q^d(m,n)),$$

whence one half of our assertion. The other half follows from the density theorem [16, Chap. 4, § 3], since the choice of our ground ring makes all our algebras semisimple.

Statement (a) is now a simple consequence of Proposition 5 (a).

Our final result presents how to recover the quantum Schur–Weyl duality from the quantum $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality (see [14] for instance in the classical case). Let n and d be integers. We consider the $U_q(\mathfrak{gl}_n)$-$H_q(\mathfrak{S}_d)$-bimodule $(V_n)^{\otimes d}$ and the $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_d)$-bimodule $S_q^d(\mathcal{M}^*_{d,n})$. By Proposition 10 (and at the expense of exchanging left and right), the $U_q(\mathfrak{gl}_d)$-weight subspace of $S_q^d(\mathcal{M}^*_{d,n})$ of weight $\omega = \varepsilon_1 + \cdots + \varepsilon_d$ becomes a $U_q(\mathfrak{gl}_n)$-$H_q(\mathfrak{S}_d)$-bimodule under the action of the q-Weyl group of $U_q(\mathfrak{gl}_d)$.

Proposition 12 *With the data above, the map*

$$u : \big((V_n)^{\otimes d} \to S_q^d(\mathcal{M}_{d,n}^*), e_{j_1,\ldots,j_d} \mapsto X_{1,j_1} \cdots X_{d,j_d}\big)$$

defines a $U_q(\mathfrak{gl}_n)$-$H_q(\mathfrak{S}_d)$-isomorphism between $(V_n)^{\otimes d}$ and the $U_q(\mathfrak{gl}_d)$-weight subspace of $S_q^d(\mathcal{M}_{d,n}^*)$ of weight ω.

Proof. By Proposition 4 and Theorem 2 (a), the map u is an isomorphism of $U_q(\mathfrak{gl}_n)$-modules between the two aforementioned spaces. Lemma 4 and induction on length show that if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of a permutation $w \in \mathfrak{S}_d$, then the relation

$$T_w \cdot f_{1,\ldots,d} = f_{1,\ldots,d} \cdot (-T_{i_1,+1}) \cdots (-T_{i_k,+1})$$

(where $T_{i,+1}$ denotes either $T'_{i,+1}$ or $T''_{i,+1}$) holds in the $H_q(\mathfrak{S}_d)$-$U_q(\mathfrak{gl}_d)$-bimodule $(V_d^*)^{\otimes d}$. It follows from Proposition 4 that u preserves the action of $H_q(\mathfrak{S}_d)$.

Remarks. (a) A version of Proposition 12 has been announced in [19, p. 249].

(b) Proposition 8 shows that the $U_q(\mathfrak{gl}_n)$-$U_q(\mathfrak{gl}_d)$-modules $S_q^d(\mathcal{M}_{d,n}^*)$ and $\mathcal{S}_q^d(n,d)$ are isomorphic in a natural way. Under this isomorphism, the subspace $u\big((V_n)^{\otimes d}\big) \subseteq S_q^d(\mathcal{M}_{d,n}^*)$ is sent to $\mathcal{S}_q^d(n,d)[I_d]$, whence an isomorphism of $U_q(\mathfrak{gl}_n)$-$H_q(\mathfrak{S}_d)$-bimodules between $(V_n)^{\otimes d}$ and $\mathcal{S}_q^d(n,d)[I_d]$, taking into account the isomorphism $H_q(\mathfrak{S}_d) \simeq [I_d]\mathcal{S}_q^d(d)[I_d]$. Up to a multiplication by the scalar $v^{d(d-1)/2}$, this is the geometric construction of quantum Schur–Weyl duality given in [10, § 1.13]. This is also the realization of the "q-tensor space" in [5, § 2.6]. These facts were already noticed by Du and R. M. Green, the new thing here is the relation with the q-Weyl group.

Finally, let us point out the difficulties that arise upon base ring change. Nearly all the constructions are valid in an integral setting: one has to replace the quantized enveloping algebra by its Lusztig form, but then Theorem 1 is still valid (see [8, Theorem 3.4]), the q-Weyl group is still defined (see [23, Chap. 41]), and Theorem 3 still holds. One can then specialize the objects in any commutative $\mathbf{Z}[v, v^{-1}]$-algebra. Theorem 4 extends to this setting (see [7, Theorem 6.3]). Proposition 11 does not hold in general, but its validity is restored if one adds the assumption that $m = n$ or that $d \leq \min(m,n)$. Although the map given in Proposition 8 is no longer an isomorphism, it maps isomorphically the subspace $u\big((V_n)^{\otimes d}\big) \subseteq S_q^d(\mathcal{M}_{d,n}^*)$ to $\mathcal{S}_q^d(n,d)[I_d]$.

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