A Solomon descent theory for the wreath products $G \wr \mathfrak{S}_n$

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Abstract

We propose an analogue of Solomon's descent theory for the case of a wreath product $G\wr\mathfrak{S}_n$, where G is a finite abelian group. Our construction mixes a number of ingredients: Mantaci-Reutenauer algebras, Specht's theory for the representations of wreath products, Okada's extension to wreath products of the Robinson-Schensted correspondence, Poirier's quasisymmetric functions. We insist on the functorial aspect of our definitions and explain the relation of our results with previous work concerning the hyperoctaedral group.

Introduction

The problem studied in this article has its roots in a discovery by Solomon in 1976. Let $(W,(s_i)_{i\in I})$ be a Coxeter system. For any subset $J\subseteq I$, call W_J the parabolic subgroup generated by the elements s_j with $j\in J$. In each left coset wW_J of W modulo W_J, there is a unique element of minimal length, called the distinguished representative of that coset. We denote the set of these distinguished representatives by X_J, and we form the sum $x_J=\sum_{w\in X_J} w$ in the group ring $\mathbb{Z}W$. Finally we denote by Σ_W the $\mathbb{Z}$-submodule of $\mathbb{Z}W$ spanned by all elements x_J.

Now let $R(W)$ be the character ring of W, and let $\varphi_J\in R(W)$ be the character of W induced from the trivial character of W_J. Given two subsets J and K of I, each double coset $C\in W_J\backslash W/W_K$ contains a unique element x of minimal length, and a result of Tits, Kilmoyer [18] and/or Solomon [33] asserts that the intersection $x^{-1}W_Jx\cap W_K$ is the parabolic subgroup $W_{L(C)}$, where $L(C)=\{k\in K\mid \exists j\in J,\ x^{-1}s_jx=s_k\}$. Joint to Mackey's tensor product theorem, this yields the multiplication rule in the representation ring $R(W)$

$$\varphi_J\varphi_K=\sum_{L\subseteq I} a_{JKL}\,\varphi_L,\quad \text{where}\quad a_{JKL}=\bigl|\{C\in W_J\backslash W/W_K\mid L=L(C)\}\bigr|.$$

With these notations, Solomon's discovery [33] is the equality $x_Jx_K=\sum_{L\subseteq I} a_{JKL}x_L$ in the ring $\mathbb{Z}W$. It implies that Σ_W is a subring of $\mathbb{Z}W$ and it shows the existence a morphism of rings $\theta_W:\Sigma_W\to R(W)$ such that $\theta_W(x_J)=\varphi_J$. This result means that (a part of) the character theory of W can be lifted to a subring of its group ring. Additional details (for instance, a more precise description of the image of θ_W) can be found in the paper [7] by F. Bergeron, N. Bergeron, Howlett and Taylor.

It is natural to look for a similar theory for groups other than Coxeter systems. The first examples that come to mind are finite groups of Lie type and finite complex reflection groups.

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Among the latter, the groups of type $G(r,1,n)$ are wreath products $(\mathbb{Z}/r\mathbb{Z})\wr\mathfrak{S}_n$ of a cyclic group $\mathbb{Z}/r\mathbb{Z}$ by the symmetric group $\mathfrak{S}_n$. One is then led to investigate the case of a general wreath product $G\wr\mathfrak{S}_n$. To build the theory, it is necessary to have some knowledge about the representation theory of G itself, and we assume in this paper that G is abelian. One of our main results explains how to construct a subring $\mathsf{MR}_n(\mathbb{Z}G)$ inside the group ring $\mathbb{Z}\bigl[G\wr\mathfrak{S}_n\bigr]$ and a surjective ring homomorphism θ_G from $\mathsf{MR}_n(\mathbb{Z}G)$ onto the representation ring $R(G\wr\mathfrak{S}_n)$ of the wreath product. Here the notation MR refers to the names of Mantaci and Reutenauer; indeed it turns out that the remarkable subring inside $\mathbb{Z}\bigl[G\wr\mathfrak{S}_n\bigr]$ discovered in 1995 by these two authors [24] is adequate to our purpose.

A usually efficient method to tackle problems with the symmetric group $\mathfrak{S}_n$ is to treat all n at the same time. For instance, Malvenuto and Reutenauer observed in 1995 [23] that the direct sum $\mathscr{F}=\bigoplus_{n\geq 0}\mathbb{Z}[\mathfrak{S}_n]$ can be endowed with the structure of a graded bialgebra in such a way that the submodule $\Sigma=\bigoplus_{n\geq 0}\Sigma_{\mathfrak{S}_n}$ is a graded subbialgebra. A similar phenomenon appears here: the direct sum $\mathscr{F}(\mathbb{Z}G)=\bigoplus_{n\geq 0}\mathbb{Z}\bigl[G\wr\mathfrak{S}_n\bigr]$ can be endowed with the structure of a graded bialgebra, of which $\mathsf{MR}(\mathbb{Z}G)=\bigoplus_{n\geq 0}\mathsf{MR}_n(\mathbb{Z}G)$ is a subbialgebra. (A particular case of this construction was previously considered by Aguiar and Mahajan; the paper [2] by Aguiar, N. Bergeron and Nyman presents an account of their result. Aguiar and his coauthors view the hyperoctaedral group of order $2^n n!$ as the wreath product $\{\pm 1\}\wr\mathfrak{S}_n$, that is, as the group of signed permutations. Then they construct the graded bialgebra $\mathscr{F}\bigl(\mathbb{Z}\bigl[\{\pm1\}\bigr]\bigr)$ and its subbialgebra $\mathsf{MR}\bigl(\mathbb{Z}\bigl[\{\pm1\}\bigr]\bigr)$. Using the morphism of group 'forgetting the signs' from $\{\pm1\}\wr\mathfrak{S}_n$ onto $\mathfrak{S}_n$, they compare these graded bialgebras with Malvenuto and Reutenauer's bialgebra $\mathscr{F}$ and its subbialgebra Σ. Our construction and its functoriality generalize Aguiar and his coauthors' results to the case of all wreath products $G\wr\mathfrak{S}_n$.) This bialgebra structure on $\mathscr{F}(\mathbb{Z}G)$ will be the starting point of our story; indeed we define a 'free quasisymmetric algebra' $\mathscr{F}(V)$ for any $\mathbb{Z}$-module V and investigate its properties.

We now present the plan and the main results of this paper.

In Section 1, we define the free quasisymmetric algebra on a module V over a commutative ground ring $\mathbb{K}$: this is a graded module $\mathscr{F}(V)=\bigoplus_{n\geq 0}\mathscr{F}_n(V)$, which we endow with an 'external product' and a coproduct to turn it into a graded bialgebra (Theorem 1). In the case where V is endowed with the structure of a coalgebra, $\mathscr{F}(V)$ contains a remarkable subbialgebra $\mathsf{MR}(V)$, the so-called Mantaci-Reutenauer bialgebra, which is a free associative algebra as soon as V is a free module (Propositions 3 and 4).

In Section 2, we show that the functor $V\rightsquigarrow\mathscr{F}(V)$ is compatible with the duality of $\mathbb{K}$-modules, in the sense that any pairing between two $\mathbb{K}$-modules V and W gives rise to a pairing of bialgebras between $\mathscr{F}(V)$ and $\mathscr{F}(W)$ (Proposition 5). In particular, the bialgebra $\mathscr{F}(V)$ is self-dual as soon as the module V is endowed with a perfect pairing.

In Section 3, we investigate the case where the module V is a $\mathbb{K}$-algebra. Then $\mathscr{F}(V)$ can be endowed with an 'internal product', which turns each of the graded components $\mathscr{F}_n(V)$ into an algebra. The interesting point here is the existence of a splitting formula that describes the compatibility between this internal product, the external product and the coproduct (Theorem 10). This formula is a generalization of the splitting formula of Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [13]; it entails that the Mantaci-Reutenauer bialgebra $\mathsf{MR}(V)$ is a subalgebra of $\mathscr{F}(V)$ for the internal product whenever V is endowed with the structure of a cocommutative bialgebra (Corollaries 11 and 12). In Section 3.5, we consider for V the case of the group algebra $\mathbb{K}\Gamma$ of a finite group Γ and justify that the graded component $\mathscr{F}_n(\mathbb{K}\Gamma)$ is canonically isomorphic to the group algebra $\mathbb{K}\bigl[\Gamma\wr\mathfrak{S}_n\bigr]$, and that the graded component $\mathsf{MR}_n(\mathbb{K}\Gamma) = \mathsf{MR}(\mathbb{K}\Gamma) \cap \mathscr{F}_n(\mathbb{K}\Gamma)$ coincides with the subalgebra defined by Mantaci and Reutenauer in [24].

In Section 4, we at last provide the link between these constructions and a Solomon descent theory for wreath products. We first recall Specht's classification of the irreducible complex characters of a wreath product $G \wr \mathfrak{S}_n$ and Zelevinsky's structure of a graded bialgebra on the direct sum $\operatorname{Rep}(G) = \bigoplus_{n \geq 0} R(G \wr \mathfrak{S}_n)$ for the induction product and the restriction coproduct (Section 4.2). We then focus on the case where G is abelian. We denote the dual group of G by Γ, we observe that the group ring $\mathbb{Z}\Gamma$ is a cocommutative bialgebra, so that the Mantaci-Reutenauer bialgebra $\mathsf{MR}(\mathbb{Z}\Gamma)$ is defined and is a subalgebra of $\mathscr{F}(\mathbb{Z}\Gamma)$ for the internal product, and we define a map $\theta_G : \mathsf{MR}(\mathbb{Z}\Gamma) \to \operatorname{Rep}(G)$. Then we show that θ_G is a surjective morphism of graded bialgebras, and that in each degree, $\theta_G : \mathsf{MR}_n(\mathbb{Z}\Gamma) \to R(G \wr \mathfrak{S}_n)$ is a surjective morphism of rings whose kernel is the Jacobson radical of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ (Theorem 16). We also show that θ_G enjoys a remarkable symmetry property analogous to the symmetry property of Solomon's homomorphisms θ_W proved by Jöllenbeck and Reutenauer [17] and by Blessenohl, Hohlweg and Schocker [8] (Theorem 19). Finally we compare our results with the work of Bonnafé and Hohlweg, who treated in [10] the case of the hyperoctaedral group $\{\pm 1\} \wr \mathfrak{S}_n$ using methods from the theory of Coxeter groups (Section 4.5).

The questions about the bialgebras $\mathscr{F}(V)$ investigated in Sections 1 to 3 are functorial in the $\mathbb{K}$-module V. As usual, the most interesting point in this assertion is the compatibility of the constructions with the homomorphisms, namely here the $\mathbb{K}$-linear maps. On the contrary the questions studied in Section 5 require that V be a free $\mathbb{K}$-module and depend on the choice of a basis B of V. Such a basis B can be viewed as the data of a structure of a pointed coalgebra on V, which yields in turn a Mantaci-Reutenauer subbialgebra $\mathsf{MR}(V)$ inside $\mathscr{F}(V)$. The choice of B also gives rise to a second subbialgebra $\mathscr{Q}(B)$, bigger than $\mathsf{MR}(V)$, which we call the coplactic bialgebra. The definition of $\mathscr{Q}(B)$ involves a combinatorial construction due to Okada [28], which extends the well-known Robinson-Schensted correspondence to 'coloured' situations; at this point, we take the opportunity to provide an analogue of Knuth relations for Okada's correspondence (Proposition 24). In the case where B is a singleton set, the bialgebra $\mathscr{Q}(B)$ is one of the 'algèbres de Hopf de tableaux' of Poirier and Reutenauer [30]. Extending the work of these authors, we define a surjective homomorphism Θ_B of graded bialgebras from $\mathscr{Q}(B)$ onto a bialgebra $\Lambda(B)$ of 'coloured' symmetric functions (Theorem 31). We then go back to the situation investigated in Section 4 and take the group algebra $\mathbb{Z}\Gamma$ for V and the group Γ for B; here Θ_Γ can be viewed as a lift of $\theta_G : \mathsf{MR}(\mathbb{Z}\Gamma) \to \operatorname{Rep}(G)$ to $\mathscr{Q}(\Gamma)$ that yields a nice description of the simple representations of all wreath products $G \wr \mathfrak{S}_n$. We recover Jöllenbeck's construction of the Specht modules [16] as the particular case where G is the group with one element; we refer the reader to Blessenohl and Schocker's survey [9] for additional details about Jöllenbeck's construction.

Finally we present in Section 6 a realization of the bialgebra $\mathscr{F}(V)$ in terms of free quasisymmetric functions. As in Section 5, the $\mathbb{K}$-module V is assumed to be free; we choose a basis B of V and endow B with a linear order. When V has rank one, our free quasisymmetric functions coincide with the usual ones [14]. In higher rank however, our free quasisymmetric functions are different from those defined by Novelli and Thibon in [27]. This disagreement has its roots in the fact that Novelli and Thibon's construction and ours were designed with different aims: roughly speaking, Novelli and Thibon's goal was to find a noncommutative version of Poirier's quasisymmetric functions [29]; on the other side, we view the dual algebra $\mathsf{MR}(V)^\vee$ as a quotient of $\mathscr{F}(V)$ and describe it in terms of commutative quasisymmetric functions.

At this point, we should mention that the assignment $(V, B) \rightsquigarrow \mathsf{MR}(V)^\vee$ enjoys a certain functoriality property; this property and the isomorphism between $\mathsf{MR}(\mathbb{K})^\vee$ and the graded bialgebra QSym of usual quasisymmetric functions yield in turn homomorphisms of graded bialgebras from $\mathscr{F}(V)$ and $\mathsf{MR}(V)^\vee$ to QSym, which amounts to say that $\mathscr{F}(V)$ and $\mathsf{MR}(V)^\vee$ are 'combinatorial Hopf algebras' in the sense of Aguiar, N. Bergeron and Sottile [3].

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We fix a commutative ground ring $\mathbb{K}$. Connected $\mathbb{N}$-graded $\mathbb{K}$-bialgebras appear everywhere in the paper. Such bialgebras are indeed automatically Hopf algebras, at least when $\mathbb{K}$ is a field. However we will neither make use of this property nor attempt to work out explicitly any antipode.

1 Free quasisymmetric bialgebras

In this section, we present our main objects of study, namely the free quasisymmetric bialgebras and the generalized descent algebras, among which the Novelli-Thibon bialgebras and the Mantaci-Reutenauer bialgebras. Before that, we introduce some notations pertaining permutations.

1.1 Notations related to permutations

For each positive integer n, we denote the symmetric group of all permutations of the set $\{1, 2, \ldots, n\}$ by $\mathfrak{S}_n$. By convention, $\mathfrak{S}_0$ is the group with one element. The unit element of $\mathfrak{S}_n$ is denoted by e_n. The group algebra over $\mathbb{K}$ of $\mathfrak{S}_n$ is denoted by $\mathbb{K}\mathfrak{S}_n$. In practice, a permutation $\sigma \in \mathfrak{S}_n$ is written as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$ with letters in $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$.

Let $\mathscr{A}$ be totally ordered set (an alphabet). The standardization of a word $w = a_1 a_2 \cdots a_n$ of length n with letters in $\mathscr{A}$ is the permutation $\sigma \in \mathfrak{S}_n$ with smallest number of inversions such that the sequence

$$\left(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)}\right)$$

is non-decreasing. In other words, the word $\sigma(1)\sigma(2)\cdots\sigma(n)$ that represents σ is obtained by putting the numbers 1, 2, ..., n in the place of the letters a_i of w; in this process of substitution, the diverse occurrences of the smallest letter of $\mathscr{A}$ get replaced first by the numbers 1, 2, etc. from left to right; then we replace the occurrences of the second-smallest element of $\mathscr{A}$ by the following numbers; and so on, up to the exhaustion of all letters of w. An example clarifies this explanation: given the alphabet $\mathscr{A} = \{a, b, c, \ldots\}$ with the usual order, the standardization of the word $w = bcbaba$ is $\sigma = 364152$.

A composition of a positive integer n is a sequence $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ of positive integers which sum up to n. The usual notation for that is to write $\mathbf{c} \models n$. Given two compositions $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_l)$ of the same integer n, we say that $\mathbf{c}$ is a refinement of $\mathbf{d}$ and we write $\mathbf{c} \succcurlyeq \mathbf{d}$ if there holds

$$\{c_1, c_1 + c_2, \ldots, c_1 + c_2 + \cdots + c_{k-1}\} \supseteq \{d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_{l-1}\}.$$

The relation $\preccurlyeq$ is a partial order on the set of compositions of n. For instance, the following chain of inequalities hold among compositions of 5:

$$(5) \prec (4,1) \prec (1,3,1) \prec (1,2,1,1) \prec (1,1,1,1,1).$$

Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a composition of n and set $t_i = c_1 + c_2 + \cdots + c_i$ for each i. Given a k-uple $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \cdots \times \mathfrak{S}_{c_k}$ of permutations, we define $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k \in \mathfrak{S}_n$ as the permutation that maps an element a belonging to the interval $[t_{i-1}+1, t_i]$ onto $t_{i-1} + \sigma_i(a - t_{i-1})$. This assignment defines an embedding $\mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \cdots \times \mathfrak{S}_{c_k} \hookrightarrow \mathfrak{S}_n$; we denote its image by $\mathfrak{S}_{\mathbf{c}}$. Such a $\mathfrak{S}_{\mathbf{c}}$ is called a Young subgroup of $\mathfrak{S}_n$. We obtain for free an embedding for the group algebras

$$\mathbb{K}\mathfrak{S}_{c_1} \otimes \mathbb{K}\mathfrak{S}_{c_2} \otimes \cdots \otimes \mathbb{K}\mathfrak{S}_{c_k} \xrightarrow{\simeq} \mathbb{K}\mathfrak{S}_{\mathbf{c}} \subseteq \mathbb{K}\mathfrak{S}_n.$$

The map $\mathbf{c} \mapsto \mathfrak{S}_{\mathbf{c}}$ is an order reversing bijection from the set of compositions of n, endowed with the refinement order, onto the set of Young subgroups of $\mathfrak{S}_n$, endowed with the inclusion order.

Let again $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a composition of n and set $t_i = c_1 + c_2 + \cdots + c_i$. The subset

$$X_{\mathbf{c}} = \big\{\sigma \in \mathfrak{S}_n \;\big|\; \forall i,\ \sigma \text{ is increasing on the interval } [t_{i-1}+1, t_i]\big\}$$

is a system of representatives of the left cosets of $\mathfrak{S}_{\mathbf{c}}$ in $\mathfrak{S}_n$. Here are some examples:

$$X_{(2,2)} = \{1234, 1324, 1423, 2314, 2413, 3412\}, \quad X_{(n)} = \{\mathrm{id}\} \quad \text{and} \quad X_{(\underbrace{1,1,\ldots,1}_{n \text{ times}})} = \mathfrak{S}_n.$$

We define an element of the group ring $\mathbb{K}\mathfrak{S}_n$ by setting $x_{\mathbf{c}} = \sum_{\sigma \in X_{\mathbf{c}}} \sigma$.

Let $\mathbf{d} = (d_1, d_2, \ldots, d_l)$ be a composition of an integer n. Then a composition $\mathbf{c}$ of n is a refinement of $\mathbf{d}$ if and only if $\mathbf{c}$ can be obtained as the concatenation $\mathbf{f}_1\mathbf{f}_2 \cdots \mathbf{f}_l$ of a composition $\mathbf{f}_1$ of d_1, a composition $\mathbf{f}_2$ of d_2, ..., and a composition $\mathbf{f}_l$ of d_l. If this holds, then the map

$$(\rho, \sigma_1, \sigma_2, \ldots, \sigma_l) \mapsto \rho \circ (\sigma_1 \times \sigma_2 \times \cdots \times \sigma_l)$$

is a bijection from $X_{\mathbf{d}} \times X_{\mathbf{f}_1} \times X_{\mathbf{f}_2} \times \cdots \times X_{\mathbf{f}_l}$ onto $X_{\mathbf{c}}$, for $X_{\mathbf{f}_1} \times \cdots \times X_{\mathbf{f}_l}$ is a set of minimal coset representatives of $\mathfrak{S}_{\mathbf{c}}$ in $\mathfrak{S}_{\mathbf{d}}$. Therefore the equality

$$x_{\mathbf{c}} = x_{\mathbf{d}}\,(x_{\mathbf{f}_1} \otimes x_{\mathbf{f}_2} \otimes \cdots \otimes x_{\mathbf{f}_l}) \tag{1}$$

holds in the group ring $\mathbb{K}\mathfrak{S}_n$. As a particular case of (1), we see that

$$x_{(n,n',n'')} = x_{(n,n'+n'')}\big(x_{(n)} \otimes x_{(n',n'')}\big) = x_{(n+n',n'')}\big(x_{(n,n')} \otimes x_{(n'')}\big) \tag{2}$$

holds true for any three positive integers n, n' and n''.

Let $\sigma \in \mathfrak{S}_n$. One may partition the word $\sigma(1)\sigma(2)\cdots\sigma(n)$ that represents σ into its longest increasing subwords; the composition of n formed by the successive lengths of these subwords is called the descent composition of σ and is denoted by $D(\sigma)$. For instance, the descent composition of $\sigma = 51243$ is $D(\sigma) = (1,3,1)$. Then for any composition $\mathbf{c}$ of n, the assertions $\sigma \in X_{\mathbf{c}}$ and $D(\sigma) \preccurlyeq \mathbf{c}$ are equivalent.

1.2 Definition of the free quasisymmetric bialgebra $\mathscr{F}(V)$

Let V be a $\mathbb{K}$-module. The group $\mathfrak{S}_n$ acts on the n-th tensor power $V^{\otimes n}$; the submodule of invariants, that is, the space of symmetric tensors, is denoted by $\mathsf{TS}^n(V)$. We may form the tensor product of $V^{\otimes n}$ by $k\mathfrak{S}_n$. To distinguish this tensor product from those used to build the tensor power $V^{\otimes n}$, we denote it with a sharp symbol. We denote the result $(V^{\otimes n})\#(\mathbb{K}\mathfrak{S}_n)$ by $\mathscr{F}_n(V)$. The actions defined by

$$\pi \cdot \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\sigma\big] = \big[(v_{\pi^{-1}(1)} \otimes v_{\pi^{-1}(2)} \otimes \cdots \otimes v_{\pi^{-1}(n)})\#(\pi\sigma)\big]$$

and

$$\big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\sigma\big] \cdot \pi = \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#(\sigma\pi)\big]$$

endow $\mathscr{F}_n(V)$ with the structure of a $\mathbb{K}\mathfrak{S}_n$-bimodule, where $(v_1, v_2, \ldots, v_n) \in V^n$ and $\pi \in \mathfrak{S}_n$. For instance, $\mathscr{F}_n(\mathbb{K})$ is the (left and right) regular $\mathbb{K}\mathfrak{S}_n$-module.

Our aim now is to endow the space $\mathscr{F}(V) = \bigoplus_{n\geq 0} \mathscr{F}_n(V)$ with the structure of a graded bialgebra. We define the product of two elements $\alpha \in \mathscr{F}_n(V)$ and $\alpha' \in \mathscr{F}_{n'}(V)$ of the form

$$\alpha = \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\sigma\big] \quad \text{and} \quad \alpha' = \big[(v'_1 \otimes v'_2 \otimes \cdots \otimes v'_{n'})\#\sigma'\big]$$

by the formula

$$\alpha * \alpha' = x_{(n,n')} \cdot \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes v'_1 \otimes v'_2 \otimes \cdots \otimes v'_{n'})\#(\sigma \times \sigma')\big].$$

(This formula can be made more concrete by noting that $x_{(n,n')}\ (\sigma \times \sigma')$ is the sum in the group algebra $\mathbb{K}\mathfrak{S}_{n+n'}$ of all permutations π such that σ is the standardization of the word $\pi(1)\pi(2)\cdots\pi(n)$ and σ' is the standardization of the word $\pi(n+1)\pi(n+2)\cdots\pi(n+n')$.) We extend this definition by multilinearity to an operation defined on the whole space $\mathscr{F}(V)$ and call this latter the external product.

We define the coproduct of an element $\alpha = \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\sigma\big]$ of $\mathscr{F}_n(V)$ as

$$\Delta\big((v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\sigma\big) = \sum_{n'=0}^{n} \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_{n'})\#\pi_{n'}\big] \otimes \big[(v_{n'+1} \otimes v_{n'+2} \otimes \cdots \otimes v_n)\#\pi'_{n-n'}\big],$$

where $\pi_{n'} \in \mathfrak{S}_{n'}$ is the inverse of the standardization of the word $\sigma^{-1}(1)\ \sigma^{-1}(2)\ \cdots\ \sigma^{-1}(n')$ and $\pi'_{n-n'} \in \mathfrak{S}_{n-n'}$ is the inverse of the standardization of the word $\sigma^{-1}(n'+1)\ \sigma^{-1}(n'+2)\ \cdots\ \sigma^{-1}(n)$. In other words, $\pi_{n'}$ and $\pi'_{n-n'}$ are such that the two sequences of letters $(\pi_{n'}(1), \pi_{n'}(2), \ldots, \pi_{n'}(n'))$ and $(n' + \pi'_{n-n'}(1), n' + \pi'_{n-n'}(2), \ldots, n' + \pi'_{n-n'}(n-n'))$ appear in this order in the word $\sigma(1)\sigma(2)\cdots\sigma(n)$. We call the map $\Delta : \mathscr{F}(V) \to \mathscr{F}(V) \otimes \mathscr{F}(V)$ the coproduct of $\mathscr{F}(V)$.

We define the unit of $\mathscr{F}(V)$ as the injection of the graded component $\mathscr{F}_0(V) = \mathbb{K}$ into $\mathscr{F}(V)$; we define the counit of $\mathscr{F}(V)$ as the projection of $\mathscr{F}(V)$ onto $\mathscr{F}_0(V) = \mathbb{K}$.

We now give an example to illustrate these definitions. Given six elements v_1, v_2, v_3, v_4, v'_1, v'_2 in V, the product of $\alpha = \big[(v_1 \otimes v_2)\#e_2\big]$ and $\alpha' = \big[(v'_2 \otimes v'_1)\#21\big] = (21) \cdot \big[(v'_1 \otimes v'_2)\#e_2\big]$ is

$$\begin{aligned}
\alpha * \alpha' &= (1243 + 1342 + 1432 + 2341 + 2431 + 3421) \cdot \big[(v_1 \otimes v_2 \otimes v'_1 \otimes v'_2)\#e_4\big] \\
&= \big[(v_1 \otimes v_2 \otimes v'_2 \otimes v'_1)\#1243\big] + \big[(v_1 \otimes v'_2 \otimes v_2 \otimes v'_1)\#1342\big] + \\
&\quad \big[(v_1 \otimes v'_2 \otimes v'_1 \otimes v_2)\#1432\big] + \big[(v'_2 \otimes v_1 \otimes v_2 \otimes v'_1)\#2341\big] + \\
&\quad \big[(v'_2 \otimes v_1 \otimes v'_1 \otimes v_2)\#2431\big] + \big[(v'_2 \otimes v'_1 \otimes v_1 \otimes v_2)\#3421\big],
\end{aligned}$$

and the coproduct of $\alpha = \big[(v_3 \otimes v_1 \otimes v_2 \otimes v_4)\#2314\big] = (2314) \cdot \big[(v_1 \otimes v_2 \otimes v_3 \otimes v_4)\#e_4\big]$ is

$$\begin{aligned}\Delta(\alpha) = {} & \big[()\#e_0\big] \otimes \alpha + \big[(v_3)\#1\big] \otimes \big[(v_1 \otimes v_2 \otimes v_4)\#123\big] + \\ & \big[(v_3 \otimes v_1)\#21\big] \otimes \big[(v_2 \otimes v_4)\#12\big] + \\ & \big[(v_3 \otimes v_1 \otimes v_2)\#231\big] \otimes \big[(v_4)\#1\big] + \alpha \otimes \big[()\#e_0\big] \\ = {} & \big[()\#e_0\big] \otimes \alpha + \big[(v_3)\#e_1\big] \otimes \big[(v_1 \otimes v_2 \otimes v_4)\#e_3\big] + \\ & + (21) \cdot \big[(v_1 \otimes v_3)\#e_2\big] \otimes \big[(v_2 \otimes v_4)\#e_2\big] \\ & + (231) \cdot \big[(v_1 \otimes v_2 \otimes v_3)\#e_3\big] \otimes \big[(v_4)\#e_1\big] + \alpha \otimes \big[()\#e_0\big].\end{aligned}$$

Theorem 1 *The unit, the counit, and the operations $*$ and Δ endow $\mathscr{F}(G)$ with the structure of a graded bialgebra.*

Proof. It is clear that the four operations respect the graduation. The associativity of $*$ follows immediately from Equation (2). A moment's thought suffices to check the coassociativity of Δ and the axioms for the unit and the counit. It remains to show the pentagon axiom, which asks that Δ be multiplicative with respect to the product $*$.

Following Malvenuto and Reutenauer's method [23], we first recall a classical construction in the theory of Hopf algebras. Let $\mathscr{A}$ be a set, let $\langle\mathscr{A}\rangle$ denote the set of words on $\mathscr{A}$, and let $\mathbb{K}\langle\mathscr{A}\rangle$ be the free $\mathbb{K}$-module with basis $\langle\mathscr{A}\rangle$. The shuffle product of two words w and w' of length n and n' respectively is the sum

$$w \sqcup\!\sqcup\, w' = \sum_{\rho \in X_{(n,n')}} b_{\rho^{-1}(1)} b_{\rho^{-1}(2)} \cdots b_{\rho^{-1}(n+n')},$$

where the word $b_1 b_2 \cdots b_{n+n'}$ is the concatenation of the words w and w'. This operation $\sqcup\!\sqcup$ is then extended bilinearly to a product on $\mathbb{K}\langle\mathscr{A}\rangle$. The deconcatenation is the coproduct δ on $\mathbb{K}\langle\mathscr{A}\rangle$ such that

$$\delta(w) = \sum_{n'=0}^{n} a_1 a_2 \cdots a_{n'} \otimes a_{n'+1} a_{n'+2} \cdots a_n$$

for any word $w = a_1 a_2 \cdots a_n$. It is known that the operations $\sqcup\!\sqcup$ and δ endow $\mathbb{K}\langle\mathscr{A}\rangle$ with the structure of a bialgebra (see Proposition 1.9 in [31] for a proof).

We are now ready to show the pentagon axiom in the case where the $\mathbb{K}$-module V is free. We take a basis B of V and we set $\mathscr{A} = \mathbb{Z}_{>0} \times B$. We observe that the elements $(b_1 \otimes b_2 \otimes \cdots \otimes b_n)\#\sigma$ form a basis of $\mathscr{F}_n(V)$, where $(b_1, b_2, \ldots, b_n) \in B^n$ and $\sigma \in \mathfrak{S}_n$. We may thus define linear maps $j_k : \mathscr{F}(G) \to \mathbb{K}\langle\mathscr{A}\rangle$ (depending on the choice of a non-negative integer k) by mapping an element $\alpha = \big[(b_1 \otimes b_2 \otimes \cdots \otimes b_n)\#\sigma\big]$ to $j_k(\alpha) = a_1 a_2 \cdots a_n$, where $a_i = (k + \sigma^{-1}(i), b_i)$. In the other direction, we define a linear map $s : \mathbb{K}\langle\mathscr{A}\rangle \to \mathscr{F}(V)$ as follows: given a word $w = a_1 a_2 \cdots a_n$ with letters in $\mathscr{A}$, we write $a_i = (p_i, b_i)$ and set $s(w) = (b_1 \otimes b_2 \otimes \cdots \otimes b_n)\#\sigma$, where σ is the inverse of the standardization of the word $p_1 p_2 \cdots p_n$.

One easily checks that $s \circ j_k = \mathrm{id}_{\mathscr{F}(G)}$ and that $(s \otimes s) \circ \delta = \Delta \circ s$. Moreover, let $w = a_1 a_2 \cdots a_n$ and $w' = a'_1 a'_2 \cdots a'_{n'}$ be two words with letters in $\mathscr{A}$. If we write $a_i = (p_i, b_i)$ and $a'_i = (p'_i, b'_i)$, then $s(w \sqcup\!\sqcup\, w') = s(w) * s(w')$ as soon as every integer p_i is strictly smaller than every integer p'_i.

We now take $\alpha \in \mathscr{F}_n(G)$ and $\alpha' \in \mathscr{F}_{n'}(G)$. We compute:

$$\begin{aligned}
\Delta(\alpha * \alpha') &= \Delta\Big[s\big(j_0(\alpha)\big) * s\big(j_n(\alpha')\big)\Big] \\
&= \big(\Delta \circ s\big)\big(j_0(\alpha) \sqcup\!\sqcup\, j_n(\alpha')\big) \\
&= (s \otimes s)\Big[\delta\big(j_0(\alpha) \sqcup\!\sqcup\, j_n(\alpha')\big)\Big] \\
&= (s \otimes s)\Big[\delta\big(j_0(\alpha)\big) \sqcup\!\sqcup\, \delta\big(j_n(\alpha')\big)\Big] \\
&= \big[(s \otimes s) \circ \delta \circ j_0(\alpha)\big] * \big[(s \otimes s) \circ \delta \circ j_n(\alpha')\big] \\
&= \big[\Delta \circ s \circ j_0(\alpha)\big] * \big[\Delta \circ s \circ j_n(\alpha')\big] \\
&= \Delta(\alpha) * \Delta(\alpha').
\end{aligned}$$

This relation proves the pentagon axiom for $\mathscr{F}(V)$ in the case where V is a free $\mathbb{K}$-module. In the general case, we may find a free $\mathbb{K}$-module $\tilde{V}$ and a surjective morphism of $\mathbb{K}$-modules $f : \tilde{V} \to V$. Then f induces a surjective map from $\mathscr{F}(\tilde{V})$ onto $\mathscr{F}(V)$ which is a morphism of algebras and of coalgebras. Since the operations $*$ and Δ on $\tilde{V}$ satisfy the pentagon axiom, their analogues on V satisfy also the pentagon axiom. This completes the proof of the theorem. $\square$

We note that the assignment $V \rightsquigarrow \mathscr{F}(V)$ is a covariant functor from the category of $\mathbb{K}$-modules to the category of $\mathbb{N}$-graded bialgebras over $\mathbb{K}$.

The algebras $\mathscr{F}(V)$ were also indirectly defined by Novelli and Thibon; in [27], they denote our $\mathscr{F}(\mathbb{K}^l)$ by $\mathbf{FQSym}^{(l)}$ and state that it is a free associative algebra, whence the name 'free quasisymmetric bialgebras.'

Remark 2. Given a $\mathbb{K}$-module V, one can endow the direct sum $\bigoplus_{n\geq 0} V^{\otimes n}$ with two structures of a graded bialgebra: the tensor algebra, denoted by $\mathsf{T}(V)$, and the cotensor algebra, sometimes denoted by $\mathsf{T}^c(V)$. (The bialgebra $\mathbb{K}\langle\mathscr{A}\rangle$ used in the proof of Theorem 1 is indeed the cotensor algebra on the free $\mathbb{K}$-module $\mathbb{K}\mathscr{A}$ with basis $\mathscr{A}$.) One checks easily that the maps

$$\iota : \mathsf{T}(V) \to \mathscr{F}(V),\ v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma\in\mathfrak{S}_n} \sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n \# e_n)$$

and

$$p : \mathscr{F}(V) \to \mathsf{T}^c(V),\ (v_1 \otimes v_2 \otimes \cdots \otimes v_n \# \sigma) \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$$

are morphisms of graded bialgebras. Moreover the composition $p \circ \iota$ is the symmetrization map

$$\mathsf{T}(V) \to \mathsf{T}^c(V),\ v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma\in\mathfrak{S}_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

For details and applications of this construction, we refer the reader to [26] and [32].

1.3 The descent subbialgebras $\Sigma(W)$

In this section, we investigate a class of graded subalgebras of $\mathscr{F}(V)$, called the descent algebras. We find a criterion for a descent algebra to be a subbialgebra of $\mathscr{F}(V)$ and give a couple of examples.

We fix here a $\mathbb{K}$-module V. To any graded submodule $W = \bigoplus_{n\geq 0} W_n$ of the tensor algebra $\mathsf{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, we associate the subalgebra $\Sigma(W)$ of $\mathscr{F}(V)$ generated by all elements of the form $(t\#e_n)$ with $t \in W_n$. We call such a subalgebra $\Sigma(W)$ a descent algebra. A descent algebra is necessarily graded, for it is generated by homogeneous elements.

Proposition 3 *Assume that V is flat and that each module W_n is free of finite rank. For each $n \geq 1$, pick a basis B_n of W_n. Then $\Sigma(W)$ is the free associative algebra on the elements $(b\#e_n)$, where $n \geq 1$ and $b \in B_n$.*

Proof. By the way of contradiction, we assume that there exists a finite family $(\mathbf{u}_i)_{i\in I}$ consisting of distinct finite sequences $\mathbf{u}_i = \big((c_1^{(i)}, b_1^{(i)}), (c_2^{(i)}, b_2^{(i)}), \ldots, (c_{k_i}^{(i)}, b_{k_i}^{(i)})\big)$ of elements in $\bigcup_{n\geq 1}\big(\{n\} \times B_n\big)$ and a finite family $(\lambda_i)_{i\in I}$ of elements of $\mathbb{K} \setminus \{0\}$ such that

$$\sum_{i\in I} \lambda_i \Big[\Big(b_1^{(i)}\#e_{c_1^{(i)}}\Big) * \Big(b_2^{(i)}\#e_{c_2^{(i)}}\Big) * \cdots * \Big(b_{k_i}^{(i)}\#e_{c_{k_i}^{(i)}}\Big)\Big] = 0. \tag{3}$$

Using the graduation, we may suppose without loss of generality that all the sequences $\mathbf{c}_i = (c_1^{(i)}, c_2^{(i)}, \ldots, c_{k_i}^{(i)})$ are compositions of the same integer n. Then (3) yields

$$\sum_{i\in I} \lambda_i \, x_{\mathbf{c}_i} \cdot \big[(b_1^{(i)} \otimes b_2^{(i)} \otimes \cdots \otimes b_{k_i}^{(i)})\#e_n\big] = 0. \tag{4}$$

We choose a maximal element $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ among the set $\{\mathbf{c}_i \mid i \in I\}$ with respect to the refinement order, we set $J = \{i \in I \mid \mathbf{c}_i = \mathbf{c}\}$, and we choose a permutation $\sigma \in \mathfrak{S}_n$ whose descent composition is $\mathbf{c}$. Then for any $i \in I$,

$$\sigma \in X_{\mathbf{c}_i} \iff \mathbf{c} \preccurlyeq \mathbf{c}_i \iff i \in J.$$

Taking the image of (4) by the linear map $p : \mathscr{F}_n(V) \to V^{\otimes n}$ defined by

$$p\big((v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\rho\big) = \begin{cases} v_{\rho(1)} \otimes v_{\rho(2)} \otimes \cdots \otimes v_{\rho(n)} & \text{if } \rho = \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$\sum_{i\in J} \lambda_i \, b_1^{(i)} \otimes b_2^{(i)} \otimes \cdots \otimes b_k^{(i)} = 0. \tag{5}$$

By assumption however, the sequences $(b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)})$ are distinct when i runs over J. Therefore the elements $b_1^{(i)} \otimes b_2^{(i)} \otimes \cdots \otimes b_k^{(i)}$ are linearly independent in $W_{c_1} \otimes W_{c_2} \otimes \cdots \otimes W_{c_k}$, for $B_{c_1} \otimes B_{c_2} \otimes \cdots \otimes B_{c_k}$ is a basis of this module. Since V and the W_{c_i} are flat modules, the images of the elements $b_1^{(i)} \otimes b_2^{(i)} \otimes \cdots \otimes b_k^{(i)}$ in $V^{\otimes n}$ are linearly independent. We then reach a contradiction with Equation (5). □

Before we look for a condition on W that would ensures that $\Sigma(W)$ is a subbialgebra of $\mathscr{F}(V)$, we introduce a piece of notation that will be needed later, especially in Section 3.3. Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a composition (possibly with parts equal to zero)[1] of n. Since

[1] It is convenient in this context to allow compositions to have parts equal to zero. We could use a special terminology, following for example Reutenauer who coined in [31] the word pseudocomposition for that purpose. To limit the advent of new words, we will however simply say 'composition (possibly with parts equal to zero).'

$V^{\otimes n} = V^{\otimes c_1} \otimes V^{\otimes c_2} \otimes \cdots \otimes V^{\otimes c_k}$, each tensor $t \in V^{\otimes n}$ can be written as a linear combination of products $t_1 \otimes t_2 \otimes \cdots \otimes t_k$, where $t_i \in V^{\otimes c_i}$ for each i. We denote such a decomposition by $t = \sum_{(t)} t_1^{(\mathbf{c})} \otimes t_2^{(\mathbf{c})} \otimes \cdots \otimes t_k^{(\mathbf{c})}$. In this equation, the symbol $t_i^{(\mathbf{c})}$ is meant as a place-holder for the actual elements t_i. With this notation, the coproduct of an element of the form $t\#e_n$ is

$$\Delta(t\#e_n) = \sum_{n'=0}^{n} \left[t_1^{((n',n-n'))}\#e_{n'}\right] \otimes \left[t_2^{((n',n-n'))}\#e_{n-n'}\right]. \tag{6}$$

Let us now return to our study of the descent algebras. We introduce the following condition on a graded submodule $W = \bigoplus_{n\geq 0} W_n$ of $\mathsf{T}(V)$:

(A) There holds $W_n \subseteq W_{c_1} \otimes W_{c_2} \otimes \cdots \otimes W_{c_k}$ for any composition (possibly with parts equal to zero) $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ of a positive integer n.[2]

In other words, for any composition $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ of a positive integer n and any $t \in W_n$, we may assume that in the writing $t = \sum_{(t)} t_1^{(\mathbf{c})} \otimes t_2^{(\mathbf{c})} \otimes \cdots \otimes t_k^{(\mathbf{c})}$, all the elements of $V^{\otimes c_i}$ represented by the place-holder $t_i^{(\mathbf{c})}$ can be picked in W_{c_i}. We can now find a sufficient condition for $\Sigma(W)$ to be a subbialgebra of $\mathscr{F}(V)$.

Proposition 4 *If W satisfies Condition (A), then $\Sigma(W)$ is a graded subbialgebra of $\mathscr{F}(V)$.*

Proof. We have already seen that $\Sigma(W)$ is a graded subalgebra of $\mathscr{F}(V)$. It remains to prove the inclusion

$$\Sigma(W) \subseteq \{x \in \mathscr{F}(V) \mid \Delta(x) \in \Sigma(W) \otimes \Sigma(W)\}.$$

The set E on the right of the symbol $\subseteq$ above is a subalgebra of $\mathscr{F}(V)$, because Δ is a morphism of algebras and $\Sigma(W) \otimes \Sigma(W)$ is a subalgebra. Moreover, Equation (6) shows that if W satisfies Condition (B), then E contains all the elements $t\#e_n$ with $t \in W_n$. Since these elements generate $\Sigma(W)$ as an algebra, it follows that E contains $\Sigma(W)$. □

Besides the trivial choice $W = \mathsf{T}(V)$, there are two main examples. The first one occurs with $W = \mathsf{TS}(V)$, the space of all symmetric tensors on V.[3] We call the corresponding subbialgebra $\Sigma(W)$ the Novelli-Thibon bialgebra and we denote it by $\mathsf{NT}(V)$. One may notice that the assignment $V \rightsquigarrow \mathsf{NT}(V)$ is functorial.

The second interesting example concerns the case where V is the underlying space of a coalgebra. We first fix two rather standard notations that are convenient for dealing with coalgebras; we will use them not only in the presentation below, but also later in Section 3.3 with the comultiplicative structure of $\mathscr{F}(V)$. Let C be a coalgebra with its coassociative coproduct δ and its counit ε. We define the iterated coproducts $\delta_n : C \to C^{\otimes n}$ by setting $\delta_0 = \varepsilon$, $\delta_1 = \mathrm{id}_C$, $\delta_2 = \delta$, and

$$\delta_n = \left(\delta \otimes (\mathrm{id}_C)^{\otimes n-2}\right) \circ \left(\delta \otimes (\mathrm{id}_C)^{\otimes n-3}\right) \circ \cdots \circ \delta$$

for all $n \geq 3$. The Sweedler notation proposes to write the image of an element $v \in C$ by δ_n as

$$\delta_n(v) = \sum_{(v)} v_{(1)} \otimes v_{(2)} \otimes \cdots \otimes v_{(n)};$$

[2]We abusively confuse $W_{c_1} \otimes W_{c_2} \otimes \cdots \otimes W_{c_k}$ with its image in $V^{\otimes c_1} \otimes V^{\otimes c_2} \otimes \cdots \otimes V^{\otimes c_k} = V^{\otimes n}$. Of course no ambiguity arises when $\mathbb{K}$ is a field or V is torsion-free module over a p.i.d.

[3]Condition (A) holds for $W = \mathsf{TS}(V)$ as soon as V is projective or $\mathbb{K}$ is a field or a Dedekind ring. We do not know if these restrictions can be lifted.

in this writing, the symbol $v_{(i)}$ is a place-holder for an actual element of C which varies from one term to the other.

Now we assume that the module V on which the free quasisymmetric algebra $\mathscr{F}(V)$ is constructed is endowed with a structure of a coalgebra, with a coproduct δ and a counit ε. In this case, we may consider the image W_n of the iterated coproduct $\delta_n : V \to V^{\otimes n}$ and we may set $W = \bigoplus_{n\geq 0} W_n$. For any composition (possibly with parts equal to zero) $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ of n and any element $v \in V$, the coassociativity of δ implies

$$\delta_n(v) = \sum_{(v)} \underbrace{\delta_{c_1}(v_{(1)})}_{(\delta_n(v))_1^{(\mathbf{c})}} \otimes \underbrace{\delta_{c_2}(v_{(2)})}_{(\delta_n(v))_2^{(\mathbf{c})}} \otimes \cdots \otimes \underbrace{\delta_{c_k}(v_{(k)})}_{(\delta_n(v))_k^{(\mathbf{c})}}, \tag{7}$$

which shows that Condition (B) holds. Therefore $\Sigma(W)$ is a subbialgebra of $\mathscr{F}(V)$. We call it the Mantaci-Reutenauer bialgebra of the coalgebra V and we denote it by $\mathsf{MR}(V)$. The assignment $V \rightsquigarrow \mathsf{MR}(V)$ is a covariant functor from the category of $\mathbb{K}$-coalgebras to the category of $\mathbb{N}$-graded bialgebras over $\mathbb{K}$. As we will see in Section 3.3, this construction is mainly useful when V is a projective $\mathbb{K}$-module and the coproduct of V is cocommutative; in this case, $\mathsf{MR}(V)$ is a subbialgebra of $\mathsf{NT}(V)$.

For convenience, we introduce the following special notation for the generators of the Mantaci-Reutenauer bialgebra $\mathsf{MR}(V)$: given any positive integer n and any element $v \in V$, we set $y_{n,v} = \big[\delta_n(v)\#e_n\big]$. Equations (6) and (7) entail that the coproduct of $y_{n,v}$ is given by

$$\Delta(y_{n,v}) = \sum_{(v)} \sum_{n'=0}^{n} y_{n',v_{(1)}} \otimes y_{n-n',v_{(2)}}. \tag{8}$$

Moreover, Proposition 3 implies that if V is a free $\mathbb{K}$-module, then the associative algebra $\mathsf{MR}(V)$ is freely generated by the elements $y_{n,v}$, where $n \geq 1$ and v is chosen in a basis of V.

2 Duality

The main result of this section says that the dual bialgebra $\mathscr{F}(V)^\vee$ of the free quasisymmetric bialgebra on V is the free quasisymmetric bialgebra $\mathscr{F}(V^\vee)$ on the dual module $V^\vee$. This result is neither deep nor difficult, but has many interesting consequences, as we will see in Sections 4 and 5. We begin by a general and easy discussion of duality for $\mathbb{K}$-modules and $\mathbb{K}$-bialgebras.

2.1 Perfect pairings

We define the duality functor $?^\vee$ as the contravariant endofunctor $\mathrm{Hom}_{\mathbb{K}}(?, \mathbb{K})$ of the category of $\mathbb{K}$-modules. In particular, this functor maps a morphism $f : M \to N$ to its transpose $f^\vee : N^\vee \to M^\vee$. Restricted to the full subcategory consisting of finitely generated projective $\mathbb{K}$-modules, the duality functor is an anti-equivalence of categories.

Given two $\mathbb{K}$-modules M and N, there is a canonical isomorphism $(M \oplus N)^\vee \cong M^\vee \oplus N^\vee$ and a canonical map $N^\vee \otimes M^\vee \to (M \otimes N)^\vee$; the latter is an isomorphism as soon as M or N is finitely generated and projective. Given a $\mathbb{K}$-module M, there is a canonical homomorphism $M \to M^{\vee\vee}$, which is an isomorphism if M is finitely generated and projective.

Let H be a $\mathbb{K}$-bialgebra whose underlying space is finitely generated and projective. Then the dual $H^\vee$ of H is also a bialgebra: the multiplication, the coproduct, the unit and the counit of $H^\vee$ are the transpose of the coproduct, the multiplication, the counit and the unit of H, respectively.

A pairing between two $\mathbb{K}$-modules M and N is a bilinear form $\varpi : M \times N \to \mathbb{K}$. It gives rive to two linear maps $\varpi^\flat : \big(M \to N^\vee,\ x \mapsto \varpi(x, ?)\big)$ and $\varpi^\# : \big(N \to M^\vee,\ y \mapsto \varpi(?, y)\big)$. The pairing ϖ is called perfect if the maps $\varpi^\flat$ and $\varpi^\#$ are isomorphisms. A pairing on a $\mathbb{K}$-module M is a pairing between M and itself; such a pairing ϖ is called symmetric if $\varpi^\flat = \varpi^\#$.

In the case where the $\mathbb{K}$-modules M and N are finitely generated and projective, we may identify M and N with their respective biduals, and for any pairing ϖ between M and N, it holds $\varpi^\# = (\varpi^\flat)^\vee$. If moreover M and N are bialgebras, then $M^\vee$ and $N^\vee$ are also bialgebras; in this situation, a pairing ϖ between M and N such that $\varpi^\flat$ and $\varpi^\#$ are morphisms of bialgebras is called a pairing of bialgebras.

The above constructions concerning biduality or bialgebras are only valid with finitely generated projective modules. We can however relax the requirement of finite generation by working with $\mathbb{N}$-graded modules. In this situation, we must adapt the definition for the dual module: the dual of $M = \bigoplus_{n\geq 0} M_n$ is the graded module $M^\vee = \bigoplus_{n\geq 0}(M_n)^\vee$, whose graded components are the dual modules in the previous sense of the graded components of M. We must also make the further assumptions that the morphisms preserve the graduation and that pairings make graded components of different degrees orthogonal to each other. Then everything works as before, and biduality and duality of bialgebras go smoothly as soon as the modules are projective with finitely generated homogeneous components.

2.2 Duality and the functor $\mathscr{F}$

The following proposition examines the relationship between the functor $\mathscr{F}$ and duality.

Proposition 5 *There is a natural transformation from the contravariant functor $\mathscr{F}(?^\vee)$ to the contravariant functor $\mathscr{F}(?)^\vee$, which is an isomorphism when the domain of these functors is restricted to the full subcategory of finitely generated projective $\mathbb{K}$-modules.*

In other words, for any $\mathbb{K}$-module V, we can define a morphism of graded algebras $c_V : \mathscr{F}(V^\vee) \xrightarrow{\simeq} \mathscr{F}(V)^\vee$, the construction being such that the assignment $V \rightsquigarrow c_V$ is natural in V, and that c_V is an isomorphism of bialgebras if V is finitely generated and projective.

Proof. Let V be a $\mathbb{K}$-module. With the help of the canonical duality bracket $\langle ?, ?\rangle : V \times V^\vee \to \mathbb{K}$ between V and $V^\vee$, we define for each $n \geq 0$ a pairing $\langle ?, ?\rangle_n$ between $\mathscr{F}_n(V)$ and $\mathscr{F}_n(V^\vee)$ by the following formula:

$$\Big\langle \big[(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\#\sigma\big], \big[(f_1 \otimes f_2 \otimes \cdots \otimes f_n)\#\pi\big] \Big\rangle_n = \begin{cases} \prod_{i=1}^n \langle v_{\sigma(i)}, f_i\rangle, & \text{if } \sigma = \pi^{-1}, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where $(v_1, v_2, \ldots, v_n) \in V^n$, $(f_1, f_2, \ldots, f_n) \in (V^\vee)^n$, and σ and π are elements of $\mathfrak{S}_n$. If V is assumed to be finitely generated and projective, the canonical duality between V and $V^\vee$ is perfect and extends to a perfect pairing between $V^{\otimes n}$ and $(V^\vee)^{\otimes n}$, which implies that the pairing $\langle ?, ?\rangle_n$ is perfect.

We combine these pieces to define a pairing $\langle ?, ?\rangle_{\text{tot}}$ between $\mathscr{F}(V)$ and $\mathscr{F}(V^\vee)$ by setting

$$\langle \alpha, \xi\rangle_{\text{tot}} = \sum_{n\geq 0} \langle \alpha_n, \xi_n\rangle_n$$

for all $\alpha = \sum_{n\geq 0} \alpha_n$ and $\xi = \sum_{n\geq 0} \xi_n$, where $\alpha_n \in \mathscr{F}_n(V)$ and $\xi_n \in \mathscr{F}_n(V^\vee)$. The map

$$c_V : \mathscr{F}(V^\vee) \to \mathscr{F}(V)^\vee, \ x \mapsto \langle ?, x\rangle_{\mathrm{tot}}$$

is a morphism of $\mathbb{K}$-modules; it is even an isomorphism if V is finitely generated and projective.

A straightforward verification shows that the product $*$ and the coproduct Δ of $\mathscr{F}(V)$ are adjoint to the coproduct Δ and to the product $*$ of $\mathscr{F}(V^\vee)$ with respect to the pairing $\langle ?, ?\rangle_{\mathrm{tot}}$. Together with a similar statement about the unit and the counits, this implies that c_V is a morphism of algebras, and even of bialgebras if $\mathscr{F}(V)$ is projective with finitely generated homogeneous components. One checks also easily the commutativity of the diagram

$$\begin{tikzcd} \mathscr{F}(V^\vee) \arrow[d, "c_V"'] & \mathscr{F}(W^\vee) \arrow[l, "\mathscr{F}(f^\vee)"'] \arrow[d, "c_W"] \\ \mathscr{F}(V)^\vee & \mathscr{F}(W)^\vee \arrow[l, "\mathscr{F}(f)^\vee"'] \end{tikzcd}$$

for any $\mathbb{K}$-linear map $f : V \to W$ of $\mathbb{K}$-modules. This means that the assignment $V \rightsquigarrow c_V$ is a natural transformation from $\mathscr{F}(?^\vee)$ to $\mathscr{F}(?)^\vee$, which completes the proof. □

Using the precise definition of the maps c_V given in the proof of Proposition 5, one may check the following additional property: the two compositions

$$\mathscr{F}(V) \longrightarrow \mathscr{F}(V^{\vee\vee}) \xrightarrow{c_{(V^\vee)}} \mathscr{F}(V^\vee)^\vee \quad \text{and} \quad \mathscr{F}(V) \longrightarrow \mathscr{F}(V)^{\vee\vee} \xrightarrow{(c_V)^\vee} \mathscr{F}(V^\vee)^\vee$$

are equal. Abusing the notations, we will write the above equality as $c_{(V^\vee)} = (c_V)^\vee$.

Now suppose that ϖ is a pairing between two $\mathbb{K}$-modules V and W. We can then define a pairing ϖ_{tot} between $\mathscr{F}(V)$ and $\mathscr{F}(W)$ by the equality ${\varpi_{\mathrm{tot}}}^\flat = c_W \circ \mathscr{F}(\varpi^\flat)$; in other words, we set

$$\varpi_{\mathrm{tot}}(x, y) = \big(c_W \circ \mathscr{F}(\varpi^\flat)\big)(x)(y),$$

where $x \in \mathscr{F}(V)$ and $y \in \mathscr{F}(W)$. Then

$${\varpi_{\mathrm{tot}}}^\# = ({\varpi_{\mathrm{tot}}}^\flat)^\vee = \mathscr{F}(\varpi^\flat)^\vee \circ (c_W)^\vee = \mathscr{F}(\varpi^\flat)^\vee \circ c_{(W^\vee)} = c_V \circ \mathscr{F}\big((\varpi^\flat)^\vee\big) = c_V \circ \mathscr{F}(\varpi^\#).$$

The equalities ${\varpi_{\mathrm{tot}}}^\flat = c_W \circ \mathscr{F}(\varpi^\flat)$ and ${\varpi_{\mathrm{tot}}}^\# = c_V \circ \mathscr{F}(\varpi^\#)$ show that ϖ_{tot} is a pairing of bialgebras. Moreover if ϖ is perfect, then so is ϖ_{tot}. In the case $V = W$, one can also see that the symmetry of ϖ entails that of ϖ_{tot}.

2.3 Orthogonals and polars

Let M be a finitely generated projective $\mathbb{K}$-module. We view it as an 'ambient' space and identify it with its bidual $M^{\vee\vee}$. We define the orthogonal of a submodule S of M as the submodule $S^\perp = \{f \in M^\vee \mid f\big|_S = 0\}$ of $M^\vee$. Then $S^\perp$ is canonically isomorphic to $(M/S)^\vee$. Likewise, the orthogonal of a submodule T of $M^\vee$ is a submodule $T^\perp$ of M.

Let $\mathscr{S}$ be the set of all submodules S of M such that M/S is projective, or in other words, that are direct summands of M. If $S \in \mathscr{S}$, then both S and M/S are finitely generated projective $\mathbb{K}$-modules. Likewise, let $\mathscr{T}$ be the set of all submodules T of $M^\vee$ that are direct summands of $M^\vee$. We endow both $\mathscr{S}$ and $\mathscr{T}$ with the partial order given by the inclusion of submodules. The following results are well-known in this context:

• The maps $\bigl(\mathscr{S} \to \mathscr{T},\ S \mapsto S^\perp\bigr)$ and $\bigl(\mathscr{T} \to \mathscr{S},\ T \mapsto T^\perp\bigr)$ are mutually inverse, order decreasing bijections.

• For any $S \in \mathscr{S}$, there is a canonical isomorphism $S^\vee \cong M^\vee/S^\perp$. Moreover for each submodule $S' \subseteq S$, there is a canonical isomorphism $(S/S')^\vee \cong S'^\perp/S^\perp$.

• Let S and S' be two elements in $\mathscr{S}$. We always have $(S+S')^\perp = S^\perp \cap S'^\perp$ and $S^\perp + S'^\perp \subseteq (S \cap S')^\perp$. If moreover $S + S'$ belongs to $\mathscr{S}$, then so does $S \cap S'$, and the equality $(S \cap S')^\perp = S^\perp + S'^\perp$ holds.

• Assume that M is endowed with the structure of a bialgebra. Then a submodule $S \in \mathscr{S}$ is a subbialgebra of M if and only if $S^\perp$ is a biideal of $M^\vee$, and a submodule $T \in \mathscr{T}$ is a subbialgebra of $M^\vee$ if and only if $T^\perp$ is a biideal of M.

Given two submodules $S \in \mathscr{S}$ and $T \in \mathscr{T}$, we have then sequences of canonical maps

$$\begin{aligned} T/(S^\perp \cap T) &\cong (S^\perp + T)/S^\perp = (S^\perp + T^{\perp\perp})/S^\perp \hookrightarrow (S \cap T^\perp)^\perp/S^\perp \cong \bigl(S/(S\cap T^\perp)\bigr)^\vee, \\ S/(S \cap T^\perp) &\cong (S + T^\perp)/T^\perp = (S^{\perp\perp} + T^\perp)/T^\perp \hookrightarrow (S^\perp \cap T)^\perp/T^\perp \cong \bigl(T/(S^\perp\cap T)\bigr)^\vee. \end{aligned} \tag{10}$$

In other words, there is a canonical pairing between $S/(S \cap T^\perp)$ and $T/(S^\perp \cap T)$, which is perfect as soon as $(S + T^\perp) \in \mathscr{S}$ and $(S^\perp + T) \in \mathscr{T}$.

We assume now that the module M is endowed with a symmetric and perfect pairing ϖ. Then to any submodule S of M we can associate its polar $P^\circ = \bigl(\varpi^\flat\bigr)^{-1}\bigl(S^\perp\bigr)$ with respect to ϖ. Using $\varpi^\flat$, one can deduce properties for polar submodules analogous to the properties for orthogonals recalled above.

One can also adapt these results to the case where the projective module M is not finitely generated, provided it is graded with finitely generated homogeneous components.

This material will prove useful in Sections 4.3 and 5, where we will meet instances of the following situation. Here V is a finitely generated projective $\mathbb{K}$-module, endowed with a symmetric and perfect pairing ϖ. Then $\mathscr{F}(V)$ is a projective $\mathbb{K}$-module, graded with finitely generated homogeneous components, and endowed with the perfect and symmetric pairing ϖ_{tot}. Let moreover S be a graded subbialgebra of $\mathscr{F}(V)$, assumed to be a direct summand of the graded $\mathbb{K}$-module $\mathscr{F}(V)$. We have then the following commutative diagram of graded bialgebras,

$$\begin{tikzcd} & \mathscr{F}(V) \arrow[rr, "\simeq"] \arrow[dr, two heads] & & \mathscr{F}(V)^\vee \arrow[dr, two heads] & \\ S \arrow[ur, hook] \arrow[dr, two heads] & & \mathscr{F}(V)/S^\circ \arrow[rr, "\simeq"] & & S^\vee. \\ & S/(S\cap S^\circ) \arrow[ur, hook] \arrow[rr, hook] & & \bigl(S/(S\cap S^\circ)\bigr)^\vee \arrow[ur, hook] & \end{tikzcd} \tag{11}$$

Here the horizontal arrows are induced by ${\varpi_{\mathrm{tot}}}^\flat$; the one at the bottom line is the pairing on $S/(S \cap S^\circ)$ defined by the sequences (10) with the choice $T = {\varpi_{\mathrm{tot}}}^\flat(S)$.

To conclude this section, we show that the framework above is general enough to accomodate the case of a Mantaci-Reutenauer bialgebra, viewed as a submodule in a free quasisymmetric bialgebra.

Proposition 6 *For any $\mathbb{K}$-coalgebra V, the submodule $\mathsf{MR}(V)$ is a direct summand of $\mathscr{F}(V)$.*

Proof. We will show two facts:

a) The submodule $\Sigma(\mathsf{T}(V))$ has a graded complement in $\mathscr{F}(V)$.

b) The Mantaci-Reutenauer bialgebra $\mathsf{MR}(V)$ has a graded complement in $\Sigma(\mathsf{T}(V))$.

Let n be a positive integer. The map

$$\mathbb{K}\mathfrak{S}_n \otimes V^{\otimes n} \to \mathscr{F}_n(V),\ \sigma \otimes t \mapsto \sigma \cdot (t\# e_n),$$

where $\sigma \in \mathfrak{S}_n$ and $t \in V^{\otimes n}$, is an isomorphism of $\mathbb{K}$-modules. The submodule $\mathscr{F}_n(V) \cap \Sigma(\mathsf{T}(V))$ is spanned by elements of the form

$$x_{\mathbf{c}} \cdot (t\# e_n) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ D(\sigma) \preccurlyeq \mathbf{c}}} \sigma \cdot (t\# e_n),$$

where $\mathbf{c}$ is a composition of n. Therefore the submodule of $\mathscr{F}_n(V)$ spanned over $\mathbb{K}$ by the elements $(\sigma - \pi) \cdot (t\# e_n)$, where $t \in V^{\otimes n}$ and σ and π are two permutations in $\mathfrak{S}_n$ with $D(\sigma) = D(\pi)$, is complementary to $\mathscr{F}_n(V) \cap \Sigma(\mathsf{T}(V))$. This proves Claim a).

Let us now denote the coproduct and the counit of V by δ and ε, respectively. Let n be a positive integer. We denote the image of the iterated coproduct $\delta_n : V \to V^{\otimes n}$ by W_n. The short exact sequence

$$0 \to V \xrightarrow{\delta_n} V^{\otimes n} \to V^{\otimes n}/W_n \to 0$$

splits, because the map $V^{\otimes n} \xrightarrow{\varepsilon^{\otimes n-1} \otimes \mathrm{id}_V} \mathbb{K}^{\otimes n-1} \otimes V \cong V$ is a retraction of δ_n. Therefore we can find a complementary submodule Z_n of W_n in $V^{\otimes n}$. Given a composition $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ of n, we set

$$W_{\mathbf{c}} = W_{c_1} \otimes W_{c_2} \otimes \cdots \otimes W_{c_k} \quad \text{and} \quad Z_{\mathbf{c}} = \sum_{i=1}^{k} V_{c_1} \otimes \cdots V_{c_{i-1}} \otimes Z_{c_i} \otimes V_{c_{i+1}} \otimes \cdots \otimes V_{c_k},$$

so that $V^{\otimes n} = W_{\mathbf{c}} \oplus Z_{\mathbf{c}}$. Then

$$\mathscr{F}_n(V) \cap \Sigma(\mathsf{T}(V)) = \bigoplus_{\mathbf{c} \models n} \big[x_{\mathbf{c}} \cdot (V^{\otimes n}\# e_n)\big] = \underbrace{\bigoplus_{\mathbf{c} \models n} \big[x_{\mathbf{c}} \cdot (W_{\mathbf{c}}\# e_n)\big]}_{\mathscr{F}_n(V) \cap \mathsf{MR}(V)} \oplus \bigoplus_{\mathbf{c} \models n} \big[x_{\mathbf{c}} \cdot (Z_{\mathbf{c}}\# e_n)\big],$$

which shows Claim b) and completes the proof. □

3 The internal product

In this section, we consider the case where V is the underlying space of an algebra A. This affords a new structure on $\mathscr{F}(A)$, called the internal product. We study ways to construct subalgebras of $\mathscr{F}(A)$ for the internal product and clarify the situation that arises when A is a symmetric algebra, that is, an algebra endowed with a symmetric, associative and perfect pairing.

3.1 The twisted group ring $\mathscr{F}_n(A)$

So let A be a $\mathbb{K}$-algebra. Then the group $\mathfrak{S}_n$ acts on the tensor power $A^{\otimes n}$ by automorphisms of algebra, which allows to construct a twisted group ring, which we denote by $(A^{\otimes n})\#(\mathbb{K}\mathfrak{S}_n)$. (In the language of Hopf algebras, one says that $A^{\otimes n}$ is a $\mathbb{K}\mathfrak{S}_n$-module algebra, and then the twisted group ring $(A^{\otimes n})\#(\mathbb{K}\mathfrak{S}_n)$ is viewed as a particular case of the smash product construction; see for instance [25].) This twisted group ring is our $\mathbb{K}\mathfrak{S}_n$-bimodule $\mathscr{F}_n(A)$ endowed additionally with the structure of an algebra. The associative product is given by the rule

$$\big[(a_1 \otimes a_2 \otimes \cdots \otimes a_n)\#\sigma\big] \cdot \big[(b_1 \otimes b_2 \otimes \cdots \otimes b_n)\#\tau\big] \\ = \big[\big(a_1 b_{\sigma^{-1}(1)} \otimes a_2 b_{\sigma^{-1}(2)} \otimes \cdots \otimes a_n b_{\sigma^{-1}(n)}\big)\#\big(\sigma\tau\big)\big] \quad (12)$$

and the unit is $1^{\otimes n}\#e_n$. The structure map $\big(\mathbb{K} \to A^{\otimes n},\ \lambda \mapsto \lambda 1^{\otimes n}\big)$ gives rise to an embedding of the group algebra $\mathbb{K}\mathfrak{S}_n = \mathscr{F}_n(\mathbb{K})$ into $\mathscr{F}_n(A)$, which allows to represent the two-sided action of $\mathbb{K}\mathfrak{S}_n$ on $\mathscr{F}_n(A)$ with the help of the product of $\mathscr{F}_n(A)$.

It is convenient to extend this product to the whole $\mathscr{F}(A)$ by linearity: if $\alpha = \sum_{n\geq 0} \alpha_n$ and $\alpha' = \sum_{n\geq 0} \alpha'_n$ with α_n and α'_n in $\mathscr{F}_n(A)$, we define $\alpha \cdot \alpha' = \sum_{n\geq 0} \alpha_n \cdot \alpha'_n$. This 'internal product' as it is called lacks a unit element.

More generally, given two $\mathbb{K}$-modules V and W, the composition

$$(V^{\otimes n}\#\mathbb{K}\mathfrak{S}_n) \otimes (W^{\otimes n}\#\mathbb{K}\mathfrak{S}_n) \twoheadrightarrow (V^{\otimes n}\#\mathbb{K}\mathfrak{S}_n) \otimes_{\mathbb{K}\mathfrak{S}_n} (W^{\otimes n}\#\mathbb{K}\mathfrak{S}_n) \xrightarrow{\simeq} (V \otimes W)^{\otimes n}\#\mathbb{K}\mathfrak{S}_n$$

defines a canonical morphism of $\mathbb{K}\mathfrak{S}_n$-modules from $\mathscr{F}_n(V)\otimes\mathscr{F}_n(W)$ into $\mathscr{F}_n(V\otimes W)$. Taking the direct sum over all $n \geq 0$, one can define an 'internal product' $\mathscr{F}(V)\otimes\mathscr{F}(W) \to \mathscr{F}(V\otimes W)$ which is natural in (V, W). Given a third $\mathbb{K}$-module X and a linear map $m : V \otimes W \to X$, we obtain an internal product $\mathscr{F}(V) \otimes \mathscr{F}(W) \to \mathscr{F}(X)$ by composition with $\mathscr{F}(m)$. We will not pursue this way for want of application, but it is worth noticing that even the apparently simple case where V or W is the ground ring $\mathbb{K}$ is not empty. We leave it to the reader to generalize the results of Section 3.3 to this wider context.

To conclude this section, we introduce two pieces of terminology that will prove convenient in Section 3.4. Let S and M be two graded submodules of $\mathscr{F}(A)$, and set $S_n = S \cap \mathscr{F}_n(A)$ and $M_n = M \cap \mathscr{F}_n(A)$. We say that S is a subalgebra of $\mathscr{F}(A)$ for the internal product if each S_n is a subalgebra of $\mathscr{F}_n(A)$. In this case, we say further that M is a left (respectively, right) internal S-submodule of $\mathscr{F}(A)$ if $S \cdot M \subseteq M$ (respectively, $M \cdot S \subseteq M$).

3.2 Double cosets in the symmetric group

In this section, we translate to the case of the symmetric group a theorem of Solomon valid in the more general context of Coxeter groups. The result will prove crucial in the proof of the splitting formula in Section 3.3.

To begin with, let $\big(W, (s_i)_{i\in I}\big)$ be a Coxeter system. Given a subset $J \subseteq I$, the parabolic subgroup W_J is the subgroup of W generated by the elements s_j with $j \in J$. In each left coset wW_J, there is a unique element with minimal length, called the distinguished representative of that coset. We denote by X_J the set of distinguished representatives of the left cosets modulo W_J. Given a second subset $K \subseteq I$, there is likewise a unique element with minimal length in each double coset $W_J w W_K$, unsurprisingly called the distinguished representative of the double coset. The set of distinguished representatives of the double cosets modulo W_J and W_K is $(X_J)^{-1} \cap X_K$. The following statement is a rephrasing of Theorem 2 of [33].

Theorem 7 *Given a double coset $C \in W_J \backslash W / W_K$, we set*

$$L_C = \{k \in K \mid \exists j \in J,\ x^{-1} s_j x = s_k\},$$

where $x \in C \cap (X_J^{-1} \cap X_K)$ is the distinguished representative of C. Then X_{L_C} is the disjoint union of the sets $X_J w$, where $w \in C \cap X_K$.

We now translate this proposition in a combinatorial language more adapted to the case of the symmetric group $\mathfrak{S}_n$. Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_l)$ be two compositions of n, and set $t_i = c_1 + c_2 + \cdots + c_i$ and $u_j = d_1 + d_2 + \cdots + d_j$. We denote by $\mathscr{M}_{\mathbf{c},\mathbf{d}}$ the set of all matrices $M = (m_{ij})$ with non-negative integral entries in k rows and l columns and with row-sum $\mathbf{c}$ and column-sum $\mathbf{d}$, that is,

$$c_i = \sum_{j=1}^{l} m_{ij} \quad \text{for all } i \text{ and } \quad d_j = \sum_{i=1}^{k} m_{ij} \quad \text{for all } j.$$

There is a well-known bijection from $\mathscr{M}_{\mathbf{c},\mathbf{d}}$ onto the double quotient $\mathfrak{S}_{\mathbf{c}} \backslash \mathfrak{S}_n / \mathfrak{S}_{\mathbf{d}}$ that maps a matrix $M = (m_{ij})$ to the double coset

$$C(M) = \bigl\{\sigma \in \mathfrak{S}_n \bigm| \forall (i,j),\ m_{ij} = \bigl| [t_{i-1}+1, t_i] \cap \sigma([u_{j-1}+1, u_j]) \bigr| \bigr\}.$$

Finally, we associate to a matrix $M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}$ its column-reading composition

$$\mathbf{cr}(M) = (m_{11}, m_{21}, \ldots, m_{k1}, m_{12}, m_{22}, \ldots, m_{k2}, \ldots, m_{1l}, m_{2l}, \ldots, m_{kl}).$$

With these notations, Theorem 7 translates to the following statement.

Corollary 8 *For any matrix $M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}$, the set $X_{\mathbf{cr}(M)}$ is the disjoint union of the sets $X_{\mathbf{c}} \sigma$, where $\sigma \in C(M) \cap X_{\mathbf{d}}$.*

Proof. We set $I = \{1, 2, \ldots, n-1\}$. For each $i \in I$, we call s_i be the transposition in $\mathfrak{S}_n$ that exchanges i and $i+1$. Endowed with the family $(s_i)_{i \in I}$, the group $\mathfrak{S}_n$ becomes a Coxeter system W.

Set $J = I \setminus \{t_1, t_2, \cdots, t_{k-1}\}$ and $K = I \setminus \{u_1, u_2, \ldots, u_{l-1}\}$. Then the Young subgroups $\mathfrak{S}_{\mathbf{c}}$ and $\mathfrak{S}_{\mathbf{d}}$ coincide with the parabolic subgroups W_J and W_K, respectively; moreover the sets $X_{\mathbf{c}}$ and $X_{\mathbf{d}}$ are the sets of distinguished representatives X_J and X_K.

We now fix a matrix $M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}$. We define a permutation $\rho \in \mathfrak{S}_n$ by the following rule: for each $a \in [1, n]$, we determine the index $j \in [1, l]$ such that $a \in [u_{j-1}+1, u_j]$ and then the index $i \in [1, k]$ such that $a - u_{j-1} \in [m_{1j} + m_{2j} + \cdots + m_{i-1,j} + 1, m_{1j} + m_{2j} + \cdots + m_{ij}]$, and we set

$$\rho(a) = a - (u_{j-1} + m_{1j} + m_{2j} + \cdots + m_{i-1,j}) + (t_{i-1} + m_{i1} + m_{i2} + \cdots + m_{i,j-1}).$$

One checks without difficulty that $\rho \in C(M) \cap (X_{\mathbf{c}})^{-1} \cap X_{\mathbf{d}}$, which implies that ρ is the distinguished representative of the double coset $C(M) \in \mathfrak{S}_{\mathbf{c}} \backslash \mathfrak{S}_n / \mathfrak{S}_{\mathbf{d}}$.

Moreover, let $a \in [1, n]$ and determine the indices i and j as above. One checks easily that

$$a - u_{j-1} \in [m_{1j} + m_{2j} + \cdots + m_{i-1,j} + 1, m_{1j} + m_{2j} + \cdots + m_{ij} - 1] \iff \begin{cases} a \in K, \\ \rho(a) \in J, \\ \rho(a+1) = \rho(a) + 1. \end{cases}$$

(The ket point here is to observe that if $a = u_{j-1} + m_{1j} + m_{2j} + \cdots + m_{ij}$ and $a \in K$, then the inequalities $\rho(a+1) > t_i \geq \rho(a)$ hold.)

For any $j \in J$, the permutation $\rho^{-1} s_j \rho$ is the transposition that exchanges $\rho^{-1}(j)$ and $\rho^{-1}(j+1)$, with necessarily $\rho^{-1}(j) < \rho^{-1}(j+1)$ because $\rho^{-1} \in X_J$. The definition

$$L_{C(M)} = \{k \in K \mid \exists j \in J,\ \rho^{-1} s_j \rho = s_k\}$$

translates therefore to the equality

$$L_{C(M)} = \bigcup_{j=1}^{l} \bigcup_{i=1}^{k} [u_{j-1} + m_{1j} + m_{2j} + \cdots + m_{i-1,j} + 1, u_{j-1} + m_{1j} + m_{2j} + \cdots + m_{ij} - 1],$$

or, in other words, to

$$L_{C(M)} = I \setminus \{f_1, f_1 + f_2, \ldots, f_1 + f_2 + \cdots + f_{m-1}\}$$

if the parts of $\mathbf{cr}(M)$ form the sequence $(f_1, f_2, \cdots, f_m)$. This implies that the sets $X_{L(C)}$ and $X_{\mathbf{cr}(M)}$ coincide.

This completes the dictionary that allows to deduce the corollary from Theorem 7. □

3.3 The splitting formula

The splitting formula, due to Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [13] in the case of $\mathscr{F}(\mathbb{K})$ and to Novelli and Thibon [27] in the general case, is the tool that enables to show that certain graded subbialgebras $\Sigma(W)$ of Section 1.3 are subalgebras of $\mathscr{F}(A)$ for the internal product. We begin with a lemma.

Lemma 9 *Let V be a projective $\mathbb{K}$-module,*[4] *let n be a positive integer, let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ and $\mathbf{d}$ be two compositions of n, and for each $i \in [1, k]$, let $a_i \in \mathsf{TS}^{c_i}(V)$ be a symmetric tensor of degree c_i. The i-th line of a matrix $M = (m_{ij})$ in $\mathscr{M}_{\mathbf{c},\mathbf{d}}$ can be seen as a composition (possibly with parts equal to zero) of c_i. According to the decomposition*

$$\mathsf{TS}^{c_i}(V) \subseteq \mathsf{TS}^{m_{i1}}(V) \otimes \mathsf{TS}^{m_{i2}}(V) \otimes \cdots \otimes \mathsf{TS}^{m_{il}}(V),$$

we write a_i as a linear combination $\sum_{(a_i)} a_{i1}^{(M)} \otimes a_{i2}^{(M)} \otimes \cdots \otimes a_{il}^{(M)}$ with $a_{ij}^{(M)} \in \mathsf{TS}^{m_{ij}}(V)$. Then in the $\mathbb{K}\mathfrak{S}_n$-bimodule $\mathscr{F}_n(V)$, there holds

$$\begin{aligned} x_{\mathbf{c}} \cdot \big[(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \# x_{\mathbf{d}}\big] \\ = \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \sum_{(a_1), (a_2), \ldots, (a_k)} x_{\mathbf{cr}(M)} \cdot \Big[\Big(& a_{11}^{(M)} \otimes a_{21}^{(M)} \otimes \cdots \otimes a_{k1}^{(M)} \\ & \otimes a_{12}^{(M)} \otimes a_{22}^{(M)} \otimes \cdots \otimes a_{k2}^{(M)} \otimes \cdots \\ & \otimes a_{1l}^{(M)} \otimes a_{2l}^{(M)} \otimes \cdots \otimes a_{kl}^{(M)}\Big) \# e_n\Big]. \end{aligned} \quad (13)$$

[4] The assuption that V is projective guarantee the existence of decompositions $a_i = \sum_{(a_i)} a_{i1}^{(M)} \otimes a_{i2}^{(M)} \otimes \cdots \otimes a_{il}^{(M)}$ below, as mentioned in the footnote 3.

Proof. We set $t_i = c_1 + c_2 + \cdots + c_i$ and $u_j = d_1 + d_2 + \cdots + u_j$. We take $M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}$ and $\sigma \in \mathfrak{S}_n$. If σ belongs to the double coset $C(M)$, then for each j, the set $\sigma([u_{j-1}+1, u_j])$ has m_{1j} elements in $[1, t_1]$, m_{2j} elements in $[t_1+1, t_2]$, ..., m_{kj} elements in $[t_{k-1}+1, t_k]$. On the other hand, if σ belongs to $X_{\mathbf{d}}$, then it is an increasing map on the interval $[u_{j-1}+1, u_j]$. Therefore, if σ belongs to $C(M) \cap X_{\mathbf{d}}$, the m_{ij} elements of the set

$$\sigma([u_{j-1} + m_{1j} + m_{2j} + \cdots + m_{i-1,j} + 1, u_{j-1} + m_{1j} + m_{2j} + \cdots + m_{ij}])$$

belong to $[t_{i-1}+1, t_i]$, so that

$$\begin{aligned}(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \# \sigma = \sum_{(a_1), (a_2), \ldots, (a_k)} \sigma \cdot \Big[\Big(a_{11}^{(M)} \otimes a_{21}^{(M)} \otimes \cdots \otimes a_{k1}^{(M)} \\ \otimes a_{12}^{(M)} \otimes a_{22}^{(M)} \otimes \cdots \otimes a_{k2}^{(M)} \otimes \cdots \\ \otimes a_{1l}^{(M)} \otimes a_{2l}^{(M)} \otimes \cdots \otimes a_{kl}^{(M)}\Big) \# e_n\Big],\end{aligned}$$

because each a_i is symmetric. Using the notations of Section 3.2, we decompose $X_{\mathbf{d}}$ as the disjoint union $\coprod_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \big(C(M) \cap X_{\mathbf{d}}\big)$. Then

$$\begin{aligned} x_{\mathbf{c}} \cdot \big[(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \# x_{\mathbf{d}}\big] \\ &= \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \sum_{\sigma \in C(M) \cap X_{\mathbf{d}}} x_{\mathbf{c}} \cdot \big[(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \# \sigma\big] \\ &= \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \sum_{\sigma \in C(M) \cap X_{\mathbf{d}}} \sum_{(a_i)} (x_{\mathbf{c}}\sigma) \cdot \Big[\Big(a_{11}^{(M)} \otimes a_{21}^{(M)} \otimes \cdots \otimes a_{kl}^{(M)}\Big) \# e_n\Big] \\ &= \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \sum_{(a_i)} x_{\mathbf{cr}(M)} \cdot \Big[\Big(a_{11}^{(M)} \otimes a_{21}^{(M)} \otimes \cdots \otimes a_{kl}^{(M)}\Big) \# e_n\Big],\end{aligned}$$

the last equality coming from Corollary 8. This calculation proves Lemma 9. □

In the remainder of this section, the letter A denotes a $\mathbb{K}$-algebra, whose underlying module is projective. We now state and prove the splitting formula.

Theorem 10 *Let y be an element in $\mathsf{NT}(A)$ and $z_1, z_2, \ldots, z_l$ be elements in $\mathscr{F}(A)$. Then*

$$y \cdot (z_1 * z_2 * \cdots * z_l) = \sum_{(y)} (y_{(1)} \cdot z_1) * (y_{(2)} \cdot z_2) * \cdots * (y_{(l)} \cdot z_l). \tag{14}$$

Proof. By linearity, it is sufficient to prove Formula (14) for elements y of the form

$$y = (a_1 \# e_{c_1}) * (a_2 \# e_{c_2}) * \cdots * (a_k \# e_{c_k}) = x_{\mathbf{c}} \cdot \big[(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \# e_n\big],$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ is a composition of a positive integer n and where $a_1, a_2, \ldots, a_k$ are symmetric tensors on A of degree $c_1, c_2, \ldots, c_k$, respectively. By Formula (6), the l-th iterated coproduct of the element $(a_i \# e_i)$ is

$$\Delta_l(a_i \# e_{c_i}) = \sum_{\mathbf{f}} \sum_{(a_i)} \big((a_i)_1^{(\mathbf{f})} \# e_{f_1}\big) \otimes \big((a_i)_2^{(\mathbf{f})} \# e_{f_2}\big) \otimes \cdots \otimes \big((a_i)_l^{(\mathbf{f})} \# e_{f_l}\big),$$

where the first sum runs over all compositions $\mathbf{f} = (f_1, f_2, \dots, f_l)$ of c_i in l parts (possibly equal to zero). Multiplying these expressions for $i = 1, 2, \dots, k$ and expanding, we obtain

$$\begin{aligned}\Delta_l(y) = \sum_{\mathbf{g}} \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{g}}} \sum_{(a_1),(a_2),\dots,(a_k)} & \Big[\Big(a_{11}^{(M)} \# e_{m_{11}}\Big) * \Big(a_{21}^{(M)} \# e_{m_{21}}\Big) * \cdots * \Big(a_{k1}^{(M)} \# e_{m_{k1}}\Big)\Big] \\ & \otimes \Big[\Big(a_{12}^{(M)} \# e_{m_{12}}\Big) * \Big(a_{22}^{(M)} \# e_{m_{22}}\Big) * \cdots * \Big(a_{k2}^{(M)} \# e_{m_{k2}}\Big)\Big] \\ & \otimes \cdots \\ & \otimes \Big[\Big(a_{1l}^{(M)} \# e_{m_{1l}}\Big) * \Big(a_{2l}^{(M)} \# e_{m_{2l}}\Big) * \cdots * \Big(a_{kl}^{(M)} \# e_{m_{kl}}\Big)\Big],\end{aligned} \tag{15}$$

where the first sum runs over all compositions (possibly with zero parts) $\mathbf{g}$ of n in l parts and where for each matrix $M \in \mathscr{M}_{\mathbf{c},\mathbf{g}}$, the tensors a_i are decomposed as in the statement of Lemma 9.

Now let $\mathbf{d} = (d_1, d_2, \dots, d_l)$ be a composition (possibly with parts equal to zero) of n and consider the equality proved in Lemma 9. The left-hand side of (13) is equal to

$$x_{\mathbf{c}} \cdot \big[(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \# e_n\big] \cdot (1^{\otimes n} \# x_{\mathbf{d}}) = y \cdot (1^{\otimes n} \# x_{\mathbf{d}}). \tag{16}$$

On the other hand, Equation (15) joint to Formula (1) shows that the right-hand side of (13) is equal to

$$\sum_{(y)} \big(y_{(1)} \cdot (1^{\otimes d_1} \# e_{d_1})\big) * \big(y_{(2)} \cdot (1^{\otimes d_2} \# e_{d_2})\big) * \cdots * \big(y_{(l)} \cdot (1^{\otimes d_l} \# e_{d_l})\big). \tag{17}$$

We conclude that the quantities (16) and (17) are equal.

By linearity, we may assume that the elements z_j are of the form $z_j = (b_j \# \sigma_j)$, where $b_j \in A^{\otimes d_j}$ and $\sigma_j \in \mathfrak{S}_{d_j}$. For degree reasons, both sides of (14) vanish unless $n = d_1 + d_2 + \cdots + d_l$. We may therefore assume without loss of generality that $\mathbf{d} = (d_1, d_2, \dots, d_l)$ is a composition of n. We now multiply both (16) and (17) on the right by $\big[(b_1 \otimes b_2 \otimes \cdots \otimes b_l) \# (\sigma_1 \times \sigma_2 \times \cdots \times \sigma_l)\big]$, using the internal product. These multiplications yield the left-hand and the right-hand side of (14), respectively. The theorem follows. □

As a first application of this formula, we consider the two following conditions for a graded submodule $B = \bigoplus_{n \geq 0} B_n$ of $\mathsf{T}(A)$.

(B) Each B_n is a subalgebra of $A^{\otimes n}$.

(C) Each space B_n consists of symmetric tensors, that is, $B \subseteq \mathsf{TS}(A)$.

Corollary 11 *For any graded submodule B of $\mathsf{T}(A)$ satisfying Conditions (A), (B) and (C), the descent bialgebra $\Sigma(B)$ is a subalgebra of $\mathscr{F}(A)$ for the internal product.*

Proof. We have to prove that for any elements y and z in $\Sigma(B)$, the product $y \cdot z$ belongs to $\Sigma(B)$. We first consider the case where z is of the form $(b \# e_n)$, where $b \in B_n$. The homogeneous components of y whose degree are different from n do not contribute to the product $y \cdot z$; they can therefore be put aside. We then write y as a linear combination of products $(a_1 \# e_{c_1}) * (a_2 \# e_{c_2}) * \cdots * (a_k \# e_{c_k})$, where $\mathbf{c} = (c_1, c_2, \dots, c_k)$ is a composition of

n and $a_1 \in B_{c_1}$, $a_2 \in B_{c_2}$, ..., $a_k \in B_{c_k}$. By Condition (A), we may find a decomposition $b = \sum_{(b)} b_1^{(\mathbf{c})} \otimes b_2^{(\mathbf{c})} \otimes \cdots \otimes b_k^{(\mathbf{c})}$ for each composition $\mathbf{c}$ that arises in the expression of y, where the elements represented by the place-holder $b_i^{(\mathbf{c})}$ belong to B_{c_i}. Therefore $y \cdot (b\#e_n)$ is a linear combination of elements of the form

$$\begin{aligned}\big(x_{\mathbf{c}} \cdot \big[(a_1 \otimes a_2 \otimes \cdots \otimes a_k)\#e_n\big]\big)\cdot&\big[(b_1 \otimes b_2 \otimes \cdots \otimes b_k)\#e_n\big] \\ &= x_{\mathbf{c}} \cdot \big[((a_1b_1) \otimes (a_2b_2) \otimes \cdots \otimes (a_kb_k))\#e_n\big] \\ &= (a_1b_1\#e_{c_1}) * (a_2b_2\#e_{c_2}) * \cdots * (a_kb_k\#e_{c_k}).\end{aligned}$$

Since each element a_ib_i appearing here belongs to B_{c_i} by Condition (B), $y \cdot (b\#e_n)$ is in $\Sigma(B)$.

In the general case, we may write z as a linear combination of products $z_1 * z_2 * \cdots * z_l$, where each z_j is of the form $b_j\#e_{d_j}$, where d_j is a positive integer and $b_j \in B_{d_j}$. We apply the splitting formula (14). Since $\Sigma(B)$ is a subcoalgebra of $\mathscr{F}(A)$ (Proposition 4), we may require that in the decomposition $\Delta_l(y) = \sum_{(y)} y_{(1)} \otimes y_{(2)} \otimes \cdots \otimes y_{(l)}$ used, all elements represented by the placeholders $y_{(j)}$ belong to $\Sigma(B)$. By the first case, each product $y_{(j)} \cdot z_j$ belongs to $\Sigma(B)$, which entails that $y \cdot (z_1 * z_2 * \cdots * z_l)$ belongs to $\Sigma(B)$. We conclude that $y \cdot z$ belongs to $\Sigma(B)$. □

Again there are two main examples to which Corollary 11 can be applied. The first one is the case of the Novelli-Thibon algebra: the sequence $B_n = \mathsf{TS}^n(A)$ satisfies Conditions (A), (B) and (C), so the submodule $\mathsf{NT}(A)$ is a subalgebra of $\mathscr{F}(A)$ for the internal product.

The second example arises when A is a cocommutative bialgebra. Each iterated coproduct $\delta_n : A \to A^{\otimes n}$ is a morphism of algebras, therefore its image is a subalgebra B_n of $A^{\otimes n}$. The submodule $B = \bigoplus_{n \geq 0} B_n$ therefore satisfies Condition (B). It also satisfies Conditions (A) and (C), because the coproduct of A is coassociative and cocommutative. It thus follows from Corollary 11 that the submodule $\Sigma(B)$, which is of course the Mantaci-Reutenauer bialgebra $\mathsf{MR}(A)$, is a subalgebra of $\mathscr{F}(A)$ for the internal product.

The following corollary gives the rule to compute internal products in a Mantaci-Reutenauer algebra. It generalizes Corollary 6.8 and Theorem 6.9 of [24].

Corollary 12 *Let A be a cocommutative bialgebra with coproduct δ, let n be a positive integer, let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ and $\mathbf{d} = (d_1, d_2, \ldots d_l)$ be two compositions of n, and let a_1, a_2, ..., a_k, b_1, b_2, ..., b_l be elements of A. Then*

$$(y_{c_1,a_1} * y_{c_2,a_2} * \cdots * y_{c_k,a_k}) \cdot (y_{d_1,b_1} * y_{d_2,b_2} * \cdots * y_{d_l,b_l}) = \sum_{(a_i),(b_j)} \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \left(\prod_{j=1}^{l} \prod_{i=1}^{k} y_{m_{ij}, a_{i(j)} b_{j(i)}} \right),$$

where the first summation symbol on the right comes from the Sweedler notation for writing the iterated coproducts $\delta_l(a_i)$ and $\delta_k(b_j)$, where the two successive symbols $\prod$ stand for the external product, and where the factors of this external product are formed by reading column by column the entries of the matrix $M = (m_{ij})$.

Proof. An easy induction based on Formula (8) implies that

$$\Delta_l(y_{c_i,a_i}) = \sum_{(a_i)} \sum_{\mathbf{f}} y_{f_1,a_{i(1)}} \otimes y_{f_2,a_{i(2)}} \otimes \cdots \otimes y_{f_l,a_{i(l)}},$$

where the second sum runs over all compositions (possibly with zero parts) $\mathbf{f} = (f_1, f_2, \ldots, f_l)$ of c_i in l parts. Setting $y = y_{c_1,a_1} * y_{c_2,a_2} * \cdots * y_{c_k,a_k}$, it follows that

$$\begin{aligned}\Delta_l(y) &= \Delta_l(y_{c_1,a_1}) * \Delta_l(y_{c_2,a_2}) * \cdots * \Delta_l(y_{c_k,a_k}) \\ &= \sum_{(a_1),(a_2),\ldots,(a_k)} \sum_M \left(\prod_{i=1}^k y_{m_{i1},a_{i(1)}}\right) \otimes \left(\prod_{i=1}^k y_{m_{i2},a_{i(2)}}\right) \otimes \cdots \otimes \left(\prod_{i=1}^k y_{m_{il},a_{i(l)}}\right), \quad (18)\end{aligned}$$

where the second sum is over all matrices M with non-negative integral entries in k rows and l columns and with row-sum $\mathbf{c}$.

We now use the splitting formula

$$y \cdot (y_{d_1,b_1} * y_{d_2,b_2} * \cdots * y_{d_l,b_l}) = \sum_{(y)} (y_{(1)} \cdot y_{d_1,b_1}) * (y_{(2)} \cdot y_{d_2,b_2}) * \cdots * (y_{(l)} \cdot y_{d_l,b_l}) \qquad (19)$$

and substitute in it the expression for the iterated coproduct $\Delta_l(y)$ found in (18). For degree reasons, each term in (18) that yields a non-zero contribution to the right-hand side of (19) corresponds to a matrix M whose column sum is equal to $\mathbf{d}$, so that we may restrict the sum to the matrices M in $\mathscr{M}_{\mathbf{c},\mathbf{d}}$. The result of the substitution is a sum of products; in each product, the j-th factor is

$$\begin{aligned}&y_{(j)} \cdot y_{d_j,b_j} \\ &\quad = \left[(y_{m_{1j},a_{1(j)}}) * (y_{m_{2j},a_{2(j)}}) * \cdots * (y_{m_{kj},a_{k(j)}})\right] \cdot y_{d_j,b_j} \\ &\quad = x_{(m_{1j},m_{2j},\ldots,m_{kj})} \cdot \left[\left(\delta_{m_{1j}}(a_{1(j)}) \otimes \delta_{m_{2j}}(a_{2(j)}) \otimes \cdots \otimes \delta_{m_{kj}}(a_{k(j)})\right) \# e_{d_j}\right] \cdot \left[\delta_{d_j}(b_j) \# e_{d_j}\right] \\ &\quad = \sum_{(b_j)} x_{(m_{1j},m_{2j},\ldots,m_{kj})} \cdot \left[\left(\delta_{m_{1j}}(a_{1(j)} b_{j(1)}) \otimes \delta_{m_{2j}}(a_{2(j)} b_{j(2)}) \otimes \cdots \otimes \delta_{m_{kj}}(a_{k(j)} b_{j(k)})\right) \# e_{d_j}\right] \\ &\quad = \sum_{(b_j)} y_{m_{1j},a_{1(j)} b_{j(1)}} * y_{m_{2j},a_{2(j)} b_{j(2)}} * \cdots * y_{m_{kj},a_{k(j)} b_{j(k)}}.\end{aligned}$$

The corollary follows immediately. □

3.4 Frobenius structures

In this section, we put together the structures defined in Sections 2.2 and 3.1.

We begin by recalling some terminology. Let A be an associative $\mathbb{K}$-algebra with unit. A pairing ϖ on A is said associative if $\varpi(ab, c) = \varpi(a, bc)$ for all $(a, b, c) \in A^3$. A trace form on A is a linear map $\tau : A \to \mathbb{K}$ such that $\tau(ab) = \tau(ba)$ for all $(a, b) \in A^2$. The data of a symmetric and associative pairing is equivalent to the data of a trace form: to the trace form τ corresponds the pairing $\varpi : (a, b) \mapsto \tau(ab)$, and conversely τ is given by $\tau = \varpi^\flat(1)$. One says that an algebra A is a Frobenius algebra if it can be endowed with an associative and perfect pairing; if one can choose this pairing symmetric, then one calls A a symmetric algebra.

Now let A be a such a symmetric algebra, endowed with a symmetric, associative and perfect pairing ϖ. Then the graded bialgebra $\mathscr{F}(A)$ is endowed with the symmetric and perfect pairing ϖ_{tot} (Section 2.2) and each graded piece $\mathscr{F}_n(A)$ is an associative algebra for the internal product $\cdot$ (Section 3.1).

Proposition 13 *For any degree n, the pairing $\varpi_{\mathrm{tot}}\big|_{\mathscr{F}_n(A)\times\mathscr{F}_n(A)}$ is associative and endows $\mathscr{F}_n(A)$ with the structure of a symmetric algebra.*

Proof. We denote the linear form $\varpi^\flat(1)$ by τ and define a linear form $\tau_{\mathrm{tot}} : \mathscr{F}(A) \to \mathbb{K}$ by setting

$$\tau_{\mathrm{tot}}\big((a_1 \otimes a_2 \otimes \cdots \otimes a_n)\#\sigma\big) = \begin{cases} \prod_{i=1}^n \tau(a_i) & \text{if } \sigma = e_n, \\ 0 & \text{otherwise,} \end{cases}$$

for any $n \in \mathbb{N}$, any $(a_1, a_2, \ldots, a_n) \in A^n$ and any $\sigma \in \mathfrak{S}_n$. A straightforward verification based on Formula (9) shows that $\varpi_{\mathrm{tot}}(x, y) = \tau_{\mathrm{tot}}(x \cdot y)$ for any $(x, y) \in \mathscr{F}(A)^2$. It follows in particular that the pairing $\varpi_{\mathrm{tot}}\big|_{\mathscr{F}_n(A)\times\mathscr{F}_n(A)}$ is associative. Since this pairing is also symmetric and perfect, the algebra $\mathscr{F}_n(A)$ is a symmetric algebra. □

We add to these ingredients the data of a graded subbialgebra S of $\mathscr{F}(A)$, assumed to be a subalgebra of it for the internal product. The polar S° of S satisfies

$$\varpi_{\mathrm{tot}}(S, S \cdot S^\circ) = \varpi_{\mathrm{tot}}(S \cdot S, S^\circ) \subseteq \varpi_{\mathrm{tot}}(S, S^\circ) = 0,$$

so that $S \cdot S^\circ \subseteq S^\circ$. A similar argument shows the inclusion $S^\circ \cdot S \subseteq S^\circ$, and we conclude that S° is a two-sided internal S-submodule of $\mathscr{F}(A)$. Assuming that A is a projective $\mathbb{K}$-module and that S is a direct summand of the graded $\mathbb{K}$-module $\mathscr{F}(A)$, we construct the diagram (11), with $\mathscr{F}(V)$ replaced by $\mathscr{F}(A)$; beside being a diagram of graded bialgebras, it is then a diagram of two-sided internal S-submodules.

3.5 The case of a group algebra

Group algebras are at the same time cocommutative bialgebras and symmetric algebras. They give therefore examples to which the constructions of Sections 3.3 and 3.4 can be applied. We study this situation here.

So let Γ be a finite group. We endow the algebra $\mathbb{K}\Gamma$ with the pairing ϖ defined by

$$\varpi(\gamma, \delta) = \begin{cases} 1 & \text{if } \gamma = \delta^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is associative, symmetric and perfect; the corresponding trace form $\tau = \varpi^\flat(1)$ is the linear form that maps an element $\gamma \in \Gamma$ to 1 if γ is the unit and to 0 otherwise. (One may observe that the familiar trace map of $\mathbb{K}\Gamma$, i.e. the regular character of Γ, is a scalar multiple of τ.)

We now construct the graded bialgebra $\mathscr{F}(\mathbb{K}\Gamma)$ and endow it with the pairing of bialgebras ϖ_{tot}. By Proposition 13, each graded component $\mathscr{F}_n(\mathbb{K}\Gamma)$ is a symmetric algebra for the pairing $\varpi_{\mathrm{tot}}\big|_{\mathscr{F}_n(\mathbb{K}\Gamma)\times\mathscr{F}_n(\mathbb{K}\Gamma)}$. This property can also be explained in the following way.

Let us first recall that the wreath product $\Gamma \wr \mathfrak{S}_n$ is the semidirect product $\Gamma^n \rtimes \mathfrak{S}_n$ for the usual permutation action of $\mathfrak{S}_n$ on Γ^n. Thus an element $\Gamma \wr \mathfrak{S}_n$ can always be written as the product of an element of $\mathfrak{S}_n$ and an element of Γ^n, and the commutation rule between these two kinds of elements is

$$\sigma \cdot (\gamma_1, \gamma_2, \ldots, \gamma_n) = \big(\gamma_{\sigma^{-1}(1)}, \gamma_{\sigma^{-1}(2)}, \ldots, \gamma_{\sigma^{-1}(n)}\big) \cdot \sigma.$$

A comparison with Equation (12) which defines the product in the twisted group ring $\mathscr{F}_n(\mathbb{K}\Gamma) = (\mathbb{K}\Gamma)^{\otimes n} \# (\mathbb{K}\mathfrak{S}_n)$ shows the existence of an isomorphism of algebras

$$\mathbb{K}\big[\Gamma \wr \mathfrak{S}_n\big] \to \mathscr{F}_n(\mathbb{K}\Gamma),\ \big[(\gamma_1, \gamma_2, \ldots, \gamma_n) \cdot \sigma\big] \mapsto \big[(\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_n) \# \sigma\big].$$

Now the group algebra $\mathbb{K}\big[\Gamma \wr \mathfrak{S}_n\big]$ has a standard structure of a symmetric algebra, whose trace form τ_n is given by

$$\tau_n\big[(\gamma_1, \gamma_2, \ldots, \gamma_n) \cdot \sigma\big] = \begin{cases} 1 & \text{if } \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n \text{ are equal to the unit of } \Gamma \text{ and } \sigma = e_n, \\ 0 & \text{otherwise.} \end{cases}$$

Under the previous isomorphism, this trace form coincides with the linear form $\tau_{\mathrm{tot}}\big|_{\mathscr{F}_n(\mathbb{K}\Gamma)}$ used in the proof of Proposition 13. We conclude that the pairing $\varpi_{\mathrm{tot}}\big|_{\mathscr{F}_n(\mathbb{K}\Gamma)\times\mathscr{F}_n(\mathbb{K}\Gamma)}$ on $\mathscr{F}_n(\mathbb{K}\Gamma)$ corresponds to the usual associative, symmetric and perfect pairing on the group algebra $\mathbb{K}\big[\Gamma \wr \mathfrak{S}_n\big]$.

4 A Solomon descent theory for the wreath products $G \wr \mathfrak{S}_n$

In this section, we study a particular case of the following problem, inspired by Solomon's article [33]: given a finite group H, is it possible to find a subalgebra of the group algebra $\mathbb{K}H$ of which the representation ring of H is a quotient? More precisely, we use the theory developed in in the previous sections to give a positive answer in the case where the group H is the wreath product $G \wr \mathfrak{S}_n$ of the symmetric group with a finite abelian group G.

4.1 Representation rings

We first set up the notation we plan to use concerning representation rings. Let H be a finite group. We denote the algebra of complex-valued functions on H by $\mathbb{C}^H$. The $\mathbb{Z}$-submodule of $\mathbb{C}^H$ spanned by the characters of H is called the ring of complex linear representations of H and is denoted by $R(H)$. The involutive map $f \mapsto f^*$ which sends a function in $\mathbb{C}^H$ to its complex-conjugate leaves $R(H)$ stable. The assignment $H \rightsquigarrow R(H)$ is a contravariant functor from the category of finite groups to the category of commutative rings with involution.

Elements of $R(H)$ are usually called virtual characters. The set $\mathrm{Irr}(H)$ of irreducible characters of H is a basis of the $\mathbb{Z}$-module $R(H)$. A virtual character is called effective if all its coordinates with respect to the basis $\mathrm{Irr}(H)$ are positive.

The linear form on $\mathbb{C}^H$ that maps a function f to the complex number $\frac{1}{|H|}\sum_{h\in H} f(h)$ restricts to a $\mathbb{Z}$-valued additive form φ on $R(H)$, which is called the fundamental linear form on $R(H)$. Its value at an irreducible character $\zeta \in \mathrm{Irr}(H)$ is 1 if ζ is the trivial character of H and 0 otherwise. We define the fundamental bilinear form $\beta : R(H) \times R(H) \to \mathbb{Z}$ by $\beta(f, g) = \varphi(fg)$ for any $(f, g) \in R(H)^2$. The usual inner product of characters is the bilinear form $(f, g) \mapsto \beta(f, g^*)$. Given two irreducible characters ζ and ψ in $\mathrm{Irr}(H)$, the number $\beta(\zeta, \psi)$ is thus 1 if $\zeta = \psi^*$ and 0 otherwise. As a consequence, the fundamental bilinear form β is an associative, symmetric and perfect pairing; endowed with it, $R(H)$ becomes a symmetric commutative algebra.

We conclude this section with a proposition which is probably well-known.

Proposition 14 *The representation ring $R(H)$ has trivial Jacobson radical.*

Proof. Let L be a number field big enough to contain all the roots of unity of order $|H|$ in $\mathbb{C}$, and let $\mathscr{O}$ be the integral closure of $\mathbb{Z}$ in L. Let X the set of maximal ideals of $\mathscr{O}$. Since $\mathscr{O}$ is a Dedekind ring, there holds

$$\bigcap_{\mathfrak{m}\in X} \mathfrak{m} = \{0\}.$$

For any $h \in H$, the evaluation $\zeta(h)$ of a virtual character $\zeta \in R(H)$ at h belongs to $\mathscr{O}$. The image of the evaluation map $\mathrm{ev}_h : \zeta \mapsto \zeta(h)$ is therefore a subring of $\mathscr{O}$, over which $\mathscr{O}$ is integral. This implies that for any $\mathfrak{m} \in X$, the intersection $(\mathrm{im}\ \mathrm{ev}_h) \cap \mathfrak{m}$ is a maximal ideal in $(\mathrm{im}\ \mathrm{ev}_h)$, and thus that the inverse image $\mathrm{ev}_h^{-1}(\mathfrak{m})$ is a maximal ideal of $R(H)$. The desired result now follows from the equality

$$\bigcap_{h\in H} \bigcap_{\mathfrak{m}\in X} \mathrm{ev}_h^{-1}(\mathfrak{m}) = \bigcap_{h\in H} \left[\mathrm{ev}_h^{-1} \left(\bigcap_{\mathfrak{m}\in X} \mathfrak{m} \right) \right] = \bigcap_{h\in H} \ker \mathrm{ev}_h = \{0\},$$

because the Jacobson radical of $R(H)$ is the intersection of all its maximal (left) ideals. □

4.2 The characters of the wreath products $G \wr \mathfrak{S}_n$

Let G be a finite group, not necessarily abelian. We present in this section Specht's results about the characters of the wreath products $G \wr \mathfrak{S}_n$. Our presentation follows the appendix of [21], Appendix B of Chap. I in [22] and and §7 in [36], to which we refer the reader for the proofs.

The wreath product $G \wr \mathfrak{S}_n$ is the semidirect product $G^n \rtimes \mathfrak{S}_n$ for the usual permutation action of $\mathfrak{S}_n$ on G^n. (By convention, the notation $G \wr \mathfrak{S}_0$ denotes the group with one element.) An element of $G \wr \mathfrak{S}_n$ can always be written in two ways as the product of an element of $\mathfrak{S}_n$ and an element of G^n, namely

$$\sigma \cdot (g_1, g_2, \ldots, g_n) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(n)}) \cdot \sigma.$$

Given a $\mathbb{C}G$-module V, we construct a complex representation $\eta_n(V)$ of $G \wr \mathfrak{S}_n$ on the space $V^{\otimes n}$ by letting a product $(g_1, \ldots, g_n) \cdot \sigma$ act on a pure tensor $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ by

$$((g_1, g_2, \ldots, g_n) \cdot \sigma) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = (g_1 \cdot v_{\sigma^{-1}(1)}) \otimes (g_2 \cdot v_{\sigma^{-1}(2)}) \otimes \cdots \otimes (g_n \cdot v_{\sigma^{-1}(n)}).$$

The character of $\eta_n(V)$ does not depend actually on V but only of its character; if ζ denotes the latter, then we will denote the former by $\eta_n(\zeta)$. Two particular cases are worth mentioning.

- If γ is a linear character of G, that is, a character of degree 1, then $\eta_n(\gamma)$ is the linear character $\big((g_1, g_2, \ldots, g_n) \cdot \sigma\big) \mapsto \gamma(g_1 g_2 \cdots g_n)$ of $G \wr \mathfrak{S}_n$.

- If ζ is the regular character of G, then $\eta_n(\zeta)$ is the character induced from the trivial representation of the subgroup $\mathfrak{S}_n$ to $G \wr \mathfrak{S}_n$.

Let $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ be a composition of n. The Young subgroup $\mathfrak{S}_{\mathbf{c}}$ of $\mathfrak{S}_n$ acts on G^n, and the semidirect product $G^n \rtimes \mathfrak{S}_{\mathbf{c}}$ can be seen as the subgroup of $G \wr \mathfrak{S}_n$ generated by G^n and $\mathfrak{S}_{\mathbf{c}}$. By analogy, we denote it by $G \wr \mathfrak{S}_{\mathbf{c}}$ and we call it a Young subgroup of $G \wr \mathfrak{S}_n$. The natural isomorphism $\mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \cdots \times \mathfrak{S}_{c_k} \cong \mathfrak{S}_{\mathbf{c}}$ gives rise to an isomorphism $(G \wr \mathfrak{S}_{c_1}) \times (G \wr \mathfrak{S}_{c_2}) \times \cdots \times (G \wr \mathfrak{S}_{c_k}) \cong (G \wr \mathfrak{S}_{\mathbf{c}})$.

A partition is an infinite non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers, all of whose terms but a finite number vanish. As usual, we denote the sum of the parts of λ by $|\lambda|$; if $|\lambda| = n$, then we say that λ is a partition of n. To a partition λ of n, we associate in the usual way an irreducible complex representation S_λ of $\mathfrak{S}_n$, the so-called Specht module. Thus for instance the characters of $S_{(n)}$ and $S_{(1,1,\ldots,1)}$ (with n terms equal to 1) are the trivial and signature characters of $\mathfrak{S}_n$, respectively.

An $\operatorname{Irr}(G)$-partition is a family $\boldsymbol{\lambda} = (\lambda_\gamma)_{\gamma\in\operatorname{Irr}(G)}$ indexed by $\operatorname{Irr}(G)$ of partitions. The size of an $\operatorname{Irr}(G)$-partition $\boldsymbol{\lambda}$ is the number $\|\boldsymbol{\lambda}\| = \sum_{\gamma\in\operatorname{Irr}(G)} |\lambda_\gamma|$. We define the dual of $\boldsymbol{\lambda}$ as the $\operatorname{Irr}(G)$-partition $\boldsymbol{\lambda}^* = \bigl(\gamma \mapsto \lambda_{\gamma^*}\bigr)$.

Given an $\operatorname{Irr}(G)$-partition $\boldsymbol{\lambda}$ of size n, one constructs a complex representation of $G \wr \mathfrak{S}_n$ as follows. One enumerates the irreducible characters $\gamma_1, \gamma_2, \ldots, \gamma_r$ of G and picks up $\mathbb{C}G$-modules $V_1, V_2, \ldots, V_r$ that afford them. Let us set $c_i = |\lambda_i|$. Since $\mathfrak{S}_{c_i}$ is a quotient of $G \wr \mathfrak{S}_{c_i}$, we may view the Specht module $S_{\lambda_{\gamma_i}}$ as a representation of $G \wr \mathfrak{S}_{c_i}$ and we may then multiply it by $\eta_{c_i}(V_i)$. The outer product

$$\bigl(S_{\lambda_{\gamma_1}} \otimes \eta_{c_1}(V_1)\bigr) \otimes \bigl(S_{\lambda_{\gamma_2}} \otimes \eta_{c_2}(V_2)\bigr) \otimes \cdots \otimes \bigl(S_{\lambda_{\gamma_r}} \otimes \eta_{c_r}(V_r)\bigr)$$

is then a representation of $(G \wr \mathfrak{S}_{c_1}) \times (G \wr \mathfrak{S}_{c_2}) \times \cdots \times (G \wr \mathfrak{S}_{c_r}) \cong \bigl(G \wr \mathfrak{S}_{(c_1,c_2,\ldots,c_r)}\bigr)$, which we can induce to $G \wr \mathfrak{S}_n$. The result of this induction does not depend up to isomorphism on the choice of the enumeration $\gamma_1, \gamma_2, \ldots, \gamma_r$. Its character depends therefore only of $\boldsymbol{\lambda}$; we denote it by $\boldsymbol{\chi}^{\boldsymbol{\lambda}}$. The map $\boldsymbol{\lambda} \mapsto \boldsymbol{\chi}^{\boldsymbol{\lambda}}$ affords a bijection from the set of $\operatorname{Irr}(G)$-partitions of size n onto the set $\operatorname{Irr}\bigl(G \wr \mathfrak{S}_n\bigr)$. The complex-conjugate of the character $\boldsymbol{\chi}^{\boldsymbol{\lambda}}$ is the character $\boldsymbol{\chi}^{\boldsymbol{\lambda}^*}$.

Each representation ring $R(G \wr \mathfrak{S}_n)$ is a ring endowed with its fundamental linear and bilinear forms φ and β, this latter being an associative, symmetric and perfect pairing. Considering all n at the same time yields however extra structures. We consider therefore the direct sum $\operatorname{Rep}(G) = \bigoplus_{n\geq 0} R(G \wr \mathfrak{S}_n)$.

We define the induction product $\psi * \psi'$ of two characters ψ of $G \wr \mathfrak{S}_n$ and ψ' of $G \wr \mathfrak{S}_{n'}$ as the induction

$$\psi * \psi' = \operatorname{Ind}_{(G\wr\mathfrak{S}_n)\times(G\wr\mathfrak{S}_{n'})}^{G\wr\mathfrak{S}_{n+n'}} \bigl(\psi \otimes \psi'\bigr),$$

where $(G \wr \mathfrak{S}_n) \times (G \wr \mathfrak{S}_{n'})$ is viewed as the subgroup $G \wr \mathfrak{S}_{(n,n')}$ of $G \wr \mathfrak{S}_{n+n'}$. The bilinear extension of the external product to $\operatorname{Rep}(G) \times \operatorname{Rep}(G)$ endows the space $\operatorname{Rep}(G)$ with the structure of a graded associative and commutative algebra.

Likewise, the restriction coproduct $\Delta(\psi)$ of a character ψ of $G \wr \mathfrak{S}_n$ is defined to be the sum over $n' \in \{0, 1, \ldots, n\}$ of the restrictions

$$\operatorname{Res}_{(G\wr\mathfrak{S}_{n'})\times(G\wr\mathfrak{S}_{n-n'})}^{G\wr\mathfrak{S}_n} \bigl(\psi\bigr).$$

This notation implicitely identifies characters of the group $(G\wr\mathfrak{S}_{n'})\times(G\wr\mathfrak{S}_{n-n'})$ with elements of $\operatorname{Rep}_{n'}(G) \otimes \operatorname{Rep}_{n-n'}(G)$, so that $\Delta(\psi) \in \operatorname{Rep}(G) \otimes \operatorname{Rep}(G)$. The linear extension of Δ to the whole space $\operatorname{Rep}(G)$ endows the latter with the structure of a graded coassociative and cocommutative coalgebra. Mackey's subgroup theorem implies that $(\operatorname{Rep}(G), *, \Delta)$ is a graded commutative cocommutative bialgebra.

In order to make the situation more alike to the structures seen in Sections 2 and 3, we extend the product and the fundamental linear and bilinear forms φ_n and β_n defined on each $R(G \wr \mathfrak{S}_n)$ to operations defined on the whole space $\operatorname{Rep}(G)$ by setting

$$fg = \sum_{n\geq 0} f_n g_n, \quad \varphi_{\mathrm{tot}}(f) = \sum_{n\geq 0} \varphi_n(f_n) \quad \text{and} \quad \beta_{\mathrm{tot}}(f,g) = \varphi_{\mathrm{tot}}(fg) = \sum_{n\geq 0} \beta_n(f_n, g_n)$$

for any $f = \sum_{n\geq 0} f_n$ and $g = \sum_{n\geq 0} g_n$, where f_n and g_n in $R(G \wr \mathfrak{S}_n)$. Then β_{tot} is a perfect symmetric pairing on $\mathrm{Rep}(G)$, with respect to which the induction product $*$ and the restriction coproduct Δ are adjoint to each other by Frobenius reciprocity. Moreover, Mackey's tensor product theorem (more precisely, the particular case stated in Corollary (10.20) of [11]) implies the following splitting formula: for any f, g_1, g_2, $\ldots$, g_l in $\mathrm{Rep}(G)$, there holds

$$f(g_1 * g_2 * \cdots * g_l) = \sum_{(f)} (f_{(1)} g_1) * (f_{(2)} g_2) * \cdots * (f_{(l)} g_l). \tag{20}$$

We denote by Λ the ring of symmetric functions. This is indeed a graded bialgebra (see I, 5, Ex. 25 in [22]). As is well-known, the complete symmetric functions h_n are algebraically independent generators of the commutative $\mathbb{Z}$-algebra Λ. On the other hand, the Schur functions s_λ, where λ is a partition, is a basis of the $\mathbb{Z}$-module Λ. Let $\Lambda(\mathrm{Irr}(G))$ be the tensor product of a family $(\Lambda(\gamma))_{\gamma\in\mathrm{Irr}(G)}$ of copies of Λ. For any $\gamma \in \mathrm{Irr}(G)$, we denote by $P(\gamma)$ the element in the tensor factor $\Lambda(\gamma)$ that corresponds to the symmetric function $P \in \Lambda$. Given an $\mathrm{Irr}(G)$-partition $\boldsymbol{\lambda} = (\lambda_\gamma)_{\gamma\in\mathrm{Irr}(G)}$, we set

$$\mathbf{s}_{\boldsymbol{\lambda}} = \prod_{\gamma\in\mathrm{Irr}(G)} s_{\lambda_\gamma}(\gamma).$$

Then the elements $h_n(\gamma)$ are algebraically independent generators of the commutative $\mathbb{Z}$-algebra $\Lambda(\mathrm{Irr}(G))$, where $n \geq 1$ and $\gamma \in \mathrm{Irr}(G)$, and the elements $\mathbf{s}_{\boldsymbol{\lambda}}$ form a basis of the $\mathbb{Z}$-module $\Lambda(\mathrm{Irr}(G))$, where $\boldsymbol{\lambda}$ is an $\mathrm{Irr}(G)$-partition. Finally we endow the graded bialgebra $\Lambda(\mathrm{Irr}(G))$ with a perfect symmetric pairing $\langle ?, ? \rangle$ defined on the basis of Schur functions by

$$\langle \mathbf{s}_{\boldsymbol{\lambda}}, \mathbf{s}_{\boldsymbol{\lambda}'} \rangle = \begin{cases} 1 & \text{if } \boldsymbol{\lambda}' = \boldsymbol{\lambda}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Our interest in $\Lambda(\mathrm{Irr}(G))$ is that it gives a model that allows to calculate in $\mathrm{Rep}(G)$. More precisely, there is an isomorphism of graded bialgebras $\mathrm{ch} : \mathrm{Rep}(G) \xrightarrow{\simeq} \Lambda(\mathrm{Irr}(G))$, called the Frobenius characteristic, such that

$$\mathrm{ch}\big(\eta_n(\gamma)\big) = h_n(\gamma) \quad \text{and} \quad \mathrm{ch}\big(\boldsymbol{\chi}^{\boldsymbol{\lambda}}\big) = \mathbf{s}_{\boldsymbol{\lambda}}$$

for any $n \geq 1$, any $\gamma \in \mathrm{Irr}(G)$ and any $\mathrm{Irr}(G)$-partition $\boldsymbol{\lambda}$. Moreover ch is compatible in the obvious sense with the perfect symmetric pairings β_{tot} on $\mathrm{Rep}(G)$ and $\langle ?, ? \rangle$ on $\Lambda(\mathrm{Irr}(G))$.

What precedes implies that

$$\mathrm{Rep}(G) \cong \bigotimes_{\gamma\in\mathrm{Irr}(G)} \mathbb{Z}\big[\eta_1(\gamma), \eta_2(\gamma), \ldots\big].$$

The following proposition, which will be used in Section 4.5, explains how to find the expression of $\eta_n(\zeta)$ as a polynomial in the $\eta_n(\gamma)$ when the effective character ζ is not irreducible. In order to state it, we introduce a last notation: viewing the signature character sgn of $\mathfrak{S}_n$ as a character of $G \wr \mathfrak{S}_n$ through the quotient map $(G \wr \mathfrak{S}_n) \to \mathfrak{S}_n$, we denote the product $\mathrm{sgn} \cdot \eta_n(\zeta)$ in the ring $R(G \wr \mathfrak{S}_n)$ by $\varepsilon_n(\zeta)$.

Proposition 15 *There exists a morphism of groups $H : R(G) \to \big(\mathrm{Rep}(G)[[u]]\big)^\times$ such that*

$$H(\zeta) = \sum_{n\geq 0} \eta_n(\zeta) u^n \quad \textit{and} \quad H(-\zeta) = \sum_{n\geq 0} (-1)^n \varepsilon_n(\zeta) u^n \tag{21}$$

for all effective characters ζ.

Proof. We extend the Frobenius characteristic ch to an isomorphism of rings from $\mathrm{Rep}(G)[[u]]$ onto $\Lambda(\mathrm{Irr}(G))[[u]]$. Since $R(G)$ is a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(G)$, there exists an homomorphism of abelian group $H : R(G) \to \big(\mathrm{Rep}(G)[[u]]\big)^{\times}$ such that for each $\gamma \in \mathrm{Irr}(G)$,

$$H(\gamma) = \sum_{n\geq 0} \eta_n(\gamma)u^n.$$

Now let $\zeta = \sum_{\gamma\in\mathrm{Irr}(G)} a_\gamma \gamma$ be an effective character of G. A slight modification of the calculation made in Appendix B of Chap. I, (8.3) in [22] yields

$$\mathrm{ch}\left(\sum_{n\geq 0} \eta_n(\zeta)u^n\right) = \prod_{\gamma\in\mathrm{Irr}(G)} \left(\sum_{n\geq 0} h_n(\gamma)u^n\right)^{a_\gamma}$$

and

$$\begin{aligned}\mathrm{ch}\left(\sum_{n\geq 0} (-1)^n\varepsilon_n(\zeta)u^n\right) &= \prod_{\gamma\in\mathrm{Irr}(G)} \left(\sum_{n\geq 0} (-1)^n e_n(\gamma)u^n\right)^{a_\gamma} \\ &= \prod_{\gamma\in\mathrm{Irr}(G)} \left(\sum_{n\geq 0} h_n(\gamma)u^n\right)^{-a_\gamma}, \end{aligned} \tag{22}$$

where $e_n \in \Lambda$ is the symmetric elementary function of degree n. It follows that

$$\begin{aligned}\mathrm{ch}\left(\sum_{n\geq 0} \eta_n(\zeta)u^n\right) &= \prod_{\gamma\in\mathrm{Irr}(G)} \left(\sum_{n\geq 0} \mathrm{ch}\big(\eta_n(\gamma)\big)u^n\right)^{a_\gamma} \\ &= \mathrm{ch}\left(\prod_{\gamma\in\mathrm{Irr}(G)} \left(\sum_{n\geq 0} \eta_n(\gamma)u^n\right)^{a_\gamma}\right) \\ &= \mathrm{ch}\big(H(\zeta)\big),\end{aligned}$$

and likewise $\mathrm{ch}\left(\sum_{n\geq 0}(-1)^n\varepsilon(\zeta)u^n\right) = \mathrm{ch}\big(H(-\zeta)\big)$. We conclude that (21) holds, as required. □

4.3 The Solomon homomorphism

The representation theory presented in Section 4.2 allows the use of the model $\Lambda(\mathrm{Irr}(G))$ to compute the character tables of all the groups $G \wr \mathfrak{S}_n$ and to study the inductions and the restrictions with respect to the Young subgroup. However $\Lambda(\mathrm{Irr}(G))$ does not make the computation of the ring structure of $R(G \wr \mathfrak{S}_n)$ particularly easy. In this section, we construct explicitly a surjective ring homomorphism from a subring of $\mathbb{Z}\big[G\wr\mathfrak{S}_n\big]$ onto $R(G\wr\mathfrak{S}_n)$. However, we must restrict ourselves to the case where G is abelian. As usual, it is convenient to do this simultaneously for all n.

In this section and in the following one, $\mathbb{K}$ is the ring $\mathbb{Z}$. Some variants are indeed possible, but this choice simplifies slightly the notation. The letter G denotes a finite abelian group. The dual group of G, denoted by $G^\wedge$ or by Γ, is the set $\mathrm{Irr}(G)$ endowed with the ordinary product of characters. Although G and Γ are isomorphic as abstract groups, we do not identify them. On the other hand, we observe that the group ring $\mathbb{Z}\Gamma$ coincides with the representation ring $R(G)$; indeed even the pairings and the trace forms which turn these rings into symmetric $\mathbb{Z}$-algebras agree.

We construct the graded bialgebra $\mathscr{F}(\mathbb{Z}\Gamma)$ with its external product $*$ and its coproduct Δ; it is further endowed with the internal product $\cdot$, the linear form τ_{tot} and the pairing ϖ_{tot} (see Section 3.5). On the other hand, we have the graded bialgebra $\mathrm{Rep}(G)$ with the induction product $*$ and the coproduct Δ, with also the fundamental linear and bilinear forms φ_{tot} and β_{tot}; moreover the graded components $R(G \wr \mathfrak{S}_n)$ of $\mathrm{Rep}(G)$ are symmetric algebras. Our aim now is to show that $\mathrm{Rep}(G)$ is a subquotient of $\mathscr{F}(\mathbb{Z}\Gamma)$.

Since $\mathbb{Z}\Gamma$ is a cocommutative bialgebra, the Mantaci-Reutenauer subbialgebra $\mathsf{MR}(\mathbb{Z}\Gamma)$ of $\mathscr{F}(\mathbb{Z}\Gamma)$ is defined. This is a graded subbialgebra, whose homogeneous component of degree n, say, will be denoted by $\mathsf{MR}_n(\mathbb{Z}\Gamma)$. By Corollary 11, each $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ is a subalgebra of $\mathscr{F}_n(\mathbb{Z}\Gamma)$ for the internal product. Moreover, it follows from Proposition 3 that with respect to the external product, the associative algebra $\mathsf{MR}(\mathbb{Z}\Gamma)$ is freely generated by the elements $y_{n,\gamma}$, where $n \geq 1$ and $\gamma \in \Gamma$. Thus there is a unique morphism of algebras $\theta_G : \mathsf{MR}(\mathbb{Z}\Gamma) \to \mathrm{Rep}(G)$ that maps $y_{n,\gamma}$ to $\eta_n(\gamma)$. We call this map θ_G the Solomon homomorphism.

Theorem 16 *(i) The Solomon homomorphism θ_G is a surjective homomorphism of graded bialgebras with respect to the products $*$ and the coproducts Δ on $\mathsf{MR}(\mathbb{Z}\Gamma)$ and $\mathrm{Rep}(G)$; its kernel is the ideal generated by the elements $(y_{m,\gamma} * y_{n,\delta} - y_{n,\delta} * y_{m,\gamma})$, where $m \geq 1$, $n \geq 1$, and $(\gamma, \delta) \in \Gamma^2$.*

(ii) For every degree n, the restriction of the Solomon map $\theta_G : \mathsf{MR}_n(\mathbb{Z}\Gamma) \to R(G \wr \mathfrak{S}_n)$ is a surjective homomorphism of rings; its kernel is the Jacobson radical of the ring $\mathsf{MR}(\mathbb{Z}\Gamma)$.

(iii) The Solomon homomorphism is compatible with the linear and bilinear forms τ_{tot} and ϖ_{tot} on $\mathsf{MR}(\mathbb{Z}\Gamma)$ and φ_{tot} and β_{tot} on $R(G \wr \mathfrak{S}_n)$, in the sense that

$$\tau_{\mathrm{tot}} = \varphi_{\mathrm{tot}} \circ \theta_G \quad \textit{and} \quad \varpi_{\mathrm{tot}} = \beta_{\mathrm{tot}}\big(\theta_G(?), \theta_G(?)\big). \tag{23}$$

The kernel of θ_G is equal to the kernel $\mathsf{MR}(\mathbb{Z}\Gamma) \cap \mathsf{MR}(\mathbb{Z}\Gamma)^\circ$ of the pairing $\varpi_{\mathrm{tot}}\big|_{\mathsf{MR}(\mathbb{Z}\Gamma)\times\mathsf{MR}(\mathbb{Z}\Gamma)}$, where the polar $\mathsf{MR}(\mathbb{Z}\Gamma)^\circ$ is defined in the ambient space $\mathscr{F}(\mathbb{Z}\Gamma)$ with respect to the perfect pairing ϖ_{tot}.

Proof. (i) The algebra $\mathsf{MR}(\mathbb{Z}\Gamma)$ is the free associative $\mathbb{Z}$-algebra generated by the elements $y_{n,\gamma}$, where $n \geq 1$ and $\gamma \in \Gamma$, whilst $\mathrm{Rep}(G)$ is the free associative commutative $\mathbb{Z}$-algebra generated by the elements $\eta_n(\gamma)$. It follows that θ_G is surjective and that its kernel is the ideal generated by the commutators $(y_{m,\gamma} * y_{n,\delta} - y_{n,\delta} * y_{m,\gamma})$. Moreover θ_G is graded, for $y_{n,\gamma}$ and $h_n(\gamma)$ have both degree n.

It is easy to see that

$$\mathrm{Res}^{G\wr\mathfrak{S}_n}_{(G\wr\mathfrak{S}_{n'})\times(G\wr\mathfrak{S}_{n-n'})}\big(\eta_n(\gamma)\big) = \eta_{n'}(\gamma) \otimes \eta_{n-n'}(\gamma)$$

for any $\gamma \in \Gamma$ and any integers n and n' with $0 \leq n' \leq n$, and therefore

$$\Delta\big(\eta_n(\gamma)\big) = \sum_{n'=0}^{n} \eta_{n'}(\gamma) \otimes \eta_{n-n'}(\gamma) \tag{24}$$

in $\mathrm{Rep}(G)$. It follows then by comparison with Equation (8) that the set

$$\{x \in \mathsf{MR}(\mathbb{Z}\Gamma) \mid \Delta \circ \theta_G(x) = (\theta_G \otimes \theta_G) \circ \Delta(x)\}$$

contains the elements $y_{n,\gamma}$. Since this set is a subalgebra, it is the whole $\mathsf{MR}(\mathbb{Z}\Gamma)$. The compatibility of θ_G with the counit is trivial, and we conclude that θ_G is a morphism of coalgebras. Assertion (i) is proved.

(ii) We first prove that θ_G maps the internal product of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ to the ordinary product of characters in $R(G \wr \mathfrak{S}_n)$. This fact may be shown by a direct computation using Mantaci and Reutenauer's rule (Corollary 12) and Mackey's tensor product theorem; it may also be obtained by the following reasoning, that is actually grounded on the same combinatorial foundations.

A straightforward calculation, based on Equations (14) and (20) and on the fact that θ_G is a morphism of bialgebras for the operations $*$ and Δ, shows that

$$E = \{z \in \mathsf{MR}(\mathbb{Z}\Gamma) \mid \forall y \in \mathsf{MR}(\mathbb{Z}\Gamma),\ \theta_G(y \cdot z) = \theta_G(y)\theta_G(z)\}$$

is a subalgebra of $\mathsf{MR}(\mathbb{Z}\Gamma)$ for the external product $*$. On the other hand, every generator $y_{n,\delta}$ of $\mathsf{MR}(\mathbb{Z}\Gamma)$ belongs to E. Indeed any element in $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ is a linear combination of elements of the form

$$y = y_{c_1,\gamma_1} * y_{c_2,\gamma_2} * \cdots * y_{c_k,\gamma_k},$$

where $\mathbf{c} = (c_1, c_2, \dots, c_k)$ is a composition of n and $\gamma_1, \gamma_2, \dots, \gamma_k$ are elements of Γ, and for such a y, Formulas (20) and (24) imply

$$\begin{aligned}
\theta_G(y \cdot y_{n,\delta}) &= \theta_G\big(x_{\mathbf{c}} \cdot \big[(\gamma_1^{\otimes c_1} \otimes \gamma_2^{\otimes c_2} \otimes \cdots \otimes \gamma_k^{\otimes c_k})\#e_n\big] \cdot (\delta^{\otimes n}\#e_n)\big) \\
&= \theta_G\big(x_{\mathbf{c}} \cdot \big[((\gamma_1\delta)^{\otimes c_1} \otimes (\gamma_2\delta)^{\otimes c_2} \otimes \cdots \otimes (\gamma_k\delta)^{\otimes c_k})\#e_n\big]\big) \\
&= \theta_G\big(y_{c_1,\gamma_1\delta} * y_{c_2,\gamma_2\delta} * \cdots * y_{c_k,\gamma_k\delta}\big) \\
&= \eta_{c_1}(\gamma_1\delta) * \eta_{c_2}(\gamma_2\delta) * \cdots * \eta_{c_k}(\gamma_k\delta) \\
&= \big(\eta_{c_1}(\gamma_1)\eta_{c_1}(\delta)\big) * \big(\eta_{c_2}(\gamma_2)\eta_{c_2}(\delta)\big) * \cdots * \big(\eta_{c_k}(\gamma_k)\eta_{c_k}(\delta)\big) \\
&= \sum_{(\eta_n(\delta))} \big(\eta_{c_1}(\gamma_1)(\eta_n(\delta))_{(1)}\big) * \big(\eta_{c_2}(\gamma_2)(\eta_n(\delta))_{(2)}\big) * \cdots * \big(\eta_{c_k}(\gamma_k)(\eta_n(\delta))_{(k)}\big) \\
&= \big(\eta_{c_1}(\gamma_1) * \eta_{c_2}(\gamma_2) * \cdots * \eta_{c_k}(\gamma_k)\big)\ \eta_n(\delta) \\
&= \theta_G(y)\ \theta_G(y_{n,\delta}).
\end{aligned}$$

Therefore $E = \mathsf{MR}(\mathbb{Z}\Gamma)$. Observing moreover that θ_G maps the unit element of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$, namely $(1^{\otimes n}\#e_n) = y_{n,1}$, to the unit of $R(G \wr \mathfrak{S}_n)$, namely the trivial character $\eta_n(1)$ of $G \wr \mathfrak{S}_n$, we conclude that the degree n part of θ_G is an homomorphism of rings from $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ to $R(G \wr \mathfrak{S}_n)$.

Assertion (i) implies that this homomorphism is surjective, which entails that the Jacobson radical of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ is contained in the preimage by $\theta_G\big|_{\mathsf{MR}_n(\mathbb{Z}\Gamma)}$ of the Jacobson radical of $R(G \wr \mathfrak{S}_n)$. By Proposition 14), this translates readily into the inclusion $\operatorname{rad}\mathsf{MR}_n(\mathbb{Z}\Gamma) \subseteq \ker\theta_G\big|_{\mathsf{MR}_n(\mathbb{Z}\Gamma)}$.

To prove the reverse inclusion, we will use the result stated in Assertion (iii). (Though its validity has not yet been established, no vicious circle arises in the reasoning.) So let us suppose that some element $x \in \mathsf{MR}_n(\mathbb{Z}\Gamma)$ belongs to the kernel of θ_G. This element x acts by left multiplication on the algebra $\mathscr{F}_n(\mathbb{Z}\Gamma)$. Since this latter is a free $\mathbb{Z}$-module, this action can be represented by a matrix with entries in $\mathbb{Z}$. For any positive integer k, the k-th power of this matrix represents the action of the left multiplication by x^k and therefore its trace is

$$\operatorname{rk} \mathscr{F}_n(\mathbb{Z}\Gamma)\ \tau_{\mathrm{tot}}(x^k)$$

by the interpretation of τ_{tot} given at the end of Section 3.5. However our assumption that $x \in \ker \theta_G$ and Assertion (iii) yield

$$\tau_{\mathrm{tot}}(x^k) = \varpi_{\mathrm{tot}}(x, x^{k-1}) = \beta_{\mathrm{tot}}\big(\theta_G(x), \theta_G\big(x^{k-1}\big)\big) = 0$$

for all $k \geq 1$. It follows that our matrix is nilpotent, and therfore that x itself is nilpotent. This argument shows that all elements of the ideal $\ker \theta_G \,\big|\, \mathsf{MR}_n(\mathbb{Z}\Gamma)$ of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ are nilpotent. This kernel is thus contained in the radical of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$, which completes the proof of Assertion (ii).

(iii) Elements of the form

$$x = y_{c_1,\gamma_1} * y_{c_2,\gamma_2} * \cdots * y_{c_k,\gamma_k}$$

span the $\mathbb{Z}$-module $\mathsf{MR}(\mathbb{Z}\Gamma)$. Putting $n = c_1 + c_2 + \cdots + c_k$ and $\mathbf{c} = (c_1, c_2, \ldots, c_k)$, we observe that

$$\tau_{\mathrm{tot}}(x) = \tau_{\mathrm{tot}}\big(x_{\mathbf{c}} \cdot \big[(\gamma_1^{\otimes c_1} \otimes \gamma_2^{\otimes c_2} \otimes \cdots \otimes \gamma_k^{\otimes c_k}) \# e_n\big]\big)$$

is 1 if all the elements γ_i are equal to 1 and is 0 otherwise. On the other hand, Frobenius reciprocity implies that

$$\begin{aligned}
\varphi_{\mathrm{tot}} \circ \theta_G(x) &= \dim \operatorname{Hom}_{G \wr \mathfrak{S}_n}\big(\eta_{c_1}(\gamma_1) * \eta_{c_2}(\gamma_2) * \cdots * \eta_{c_k}(\gamma_k), 1_{G \wr \mathfrak{S}_n}\big) \\
&= \dim \operatorname{Hom}_{G \wr \mathfrak{S}_n}\big(\operatorname{Ind}_{G \wr \mathfrak{S}_{\mathbf{c}}}^{G \wr \mathfrak{S}_n}\big(\eta_{c_1}(\gamma_1) \otimes \eta_{c_2}(\gamma_2) \otimes \cdots \otimes \eta_{c_k}(\gamma_k)\big), 1_{G \wr \mathfrak{S}_n}\big) \\
&= \dim \operatorname{Hom}_{G \wr \mathfrak{S}_{\mathbf{c}}}\big(\eta_{c_1}(\gamma_1) \otimes \eta_{c_2}(\gamma_2) \otimes \cdots \otimes \eta_{c_k}(\gamma_k), 1_{G \wr \mathfrak{S}_{\mathbf{c}}}\big).
\end{aligned}$$

The character $\eta_{c_1}(\gamma_1) \otimes \eta_{c_2}(\gamma_2) \otimes \cdots \otimes \eta_{c_k}(\gamma_k)$ of $G \wr \mathfrak{S}_{\mathbf{c}}$ is one-dimensional. Therefore $\varphi_{\mathrm{tot}} \circ \theta_G(x)$ is 1 if this character is trivial, that is, if all the elements γ_i are equal to 1, and is 0 otherwise. The equality $\tau_{\mathrm{tot}}(x) = \varphi_{\mathrm{tot}} \circ \theta_G(x)$ being valid for each x in a spanning set for $\mathsf{MR}(\mathbb{Z}\Gamma)$, we conclude that $\tau_{\mathrm{tot}} = \varphi_{\mathrm{tot}} \circ \theta_G$. In turn, this implies that

$$\varpi_{\mathrm{tot}}(x, y) = \tau_{\mathrm{tot}}(x \cdot y) = \varphi_{\mathrm{tot}} \circ \theta_G(x \cdot y) = \varphi_{\mathrm{tot}}(\theta_G(x)\theta_G(y)) = \beta_{\mathrm{tot}}(\theta_G(x), \theta_G(y))$$

for any $(x, y) \in \mathsf{MR}(\mathbb{Z}\Gamma)^2$.

An immediate consequence of this last equality is that $\ker \theta_G$ is contained in the kernel $\mathsf{MR}(\mathbb{Z}\Gamma) \cap \mathsf{MR}(\mathbb{Z}\Gamma)^\circ$ of the symmetric pairing $\varpi_{\mathrm{tot}}\big|_{\mathsf{MR}(\mathbb{Z}\Gamma) \times \mathsf{MR}(\mathbb{Z}\Gamma)}$. The reverse inclusion holds also because θ_G is surjective and β_{tot} is a perfect pairing on $\operatorname{Rep}(G)$. Assertion (iii) is proved. $\square$

Assertion (ii) of Theorem 16 says that the representation ring $R(G \wr \mathfrak{S}_n)$ can be obtained as a quotient of the subring $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ of $\mathscr{F}_n(\mathbb{Z}\Gamma) \cong \mathbb{Z}\big[\Gamma \wr \mathfrak{S}_n\big]$. Since Γ and G are isomorphic, this entails that the representation ring of the group $G \wr \mathfrak{S}_n$ can be realized as a quotient of a subring of its group algebra. In other words, there exists a Solomon descent theory for the wreath product $G \wr \mathfrak{S}_n$. However it is not canonical, for it depends on the choice of an isomorphism between G and its dual.

The notation used above suggests the existence of some kind of functoriality. In order to state a precise statement, we define a category $\mathscr{V}$. Objects of $\mathscr{V}$ are $\mathbb{N}$-graded abelian groups $A = \bigoplus_{n \geq 0} A_n$; each graded piece A_n is further endowed with the structure of a ring, and the whole space A is endowed with the structure of a graded $\mathbb{Z}$-bialgebra through another, graded product, a unit, a coproduct and a counit. Morphisms in $\mathscr{V}$ are maps that respect the $\mathbb{N}$-graduation, all products with their units and the coproduct with its counit. In the statement below, we denote the dual of a finite abelian group G by $G^\wedge$; the assignment $G \rightsquigarrow G^\wedge$ is a contravariant endofunctor of the category of finite abelian groups.

Proposition 17 *The assignments $G \rightsquigarrow \mathscr{F}(\mathbb{Z}[G^\wedge])$, $G \rightsquigarrow \mathsf{MR}(\mathbb{Z}[G^\wedge])$ and $G \rightsquigarrow \operatorname{Rep}(G)$ are contravariant functors from the category of finite abelian groups to the category $\mathscr{V}$. The assignment $G \rightsquigarrow \theta_G$ is a natural transformation from $\mathsf{MR}(\mathbb{Z}[?^\wedge])$ to $\operatorname{Rep}(?)$.*

We leave the proof as a (rather tedious) exercise. The naturality of $G \rightsquigarrow \theta_G$ means that for each morphism $f : G \to G'$ between two finite abelian groups, the diagram

$$\begin{tikzcd}
\mathscr{F}(\mathbb{Z}[G^\wedge]) & & \mathscr{F}(\mathbb{Z}[(G')^\wedge]) \arrow[ll, "\mathscr{F}(\mathbb{Z}[f^\wedge])"'] \\
\mathsf{MR}(\mathbb{Z}[G^\wedge]) \arrow[u, hook] \arrow[d, two heads, "\theta_G"'] & & \mathsf{MR}(\mathbb{Z}[(G')^\wedge]) \arrow[ll, "\mathsf{MR}(\mathbb{Z}[f^\wedge])"'] \arrow[u, hook] \arrow[d, two heads, "\theta_{G'}"] \\
\operatorname{Rep}(G) & & \operatorname{Rep}(G') \arrow[ll, "\operatorname{Rep}(f)"']
\end{tikzcd}$$

is commutative.

To conclude this section, let us observe that Formula (23) implies the commutativity of the diagram

$$\begin{tikzcd}
& \mathscr{F}(\mathbb{Z}\Gamma) \arrow[r, "\simeq", "{\varpi_{\mathrm{tot}}}^\flat"'] & \mathscr{F}(\mathbb{Z}\Gamma)^\vee \arrow[dr, two heads] & \\
\mathsf{MR}(\mathbb{Z}\Gamma) \arrow[ur, hook] \arrow[dr, two heads, "\theta_G"'] & & & \mathsf{MR}(\mathbb{Z}\Gamma)^\vee. \\
& \operatorname{Rep}(G) \arrow[r, "\simeq", "{\beta_{\mathrm{tot}}}^\flat"'] & \operatorname{Rep}(G)^\vee \arrow[ur, hook, "(\theta_G)^\vee"'] &
\end{tikzcd} \tag{25}$$

(The surjectivity of the map $\mathscr{F}(\mathbb{Z}\Gamma)^\vee \to \mathsf{MR}(\mathbb{Z}\Gamma)^\vee$ comes from Proposition 6.) By Theorem 16, θ_G is surjective with kernel $\mathsf{MR}(\mathbb{Z}\Gamma) \cap \mathsf{MR}(\mathbb{Z}\Gamma)^\circ$, which implies that θ_G defines an isomorphism of graded bialgebras

$$\overline{\theta_G} : \mathsf{MR}(\mathbb{Z}\Gamma)/(\mathsf{MR}(\mathbb{Z}\Gamma) \cap \mathsf{MR}(\mathbb{Z}\Gamma)^\circ) \xrightarrow{\simeq} \operatorname{Rep}(G).$$

We can therefore add an horizontal line in the middle of the diagram (25) and get

$$\begin{tikzcd}
 & \mathscr{F}(\mathbb{Z}\Gamma) \arrow[r, "\simeq", "{\varpi_{\mathrm{tot}}}^{\flat}"'] & \mathscr{F}(\mathbb{Z}\Gamma)^{\vee} \arrow[dr, two heads] & \\
\mathsf{MR}(\mathbb{Z}\Gamma) \arrow[ur, hook] \arrow[r, two heads] \arrow[dr, two heads, "\theta_G"'] & \mathsf{MR}(\mathbb{Z}\Gamma)/(\mathsf{MR}(\mathbb{Z}\Gamma)\cap\mathsf{MR}(\mathbb{Z}\Gamma)^{\circ}) \arrow[d, "\overline{\theta_G}", "\wr"'] & \big(\mathsf{MR}(\mathbb{Z}\Gamma)/(\mathsf{MR}(\mathbb{Z}\Gamma)\cap\mathsf{MR}(\mathbb{Z}\Gamma)^{\circ})\big)^{\vee} \arrow[r, hook] & \mathsf{MR}(\mathbb{Z}\Gamma)^{\vee}. \\
 & \mathrm{Rep}(G) \arrow[r, "\simeq", "{\beta_{\mathrm{tot}}}^{\flat}"'] & \mathrm{Rep}(G)^{\vee} \arrow[u, "\overline{\theta_G}^{\vee}", "\wr"'] \arrow[ur, hook, "(\theta_G)^{\vee}"'] &
\end{tikzcd}$$

This is of course an occurence of the diagram (11) with $V = \mathbb{Z}\Gamma$ and $S = \mathsf{MR}(\mathbb{Z}\Gamma)$. As a bonus, we see that the pairing induced by ϖ_{tot} on $S/(S \cap S^{\circ})$ is perfect in the present situation.

Remark 18. In this remark, we consider the case $G = \mathbb{Z}/r\mathbb{Z}$. Hiver, Novelli and Thibon [15] have found an embodiment of the lower half of the diagram (25) in terms of the representation theory of a suitable limit at $q = 0$ of the Ariki-Koike algebra $\mathscr{H}_{n,r}(q)$. More precisely, these authors propose to identify as $\mathbb{Z}$-modules the degree n components $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ and $\mathsf{MR}_n(\mathbb{Z}\Gamma)^{\vee}$ with the Grothendieck groups $K_0(\mathscr{H}_{n,r}(0))$ and $G_0(\mathscr{H}_{n,r}(0))$, respectively. They claim that in this identification, the map $\theta_G^{\vee} \circ \theta_G$ coincides with the Cartan homomorphism $c : K_0(\mathscr{H}_{n,r}(0)) \to G_0(\mathscr{H}_{n,r}(0))$, which describes the Jordan-Hölder multiplicities of the simple modules in a projective module. They also assert that the maps θ_G and $\theta_G^{\vee}$ can be interpreted as arrows in a Cartan-Brauer *cde* triangle

$$\begin{tikzcd}
K_0(\mathscr{H}_{n,r}(0)) \arrow[rr, "c"] \arrow[dr, "e"'] & & G_0(\mathscr{H}_{n,r}(0)) \;, \\
 & K_0(\mathscr{H}_{n,r}(q)) \arrow[ur, "d"'] &
\end{tikzcd}$$

the bottom vertex being the Grothendieck group of the semisimple category of finitely generated $\mathscr{H}_{n,r}(q)$-modules, where q is generic. Yet the fact, apparent in our constructions, that the Cartan map c is a morphism of $K_0(\mathscr{H}_{n,r}(0))$-bimodules is missing in this picture.

4.4 Symmetry property of the Solomon homomorphism

Given any finite group H, the data of a complex-valued function on H is the same thing as the data of a $\mathbb{Z}$-linear map from $\mathbb{Z}H$ into $\mathbb{C}$; we can therefore evaluate an element of $R(H)$ on an element of $\mathbb{Z}H$. Applying this remark to the case of the group $G \wr \mathfrak{S}_n$, we can evaluate an element $\theta_G(y)$ with $y \in \mathsf{MR}_n(\mathbb{Z}\Gamma)$ on an element of $\mathbb{Z}\big[G \wr \mathfrak{S}_n\big] = \mathscr{F}_n(\mathbb{Z}G)$, and in particular on an element x of $\mathsf{MR}_n(\mathbb{Z}G)$. Now G and Γ play symmetric roles, so that the problem of comparing $\theta_G(y)(x)$ and $\theta_\Gamma(x)(y)$ arises.

Theorem 19 *For any $n \geq 1$, $x \in \mathsf{MR}_n(\mathbb{Z}G)$ and $y \in \mathsf{MR}_n(\mathbb{Z}\Gamma)$, there holds $\theta_G(y)(x) = \theta_\Gamma(x)(y)$.*

Proof. In order to better put in evidence the symmetry between G and Γ, we denote the evaluation of a character $\gamma \in \Gamma$ at a point $g \in G$ by a bracket $\langle \gamma, g \rangle$; the same notation can then also be used to denote the evaluation of g, viewed as a character of Γ, at the point γ.

We check the property asserted by the theorem for $x = y_{c_1,g_1} * y_{c_2,g_2} * \cdots * y_{c_k,g_k}$ and $y = y_{d_1,\gamma_1} * y_{d_2,\gamma_2} * \cdots * y_{d_k,\gamma_k}$, where $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ and $\mathbf{d} = (d_1, d_2, \ldots, d_l)$ are two compositions of n, $(g_1, g_2, \ldots, g_k) \in G^k$ and $(\gamma_1, \gamma_2, \ldots, \gamma_l) \in \Gamma^l$. Let us set

$$\tilde{\mathbf{g}} = \tilde{g}_1 \otimes \tilde{g}_2 \otimes \cdots \otimes \tilde{g}_n = (g_1^{\otimes c_1}) \otimes (g_2^{\otimes c_2}) \otimes \cdots \otimes (g_k^{\otimes c_k}),$$

$$\tilde{\boldsymbol{\gamma}} = \tilde{\gamma}_1 \otimes \tilde{\gamma}_2 \otimes \cdots \otimes \tilde{\gamma}_n = (\gamma_1^{\otimes d_1}) \otimes (\gamma_2^{\otimes d_2}) \otimes \cdots \otimes (\gamma_l^{\otimes d_l}).$$

We first compute $\theta_G(y)(x)$. Let $\rho \in X_{\mathbf{c}}$. Noting that the composed map

$$\mathfrak{S}_n \hookrightarrow G \wr \mathfrak{S}_n \twoheadrightarrow (G \wr \mathfrak{S}_n)/(G \wr \mathfrak{S}_{\mathbf{d}})$$

induces a bijection from $\mathfrak{S}_n/\mathfrak{S}_{\mathbf{d}}$ onto $(G \wr \mathfrak{S}_n)/(G \wr \mathfrak{S}_{\mathbf{d}})$ and setting $u_j = d_1 + d_2 + \cdots + d_j$, we compute

$$\begin{aligned}
\theta_G(y)(\rho \cdot (\tilde{\mathbf{g}} \# e_n)) &= \operatorname{Ind}_{G\wr\mathfrak{S}_{\mathbf{d}}}^{G\wr\mathfrak{S}_n} \big(\eta_{d_1}(\gamma_1) \otimes \eta_{d_2}(\gamma_2) \otimes \cdots \otimes \eta_{d_l}(\gamma_l)\big)\big(\rho \cdot (\tilde{\mathbf{g}} \# e_n)\big) \\
&= \sum_{\substack{\pi \in \mathfrak{S}_n/\mathfrak{S}_{\mathbf{d}} \\ \pi^{-1}\rho\pi \in \mathfrak{S}_{\mathbf{d}}}} \big(\eta_{d_1}(\gamma_1) \otimes \eta_{d_2}(\gamma_2) \otimes \cdots \otimes \eta_{d_l}(\gamma_l)\big)\big(\pi^{-1}\rho \cdot (\tilde{\mathbf{g}} \# e_n) \cdot \pi\big) \\
&= \sum_{\substack{\pi \in \mathfrak{S}_n/\mathfrak{S}_{\mathbf{d}} \\ \pi^{-1}\rho\pi \in \mathfrak{S}_{\mathbf{d}}}} \prod_{j=1}^{l} \langle \gamma_j, \tilde{g}_{\pi(u_{j-1}+1)} \tilde{g}_{\pi(u_{j-1}+2)} \cdots \tilde{g}_{\pi(u_j)} \rangle \\
&= \sum_{\substack{\pi \in X_{\mathbf{d}} \\ \rho\pi\mathfrak{S}_{\mathbf{d}} = \pi\mathfrak{S}_{\mathbf{d}}}} \langle \tilde{\gamma}_1, \tilde{g}_{\pi(1)} \rangle \langle \tilde{\gamma}_2, \tilde{g}_{\pi(2)} \rangle \cdots \langle \tilde{\gamma}_n, \tilde{g}_{\pi(n)} \rangle.
\end{aligned}$$

Taking the sum for all $\rho \in X_{\mathbf{c}}$, we find

$$\theta_G(y)(x) = \sum_{\rho \in X_{\mathbf{c}}} \theta_G(y)(\rho \cdot (\tilde{\mathbf{g}} \# e_n)) = \sum_{\substack{\rho \in X_{\mathbf{c}},\ \pi \in X_{\mathbf{d}} \\ \rho\pi\mathfrak{S}_{\mathbf{d}} = \pi\mathfrak{S}_{\mathbf{d}}}} \langle \tilde{\gamma}_1, \tilde{g}_{\pi(1)} \rangle \langle \tilde{\gamma}_2, \tilde{g}_{\pi(2)} \rangle \cdots \langle \tilde{\gamma}_n, \tilde{g}_{\pi(n)} \rangle. \tag{26}$$

In Section 3.2, we have parametrized double cosets $C \in \mathfrak{S}_{\mathbf{c}} \backslash \mathfrak{S}_n / \mathfrak{S}_{\mathbf{d}}$ by matrices $M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}$: to the matrix $M = (m_{ij})$ corresponds the double coset $C(M)$. We aim now at splitting the sum in the right-hand side of (26) according to the double coset C containing π. For that, we set

$$\mathcal{F}(\mathbf{c}, \mathbf{d}, M) = \big\{(\rho, \pi) \in X_{\mathbf{c}} \times (X_{\mathbf{d}} \cap C(M)) \bigm| \rho\pi\mathfrak{S}_{\mathbf{d}} = \pi\mathfrak{S}_{\mathbf{d}}\big\}$$

and we observe that

$$\langle \tilde{\gamma}_1, \tilde{g}_{\pi(1)} \rangle \langle \tilde{\gamma}_2, \tilde{g}_{\pi(2)} \rangle \cdots \langle \tilde{\gamma}_n, \tilde{g}_{\pi(n)} \rangle = \prod_{i=1}^{k} \prod_{j=1}^{l} \langle g_i, \gamma_j \rangle^{m_{ij}}$$

if $\pi \in C(M)$. Then (26) reads

$$\theta_G(y)(x) = \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \left[\big|\mathcal{F}(\mathbf{c}, \mathbf{d}, M)\big| \prod_{i=1}^{k} \prod_{j=1}^{l} \langle g_i, \gamma_j \rangle^{m_{ij}} \right],$$

and by symmetry,

$$\theta_\Gamma(x)(y) = \sum_{N \in \mathscr{M}_{\mathbf{d},\mathbf{c}}} \left[\big|\mathcal{F}(\mathbf{d},\mathbf{c},N)\big| \prod_{j=1}^{l} \prod_{i=1}^{k} \langle \gamma_j, g_i \rangle^{n_{ji}} \right]$$
$$= \sum_{M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}} \left[\big|\mathcal{F}(\mathbf{d},\mathbf{c},M^T)\big| \prod_{i=1}^{k} \prod_{j=1}^{l} \langle g_i, \gamma_j \rangle^{m_{ij}} \right],$$

where M^T denote the transpose of the matrix M. Observing now that the double coset $C(M^T) \in \mathfrak{S}_{\mathbf{d}} \backslash \mathfrak{S}_n / \mathfrak{S}_{\mathbf{c}}$ is equal to $C(M)^{-1} = \{\pi^{-1} \mid \pi \in C(M)\}$, we deduce from Theorem 1.2 and Corollary 2.2 of [8], applied to the group $\mathfrak{S}_n$, that

$$\big|\mathcal{F}(\mathbf{c},\mathbf{d},M)\big| = \big|\mathcal{F}(\mathbf{d},\mathbf{c},M^T)\big|$$

for each matrix $M \in \mathscr{M}_{\mathbf{c},\mathbf{d}}$. The theorem follows. $\square$

This kind of question was first investigated by Jöllenbeck and Reutenauer in [17]; their result corresponds to the (already non-trivial) case where G is the group with one element. A similar symmetry result holds also for the original Solomon descent algebra and the original Solomon homomorphism of an arbitrary finite Coxeter group (see [8]); the critical point in the proof above is a theorem from this latter work.

4.5 The particular case $G = \mathbb{Z}/2\mathbb{Z}$

In this section, we apply our results to the case where $G = \{\pm 1\}$ is the group with two elements. The peculiarity of this case is that $W_n = G \wr \mathfrak{S}_n$ is then the Coxeter group of type B_n. Thus Solomon's constructions [33] can be applied to it: there is a certain subring $\tilde{\Sigma}$ of the group ring $\mathbb{Z}W_n$ and a certain homomorphism of rings $\tilde{\theta}$ from $\tilde{\Sigma}$ to the representation ring $R(W_n)$. This map $\tilde{\theta}$ is not surjective, but Bonnafé and Hohlweg [10] manage to correct the situation. They notice that the subring $\tilde{\Sigma}$ of $\mathbb{Z}W_n \cong \mathscr{F}_n(\mathbb{Z}G)$ is contained in the Mantaci-Reutenauer algebra $\mathsf{MR}_n(\mathbb{Z}G)$ and show how to extend $\tilde{\theta}$ to $\mathsf{MR}_n(\mathbb{Z}G)$. The resulting map, still denoted by $\tilde{\theta}$, is a surjective homomorphism of rings from $\mathsf{MR}_n(\mathbb{Z}G)$ onto $R(W_n)$. The situation now looks like our Theorem 16 (ii), which says that the homomorphism θ_G is a surjective ring homomorphism from $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ onto $R(G \wr \mathfrak{S}_n) = R(W_n)$. Indeed we may identify G and Γ in the present case $G = \{\pm 1\}$, because there is a unique isomorphism between G and Γ. Then both $\tilde{\theta}$ and θ_G are surjective ring homomorphisms from $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ onto $R(W_n)$; our aim in this section is to explain the relationship between them.

We begin by setting the notation, following [10]. Let n be a non-negative integer. We set $G = \{\pm 1\}$ and $W_n = G \wr \mathfrak{S}_n$. The group W_n contains $\mathfrak{S}_n$ as a subgroup; it is generated by the transpositions $s_i \in \mathfrak{S}_n$ (see the proof of Corollary 8) and the element $(-1, 1, 1, \ldots, 1) \in G^n$. Endowed with this system of generators, W_n becomes a Coxeter system.

We agree to denote the subgroup $\mathfrak{S}_n$ of W_n by the somewhat strange convention W_{-n}. Likewise, we denote the trivial subgroup with one element of G^c by G^{-c}, for any positive integer c. We define a signed composition of n as a finite sequence $C = (c_1, c_2, \ldots, c_k)$ of non-zero integers such that $|c_1| + |c_2| + \cdots + |c_k| = n$; then the sequence $C^+ = (|c_1|, |c_2|, \ldots, |c_k|)$ is a composition of n. Given such a sequence C, we observe that the Young subgroup $\mathfrak{S}_{C^+}$ of $\mathfrak{S}_n$, acting on G^n, leaves stable the subgroup $G^C = G^{c_1} \times G^{c_2} \times \cdots \times G^{c_k}$. We can thus make the semidirect product $W_C = G^C \rtimes \mathfrak{S}_{C^+}$; this is a subgroup of $G^n \rtimes \mathfrak{S}_{C^+} = G \wr \mathfrak{S}_{C^+}$, hence of W_n. For instance, $W_{(-n)} = \mathfrak{S}_n = W_{-n}$.

Let C be a signed composition of n. By Proposition 2.8 of [10], each left coset wW_C of W_n modulo W_C contains a unique element of minimal length, called the distinguished representative of this coset. Following [10], we denote the set of all these distinguished representatives by X_C and we define the element $\tilde{x}_C = \sum_{w \in X_C} w$ in the group ring $\mathbb{Z}W_n$.

The dual group Γ of G has also two elements, namely the trivial character t and the sign character s. Since Γ is canonically isomorphic to G, the group ring $\mathbb{Z}\big[\Gamma \wr \mathfrak{S}_n\big] = \mathscr{F}_n(\mathbb{Z}\Gamma)$ is canonically isomorphic to $\mathbb{Z}W_n$. The elements $\tilde{x}_C$ can therefore be viewed as elements in $\mathscr{F}_n(\mathbb{Z}\Gamma)$. To complete the notation, we set

$$z_n = (\underbrace{s, s, \ldots, s}_{n \text{ times}}) \cdot [n(n-1)\cdots 1]$$

for any $n \geq 1$, where $[n(n-1)\cdots 1]$ is the longest permutation in $\mathfrak{S}_n$, and we agree that $\tilde{x}_{(0)}$, $y_{0,s}$, $y_{0,t}$ and z_0 are all equal to the unit of $\mathscr{F}_0(\mathbb{Z}\Gamma) = \mathbb{Z}$.

Proposition 20 *(i) We have the following relations:*

$$\tilde{x}_{(n)} = y_{n,t}, \tag{27}$$

$$\tilde{x}_{(-n)} = \sum_{i=0}^{n} z_i * y_{n-i,t}, \tag{28}$$

$$\sum_{i=0}^{n} (-1)^i \, z_i * y_{n-i,s} = \sum_{i=0}^{n} (-1)^i \, y_{n-i,s} * z_i = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{29}$$

$$\tilde{x}_C = \tilde{x}_{(c_1)} * \tilde{x}_{(c_2)} * \cdots * \tilde{x}_{(c_k)}. \tag{30}$$

for any non-negative integer n and any signed composition $C = (c_1, c_2, \ldots, c_k)$ of n.

(ii) The elements $\tilde{x}_C$ form a basis of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$, where C is a signed composition of n.

Proof. (i) Formula (27) holds because both members are equal to the unit of the ring $\mathbb{Z}W_n$, by definition.

By Example 2.23 in [10], we know that $X_{(-n)}$ is the set of all elements $w \in W_n$ of the form

$$w = \sigma \cdot (\underbrace{-1, -1, \ldots, -1}_{i \text{ times}}, \underbrace{1, 1, \ldots, 1}_{n-i \text{ times}}),$$

where $\sigma \in \mathfrak{S}_n$ is decreasing on the interval $[1, i]$ and increasing on the interval $[i+1, n]$. This entails Formula (28), since in the identification of G with Γ, the elements 1 and -1 correspond to t and s, respectively.

Let n be a positive integer. The set of all compositions of n is a ranked poset when endowed with the refinement order $\preccurlyeq$; here the rank function is the map which associates to a composition $\mathbf{d}$ its number of parts $l(\mathbf{d})$. The equality

$$x_{\mathbf{c}} = \sum_{\substack{\mathbf{d} \models n \\ \mathbf{d} \preccurlyeq \mathbf{c}}} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ D(\sigma) = \mathbf{d}}} \sigma,$$

valid for each composition $\mathbf{c}$ of n, entails by Möbius inversion

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ D(\sigma)=\mathbf{c}}} \sigma = \sum_{\substack{\mathbf{d}\models n\\ \mathbf{d}\preccurlyeq\mathbf{c}}} (-1)^{l(\mathbf{c})-l(\mathbf{d})}\ x_{\mathbf{d}}. \tag{31}$$

Taking $\mathbf{c} = (1, 1, \dots, 1)$ (n times) in Formula (31) and multiplying by $(s, s, \dots, s)$, we obtain

$$z_n = \sum_{\substack{\mathbf{d}\models n\\ \mathbf{d}=(d_1,d_2,\dots,d_l)}} (-1)^{n-l}\ y_{d_1,s} * y_{d_2,s} * \cdots * y_{d_l,s}.$$

From there, one deduces easily Formula (29).

Finally Formula (30) is Example 5.3 in [10].

(ii) Formula (29) implies that each element z_n belongs to $\mathsf{MR}(\mathbb{Z}\Gamma)$. Using Formulas (27), (28) and (30), we then deduce that each element $\tilde{x}_C$ belongs to $\mathsf{MR}(\mathbb{Z}\Gamma)$, where C is a signed composition. In other words, the submodule MR' of $\mathscr{F}(\mathbb{Z}\Gamma)$ spanned over $\mathbb{Z}$ by the elements $\tilde{x}_C$ is contained in $\mathsf{MR}(\mathbb{Z}\Gamma)$. Formula (30) shows furthermore that MR' is a subalgebra for the external product $*$. Observing then that MR' contains all the elements $y_{n,t}$ and $\tilde{x}_{(-n)}$, an easy induction based on Formulas (28) and (29) shows that each z_n and each $y_{n,s}$ is in MR'. This implies that MR' contains $\mathsf{MR}(\mathbb{Z}\Gamma)$ because the latter is generated as an algebra by the elements $y_{n,t}$ and $y_{n,s}$. It follows that the $\mathbb{Z}$-module $\mathsf{MR}(\mathbb{Z}\Gamma) = \mathsf{MR}'$ is spanned by the elements $\tilde{x}_C$.

Now Proposition 3 (or more precisely, its consequence stated at the end of Section 1.3) implies that $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ is a free $\mathbb{Z}$-module whose rank r is equal to the number of words $y_{c_1,\gamma_1} * y_{c_2,\gamma_2} * \cdots * y_{c_k,\gamma_k}$, where $(c_1, c_2, \dots, c_k)$ is a composition of n and each $\gamma_i \in \{t, s\}$. Then any generating family of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ with r elements is a basis thereof. We conclude that the family of elements $\tilde{x}_C$, where C is a signed composition of n, is a basis of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$.
□

Bonnafé and Hohlweg call the submodule spanned by the elements $\tilde{x}_C$ the 'generalized descent algebra' and observe that it coincides with the Mantaci-Reutenauer algebra $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ (see §3.1 in [10]). Assertion (ii) of Proposition 20 is roughly equivalent to this observation, and indeed our proof follows closely the analysis in [10].

The associative algebra $\mathsf{MR}(\mathbb{Z}\Gamma)$ is freely generated by the elements $y_{n,t}$ and $y_{n,s}$, where $n \geq 1$. On the other side, we have defined in Section 4.2 the characters $\eta_n(t)$ and $\varepsilon_n(s)$. Thus there exists a unique morphism of algebras $\tilde{\theta} : \mathsf{MR}(\mathbb{Z}\Gamma) \to \mathrm{Rep}(G)$ that maps $y_{n,t}$ and $y_{n,s}$ to $\eta_n(t)$ and $\varepsilon_n(s)$, respectively.

Proposition 21 *(i) The map $\tilde{\theta}$ enjoys all the properties stated in Theorem 16 for the map θ_G.*

(ii) For any signed composition C of a positive integer n, $\tilde{\theta}(\tilde{x}_C)$ is the character of W_n induced from the trivial character of W_C.

Proof. (i) The graded bialgebra Λ of symmetric functions has a canonical involution ω, which exchanges the complete symmetric function h_n with the elementary symmetric function e_n of the same degree (see I, (2.7) in [22]). Now $\Lambda(\mathrm{Irr}(G))$ is the tensor product $\Lambda(t) \otimes \Lambda(s)$ of two copies of Λ, so that $\mathrm{id}_\Lambda(t) \otimes \omega(s)$ is an involutive automorphism of $\Lambda(\mathrm{Irr}(G))$. Equation (22) shows that the Frobenius characteristic ch maps $\varepsilon_n(s)$ to the element $e_n(s)$ of $\Lambda(\mathrm{Irr}(G))$. Therefore the homomorphism $\mathrm{ch}\circ\tilde\theta$ maps the two elements $y_{n,t}$ and $y_{n,s}$ to $h_n(t)$ and $e_n(s)$, respectively, while $\mathrm{ch}\circ\theta_G$ maps these elements to $h_n(t)$ and $h_n(s)$. Thus the diagram

$$\begin{tikzcd}
 & \mathrm{Rep}(G) \arrow[r, "\mathrm{ch}", "\simeq"'] & \Lambda(\mathrm{Irr}(G)) \arrow[dd, "\mathrm{id}_\Lambda(t)\otimes\omega(s)"] \\
\mathsf{MR}(\mathbb{Z}\Gamma) \arrow[ur, "\theta_G"] \arrow[dr, "\tilde\theta"'] & & \\
 & \mathrm{Rep}(G) \arrow[r, "\mathrm{ch}", "\simeq"'] & \Lambda(\mathrm{Irr}(G))
\end{tikzcd}$$

is commutative. Since ch and $\mathrm{id}_\Lambda(t)\otimes\omega(s)$ are isomorphisms of graded bialgebras, $\tilde\theta$ inherits from θ_G the properties stated in Assertion (i) of Theorem 16. The proof of Assertions (ii) and (iii) of Theorem 16 presented in Section 4.3 can be repeated with evident adjustments to the case of $\tilde\theta$; the main difference lies in the proof of the multiplicativity of $\tilde\theta$ with respect to the internal product of $\mathsf{MR}_n(\mathbb{Z}\Gamma)$ and the ordinary product of $R(W_n)$, where one must use the equalities $\eta_n(t)\varepsilon_n(s)=\varepsilon_n(s)$ and $\varepsilon_n(s)\varepsilon_n(s)=\eta_n(t)$.

(ii) Let u be an indeterminate. Applying $\tilde\theta$ to Formulas (28) and (29) and summing over n, we find

$$\left(\sum_{n\geq 0}\tilde\theta(z_n)u^n\right)*\left(\sum_{n\geq 0}(-1)^n\,\tilde\theta(y_{n,s})u^n\right)=\left(\sum_{n\geq 0}(-1)^n\,\tilde\theta(y_{n,s})u^n\right)*\left(\sum_{n\geq 0}\tilde\theta(z_n)u^n\right)=1,$$

$$\sum_{n\geq 0}\tilde\theta(\tilde x_{(-n)})u^n=\left(\sum_{n\geq 0}\tilde\theta(z_n)u^n\right)*\left(\sum_{n\geq 0}\tilde\theta(y_{n,t})u^n\right).$$

In Proposition 15, we have constructed an homomorphism H from the additive group $R(G)$ into $\big(\mathrm{Rep}(G)[[u]]\big)^\times$ such that

$$H(t)=\sum_{n\geq 0}\eta_n(t)u^n=\sum_{n\geq 0}\tilde\theta(y_{n,t})u^n,$$

$$H(-s)=\sum_{n\geq 0}(-1)^n\varepsilon_n(s)u^n=\sum_{n\geq 0}(-1)^n\tilde\theta(y_{n,s})u^n.$$

Then

$$\sum_{n\geq 0}\tilde\theta(z_n)u^n=\left(\sum_{n\geq 0}(-1)^n\tilde\theta(y_{n,s})u^n\right)^{-1}=H(-s)^{-1}=H(s),$$

which implies in turn

$$\sum_{n\geq 0}\tilde\theta(\tilde x_{(-n)})u^n=H(s)*H(t)=H(t+s)=\sum_{n\geq 0}\eta_n(t+s)u^n.$$

It follows that $\tilde\theta(\tilde x_{(-n)})=\eta_n(t+s)$. Now the character $\eta_n(t+s)$ of $G\wr\mathfrak{S}_n$ is induced from the trivial representation of $\mathfrak{S}_n$, because $t+s$ is the regular character of G. Therefore $\tilde\theta(\tilde x_{(-n)})$ is the character of W_n induced from the trivial character of W_{-n}.

On the other side, $\tilde\theta(\tilde x_{(n)}) = \tilde\theta(y_{n,t}) = \eta_n(t)$ is the trivial character of W_n. Using the transitivity of induction, we thus find that for any signed composition $C = (c_1, c_2, \ldots, c_k)$ of n,

$$\begin{aligned}\tilde\theta(\tilde x_C) &= \tilde\theta\big(\tilde x_{(c_1)}\big) * \tilde\theta\big(\tilde x_{(c_2)}\big) * \cdots * \tilde\theta\big(\tilde x_{(c_k)}\big)\\ &= \operatorname{Ind}_{G\wr\mathfrak{S}_{C^+}}^{G\wr\mathfrak{S}_n}\left(\operatorname{Ind}_{W_{c_1}}^{W_{|c_1|}} 1 \otimes \operatorname{Ind}_{W_{c_2}}^{W_{|c_2|}} 1 \otimes \cdots \otimes \operatorname{Ind}_{W_{c_k}}^{W_{|c_k|}} 1\right)\\ &= \operatorname{Ind}_{W_C}^{W_n} 1,\end{aligned}$$

taking into account the identifications

$$G\wr\mathfrak{S}_n = W_n, \quad G\wr\mathfrak{S}_{C^+} \cong W_{|c_1|}\times W_{|c_2|}\times\cdots\times W_{|c_k|} \quad \text{and} \quad W_{c_1}\times W_{c_2}\times\cdots\times W_{c_k} \cong W_C.$$

This concludes the proof.
□

Assertion (ii) of this proposition says that our homomorphism $\tilde\theta$ is equal to the homomorphism defined by Bonnafé and Hohlweg §3.1 in [10]. It follows then from the results of these authors that $\tilde\theta$ extends Solomon's original homomorphism.

On the contrary, θ_G does not extend Solomon's original homomorphism. Indeed we observe that the parabolic subgroups of the Coxeter system W_n are the subgroups W_C, where the signed composition $C = (c_1, c_2, \ldots, c_k)$ has all its parts negative with the possible exception of c_1. Therefore the original Solomon algebra of W_n is the submodule $\tilde\Sigma$ of the group ring $\mathbb{Z}W_n$ spanned by the elements $\tilde x_C$ for such signed compositions. Taking $n = 2$ and using Relations (27)–(30), we check that the element $y_{2,s} = \tilde x_{(-1,-1)} - \tilde x_{(1,-1)} + \tilde x_{(2)} - \tilde x_{(-2)}$ belongs to $\tilde\Sigma$. Its image under $\operatorname{ch}\circ\theta_G$, namely $\operatorname{ch}(\eta_2(s)) = h_2(s)$, is different from its image under $\operatorname{ch}\circ\tilde\theta$, namely $\omega(h_2(s)) = e_2(s)$; it follows that $\theta_G\big|_{\tilde\Sigma}$ does not coincide with Solomon's original homomorphism $\tilde\theta\big|_{\tilde\Sigma}$.

On the other side, our map θ_G shares with Solomon's original homomorphism a property that Bonnafé and Hohlweg's extension $\tilde\theta$ does not have, namely the symmetry property of Theorem 19. Again a counterexample can be found already for $n = 2$: one can indeed check that the value of the character $\tilde\theta(\tilde x_{(-2)})$ on the element $\tilde x_{(1,1)}$ is 6, while the value of $\tilde\theta(\tilde x_{(1,1)})$ on $\tilde x_{(-2)}$ is 4.

5 Coloured combinatorial Hopf algebras

In the previous sections, we have presented our main constructions and the applications which have motivated them. In the case where G is the group with one element, the diagram (25) is the usual diagram which relates the different kinds of symmetric functions: ordinary, noncommutative, quasisymmetric, free quasisymmetric. This diagram can be enriched with other bialgebras: the plactic and the coplactic bialgebras [30], the Loday-Ronco bialgebra [20], the peak algebra [34], etc.

Here we define analogues of some of the plactic and the coplactic bialgebras and we insert them in (25). The analogue of the coplactic bialgebra presents two interests: first, the Solomon homomorphism θ_G can be extended to it in a natural way; second it is related to a construction already present in the litterature, which we call the Robinson-Schensted-Okada correspondence.

5.1 Categorical framework

We start by setting up quickly a clean framework adapted to our goal. We define a category $\mathscr{E}$ as follows. The objects of $\mathscr{E}$ are pairs $(B, ?^*)$, where B is a finite set and $?^* : b \mapsto b^*$ is an involutive map from B to B. We generally omit the involution in the notation, writing just B. Given two objects B and C of $\mathscr{E}$, an homomorphism from B to C is the data of a bijection φ from a subset B' of B onto a subset C' of C.

Any finite group Γ can be considered as an object of $\mathscr{E}$, where the involution $?^*$ is the map $\gamma \mapsto \gamma^{-1}$. The set $\mathrm{Irr}(H)$ of irreducible characters of a finite group H can also be considered as an object of $\mathscr{E}$, with the complex conjugation as involution $?^*$.

An object $(B, ?^*)$ of $\mathscr{E}$ is viewed as the basis of the free $\mathbb{K}$-module $\mathbb{K}B$. We define a pairing ϖ on $\mathbb{K}B$ by setting $\varpi(b, b')$ equal to 1 if $b' = b^*$ and equal to 0 otherwise; this pairing is perfect and symmetric, for $b \mapsto b^*$ is involutive. A morphism $f : B \to C$ induces a linear map $\mathbb{K}f : \mathbb{K}B \to \mathbb{K}C$ as follows: if f is defined by the bijection $\varphi : B' \to C'$, then $\mathbb{K}f$ maps an element b of the basis B to $\varphi(b)$ if $b \in B'$ and to 0 otherwise.

Let B be an object of $\mathscr{E}$. We can trace the constructions of Section 1 at the level of bases. In more details, the group $\mathfrak{S}_n$ acts by permutation on B^n. We denote the cartesian product of B^n with $\mathfrak{S}_n$ by $B \wr \mathfrak{S}_n$ and endow it with the following two-sided action of $\mathfrak{S}_n$:

$$\pi \cdot (b_1, b_2, \ldots, b_n; \sigma) = (b_{\pi^{-1}(1)}, b_{\pi^{-1}(2)}, \ldots, b_{\pi^{-1}(n)}); \pi\sigma)$$
$$(b_1, b_2, \ldots, b_n; \sigma) \cdot \pi = (b_1, b_2, \ldots, b_n; \sigma\pi).$$

We can view $B \wr \mathfrak{S}_n$ as a basis of the free $\mathbb{K}$-module $\mathscr{F}_n(\mathbb{K}B)$ by identifying the element $(b_1, b_2, \ldots, b_n; \sigma)$ of $B \wr \mathfrak{S}_n$ with the element $\left[(b_1 \otimes b_2 \otimes \cdots \otimes b_n)\#\sigma\right]$ of $\mathscr{F}_n(\mathbb{K}B)$.

We can then continue the construction and obtain from B the free quasisymmetric graded bialgebra $\mathscr{F}(\mathbb{K}B)$ and the Novelli-Thibon algebra $\mathsf{NT}(\mathbb{K}B)$. Now the construction of the Mantaci-Reutenauer bialgebra requires additionnally the data of a coalgebra structure. But given a finite set B, one can always define a structure of a coalgebra on $\mathbb{K}B$ by requiring that the elements of B are group-like; in other words, one agrees that the coproduct δ and the counit ε are defined by

$$\delta(b) = b \otimes b \quad \text{and} \quad \varepsilon(b) = 1$$

for any $b \in B$. Endowing $\mathbb{K}B$ with this structure, we can construct the Mantaci-Reutenauer bialgebra $\mathsf{MR}(\mathbb{K}B)$; to translate into the notation the fact that this bialgebra depends on the choice of the basis B of $\mathbb{K}B$, we denote it by $\mathscr{D}(B)$. By Proposition 3, the associative algebra $\mathscr{D}(B)$ is freely generated by the elements $y_{n,b}$ with $n \geq 1$ and $b \in B$. The assignments $B \rightsquigarrow \mathscr{F}(\mathbb{K}B)$ and $B \rightsquigarrow \mathscr{D}(B)$ are covariant functors from the category $\mathscr{E}$ to the category of graded bialgebras.

Finally, given an object B of $\mathscr{E}$, the perfect symmetric pairing ϖ on $\mathbb{K}B$ can be extended to a perfect symmetric pairing ϖ_{tot} on $\mathscr{F}(\mathbb{K}B)$ (see Section 2.2). The basis $B \wr \mathfrak{S}_n$ is dual to itself with respect to ϖ_{tot}; more precisely, the basis element dual to $\alpha = (b_1, b_2, \ldots, b_n; \sigma)$ is $\alpha^* = \left[\sigma^{-1} \cdot (b_1^*, b_2^*, \ldots, b_n^*; e_n)\right]$.

5.2 Coloured descent compositions

Let B be an object of $\mathscr{E}$. Since $\mathscr{D}(B)$ is a direct summand of $\mathscr{F}(\mathbb{K}B)$ by Proposition 6, the dual bialgebra $\mathscr{D}(B)^\vee$ is canonically isomorphic to the quotient $\mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$. Our aim is to study the subbialgebra $\mathscr{D}(B)$ and the quotient bialgebra $\mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$ of $\mathscr{F}(\mathbb{K}B)$ on the level of basis in a combinatorial way.

We begin with definitions. A B-composition is a finite sequence $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \dots, (c_k, b_k))$ of elements of $\mathbb{Z}_{>0} \times B$. The size of $\mathbf{c}$ is the integer $\|\mathbf{c}\| = c_1 + c_2 + \cdots + c_k$. The dual of $\mathbf{c}$ is the B-composition $\mathbf{c}^* = ((c_1, b_1^*), (c_2, b_2^*), \dots, (c_k, b_k^*))$. Given two B-compositions $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \dots, (c_k, b_k))$ and $\mathbf{d} = ((d_1, b_1'), (d_2, b_2'), \dots, (d_l, b_l'))$ of the same size n, we say that $\mathbf{c}$ is a refinement of $\mathbf{d}$ and we write $\mathbf{c} \succcurlyeq \mathbf{d}$ if there holds

$$(c_1, c_2, \dots, c_k) \succcurlyeq (d_1, d_2, \dots, d_l),$$
$$(\underbrace{b_1, \dots, b_1}_{c_1 \text{ times}}, \underbrace{b_2, \dots, b_2}_{c_2 \text{ times}}, \dots, \underbrace{b_k, \dots, b_k}_{c_k \text{ times}}) = (\underbrace{b_1', \dots, b_1'}_{d_1 \text{ times}}, \underbrace{b_2', \dots, b_2'}_{d_2 \text{ times}}, \dots, \underbrace{b_k', \dots, b_k'}_{d_k \text{ times}}).$$

The relation $\preccurlyeq$ is a partial order on the set of B-compositions of n.

We associate to each element $\alpha \in B \wr \mathfrak{S}_n$ two B-compositions $D(\alpha)$ and $R(\alpha)$ of size n. The 'descent composition' $D(\alpha)$ is constructed by the following procedure, due to Mantaci and Reutenauer [24]. We first write α as $\big(b_{\sigma^{-1}(1)}, b_{\sigma^{-1}(2)}, \dots, b_{\sigma^{-1}(n)}; \sigma\big) = \big[\sigma \cdot (b_1, b_2, \dots, b_n; e_n)\big]$, as before. Then one decomposes the interval $[1, n]$ into the largest subintervals on which the map $i \mapsto b_i$ is constant, and after that, one decomposes each such subinterval into the largest subsubintervals on which the map $i \mapsto \sigma(i)$ is increasing. Each subsubinterval yields a pair formed by its length and the value taken by the map $i \mapsto b_i$. Then $D(\alpha)$ is the ordered list of all these pairs. We define the 'receding composition' of α by the equality $R(\alpha) = D(\alpha^*)^*$. An example illustrates these definitions. We take a and b in B, $n = 7$ and $\alpha = (a, a, b, a, b, b, a; 1426735) = \big[1426735 \cdot (a, a, a, b, a, b, b; e_7)\big]$; then $D(\alpha) = ((2, a), (1, a), (1, b), (1, a), (2, b))$ and $R(\alpha) = ((2, a), (1, b), (1, a), (1, b), (1, b), (1, a))$.

For any $i \in \{1, 2, \dots, n-1\}$, we denote by s_i the transposition in $\mathfrak{S}_n$ that exchanges i and $i + 1$. We say that two elements α and α' of $B \wr \mathfrak{S}_n$ are related by an Atkinson relation and we write $\alpha \underset{A}{\sim} \alpha'$ if there exists an index i such that:

- $\alpha' = \alpha \cdot s_i$;
- writing $\alpha = (b_1, b_2, \dots, b_n; \sigma)$, the map $j \mapsto b_j$ is not constant on the interval $[\sigma(i), \sigma(i+1)]$ or the inequality $|\sigma(i+1) - \sigma(i)| > 1$ holds.

(In the case where $\sigma(i+1) < \sigma(i)$, the notation $[\sigma(i), \sigma(i+1)]$ means the interval $[\sigma(i+1), \sigma(i)]$.) The Atkinson relation is clearly symmetric.

The following proposition explains the relation between these combinatorial definitions and the maps $\mathscr{D}(B) \hookrightarrow \mathscr{F}(\mathbb{K}B)$ and $\mathscr{F}(\mathbb{K}B) \twoheadrightarrow \mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$.

Proposition 22 *Let B be an object of $\mathscr{E}$ and let n be a non-negative integer.*

(i) The submodule $\mathscr{D}(B) \cap \mathscr{F}_n(\mathbb{K}B)$ of $\mathscr{F}_n(\mathbb{K}B)$ is spanned over $\mathbb{K}$ by the elements

$$\sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ D(\alpha) = \mathbf{c}}} \alpha,$$

where $\mathbf{c}$ is a B-composition.

(ii) Two elements α and α' in $B \wr \mathfrak{S}_n$ have the same receding composition $R(\alpha) = R(\alpha')$ if and only if there exists a sequence of elements α_1, α_2, ..., α_k such that

$$\alpha = \alpha_1 \underset{A}{\sim} \alpha_2 \underset{A}{\sim} \cdots \underset{A}{\sim} \alpha_k = \alpha'.$$

(iii) The module $\mathscr{D}(B)^\circ \cap \mathscr{F}_n(\mathbb{K}B)$ is spanned over $\mathbb{K}$ by the set

$$\{\alpha - \alpha' \mid \alpha \text{ and } \alpha' \text{ in } B \wr \mathfrak{S}_n \text{ with } \alpha \underset{A}{\sim} \alpha'\}.$$

Proof. (i) We observe that for any B-composition $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \ldots, (c_k, b_k))$ of n,

$$\begin{aligned} y_{c_1,b_1} * y_{c_2,b_2} * \cdots * y_{c_k,b_k} &= x_{(c_1,c_2,\ldots,c_k)} \cdot (b_1^{\otimes c_1} \otimes b_2^{\otimes c_1} \otimes \cdots \otimes b_k^{\otimes c_k} \# e_n) \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ D(\sigma) \preccurlyeq (c_1,c_2,\ldots,c_k)}} \sigma(b_1^{\otimes c_1} \otimes b_2^{\otimes c_1} \otimes \cdots \otimes b_k^{\otimes c_k} \# e_n) \\ &= \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ D(\alpha) \preccurlyeq \mathbf{c}}} \alpha. \end{aligned} \tag{32}$$

Assertion (i) follows easily from (32) and from the fact that $\mathscr{D}(B) \cap \mathscr{F}_n(\mathbb{K}B)$ is spanned over $\mathbb{K}$ by such products $y_{c_1,b_1} * y_{c_2,b_2} * \cdots * y_{c_k,b_k}$.

(ii) For any two elements α and α' in $B \wr \mathfrak{S}_n$, the relation $\alpha^* \underset{A}{\sim} \alpha'^*$ is equivalent to the existence of an index $i \in \{1, 2, \ldots, n-1\}$ such that:

- $\alpha' = s_i \cdot \alpha$;
- writing $\alpha = \big[\sigma \cdot (b_1, b_2, \ldots, b_n; e_n)\big]$, the map $j \mapsto b_j$ is not constant on the interval $[\sigma^{-1}(i), \sigma^{-1}(i+1)]$ or the inequality $|\sigma^{-1}(i+1) - \sigma^{-1}(i)| > 1$ holds.

An easy verification shows then that $D(\alpha) = D(\alpha')$ as soon as $\alpha^* \underset{A}{\sim} \alpha'^*$, and therefore as soon as α^* and α'^* are related by a sequence of Atkinson relations.

Let now $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \ldots, (c_k, b_k))$ be a B-composition of n and set

$$(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) = (\underbrace{b_1, b_1, \ldots, b_1}_{c_1 \text{ times}}, \underbrace{b_2, b_2, \ldots, b_2}_{c_2 \text{ times}}, \ldots, \underbrace{b_k, b_k, \ldots, b_k}_{c_k \text{ times}}).$$

Each element α in $B \wr \mathfrak{S}_n$ such that $D(\alpha) = \mathbf{c}$ can be written $\alpha = \big[\sigma \cdot (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n; e_n)\big]$. We now apply successive Atkinson relations to α^* to reduce as much as possible the number of inversions of σ, obtaining eventually an element α_0^*. By the previous paragraph, the descent composition is preserved at each step of the process, so that $D(\alpha_0) = \mathbf{c}$.

We now observe that α_0 depends only on $\mathbf{c}$ and not on the element α from which we started or on the choices made during the reduction process. Indeed let us write $\alpha_0 = \big[\sigma_0 \cdot (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n; e_n)\big]$. The equality $D(\alpha_0) = \mathbf{c}$ holds, and there is no permutation $\sigma' \in \mathfrak{S}_n$ with smaller number of inversions than σ_0 such that $\alpha^* \underset{A}{\sim} \alpha'^*$, where $\alpha' = \big[\sigma' \cdot (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n; e_n)\big]$. Setting $t_j = c_1 + c_2 + \cdots + c_j$ for each $j \in \{1, 2, \ldots, k-1\}$, these constraints imply in turn the equivalence of the three following assertions for each $i \in [1, n-1]$:

- there exists an index $j \in \{1, 2, \ldots, k-1\}$ such that $i = t_j$ and $b_j = b_{j+1}$;
- $\sigma(i) > \sigma(i+1)$;
- $\sigma(i) = \sigma(i+1) + 1$.

The uniqueness of σ_0, hence of α_0, can be easily derived from this.

Summarizing, we have seen that for any two elements α and α' in $B \wr \mathfrak{S}_n$:

- If α^* and α'^* are related by a sequence of Atkinson relations, then $D(\alpha) = D(\alpha')$.
- If $D(\alpha) = D(\alpha')$, then starting from α^* as well as α'^*, one may reach the same element α_0^* by applying a sequence of Atkinson relations.

Therefore α^* and α'^* are related by a sequence of Atkinson relations if and only if $D(\alpha) = D(\alpha')$. This fact is equivalent to Assertion (ii).

(iii) Let $x = \sum_{\alpha \in B \wr \mathfrak{S}_n} a_\alpha \, \alpha$ be an element of $\mathscr{F}_n(\mathbb{K}B)$, where $a_\alpha \in \mathbb{K}$. Then for any B-composition $\mathbf{c}$ of size n,

$$\varpi_{\mathrm{tot}}\left(x, \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ D(\alpha) = \mathbf{c}}} \alpha\right) = \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ D(\alpha) = \mathbf{c}}} a_{\alpha^*} = \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ R(\alpha) = \mathbf{c}^*}} a_\alpha.$$

The element x is orthogonal to $\mathscr{D}(B)$ if and only if this quantity vanishes for all $\mathbf{c}$. Assertion (iii) is then a direct consequence of Assertion (ii).
□

The result stated in Proposition 22 (ii) above was first obtained by Atkinson (see [4], Corollary on p. 352) for the case where B has only one element.

We already mentioned in Section 5.1 that the assignments $B \rightsquigarrow \mathscr{F}(\mathbb{K}B)$ and $B \rightsquigarrow \mathscr{D}(B)$ are covariant functors from the category $\mathscr{E}$ to the category of graded bialgebras. By Proposition 22 (iii), the biideal $\mathscr{D}(B)^\circ$ of $\mathscr{F}(\mathbb{K}B)$ is functorial in B, which implies that $B \rightsquigarrow \mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$ is a covariant functor from $\mathscr{E}$ to the category of graded bialgebras. One may also observe that $B \rightsquigarrow \mathscr{D}(B)^\vee$ is a contravariant functor between the same categories, and that the two graded bialgebras $\mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$ and $\mathscr{D}(B)^\vee$ are isomorphic.

5.3 Tableaux and the Robinson-Schensted-Knuth correspondence

In this section, we recall some classical stuff to fix the notations needed to present the Robinson-Schensted-Okada correspondence.

Let $\mathscr{A}$ be a totally ordered set (an alphabet). An $\mathscr{A}$-weight is a finite multiset of $\mathscr{A}$, that is, a map $\mu : \mathscr{A} \to \mathbb{N}$ with finite support. Thus for instance a $\mathbb{Z}_{>0}$-weight is an infinite sequence $\mu = (\mu_1, \mu_2, \ldots)$ of non-negative integers, all of whose terms but a finite number vanish. The size of a weight μ is the sum of its values; we denote it by $|\mu|$. The weight of a word $w = a_1 a_2 \cdots a_n$ with letters in $\mathscr{A}$ is the $\mathscr{A}$-weight μ such that any letter $a \in \mathscr{A}$ occurs $\mu(a)$ times in w; we denote it by $\mathrm{wt}(w)$.

A semistandard tableau T with entries in $\mathscr{A}$ is a Young diagram whose boxes are labelled by letters in $\mathscr{A}$ in such a way that the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. The shape of T is the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that T has λ_1 boxes in the first row, λ_2 boxes in the second row, and so on; we denote it by $\mathrm{sh}(T)$. The weight of T is the $\mathscr{A}$-weight μ such that any letter $a \in \mathscr{A}$

occurs $\mu(a)$ times as the label in a box of T; we denote it by $\mathrm{wt}(T)$. A tableau T filled with positive integers is said to be standard if its weight is

$$(\underbrace{1,1,\ldots,1}_{|\operatorname{sh}(T)| \text{ times}},0,0,\ldots).$$

To a word $w = a_1a_2\cdots a_n$ with letters in $\mathscr{A}$, the Robinson-Schensted correspondence associates a pair (P,Q) of tableaux with the same shape, such that $\mathrm{wt}(P) = \mathrm{wt}(w)$ and Q is standard. The insertion tableau P is constructed inductively using the well-known 'bump' procedure; the label in a box of the record tableau Q indicate the number of the step at which this box appears during the making of P.

One says that two words $w = a_1a_2\cdots a_n$ and $w' = a'_1a'_2\cdots a'_n$ with letters in $\mathscr{A}$ and of the same length are related by a Knuth relation and one writes $w \underset{K}{\sim} w'$ if one can find two decompositions $w = xuy$ and $w' = xu'y$ of w and w' as the concatenation of subwords in such a way that one of the two following conditions holds:

(a) There exist three letters $a \le b < c$ in $\mathscr{A}$ such that $\{u,u'\} = \{acb, cab\}$.

(b) There exist three letters $a < b \le c$ in $\mathscr{A}$ such that $\{u,u'\} = \{bac, bca\}$.

The following results can be found in [19].

Proposition 23 *(i) Let (P,Q) be the image of the word $w = a_1a_2\cdots a_n$ under the Robinson-Schensted correspondence. Then $a_i > a_{i+1}$ if and only if the box of Q that contains the label $i+1$ appears south or south-west to the box that contains the label i.*

(ii) Two words w and w' with letters in $\mathscr{A}$ have the same insertion tableau P under the Robinson-Schensted correspondence if and only if there exists a sequence of words w_1, w_2, …, w_k such that

$$w = w_1 \underset{K}{\sim} w_2 \underset{K}{\sim} \cdots \underset{K}{\sim} w_k = w'.$$

Knuth has extended the scope of the Robinson-Schensted correspondence to a slightly more general situation, which we recall now. Let $\mathscr{B}$ be a second alphabet. Given an $\mathscr{A}$-weight μ and a $\mathscr{B}$-weight ν, we denote by $\mathscr{M}_{\mu,\nu}$ the set of matrices $M = (m_{ab})_{(a,b)\in\mathscr{A}\times\mathscr{B}}$ with non-negative integral entries and with row-sum μ and column-sum ν, that is,

$$\mu_a = \sum_{b\in\mathscr{B}} m_{ab} \quad \text{for all } a \text{ and } \quad \nu_b = \sum_{a\in\mathscr{A}} m_{ab} \quad \text{for all } b.$$

(This condition tacitely implies that all but a finite number of entries of M vanish and that μ and ν have the same size. The notation $\mathscr{M}_{\mathbf{c},\mathbf{d}}$ used in Section 3.2 is a particular case of this one.)

We order the product $\mathscr{B}\times\mathscr{A}$ lexicographically. An element $M \in \mathscr{M}_{\mu,\nu}$ can be seen as a finite multiset of $\mathscr{B}\times\mathscr{A}$, whose elements can be listed in increasing order: $((b_1,a_1),(b_2,a_2),\ldots,$ $(b_n,a_n))$. In this way, M determines two words $w_{\mathscr{A}} = a_1a_2\cdots a_n$ and $w_{\mathscr{B}} = b_1b_2\cdots b_n$, with the obvious property that $\mu = \mathrm{wt}(w_{\mathscr{A}})$ and $\nu = \mathrm{wt}(w_{\mathscr{B}})$. The Robinson-Schensted correspondence applied to $w_{\mathscr{A}}$ yields a pair of tableaux $(P,\tilde{Q})$. Substituting in each box of $\tilde{Q}$

the label j by the letter b_j, we obtain a tableau Q. With these notations, Knuth has shown in [19] that the map $T \mapsto (P, Q)$ is a bijection from $\mathscr{M}_{\mu,\nu}$ onto

$$\left\{ (P,Q) \;\middle|\; \begin{array}{c} P \text{ and } Q \text{ tableaux with } \operatorname{sh}(P) = \operatorname{sh}(Q), \\ \operatorname{wt}(P) = \mu \text{ and } \operatorname{wt}(Q) = \nu \end{array} \right\}.$$

Furthermore the transposition of M corresponds to the exchange of P and Q. It is usual to call this map the RSK correspondence.

5.4 The Robinson-Schensted-Okada correspondence

Let B be an object of $\mathscr{E}$. We define a B-partition as a family $\boldsymbol{\lambda} = (\lambda_b)_{b\in B}$ of partitions. The size of $\boldsymbol{\lambda}$ is the integer $\|\boldsymbol{\lambda}\| = \sum_{b\in B} |\lambda_b|$. The dual of $\boldsymbol{\lambda}$ is the B-partition $\boldsymbol{\lambda}^* = \big(b \mapsto \lambda_{b^*}\big)$.

Let now $\mathscr{A}$ be an alphabet. We define a B-tableau with entries in $\mathscr{A}$ as a family $\mathbf{T} = (T_b)_{b\in B}$ of tableaux whose boxes are filled by elements of $\mathscr{A}$. The shape of $\mathbf{T}$ is the B-partition $\operatorname{sh}(\mathbf{T}) = (\operatorname{sh}(T_b))_{b\in B}$. A B-tableau $\mathbf{T}$ with entries in $\mathbb{Z}_{>0}$ is said to be standard if

$$\sum_{b\in B} \operatorname{wt}(T_b) = (\underbrace{1, 1, \ldots, 1}_{\|\operatorname{sh}(\mathbf{T})\| \text{ times}}, 0, 0, \ldots).$$

In other words, all the labels 1, 2, …, n are used once and only once to fill the boxes of the tableaux T_b, where $n = \|\operatorname{sh}(\mathbf{T})\|$ is the total number of boxes in $\mathbf{T}$.

Now let $w = x_1 x_2 \cdots x_n$ be a word whose letters $x_i = (a_i, b_i)$ belong to $\mathscr{A} \times B$. For each $b \in B$, we form a matrix $M^{(b)} = \Big(m^{(b)}_{aj}\Big)_{(a,j)\in\mathscr{A}\times[1,n]}$ by setting $m^{(b)}_{aj}$ equal to 1 if $(a_j, b_j) = (a, b)$ and equal to 0 otherwise. From the matrix $M^{(b)}$, the RSK correspondence produces a pair of tableaux (P_b, Q_b) with the same shape. The family $\mathbf{P} = (P_b)_{b\in B}$ is a B-tableau with entries in $\mathscr{A}$ such that $\sum_{b\in B} \operatorname{wt}(P_b)$ is the weight of the word $a_1 a_2 \cdots a_n$; the family $\mathbf{Q} = (Q_b)_{b\in B}$ is a standard B-tableau; the tableaux $\mathbf{P}$ and $\mathbf{Q}$ have the same shape. We say that $\mathbf{P}$ and $\mathbf{Q}$ are the insertion and record tableaux of w, respectively, and we call the map $w \mapsto (\mathbf{P}, \mathbf{Q})$ the RSO correspondence (for Robinson-Schensted-Okada).

One can adapt the Knuth relations to the RSO correspondence in the following way. We say that two words $w = x_1 x_2 \cdots x_n$ and $w' = x'_1 x'_2 \cdots x'_n$ of the same length with letters $x_i = (a_i, b_i)$ and $x'_i = (a'_i, b'_i)$ are related by a Knuth relation and we write $w \underset{K}{\sim} w'$ if there exists an index i such that one of the following two conditions holds:

(c) $b_i \neq b_{i+1}$, $x_i = x'_{i+1}$, $x_{i+1} = x'_i$, and $x_j = x'_j$ for all $j \notin \{i, i+1\}$.

(d) $b_i = b_{i+1} = b_{i+2} = b'_i = b'_{i+1} = b'_{i+2}$, the two words $u = a_i a_{i+1} a_{i+2}$ and $u' = a'_i a'_{i+1} a'_{i+2}$ are as in Condition (a) or (b), and $x_j = x'_j$ for all $j \notin \{i, i+1, i+2\}$.

Then we have the following analogue of Knuth's theorem.

Proposition 24 *Two words w and w' with letters in $\mathscr{A} \times B$ have the same insertion tableau $\mathbf{P}$ under the RSO correspondence if and only if there exists a sequence of words w_1, w_2, …, w_k such that*

$$w = w_1 \underset{K}{\sim} w_2 \underset{K}{\sim} \cdots \underset{K}{\sim} w_k = w'.$$

Proof. Let $w = x_1 x_2 \cdots x_n$ be a word with the letters $x_i = (a_i, b_i)$, and let $\mathbf{P} = (P_b)_{b \in B}$ be the insertion tableau of w. For each $B \in B$, we form the word $w^{(b)} = a_{j_1} a_{j_2} \cdots a_{j_k}$, where $(j_1, j_2, \ldots, j_k)$ is the list in increasing order of all indices j for which $b_j = b$. By construction, P_b is the insertion tableau in the RSK image of the matrix $M^{(b)}$, so P_b is the insertion tableau of the word $w^{(b)}$. We fix an enumeration $b_1, b_2, \ldots, b_l$ of the elements of B and we form the word $\overline{w} = w^{(b_1)} w^{(b_2)} \cdots w^{(b_l)}$ by concatenation. Obviously w and $\overline{w}$ are related by a sequence of Knuth relations of type (c).

Let now w' be a word with the same length as w. We produce the words $w'^{(b)}$ and $\overline{w'} = w'^{(b_1)} w'^{(b_2)} \cdots w'^{(b_l)}$ in the same way as we formed $w^{(b)}$ and $\overline{w}$ from w. The words w and w' are related by a sequence of Knuth relations of type (c) or (d) if and only if the words $\overline{w}$ and $\overline{w'}$ are related by a sequence of Knuth relations of type (d). By definition, this happens if and only if for each $b \in B$, the words $w^{(b)}$ and $w'^{(b)}$ are related by a sequence of Knuth relations as in Section 5.3. On the other hand, w and w' have the same insertion tableau $\mathbf{P}$ if and only if for each $b \in B$, the words $w^{(b)}$ and $w'^{(b)}$ have the same insertion tableau. The desired result now follows directly from Proposition 23 (ii). $\square$

We now explain why we have added Okada's name after those of Robinson and Schensted. Any element $\alpha \in B \wr \mathfrak{S}_n$ can be written uniquely in the form $\alpha = \big[\sigma \cdot (b_1, b_2, \ldots, b_n; e_n)\big]$, where $\sigma \in \mathfrak{S}_n$ and $(b_1, b_2, \ldots, b_n) \in B^n$. It thus determines the word

$$w(\alpha) = (\sigma(1), b_1)(\sigma(2), b_2) \cdots (\sigma(n), b_n)$$

with letters in $[1, n] \times B$. We denote the RSO correspondent of $w(\alpha)$ by $(\mathbf{P}(\alpha), \mathbf{Q}(\alpha))$. The element α can be recovered from the data of $w(\alpha)$; it is therefore characterized by $(\mathbf{P}(\alpha), \mathbf{Q}(\alpha))$. Finally, we define the dual of a B-tableau $\mathbf{T} = (T_b)_{b \in B}$ as the B-tableau $\mathbf{T}^* = \big(b \mapsto T_{b^*}\big)$, where $b \mapsto b^*$ is the involution on B. The following result is in substance a theorem of Okada [28].

Proposition 25 *The map $\alpha \mapsto (\mathbf{P}(\alpha), \mathbf{Q}(\alpha))$ is a bijection from $B \wr \mathfrak{S}_n$ onto the set of pairs of standard B-tableaux with the same shape. For any element α of $B \wr \mathfrak{S}_n$, there holds $\mathbf{Q}(\alpha^*) = \mathbf{P}(\alpha)^*$.*

As an example, we consider the same situation as in Section 5.2, that is, we take a, b in B, $n = 7$ and $\alpha = \big[1426735 \cdot (a, a, a, b, a, b, b; e_7)\big]$. Then the matrices $M^{(a)}$ and $M^{(b)}$ are

$$M^{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M^{(b)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we find

$$P_a = \begin{array}{|c|c|c|} \hline 1 & 2 & 7 \\ \hline 4 \\ \cline{1-1} \end{array}\,, \quad Q_a = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}\,, \quad P_b = \begin{array}{|c|c|} \hline 3 & 5 \\ \hline 6 \\ \cline{1-1} \end{array}\,, \quad Q_b = \begin{array}{|c|c|} \hline 4 & 7 \\ \hline 6 \\ \cline{1-1} \end{array}\,.$$

We will write $\alpha \underset{K}{\sim} \alpha'$ whenever the words $w(\alpha)$ and $w(\alpha')$ are related by a Knuth relation. Writing $\alpha = (b_1, b_2, \ldots, b_n; \sigma)$, one checks easily that $\alpha \underset{K}{\sim} \alpha'$ if and only if $\alpha' = \alpha \cdot s_i$ for an index i such that at least one of the following three conditions is satisfied:

- $b_{\sigma(i)} \neq b_{\sigma(i+1)}$;
- $\sigma(i-1) \in [\sigma(i), \sigma(i+1)]$ and $b_{\sigma(i-1)} = b_{\sigma(i)} = b_{\sigma(i+1)}$;
- $\sigma(i+2) \in [\sigma(i), \sigma(i+1)]$ and $b_{\sigma(i+2)} = b_{\sigma(i)} = b_{\sigma(i+1)}$.

(Here again the notation $[\sigma(i), \sigma(i+1)]$ means the interval $[\sigma(i+1), \sigma(i)]$ if ever $\sigma(i+1) < \sigma(i)$.) It follows then from Proposition 24 that the insertion tableaux $\mathbf{P}(\alpha)$ and $\mathbf{P}(\alpha')$ of two elements α and α' of $B \wr \mathfrak{S}_n$ are equal if and only if there exists a sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\alpha = \alpha_1 \underset{K}{\sim} \alpha_2 \underset{K}{\sim} \cdots \underset{K}{\sim} \alpha_k = \alpha'.$$

5.5 The plactic and the coplactic bialgebras

In this section, we fix an object B of the category $\mathscr{E}$ and we use the Robinson-Schensted-Okada correspondence to define a subbialgebra and a quotient bialgebra of $\mathscr{F}(\mathbb{K}B)$, called respectively the coplactic and the plactic bialgebra.

Given a standard B-tableau $\mathbf{T} = (T_b)_{b \in B}$, we define an element $t_{\mathbf{T}}$ of $\mathscr{F}(\mathbb{K}B)$ by setting

$$t_{\mathbf{T}} = \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ \mathbf{Q}(\alpha) = \mathbf{T}}} \alpha,$$

where $n = \|\operatorname{sh}(\mathbf{T})\|$ is the total number of boxes in $\mathbf{T}$. Clearly, the elements $t_{\mathbf{T}}$ are linearly independent and the $\mathbb{K}$-submodule $\mathscr{Q}(B)$ that they span is a direct summand of $\mathscr{F}(\mathbb{K}B)$. This submodule depends on B and not only on $\mathbb{K}B$. The following result is the analogue of Proposition 22 (iii).

Proposition 26 *Let B be an object of $\mathscr{E}$ and let n be a non-negative integer.*

(i) The module $\mathscr{Q}(B)^\circ \cap \mathscr{F}_n(\mathbb{K}B)$ is spanned over $\mathbb{K}$ by the set

$$\{\alpha - \alpha' \mid \alpha \text{ and } \alpha' \text{ in } B \wr \mathfrak{S}_n \text{ with } \alpha \underset{K}{\sim} \alpha'\}.$$

(ii) The submodules $\mathscr{Q}(B)$ and $\mathscr{Q}(B)^\circ$ are respectively a graded subbialgebra and a graded biideal of the graded bialgebra $(\mathscr{F}(\mathbb{K}B), *, \Delta)$.

(iii) The submodule $\mathscr{Q}(B) \cap \mathscr{Q}(B)^\circ$ is spanned over $\mathbb{K}$ by the set

$$\{t_{\mathbf{T}} - t_{\mathbf{T}'} \mid \mathbf{T} \text{ and } \mathbf{T}' \text{ standard } B\text{-tableaux with } \operatorname{sh}(\mathbf{T}) = \operatorname{sh}(\mathbf{T}')\}.$$

(iv) The submodule $\mathscr{Q}(B) + \mathscr{Q}(B)^\circ$ is a direct summand of $\mathscr{F}(\mathbb{K}B)$.

Proof. Let $x = \sum_{\alpha \in B \wr \mathfrak{S}_n} a_\alpha \alpha$ be an element of $\mathscr{F}_n(\mathbb{K}B)$, where each $a_\alpha \in \mathbb{K}$. Then for any standard B-tableau $\mathbf{T}$,

$$\varpi_{\mathrm{tot}}(x, t_{\mathbf{T}}) = \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ \mathbf{Q}(\alpha^*) = \mathbf{T}}} a_\alpha = \sum_{\substack{\alpha \in B \wr \mathfrak{S}_n \\ \mathbf{P}(\alpha) = \mathbf{T}^*}} a_\alpha.$$

The element x is orthogonal to $\mathscr{Q}(B)$ if and only if these quantities vanish for all $\mathbf{T}$. Assertion (i) now follows from Proposition 24, or more precisely, from its consequence explained at the end of Section 5.4.

Now let $\alpha = (b_1, b_2, \ldots, b_n; \sigma)$ and $\alpha' = (b'_1, b'_2, \ldots, b'_n; \sigma')$ be two elements in $B \wr \mathfrak{S}_n$ that are related by a Knuth relation. Then for each element $\alpha'' = (b''_1, b''_2, \ldots, b''_{n'}; \sigma'')$ in $B \wr \mathfrak{S}_{n'}$ and each permutation $\rho \in X_{n,n'}$, the two elements

$$\rho \cdot (b_1, b_2, \ldots, b_n, b''_1, b''_2, \ldots, b''_{n'}; \sigma \times \sigma'') \quad \text{and} \quad \rho \cdot (b'_1, b'_2, \ldots, b'_n, b''_1, b''_2, \ldots, b''_{n'}; \sigma' \times \sigma'')$$

of $B \wr \mathfrak{S}_{n+n'}$ are related by a Knuth relation, because ρ is increasing on the interval $[1, n]$. Therefore $(\alpha - \alpha') * \alpha''$, which is equal to the sum

$$\sum_{\rho \in X_{n,n'}} \left[\rho \cdot (b_1, b_2, \ldots, b_n, b''_1, b''_2, \ldots, b''_{n'}; \sigma \times \sigma'') - \rho \cdot (b'_1, b'_2, \ldots, b'_n, b''_1, b''_2, \ldots, b''_{n'}; \sigma' \times \sigma'')\right],$$

belongs to $\mathscr{Q}(B)^\circ$. Since $\mathscr{Q}(B)^\circ$ is spanned by such differences $\alpha - \alpha'$, we conclude that $\mathscr{Q}(B)^\circ$ is a left ideal of $\mathscr{F}(\mathbb{K}B)$. A similar reasoning shows that $\mathscr{Q}(B)^\circ$ is a right ideal.

Consider again an element $\alpha = (b_1, b_2, \ldots, b_n; \sigma)$ in $B \wr \mathfrak{S}_n$. Given an integer $n' \in [0, n]$, we denote the standardizations of the words $\sigma^{-1}(1)\, \sigma^{-1}(2)\, \cdots\, \sigma^{-1}(n')$ and $\sigma^{-1}(n'+1)\, \sigma^{-1}(n'+2)\, \cdots\, \sigma^{-1}(n)$ by $\pi_{n'} \in \mathfrak{S}_{n'}$ and $\pi'_{n-n'} \in \mathfrak{S}_{n-n'}$, respectively. A straightforward but tedious verification shows that whenever α undergoes a Knuth relation, either both of

$$(b_1, b_2, \ldots, b_{n'}; \pi_{n'}) \quad \text{and} \quad (b_{n'+1}, b_{n'+2}, \ldots, b_n; \pi'_{n-n'})$$

are left unchanged, or one of them remains the same and the other undergoes a Knuth relation. This fact implies that the class modulo $\mathscr{Q}(B)^\circ \otimes \mathscr{F}(\mathbb{K}B) + \mathscr{F}(\mathbb{K}B) \otimes \mathscr{Q}(B)^\circ$ of

$$\Delta(\alpha) = \sum_{n'=0}^{n} (b_1, b_2, \ldots, b_{n'}; \pi_{n'}) \otimes (b_{n'+1}, b_{n'+2}, \ldots, b_n; \pi'_{n-n'})$$

does not change when α undergoes a Knuth relation. We conclude that

$$\Delta\big(\mathscr{Q}(B)^\circ\big) \subseteq \mathscr{Q}(B)^\circ \otimes \mathscr{F}(\mathbb{K}B) + \mathscr{F}(\mathbb{K}B) \otimes \mathscr{Q}(B)^\circ. \tag{33}$$

Observing then that all homogeneous elements of $\mathscr{Q}(B)^\circ$ have positive degree, we see that the counit of $\mathscr{F}(\mathbb{K}B)$ vanishes on $\mathscr{Q}(B)^\circ$. Jointly with Equation (33), this means that $\mathscr{Q}(B)^\circ$ is a coideal of $\mathscr{F}(\mathbb{K}B)$.

We have therefore proved that $\mathscr{Q}(B)^\circ$ is a graded biideal of $\mathscr{F}(\mathbb{K}B)$. Since $\mathscr{Q}(B)$ is a direct summand of $\mathscr{F}(\mathbb{K}B)$, this is equivalent to the fact that $\mathscr{Q}(B)$ is a subbialgebra of $\mathscr{F}(\mathbb{K}B)$, which concludes the proof of Assertion (ii).

Proposition 25 implies that for each positive integer n and each pair $(\mathbf{T}, \mathbf{T}')$ of standard B-tableaux with n boxes,

$$\begin{aligned}
\varpi_{\text{tot}}(t_{\mathbf{T}}, t_{\mathbf{T}'}) &= \big|\{\alpha \in B \wr \mathfrak{S}_n \mid \mathbf{Q}(\alpha) = \mathbf{T},\ \mathbf{Q}(\alpha^*) = \mathbf{T}'\}\big| \\
&= \big|\{\alpha \in B \wr \mathfrak{S}_n \mid \mathbf{Q}(\alpha) = \mathbf{T},\ \mathbf{P}(\alpha) = \mathbf{T}'^*\}\big| \\
&= \begin{cases} 1 & \text{if } \mathbf{T} \text{ and } \mathbf{T}' \text{ have the same shape,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Assertion (iii) follows easily from this fact.

Given an finite index set I, we denote by $\mathrm{Mat}_I(\mathbb{K})$ the set of matrices with lines and columns indexed by I and with entries in $\mathbb{K}$. The subspace $\mathrm{Mat}_I^r(\mathbb{K})$ of matrices $(m_{ij})_{(i,j)\in I^2}$ such that all row sums $\sum_{j\in I} m_{ij}$ are equal is a direct summand of $\mathrm{Mat}_I(\mathbb{K})$.

For each B-partition $\boldsymbol{\lambda} = (\lambda_b)_{b\in B}$, let $\mathscr{T}_{\boldsymbol{\lambda}}$ be the set of all standard B-tableaux of shape $\boldsymbol{\lambda}$. Let n be a positive integer, and let $\mathrm{Part}_B(n)$ be the set of B-partitions of size n. Using the RSO correspondence, we define a linear bijection between $\mathscr{F}_n(\mathbb{K}B)$ and $\prod_{\boldsymbol{\lambda}\in\mathrm{Part}_B(n)} \mathrm{Mat}_{\mathscr{T}_{\boldsymbol{\lambda}}}(\mathbb{K})$ as follows: an element $\sum_{\alpha\in B\wr\mathfrak{S}_n} a_\alpha \alpha$ of $\mathscr{F}_n(\mathbb{K}B)$ corresponds to a family of matrices $(M_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda}\in\mathrm{Part}_B(n)}$ if and only if for each $\alpha \in B\wr\mathfrak{S}_n$, the coefficient a_α is equal to the entry in $M_{\boldsymbol{\lambda}}$ with row index $\mathbf{P}(\alpha)$ and column index $\mathbf{Q}(\alpha)$, where $\boldsymbol{\lambda} = \mathrm{sh}(\mathbf{P}(\alpha))$. One checks without much difficulty that $\big(\mathscr{Q}(B)+\mathscr{Q}(B)^\circ\big)\cap\mathscr{F}_n(\mathbb{K}B)$ is mapped by this bijection to the product $\prod_{\boldsymbol{\lambda}\in\mathrm{Part}_B(n)} \mathrm{Mat}^r_{\mathscr{T}_{\boldsymbol{\lambda}}}(\mathbb{K})$. Assertion (iv) follows. □

The subbialgebra $\mathscr{Q}(B)$ is called the coplactic bialgebra. We denote the quotient $\mathscr{F}(\mathbb{K}B)/\mathscr{Q}(B)^\circ$ by $\mathscr{P}(B)$ and we name it the plactic bialgebra. Both assignments $B \rightsquigarrow \mathscr{Q}(B)$ and $B \rightsquigarrow \mathscr{P}(B)$ are covariant functors from $\mathscr{E}$ to the category of graded bialgebras, and moreover the graded bialgebras $\mathscr{P}(B)$ and $\mathscr{Q}(B)^\vee$ are isomorphic for each B.

Remark 27. It turns out that $\mathscr{Q}(B)$ is neither a left nor a right internal $\mathscr{D}(B)$-submodule of $\mathscr{F}(\mathbb{K}B)$. Indeed there is the following counterexample in degree 4. The set B does not play any role here; we take it reduced to one element and abbreviate $\mathscr{D}(B)$ and $\mathscr{Q}(B)$ to $\mathscr{D}$ and $\mathscr{Q}$, respectively. We consider the standard tableau $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ and the elements

$$t_T = 3142+2143 \quad \text{and} \quad y = x_{(1,2,1)} - x_{(3,1)} - x_{(1,3)} + x_{(4)} = 3142+2143+4132+4231+3241.$$

Then t_T belongs to $\mathscr{Q}$ and y belongs to $\mathscr{D}$. A direct computation yields

$$y\cdot t_T = 4321 + 4231 + 1324 + 1234 + 3421 + 3412 + 4312 + 1423 + 2413 + 2314.$$

We observe that the permutation 3421 appears with a positive coefficient in $y \cdot t_T$, which is not the case of the permutation 1432, although they have the same record tableau. Therefore $y \cdot t_T$ does not belong to $\mathscr{Q}$. One checks similarly that $t_T \cdot y$ does not either belong to $\mathscr{Q}$.

5.6 An homomorphism onto a bialgebra of coloured symmetric functions

Our aim now is to extend the work of Poirier and Reutenauer [30] to the present framework. We compare the coplactic bialgebra $\mathscr{Q}(B)$ and the plactic bialgebra $\mathscr{P}(B)$ with the Mantaci-Reutenauer algebra $\mathscr{D}(B)$ and its dual $\mathscr{D}(B)^\vee \cong \mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$, and we insert them in a commutative diagram similar to (25).

We need some preparation, and to begin with, we define the descent composition $D(\mathbf{T})$ of a standard B-tableau $\mathbf{T} = (T_b)_{b\in B}$ in the following way. Let n be the total number of boxes in $\mathbf{T}$ and let $\beta : \{1,2,\ldots,n\} \to B$ the map which sends a label i to the element b such that i appears in a box of T_b. We decompose the interval $[1,n]$ into the largest subintervals on which the map β takes a constant value; in turn we decompose each subinterval into the largest possible subsubintervals, so that for any two numbers i and j located in the same subsubinterval, $i < j$ if and only if i is located west or south-west to j. For each subsubinterval, we form the pair consisting of its length c and the value b taken on it by the map β. The ordered list of all these pairs is the B-composition $D(\mathbf{T})$. For instance, with $B = \{a, b\}$, the descent composition of the B-tableau $\mathbf{T}$ given by

$$T_a = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad T_b = \begin{array}{|c|c|}\hline 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}$$

is $D(\mathbf{T}) = ((2, a), (1, a), (1, b), (1, a), (2, b))$.

Lemma 28 *(i) The descent composition of an element $\alpha \in B \wr \mathfrak{S}_n$ coincides with the descent composition of its record tableau $\mathbf{Q}(\alpha)$*

(ii) Let $\boldsymbol{\lambda} = (\lambda_b)_{b\in B}$ be a B-partition and $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \dots, (c_k, b_k))$ be a B-composition, both of the same size. For each $b \in B$, we define a $\mathbb{Z}_{>0}$-weight $\mu^{(b)} = (\mu_1^{(b)}, \mu_2^{(b)}, \dots, \mu_k^{(b)}, 0, 0, \dots)$ by setting $\mu_j^{(b)} = c_j$ if $b_j = b$ and $\mu_j^{(b)} = 0$ otherwise. Then the two sets

$$\left\{ \mathbf{T} \;\middle|\; \begin{array}{c} \mathbf{T} \textit{ standard } B\textit{-tableau with} \\ \operatorname{sh}(\mathbf{T}) = \boldsymbol{\lambda} \textit{ and } D(\mathbf{T}) \preccurlyeq \mathbf{c} \end{array} \right\} \quad \textit{and} \quad \left\{ \mathbf{U} \;\middle|\; \begin{array}{c} \mathbf{U} = (U_b)_{b\in B} \;\; B\textit{-tableau with entries in } \mathbb{Z}_{>0} \\ \textit{such that } \operatorname{sh}(\mathbf{U}) = \boldsymbol{\lambda} \textit{ and } \;\forall b,\; \operatorname{wt}(U_b) = \mu^{(b)} \end{array} \right\}$$

are equipotent.

Proof. Assertion (i) is a direct consequence of Proposition 23 (i). Let us prove Assertion (ii). We set $t_i = c_1 + c_2 + \cdots + c_i$; we denote the first set by X and the second set by Y. Our aim is to construct mutually inverse bijections from X onto Y and from Y onto X.

Let first $\mathbf{T} = (T_b)_{b\in B}$ be an element of X. For each b, we construct a tableau U_b by substituting in each box of T_b the label j it contains by the index i such that $j \in [t_{i-1} + 1, t_i]$. Since $D(\mathbf{T}) \preccurlyeq \mathbf{c}$, each index i appears c_i times in U_{b_i} and does not appear in the other tableaux U_b. Therefore each tableau U_b has $\mu^{(b)}$ for weight. It follows that the B-tableau $\mathbf{U} = (U_b)_{b\in B}$, which has visibly the same shape as $\mathbf{T}$, namely $\boldsymbol{\lambda}$, belongs to Y.

In the other direction, let $\mathbf{U} = (U_b)_{b\in B}$ be an element of Y. By definition, any label $i \in \{1, 2, \dots, k\}$ appears c_i times in U_{b_i}. We replace these entries i in the boxes of U_{b_i} by the numbers $t_{i-1} + 1$, $t_{i-1} + 2$, $\dots$, t_i, proceeding in increasing order whilst going south-west to north-east. These substitutions transform the B-tableau $\mathbf{U}$ in a standard B-tableau $\mathbf{T}$. By construction, $\mathbf{T}$ has the same shape as $\mathbf{U}$, namely $\boldsymbol{\lambda}$, and satisfies $D(\mathbf{T}) \preccurlyeq \mathbf{c}$; it thus belongs to X.

Routine verifications show that these correspondences are inverse bijections, which entails Assertion (ii). $\square$

Corollary 29 *The inclusion $\mathscr{D}(B) \subseteq \mathscr{Q}(B)$ holds.*

Proof. By Proposition 22 (i), the module $\mathscr{D}(B)$ is spanned by elements of the form

$$\sum_{\substack{\alpha\in B\wr\mathfrak{S}_n \\ D(\alpha)=\mathbf{c}}} \alpha,$$

where n is a positive integer and $\mathbf{c}$ is a B-composition of n. By Lemma 28 (i), such a sum may be rewritten as

$$\sum_{\substack{\alpha\in B\wr\mathfrak{S}_n \\ D(\alpha)=\mathbf{c}}} \alpha = \sum_{\substack{\mathbf{T} \text{ standard } B\text{-tableau} \\ D(\mathbf{T})=\mathbf{c}}} \sum_{\substack{\alpha\in B\wr\mathfrak{S}_n \\ \mathbf{Q}(\alpha)=\mathbf{T}}} \alpha = \sum_{\substack{\mathbf{T} \text{ standard } B\text{-tableau} \\ D(\mathbf{T})=\mathbf{c}}} t_{\mathbf{T}}. \tag{34}$$

It belongs therefore to $\mathscr{Q}(B)$. The corollary follows. $\square$

Changing slightly the notation used in Section 4.2, we use now the symbol Λ to denote the algebra of symmetric functions with coefficients in $\mathbb{K}$. It is indeed a bialgebra (see I, 5, Ex. 25 in [22]). We keep the notation h_n and s_λ to denote the complete symmetric functions and the Schur functions, where n is a positive integer and λ is a partition. We consider a family $(\Lambda(b))_{b\in B}$ of copies of Λ: given $b \in B$, we denote by $P(b)$ the image in $\Lambda(b)$ of an element $P \in \Lambda$. We carry out the tensor product $\Lambda(B) = \bigotimes_{b\in B} \Lambda(b)$. Given a B-partition $\boldsymbol{\lambda} = (\lambda_b)_{b\in B}$, we set $\mathbf{s}_{\boldsymbol{\lambda}} = \prod_{b\in B} s_{\lambda_b}(b)$; these elements $\mathbf{s}_{\boldsymbol{\lambda}}$ form a basis of the $\mathbb{K}$-module $\Lambda(B)$. The pairing $\langle ?, ? \rangle$ on $\Lambda(B)$ defined on this basis by

$$\langle \mathbf{s}_{\boldsymbol{\lambda}}, \mathbf{s}_{\boldsymbol{\lambda}'} \rangle = \begin{cases} 1 & \text{if } \boldsymbol{\lambda}' = \boldsymbol{\lambda}^*, \\ 0 & \text{otherwise,} \end{cases}$$

is then a perfect and symmetric pairing.

Let $\Theta_B : \mathscr{Q}(B) \to \Lambda(B)$ be the $\mathbb{K}$-linear map such that $\Theta_B(t_{\mathbf{T}}) = \mathbf{s}_{\mathrm{sh}(\mathbf{T})}$, for each standard B-tableau $\mathbf{T}$. The following lemma will help us to understand the behaviour of Θ_B on the subspace $\mathscr{D}(B)$ of $\mathscr{Q}(B)$.

Lemma 30 *For any B-composition $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \ldots, (c_k, b_k))$, there holds*

$$\Theta_B(y_{c_1,b_1} * y_{c_2,b_2} * \cdots * y_{c_k,b_k}) = h_{c_1}(b_1) h_{c_2}(b_2) \cdots h_{c_k}(b_k).$$

Proof. It is known (see I, (6.4) in [22] for a proof) that in the ring Λ of symmetric functions,

$$h_{\mu_1} h_{\mu_2} \cdots = \sum_{\lambda \text{ partition}} \left| \left\{ U \;\middle|\; \begin{array}{c} U \text{ tableau with entries in } \mathbb{Z}_{>0} \\ \text{such that } \mathrm{sh}(U) = \lambda \text{ and } \mathrm{wt}(U) = \mu \end{array} \right\} \right| s_\lambda \tag{35}$$

for any $\mathbb{Z}_{>0}$-weight $\mu = (\mu_1, \mu_2, \ldots)$.

We fix a B-composition $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \ldots, (c_k, b_k))$ as in the statement of the lemma, of size say n, and we construct a family $(\mu^{(b)})_{b\in B}$ of $\mathbb{Z}_{>0}$-weights as in Lemma 28 (ii). Regrouping the factors in the product $h_{c_1}(b_1) h_{c_2}(b_2) \cdots h_{c_k}(b_k)$ that correspond to the different indices b and applying Formula (35), we find

$$\begin{aligned} & h_{c_1}(b_1) h_{c_2}(b_2) \cdots h_{c_k}(b_k) \\ & \qquad = \sum_{\boldsymbol{\lambda}\ B\text{-partition}} \left| \left\{ \mathbf{U} \;\middle|\; \begin{array}{c} \mathbf{U} = (U_b)_{b\in B} \ B\text{-tableau with entries in } \mathbb{Z}_{>0} \\ \text{such that } \mathrm{sh}(\mathbf{U}) = \boldsymbol{\lambda} \text{ and } \ \forall b,\ \mathrm{wt}(U_b) = \mu^{(b)} \end{array} \right\} \right| \mathbf{s}_{\boldsymbol{\lambda}}. \end{aligned}$$

On the other hand, Equation 32 and Lemma 28 (i) imply that

$$y_{c_1,b_1} * y_{c_2,b_2} * \cdots * y_{c_k,b_k} = \sum_{\substack{\alpha \in B\wr\mathfrak{S}_n \\ D(\alpha) \preccurlyeq \mathbf{c}}} \alpha = \sum_{\substack{\mathbf{T} \text{ standard } B\text{-tableau} \\ D(\mathbf{T}) \preccurlyeq \mathbf{c}}} t_{\mathbf{T}},$$

so that

$$\Theta_B(y_{c_1,b_1} * y_{c_2,b_2} * \cdots * y_{c_k,b_k}) = \sum_{\boldsymbol{\lambda}\ B\text{-partition}} \left| \left\{ \mathbf{T} \;\middle|\; \begin{array}{c} \mathbf{T} \text{ standard } B\text{-tableau with} \\ \mathrm{sh}(\mathbf{T}) = \boldsymbol{\lambda} \text{ and } D(\mathbf{T}) \preccurlyeq \mathbf{c} \end{array} \right\} \right| \mathbf{s}_{\boldsymbol{\lambda}}.$$

The desired result follows now from Lemma 28 (ii). $\square$

We can now state and prove the main properties of Θ_B.

Theorem 31 *The map $\Theta_B : \mathscr{Q}(B) \to \Lambda(B)$ is a surjective morphism of graded bialgebras, with kernel $\mathscr{Q}(B) \cap \mathscr{Q}(B)^\circ$. It is compatible with the pairings ϖ_{tot} on $\mathscr{Q}(B)$ and $\langle ?, ? \rangle$ on $\Lambda(B)$, in the sense that*

$$\varpi_{\mathrm{tot}} = \langle \Theta_B(?), \Theta_B(?) \rangle.$$

The restriction of Θ_B to $\mathscr{D}(B)$ is the unique algebra homomorphism that maps $y_{n,b}$ to $h_n(b)$, where n is a positive integer and $b \in B$; this restriction also is surjective, with kernel $\mathscr{D}(B) \cap \mathscr{D}(B)^\circ$.

Proof. Lemma 30 implies that the restriction of Θ_B to $\mathscr{D}(B)$ is a morphism of graded algebras, for the associative algebra $\mathscr{D}(B)$ is generated by the elements $y_{n,b}$, and that this restriction is surjective, for the algebra $\Lambda(B)$ is generated by the elements $h_n(b)$. The coproducts of $\mathscr{D}(B)$ and $\Lambda(B)$ being characterized by the equations

$$\Delta(y_{n,b}) = \sum_{n'=0}^{n} y_{n',b} \otimes y_{n-n',b} \quad \text{and} \quad \Delta(h_n(b)) = \sum_{n'=0}^{n} h_{n'}(b) \otimes h_{n-n'}(b)$$

(with the convention that $y_{0,b}$ and $h_0(b)$ are the unit of the algebras $\mathscr{D}(B)$ and $\Lambda(B)$, respectively), we also see that $\Theta_B\big|_{\mathscr{D}(B)}$ preserves the coproducts. To sum up, Θ_B is a surjective morphism of graded bialgebras.

We have seen in the proof of Proposition 26 (iii) that for each pair $(\mathbf{T}, \mathbf{T}')$ of standard B-tableaux with the same number of boxes, there holds

$$\varpi_{\mathrm{tot}}(t_{\mathbf{T}}, t_{\mathbf{T}'}) = \begin{cases} 1 & \text{if } \mathrm{sh}(\mathbf{T}) = \mathrm{sh}(\mathbf{T}'), \\ 0 & \text{otherwise}, \end{cases}$$

which implies that $\varpi_{\mathrm{tot}}(t_{\mathbf{T}}, t_{\mathbf{T}'}) = \langle \mathbf{s}_{\mathrm{sh}(\mathbf{T})}, \mathbf{s}_{\mathrm{sh}(\mathbf{T}')} \rangle = \langle \Theta_B(t_{\mathbf{T}}), \Theta_B(t_{\mathbf{T}'}) \rangle$. Therefore Θ_B is compatible with the pairings ϖ_{tot} and $\langle ?, ? \rangle$. In turn, this assertion, the fact that $\langle ?, ? \rangle$ is a perfect pairing on $\Lambda(B)$ and the surjectivity of Θ_B imply that the kernel of Θ_B is equal to $\mathscr{Q}(B) \cap \mathscr{Q}(B)^\circ$. The surjectivity of the restriction $\Theta_B\big|_{\mathscr{D}(B)}$ implies likewise that the kernel of $\Theta_B\big|_{\mathscr{D}(B)}$ is equal to $\mathscr{D}(B) \cap \mathscr{D}(B)^\circ$.

We then arrive at the following commutative diagram of graded bialgebras

$$\begin{tikzcd}
 & \mathscr{D}(B) \arrow[r, hook] \arrow[dl, two heads, "\Theta_B|_{\mathscr{D}(B)}"'] \arrow[d, two heads] & \mathscr{Q}(B) \arrow[d, two heads] \\
\Lambda(B) & \mathscr{D}(B)/(\mathscr{D}(B) \cap \mathscr{D}(B)^\circ) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathscr{Q}(B)/(\mathscr{Q}(B) \cap \mathscr{Q}(B)^\circ).
\end{tikzcd}$$

An easy chase in this diagram shows that there exists a unique homomorphism of $\mathbb{K}$-modules from $\mathscr{Q}(B)$ to $\Lambda(B)$ which factorizes through $\mathscr{Q}(B)/(\mathscr{Q}(B) \cap \mathscr{Q}(B)^\circ)$ and which extends $\Theta_B\big|_{\mathscr{D}(B)}$, and that this homomorphism is a morphism of graded bialgebras. This isomorphism is of course Θ_B, which concludes the proof of the theorem. □

Corollary 32 *There holds*

$$\mathscr{D}(B) \subseteq \mathscr{Q}(B) \subseteq \mathscr{D}(B) + \mathscr{Q}(B)^\circ \quad \textit{and} \quad \mathscr{D}(B) \cap \mathscr{Q}(B)^\circ = \mathscr{D}(B) \cap \mathscr{D}(B)^\circ.$$

Proof. The inclusion $\mathscr{D}(B) \subseteq \mathscr{Q}(B)$ gives rise to an injective map

$$\mathscr{D}(B)/\ker\big(\Theta_B\big|_{\mathscr{D}(B)}\big) \hookrightarrow \mathscr{Q}(B)/\ker\Theta_B.$$

This latter is surjective, for the restriction $\Theta_B\big|_{\mathscr{D}(B)}$ has the same image as Θ_B. Using Theorem 31, we arrive at the isomorphism $\mathscr{D}(B)/(\mathscr{D}(B)\cap\mathscr{D}(B)^\circ) \xrightarrow{\simeq} \mathscr{Q}(B)/(\mathscr{Q}(B)\cap\mathscr{Q}(B)^\circ)$. The corollary follows from this by standard arguments. □

We now have a big commutative diagram of graded bialgebras

$$\begin{tikzcd}
 & & \mathscr{F}(\mathbb{K}B) \arrow[rr, "\simeq", "{\varpi_{\mathrm{tot}}}^\flat"'] \arrow[dr, two heads] & & \mathscr{F}(\mathbb{K}B)^\vee \arrow[dr, two heads] & & \\
 & \mathscr{Q}(B) \arrow[ur, hook] \arrow[dr, two heads, "\Theta_B"'] & & \mathscr{P}(B) \arrow[rr, "\simeq"] & & \mathscr{Q}(B)^\vee \arrow[dr, two heads] & \\
\mathscr{D}(B) \arrow[ur, hook] \arrow[rr, two heads] & & \Lambda(B) \arrow[ur, hook] \arrow[rr, "\simeq", "\langle ?,?\rangle^\flat"'] & & \Lambda(B)^\vee \arrow[ur, hook, "\Theta_B{}^\vee"] \arrow[rr, hook] & & \mathscr{D}(B)^\vee.
\end{tikzcd} \tag{36}$$

Given a standard B-tableau $\mathbf{T}$, let us denote by $u_{\mathbf{T}}$ the class modulo $\mathscr{Q}(B)^\circ$ of an $\alpha \in B\wr\mathfrak{S}_n$ such that $\mathbf{P}(\alpha) = \mathbf{T}$ (this class does not depends on the choice of α). Using the pairings, one checks rather easily that for any B-partition $\boldsymbol{\lambda}$, the map from $\Lambda(B)$ to $\mathscr{P}(B)$ in the diagram (36) sends an element $\mathbf{s}_{\boldsymbol{\lambda}}$ to

$$\sum_{\substack{\mathbf{T}\text{ standard }B\text{-tableau}\\ \mathrm{sh}(\mathbf{T})=\boldsymbol{\lambda}}} u_{\mathbf{T}}.$$

Finally, one may observe that the sequences (10) of homomorphisms, applied to the case $M = \mathscr{F}(\mathbb{K}B)$, $S = \mathscr{Q}(B)$ and $T = {\varpi_{\mathrm{tot}}}^\flat(\mathscr{Q}(B))$, show the existence of a symmetric pairing on $\mathscr{Q}(B)/(\mathscr{Q}(B)\cap\mathscr{Q}(B)^\circ)$, which is perfect thanks to Proposition 26 (iv). This pairing is of course equal to $\langle ?, ?\rangle$ under the isomorphism $\mathscr{Q}(B)/(\mathscr{Q}(B)\cap\mathscr{Q}(B)^\circ) \cong \Lambda(B)$ defined by Θ_B.

5.7 Consequences for the Solomon descent theory

We now use the construction presented in the previous section to complement the results of Section 4.3. We consider a finite abelian group G, we call $\Gamma = \mathrm{Irr}(G)$ its dual, and we view Γ as an object of $\mathscr{E}$ as explained in Section 5.1. The Frobenius characteristic ch is an isomorphism of graded bialgebras from $\mathrm{Rep}(G)$ onto $\Lambda(\Gamma)$, and there holds $\Theta_\Gamma\big|_{\mathscr{D}(\Gamma)} = \mathrm{ch}\circ\theta_G$, because both members are homomorphisms of algebras which map $y_{n,\gamma}$ to $h_n(\gamma)$, where n is a positive integer and $\gamma \in \Gamma$. Therefore the diagrams (25) and (36) agree and can be fused together.

We have recalled in Section 4.2 the construction of the irreducible characters $\boldsymbol{\chi}^{\boldsymbol{\lambda}}$ of the wreath product $G \wr \mathfrak{S}_n$, indexed by the Γ-partitions $\boldsymbol{\lambda}$ of n. From the diagram

$$\begin{tikzcd}
\mathsf{MR}(\mathbb{K}\Gamma) \arrow[r, equal] \arrow[d, two heads, "\theta_G"'] & \mathscr{D}(\Gamma) \arrow[r, hook] & \mathscr{Q}(\Gamma) \arrow[d, two heads, "\Theta_\Gamma"'] \\
\mathrm{Rep}(G) \arrow[rr, "\simeq", "\mathrm{ch}"'] & & \Lambda(\Gamma),
\end{tikzcd}$$

we see that the homomorphism of $\mathbb{K}$-modules $\tilde{\theta}_G$ from $\mathscr{Q}(\Gamma)$ to $\mathrm{Rep}(G)$, defined by $\tilde{\theta}_G(t_{\mathbf{T}}) = \boldsymbol{\chi}^{\mathrm{sh}(\mathbf{T})}$ for any standard Γ-tableau $\mathbf{T}$, is a graded morphism of bialgebras which extends θ_G and which is compatible with the pairings ϖ_{tot} on $\mathscr{Q}(\Gamma)$ and β_{tot} on $\mathrm{Rep}(G)$.

Let n be a positive integer and $\mathbf{c} = ((c_1,\gamma_1),(c_2,\gamma_2),\ldots,(c_k,\gamma_k))$ be a Γ-composition of size n. Set $\mathbf{c}^+ = (c_1, c_2, \ldots, c_k)$ and $\tilde{\boldsymbol{\gamma}} = \gamma_1^{\otimes c_1} \otimes \gamma_2^{\otimes c_2} \otimes \cdots \otimes \gamma_k^{\otimes c_k}$. The character

$$\theta_G\big(y_{c_1,\gamma_1} * y_{c_2,\gamma_2} * \cdots * y_{c_k,\gamma_k}\big) = \mathrm{Ind}_{G\wr\mathfrak{S}_{\mathbf{c}^+}}^{G\wr\mathfrak{S}_n}\big(\eta_{c_1}(\gamma_1)\otimes\eta_{c_2}(\gamma_2)\otimes\cdots\otimes\eta_{c_k}(\gamma_k)\big)$$

of $G\wr\mathfrak{S}_n$ is induced from a linear character of $G\wr\mathfrak{S}_{\mathbf{c}^+}$. It can therefore be realized by a representation on the $\mathbb{C}$-vector space with basis $(G\wr\mathfrak{S}_n)/(G\wr\mathfrak{S}_{\mathbf{c}^+})$. As we saw during the proof of Theorem 19, this set is in natural bijection with $X_{\mathbf{c}^+} \cong \{\rho\cdot(\tilde{\boldsymbol{\gamma}}\#e_n) \mid \rho\in X_{\mathbf{c}^+}\}$. After translation in the notation of Section 5.2, this result means that

$$\theta_G\left(\sum_{\substack{\alpha\in\Gamma\wr\mathfrak{S}_n\\ D(\alpha)\preccurlyeq\mathbf{c}}}\alpha\right)$$

is the character of a representation of $G\wr\mathfrak{S}_n$ on the $\mathbb{C}$-vector space with basis

$$\{\alpha\in\Gamma\wr\mathfrak{S}_n \mid D(\alpha)\preccurlyeq\mathbf{c}\}.$$

Using a quotient construction, we may substitute equalities to the inequalities $D(\alpha)\preccurlyeq\mathbf{c}$ in both formulas above. A representation of $G\wr\mathfrak{S}_n$ whose character is

$$\theta_G\left(\sum_{\substack{\alpha\in\Gamma\wr\mathfrak{S}_n\\ D(\alpha)=\mathbf{c}}}\alpha\right)$$

is called a descent representation. Descent representations are studied in [1] by Adin, Brenti and Roichman in the case $G=\{\pm1\}$ and in the forthcoming paper [6] by Bagno and Biagioli in the case $G=\mathbb{Z}/r\mathbb{Z}$. Our methods provide alternative proofs for some of their results; for instance, Formula (34) and the equality $\tilde{\theta}_G(t_{\mathbf{T}}) = \boldsymbol{\chi}^{\mathrm{sh}(\mathbf{T})}$ imply the decomposition into irreducible characters

$$\theta_G\left(\sum_{\substack{\alpha\in\Gamma\wr\mathfrak{S}_n\\ D(\alpha)=\mathbf{c}}}\alpha\right) = \sum_{\substack{\boldsymbol{\lambda}\ \Gamma\text{-partition}\\ \|\boldsymbol{\lambda}\|=\|\mathbf{c}\|}}\left|\left\{\mathbf{T}\ \middle|\ \begin{matrix}\mathbf{T}\text{ standard }\Gamma\text{-tableau with}\\ \mathrm{sh}(\mathbf{T})=\boldsymbol{\lambda}\text{ and }D(\mathbf{T})=\mathbf{c}\end{matrix}\right\}\right|\ \boldsymbol{\chi}^{\boldsymbol{\lambda}},$$

which generalizes Theorems 4.1 and 5.9 in [1].

6 Coloured quasisymmetric functions

Motivated by problems of enumeration of permutations having a given descent type, Gessel discovered in 1984 a link between Solomon's descent algebra for the symmetric group and symmetric functions. More precisely, he introduces in [14] an algebra QSym of 'quasisymmetric functions,' which are polynomials in a countable and totally ordered set of variables enjoying a certain symmetry property. The algebra QSym is graded by the degree of polynomials (that is, the homogeneous components of a quasisymmetric function are quasisymmetric), which we write $\mathrm{QSym} = \bigoplus_{n \geq 0} \mathrm{QSym}_n$. Gessel endows each graded component QSym_n with the structure of a coalgebra and observes that the dual algebra QSym_n^* can be identified with Solomon's descent algebra $\Sigma_{\mathfrak{S}_n}$ for the symmetric group. Gessel observes further that QSym_n contains the set Λ_n of homogeneous symmetric polynomials of degree n. Now Λ_n is isomorphic to its dual thanks to the usual inner product on symmetric polynomials, and it is also isomorphic to the character ring $R(\mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$ thanks to the characteristic map. The inclusion $\Lambda_n \hookrightarrow \mathrm{QSym}_n$ gives then by duality a surjection $\Sigma_{\mathfrak{S}_n} \cong \mathrm{QSym}_n^\vee \twoheadrightarrow \Lambda_n^\vee \cong R(\mathfrak{S}_n)$, which Gessel identifies with the Solomon map $\theta_{\mathfrak{S}_n}$.

$$\begin{tikzcd}
 & \mathbb{Z}\mathfrak{S}_n & & & & \\
\Sigma_{\mathfrak{S}_n} \arrow[ur, hook] \arrow[rr, "\simeq"] \arrow[dr, two heads, "\theta_{\mathfrak{S}_n}"'] & & \mathrm{QSym}_n^\vee \arrow[dr, two heads] & & & \mathrm{QSym}_n. \\
 & R(\mathfrak{S}_n) \arrow[rr, "\simeq"] & & \Lambda_n^\vee \arrow[r, equal, "\sim"] & \Lambda_n \arrow[ur, hook] &
\end{tikzcd}$$

This picture was completed in 1995 by two independent groups of people. On the one hand, Malvenuto and Reutenauer [23] endow the space $\mathscr{F} = \bigoplus_{n \geq 0} \mathbb{Z}\mathfrak{S}_\mathfrak{n}$ with the structure of a graded bialgebra by defining the external product and the coproduct. They show that $\Sigma = \bigoplus_{n \geq 0} \Sigma_{\mathfrak{S}_n}$ is a graded subbialgebra of $\mathscr{F}$. They endow Gessel's algebra QSym with a second coproduct, different from Gessel's one, and they observe that this operation turns the algebra QSym into a graded bialgebra, which they identify with the graded dual of Σ.

On the other hand, Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon [13] introduce a graded module $\mathrm{Sym} = \bigoplus_{n \geq 0} \mathrm{Sym}_n$ of 'non-commutative symmetric functions.' They endow each graded component Sym_n with an associative product with unit, which they call the internal product, and find an explicit isomorphism between the resulting algebra Sym_n and Solomon's descent algebra $\Sigma_{\mathfrak{S}_n}$. Defining an external product and a coproduct, they also endow Sym with the structure of a graded bialgebra, in such a way that Sym can be identified as a graded bialgebra to Σ and to the graded dual of QSym. The pairing between Sym and QSym is made explicit through the use of bases; it reminds of the inner product on the bialgebra Λ of symmetric functions. Finally Λ can be recovered as the quotient of Sym obtained by making commutative the variables.

We want to generalize these works to the multidimensional case. To this aim, we fix a finite set B endowed with a linear order. As in Section 5.1, we denote the free $\mathbb{K}$-module with basis B by $\mathbb{K}B$ and define a Mantaci-Reutenauer subbialgebra $\mathscr{D}(B)$ in $\mathscr{F}(\mathbb{K}B)$. In Section 6.1, we present a realization of the algebra $\mathscr{F}(\mathbb{K}B)$ in terms of 'coloured' free quasisymmetric functions. The dependence of our realization on the linear order on B may seem cumbersome, but is a necessary step so that the quotient map $\mathscr{F}(\mathbb{K}B) \to \mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$ corresponds to make the variables commutative. In Section 6.2, we show that our construction yields some of Poirier's quasisymmetric functions.

We fix for the whole Section 6 an infinite alphabet $\mathscr{A}$ and we endow the product $\mathscr{A} \times B$ with the lexicographical order.

6.1 The word realization of $\mathscr{F}(k\Gamma)$

Let $w = x_1x_2\cdots x_n$ be a word with letters $x_i = (a_i, b_i)$ in $\mathscr{A} \times B$. Denoting by $\sigma \in \mathfrak{S}_n$ the standardization of w, we may form the element

$$\mathrm{std}_B(w) = \sigma \cdot (b_1, b_2, \ldots, b_n; e_n) = (b_{\sigma^{-1}(1)}, b_{\sigma^{-1}(2)}, \ldots, b_{\sigma^{-1}(n)}; \sigma)$$

of $B \wr \mathfrak{S}_n$; we call it the B-standardization of w. (This element $\mathrm{std}_B(w)$ is called 'standard signed permutation' of w by Poirier; see [29], p. 322.) As an example, let $B = \{^-, ^=\}$ with $^- < ^=$, and let $\mathscr{A} = \{u, v, w, \ldots\}$ with the usual alphabetical order. We denote the letters $(u, ^-)$, $(u, ^=)$, etc. by $\bar{u}$, $\bar{\bar{u}}$, etc. Then the standardization of the word $w = \bar{u}\bar{v}\bar{u}\bar{\bar{v}}\bar{w}\bar{\bar{u}}\bar{v}$ is $\mathrm{std}_B(w) = \big[(1426735) \cdot (^-, ^-, ^-, ^=, ^-, ^=, ^-; e_7)\big]$.

We denote the algebra of non-commutative formal power series on the set $\mathscr{A} \times B$ with coefficients in $\mathbb{K}$ by $\mathbb{K}\langle\langle \mathscr{A} \times B\rangle\rangle$; thus elements of $\mathbb{K}\langle\langle \mathscr{A} \times B\rangle\rangle$ are (possibly infinite) linear combinations of words on the alphabet $\mathscr{A} \times B$. We denote the algebra of commutative formal power series on the set $\mathscr{A} \times B$ with coefficients in $\mathbb{K}$ by $\mathbb{K}[[\mathscr{A} \times B]]$; elements of this algebra may be viewed as (possibly infinite) linear combinations of $\mathscr{A} \times B$-weights. There is an obvious morphism of $\mathbb{K}$-algebras from $\mathbb{K}\langle\langle \mathscr{A} \times B\rangle\rangle$ onto $\mathbb{K}[[\mathscr{A} \times B]]$, which maps each word w on the alphabet $\mathscr{A} \times B$ to its weight.

We denote by $\Phi : \mathscr{F}(\mathbb{K}B) \to \mathbb{K}\langle\langle \mathscr{A} \times B\rangle\rangle$ the map which sends an element $\alpha \in (B \wr \mathfrak{S}_n)$ to the sum of all words w such that α is the B-standardization of w:

$$\Phi(\alpha) = \sum_{\substack{w \in \langle \mathscr{A} \times B\rangle \\ \mathrm{std}_B(w) = \alpha}} w.$$

Theorem 33 *(i) The map Φ is an injective morphism of algebras from $\mathscr{F}(\mathbb{K}B)$ to $\mathbb{K}\langle\langle \mathscr{A} \times B\rangle\rangle$.*

(ii) Let I be the kernel of the canonical morphism from $\mathbb{K}\langle\langle \mathscr{A} \times B\rangle\rangle$ onto $\mathbb{K}[[\mathscr{A} \times B]]$. Then $\Phi^{-1}(I) = \mathscr{D}(B)^\circ$.

Proof. (i) Let n and n' be two positive integers and let w and w' be two words on the alphabet $\mathscr{A} \times B$ of length n and n', respectively. If we denote by $\sigma \in \mathfrak{S}_n$, $\sigma' \in \mathfrak{S}_{n'}$ and $\pi \in \mathfrak{S}_{n+n'}$ the standardizations of the words w, w' and ww', respectively, then σ is the standardization of the word $\pi(1)\pi(2)\cdots\pi(n)$ and σ' is the standardization of the word $\pi(n+1)\pi(n+2)\cdots\pi(n+n')$; in other words, there exists $\rho \in X_{(n,n')}$ such that $\pi = \rho(\sigma \times \sigma')$.

Now let $\alpha \in B \wr \mathfrak{S}_n$ and $\alpha' \in B \wr \mathfrak{S}_{n'}$. We write $\alpha = \sigma \cdot (b_1, b_2, \ldots, b_n; e_n)$, $\alpha' = \sigma' \cdot (b'_1, b'_2, \ldots, b'_{n'}; e_{n'})$, $w = x_1x_2\cdots x_n$ and $w' = x'_1x'_2\cdots x_{n'}$. Given a letter $x = (a, b)$ in $\mathscr{A} \times B$, we say that b is the colour of x. Then

$$\begin{aligned}
&\alpha = \mathrm{std}_B(w) \ \text{ and } \ \alpha' = \mathrm{std}_B(w') \\
&\qquad\iff \begin{cases} \sigma \text{ is the standardization of } w,\ \sigma' \text{ is the standardization of } w', \\ b_i \text{ is the colour of } x_i \text{ and } b'_j \text{ is the colour of } x'_j, \end{cases} \\
&\qquad\iff \begin{cases} \exists \rho \in X_{(n,n')} \text{ such that } \rho(\sigma \times \sigma') \text{ is the standardization of } ww', \\ b_i \text{ is the colour of } x_i \text{ and } b'_j \text{ is the colour of } x'_j, \end{cases} \\
&\qquad\iff \begin{cases} \exists \rho \in X_{(n,n')} \text{ such that} \\ \mathrm{std}_B(ww') = \rho(\sigma \times \sigma') \cdot (b_1, b_2, \ldots, b_n, b'_1, b'_2, \ldots, b'_{n'}; e_{n+n'}). \end{cases}
\end{aligned}$$

This proves that Φ is a morphism of algebras. The injectivity of Φ is an obvious consequence of the fact that $\mathscr{A}$ was chosen infinite.

(ii) Let n be a positive integer and let $\alpha = (b_1, b_2, \ldots, b_n; \sigma)$ and α' be two elements in $B \wr \mathfrak{S}_n$ such that $\alpha \underset{A}{\sim} \alpha'$. Then there exists a simple transposition $s_i \in \mathfrak{S}_n$ such that $\alpha' = \alpha \cdot s_i$, the index $i \in \{1, 2, \ldots, n-1\}$ enjoying moreover the property that the map $j \mapsto b_j$ is not constant on the interval $[\sigma(i), \sigma(i+1)]$ or that the inequality $|\sigma(i+1) - \sigma(i)| > 1$ holds.

Then in each word $w = x_1 x_2 \cdots x_n$ of length n on the alphabet $\mathscr{A} \times B$ whose B-standardization is α, the letters x_i and x_{i+1} differ. The word $w' = x_1 x_2 \cdots x_{i-1} x_{i+1} x_i x_{i+2} \cdots x_n$ obtained from w by exchanging the letters x_i and x_{i+1} has thus $\alpha \cdot s_i = \alpha'$ for B-standardization, and the map $w \mapsto w'$ is a bijective correspondence

$$\left\{ w \,\middle|\, \begin{array}{c} w \text{ word on } \mathscr{A} \times B \text{ such} \\ \text{that } \operatorname{std}_B(w) = \alpha \end{array} \right\} \xrightarrow{\simeq} \left\{ w' \,\middle|\, \begin{array}{c} w' \text{ word on } \mathscr{A} \times B \text{ such} \\ \text{that } \operatorname{std}_B(w') = \alpha' \end{array} \right\}.$$

Therefore $\Phi(\alpha)$ and $\Phi(\alpha')$ have the same image in $\mathbb{K}[[\mathscr{A} \times B]]$, for w and w' have the same weight. By Proposition 22 (iii), this implies that $\Phi\big(\mathscr{D}(B)^\circ\big) \subseteq I$.

The morphism Φ defines therefore a map $\overline{\Phi}$ from $\mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$ to $\mathbb{K}[[\mathscr{A} \times B]]$. Assertion (ii) will then be proved as soon as the injectivity of $\overline{\Phi}$ is established.

We associate a B-composition $C(\mu)$ to each $(\mathscr{A} \times B)$-weight μ as follows: we list in increasing order $(a_1, b_1) < (a_2, b_2) < \cdots < (a_k, b_k)$ the elements (a, b) in the support of the multiset μ, and we then define $C(\mu)$ as the sequence $((\mu(a_1, b_1), b_1), (\mu(a_2, b_2), b_2), \ldots, (\mu(a_k, b_k), b_k))$. One checks that $R(\operatorname{std}_B(w)) \preccurlyeq C(\operatorname{wt}(w))$ for any word w on the alphabet $\mathscr{A} \times B$.

Let z be a non-zero element in $\mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ$. By Proposition 22, z has an antecedent in $\mathscr{F}(\mathbb{K}B)$ of the form $\sum_{j \in J} a_j \alpha_j$, where J is a finite index set, $a_j \in \mathbb{K} \setminus \{0\}$, and the elements $\alpha_j \in B \wr \mathfrak{S}_n$ are such that all B-compositions $R(\alpha_j)$ are different. We may then find $j_0 \in J$ such that $R(\alpha_{j_0})$ is a minimal element of the set $\{R(\alpha_j) \mid j \in J\}$ with respect to the refinement order $\preccurlyeq$, and we may find a word w on the alphabet $\mathscr{A} \times B$ such that $\operatorname{std}_B(w) = \alpha_{j_0}$ and $C(\operatorname{wt}(w)) = R(\alpha_{j_0})$. Then $\operatorname{wt}(w)$ appears in $\overline{\Phi}(z)$ with the coefficient $a_{j_0} \neq 0$, which entails that $\overline{\Phi}(z) \neq 0$.

Therefore $\overline{\Phi}$ is injective, which completes the proof.
□

Assertion (i) of Theorem 33 says that we can find a realization of the algebra $\mathscr{F}(\mathbb{K}B)$ in terms of free (non-commutative) quasisymmetric functions. Assertion (ii) says that the quotient map from $\mathscr{F}(\mathbb{K}B)$ onto $\mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ \cong \mathscr{D}(B)^\vee$ is obtained in this realization by making commutative all words $w \in \langle \mathscr{A} \times B \rangle$. This can be translated into the commutative diagram

$$\begin{tikzcd} \mathscr{F}(\mathbb{K}B) \arrow[r, hook, "\Phi"] \arrow[d, two heads] & \mathbb{K}\langle\langle \mathscr{A} \times B \rangle\rangle \arrow[d, two heads] \\ \mathscr{F}(\mathbb{K}B)/\mathscr{D}(B)^\circ \arrow[r, hook, "\overline{\Phi}"] & \mathbb{K}[[\mathscr{A} \times B]]. \end{tikzcd} \tag{37}$$

One can find a similar description of all the algebras that appear in the diagram (36); for instance, the quotient map from $\mathscr{F}(\mathbb{K}B)$ onto $\mathscr{F}(\mathbb{K}B)/\mathscr{Q}(B)^\circ = \mathscr{P}(B)$ amounts to look at the words $w \in \langle \mathscr{A} \times B \rangle$ modulo the Knuth relation $\underset{K}{\sim}$ of Section 5.4.

6.2 Poirier's quasisymmetric functions

Let $\mathrm{QSym}(B)$ denote the image in $\mathbb{K}[[\mathscr{A}\times B]]$ of the map $\overline{\Phi}$ in the diagram (37). In this section, we describe $\mathrm{QSym}(B)$ explicitly and compare it with Poirier's algebra of quasisymmetric functions.

By Proposition 22, the class modulo $\mathscr{D}(B)^\circ$ of an element $\alpha \in B \wr \mathfrak{S}_n$ is determined by its receding composition $R(\alpha)$. A stronger assertion holds: it is possible to find a combinatorial description of $\overline{\Phi}\big(\alpha + \mathscr{D}(B)^\circ\big)$ based on the sole data of $R(\alpha)$.

Indeed let $\mathbf{c} = ((c_1, b_1), (c_2, b_2), \ldots, (c_k, b_k))$ be a B-composition of size say n, set $t_i = c_1 + c_2 + \cdots + c_i$ for each i, and set

$$(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) = (\underbrace{b_1, b_1, \ldots, b_1}_{c_1 \text{ times}}, \underbrace{b_2, b_2, \ldots, b_2}_{c_2 \text{ times}}, \ldots, \underbrace{b_k, b_k, \ldots, b_k}_{c_k \text{ times}}).$$

From $\mathbf{c}$, we construct the set $S_{\mathbf{c}}$ of all n-uples $(x_1, x_2, \ldots, x_n) \in (\mathscr{A} \times B)^n$ satisfying the three following conditions: the sequence $(x_1, x_2, \ldots, x_n)$ is non-decreasing; $x_{t_i} < x_{t_i+1}$ for each $i \in \{1, 2, \ldots, k-1\}$; the second component of $x_i \in \mathscr{A} \times B$ is $\tilde{b}_i$. In other words, a n-uple $(x_1, x_2, \ldots, x_n)$ belongs to $S_{\mathbf{c}}$ if and only if each x_i can be written $(a_i, \tilde{b}_i)$, where $(a_1, a_2, \ldots, a_n)$ is a non-decreasing sequence of elements of $\mathscr{A}$ such that

$$\forall i \in \{1, 2, \ldots, k-1\}, \quad b_i \geq b_{i+1} \Longrightarrow a_{t_i} < a_{t_i+1}.$$

By analogy with Formula (2) on p. 324 in [29], we define the formal series in $\mathbb{K}[[\mathscr{A} \times B]]$

$$F_{\mathbf{c}} = \sum_{(x_1, x_2, \ldots, x_n) \in S_{\mathbf{c}}} x_1 x_2 \cdots x_n.$$

For instance if B is the set $\{^-, ^=\}$ with the order $^- < ^=$, then

$$F_{((2,^-))} = \sum_{\substack{(x,y)\in\mathscr{A}^2 \\ x \leq y}} \bar{x}\bar{y}, \qquad F_{((2,^=))} = \sum_{\substack{(x,y)\in\mathscr{A}^2 \\ x \leq y}} \bar{\bar{x}}\bar{\bar{y}}, \qquad F_{((1,^-),(1,^-))} = \sum_{\substack{(x,y)\in\mathscr{A}^2 \\ x < y}} \bar{x}\bar{y},$$

$$F_{((1,^-),(1,^=))} = \sum_{\substack{(x,y)\in\mathscr{A}^2 \\ x \leq y}} \bar{x}\bar{\bar{y}}, \qquad F_{((1,^=),(1,^-))} = \sum_{\substack{(x,y)\in\mathscr{A}^2 \\ x < y}} \bar{\bar{x}}\bar{y}, \qquad F_{((1,^=),(1,^=))} = \sum_{\substack{(x,y)\in\mathscr{A}^2 \\ x < y}} \bar{\bar{x}}\bar{\bar{y}}.$$

The following result is a rewriting of Lemma 11 in [29]; it implies that the elements $F_{\mathbf{c}}$ form a basis of the $\mathbb{K}$-module $\mathrm{QSym}(B)$, where $\mathbf{c}$ is a B-composition.

Proposition 34 *For each element $\alpha \in B \wr \mathfrak{S}_n$, there holds $F_{R(\alpha)} = \overline{\Phi}\big(\alpha + \mathscr{D}(B)^\circ\big)$.*

Proof. We take an element $\alpha \in B \wr \mathfrak{S}_n$, we write

$$\alpha = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n; \sigma) \qquad \text{and} \qquad R(\alpha) = ((c_1, b_1), (c_2, b_2), \ldots, (c_k, b_k)),$$

and we set $t_i = c_1 + c_2 + \cdots + c_i$ for each i. The definition of $R(\alpha)$ implies that

$$(\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n) = (\underbrace{b_1, b_1, \ldots, b_1}_{c_1 \text{ times}}, \underbrace{b_2, b_2, \ldots, b_2}_{c_2 \text{ times}}, \ldots, \underbrace{b_k, b_k, \ldots, b_k}_{c_k \text{ times}}),$$

that the permutation σ^{-1} is increasing on each interval $[t_{i-1}+1, t_i]$, and that

$$\forall i \in \{1, 2, \ldots, k-1\}, \quad b_i = b_{i+1} \Longrightarrow \sigma(t_i) > \sigma(t_i + 1).$$

Each sequence $(x_1, x_2, \ldots, x_n) \in S_{R(\alpha)}$ yields a word $w = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ with letters in $\mathscr{A} \times B$. The definition of $S_{R(\alpha)}$ is so shaped that the standardization of w is σ; it follows that the B-standardization of w is $\sigma \cdot \left(\tilde{b}_{\sigma(1)}, \tilde{b}_{\sigma(2)}, \ldots, \tilde{b}_{\sigma(n)}; e_n\right) = \alpha$. Conversely, each word w with letters in $\mathscr{A} \times B$ whose B-standardization is α can be written $w = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$, where the sequence $(x_1, x_2, \ldots, x_n)$ belongs to $S_{R(\alpha)}$.

We conclude that the image of

$$\Phi(\alpha) = \sum_{\substack{w \in \langle \mathscr{A} \times B \rangle \\ \mathrm{std}_B(w) = \alpha}} w$$

under the canonical map from $\mathbb{K}\langle\langle \mathscr{A} \times B \rangle\rangle$ to $\mathbb{K}[[\mathscr{A} \times B]]$ is equal to

$$\sum_{(x_1, x_2, \ldots, x_n) \in S_{R(\alpha)}} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} = \sum_{(x_1, x_2, \ldots, x_n) \in S_{R(\alpha)}} x_1 x_2 \cdots x_n = F_{R(\alpha)}.$$

The proposition follows. $\square$

Now let us enumerate the elements of B in increasing order: $\bar{b}_1$, $\bar{b}_2$, $\ldots$, $\bar{b}_l$, where l is the cardinality of B, and let us review the definitions of a combinatorial nature that are needed to introduce Poirier's theory of coloured quasisymmetric functions. Since Poirier made a slight mistake (in [29], Lemma 8 does not always agree with Formulas (1) and (2) on p. 324), we will follow Novelli and Thibon's presentation [27].

A l-partite number is an element of $\mathbb{N}^l$; we view it as a column matrix. Given a positive integer k, a l-vector composition of length k is a k-uple of non-zero l-partite numbers; it can be viewed as a sequence of column matrices, or more simply as a matrix with non-negative integral entries in l rows and k columns which has at least one non-zero element in each column.

Each l-vector composition $\mathbf{I}$ produces a formal power series in $\mathbb{K}[[\mathscr{A} \times B]]$ called a monomial quasisymmetric function of level l and defined by

$$M_{\mathbf{I}} = \sum_{\substack{(a_1, a_2, \ldots, a_k) \in \mathscr{A}^k \\ a_1 < a_2 < \cdots < a_k}} \left(\prod_{i=1}^{l} \prod_{j=1}^{k} (a_j, \bar{b}_i)^{m_{ij}} \right),$$

where (m_{ij}) is the matrix that represents $\mathbf{I}$. For instance in the case where B is the set $\{\bar{\ }, \bar{\bar{\ }}\}$ with the order $\bar{\ } < \bar{\bar{\ }}$, the monomial quasisymmetric functions of level $l = 2$ and of degree 2 are

$$M_{\binom{2}{0}} = \sum_{x \in \mathscr{A}} \bar{x}^2, \qquad M_{\binom{1}{1}} = \sum_{x \in \mathscr{A}} \bar{x}\bar{\bar{x}}, \qquad M_{\binom{0}{2}} = \sum_{x \in \mathscr{A}} \bar{\bar{x}}^2,$$

$$M_{\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)} = \sum_{\substack{(x,y)\mathscr{A}^2 \\ x<y}} \bar{x}\bar{y}, \qquad M_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)} = \sum_{\substack{(x,y)\mathscr{A}^2 \\ x<y}} \bar{x}\bar{\bar{y}}, \qquad M_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} = \sum_{\substack{(x,y)\mathscr{A}^2 \\ x<y}} \bar{\bar{x}}\bar{y},$$

$$M_{\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)} = \sum_{\substack{(x,y)\mathscr{A}^2 \\ x<y}} \bar{\bar{x}}\bar{\bar{y}}.$$

Let $\mathbf{I}$ be a l-vector composition, represented by the matrix (m_{ij}). We form the list of all pairs $(m_{ij}, \bar{b}_i)$, reading columnwise the entries of (m_{ij}) from top to bottom and from left to right. Erasing in this list all the pairs whose first component m_{ij} is zero, we obtain a B-composition, which we call the sequential reading of $\mathbf{I}$ and which we denote by $\mathbf{sr}(\mathbf{I})$. For instance with our favorite set $B = \{^-, ^=\}$ with the order $^- < ^=$, the l-vector compositions represented by the matrices

$$\begin{pmatrix} 1 & 0 & 4 \\ 3 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 3 & 2 & 1 \end{pmatrix}$$

have both sequential reading $((1, ^-), (3, ^=), (2, ^=), (4, ^-), (1, ^=))$. We see therefore that the map $\mathbf{I} \mapsto \mathbf{sr}(\mathbf{I})$ is not injective.

This definition allows us to express each formal power series $F_{\mathbf{c}}$ as a linear combination of monomial quasisymmetric functions.

Proposition 35 *For each B-composition $\mathbf{c}$, there holds*

$$F_{\mathbf{c}} = \sum_{\substack{\mathbf{I}\ l\text{-vector composition} \\ \mathbf{c} \preccurlyeq \mathbf{sr}(\mathbf{I})}} M_{\mathbf{I}}.$$

Proof. Let S be the set of all non-decreasing finite sequences $(x_1, x_2, \ldots, x_n)$ of elements of $\mathscr{A} \times B$. We define a map ψ from S to the set of all l-vector compositions by the following recipe. Let $(x_1, x_2, \ldots, x_n)$ in S; write $x_i = (a_i, b_i)$ for each i; let $\tilde{a}_1$, $\tilde{a}_2$, $\ldots$, $\tilde{a}_k$ be the (distinct) elements of $\{a_i \mid 1 \le i \le n\}$ enumerated in increasing order. Then $\psi(x_1, x_2, \ldots, x_n)$ is the l-vector composition of length k represented by the matrix (m_{ij}), where each m_{ij} counts the number of times that the element $(\tilde{a}_j, \bar{b}_i)$ appears in the sequence $(x_1, x_2, \ldots, x_n)$.

Given a l-vector composition $\mathbf{I}$, we set $T_{\mathbf{I}} = \psi^{-1}\big(\{\mathbf{I}\}\big)$. Then by definition

$$M_{\mathbf{I}} = \sum_{(x_1, x_2, \ldots, x_n) \in T_{\mathbf{I}}} x_1 x_2 \cdots x_n.$$

Now let $\mathbf{c}$ be a B-composition. Routine arguments show that

$$\forall \mathbf{x} \in S, \quad \mathbf{x} \in S_{\mathbf{c}} \Longleftrightarrow \mathbf{c} \preccurlyeq \mathbf{sr}(\psi(\mathbf{x})).$$

In other words, $S_{\mathbf{c}}$ is the disjoint union of the sets $T_{\mathbf{I}}$, where $\mathbf{I}$ is an l-vector composition such that $\mathbf{c} \preccurlyeq \mathbf{sr}(\mathbf{I})$. It follows that

$$\begin{aligned} F_{\mathbf{c}} &= \sum_{(x_1, x_2, \ldots, x_n) \in S_{\mathbf{c}}} x_1 x_2 \cdots x_n \\ &= \sum_{\substack{\mathbf{I}\ l\text{-vector composition} \\ \mathbf{c} \preccurlyeq \mathbf{sr}(\mathbf{I})}} \left(\sum_{(x_1, x_2, \ldots, x_n) \in T_{\mathbf{I}}} x_1 x_2 \cdots x_n \right) \\ &= \sum_{\substack{\mathbf{I}\ l\text{-vector composition} \\ \mathbf{c} \preccurlyeq \mathbf{sr}(\mathbf{I})}} M_{\mathbf{I}}, \end{aligned}$$

which proves the proposition. $\square$

Paraphrasing a construction of Poirier, Novelli and Thibon endow the set of l-vector compositions with a partial order $\leq$ and define for each l-vector composition $\mathbf{I}$ the formal power series

$$F_{\mathbf{I}} = \sum_{\substack{\mathbf{J}\ l\text{-vector composition} \\ \mathbf{I} \leq \mathbf{J}}} M_{\mathbf{J}},$$

which they call a quasi-ribbon function of level l. On the other side, one can show quite easily that for each B-composition $\mathbf{c}$, there exists a unique l-vector composition $\mathbf{K}(\mathbf{c})$ such that

$$\left\{ \mathbf{J} \;\middle|\; \begin{matrix} \mathbf{J}\ l\text{-vector composition} \\ \text{such that } \mathbf{c} \preccurlyeq \mathbf{sr}(\mathbf{J}) \end{matrix} \right\} = \left\{ \mathbf{J} \;\middle|\; \begin{matrix} \mathbf{J}\ l\text{-vector composition} \\ \text{such that } \mathbf{K}(\mathbf{c}) \leq \mathbf{J} \end{matrix} \right\}.$$

With these notations, Proposition 35 asserts that the formal power series $F_{\mathbf{c}}$ coincides with the quasi-ribbon function $F_{\mathbf{K}(\mathbf{c})}$.

Let us denote by $QSym^{(l)}$ the submodule of $\mathbb{K}[[\mathscr{A} \times B]]$ spanned by the monomial quasisymmetric functions of level l. Novelli and Thibon claim in [27] that $QSym^{(l)}$ is a subalgebra of $\mathbb{K}[[\mathscr{A} \times B]]$, and moreover that $QSym^{(l)}$ has the structure of a graded bialgebra, whose dual can be identified to the Novelli-Thibon bialgebra $\mathsf{NT}(\mathbb{K}B)$. In this context, Propositions 34 and 35 imply that $\mathrm{QSym}(B)$ is a subalgebra of $QSym^{(l)}$. It is amusing to note here that the graded algebra $\mathrm{QSym}(B)$, which is isomorphic to the dual of the graded bialgebra $\mathscr{D}(B)$, can also be viewed as a quotient of $QSym^{(l)}$, since $\mathscr{D}(B)$ is a graded subbialgebra of $\mathsf{NT}(\mathbb{K}B)$.

We conclude this paper by mentioning that Aval, F. Bergeron and N. Bergeron recently observed that coloured quasisymmetric functions of level $l = 2$ appear in a completely different context. We refer the reader to their paper [5] for additional details.

References

[1] R. M. Adin, F. Brenti and Y. Roichman, *Descent representations and multivariate statistics,* to appear in Trans. Amer. Math. Soc.

[2] M. Aguiar, N. Bergeron and K. Nyman, *The peak algebra and the descent algebras of type B and D,* Trans. Amer. Math. Soc. **356** (2004), 2781–2824.

[3] M. Aguiar, N. Bergeron and F. Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations,* preprint arXiv:math.CO/0310016.

[4] M. D. Atkinson, *A new proof of a theorem of Solomon,* Bull. London Math. Soc. **18** (1986), 351–354.

[5] J. C. Aval, F. Bergeron and N. Bergeron, *Diagonal Temperley-Lieb invariants and harmonics,* preprint arXiv:math.CO/0411568.

[6] E. Bagno and R. Biagioli, *Colored descent representations for complex reflections groups $G(r, p, n)$,* in preparation.

[7] F. Bergeron, N. Bergeron, R. B. Howlett and D. E. Taylor, *A decomposition of the descent algebra of a finite Coxeter group,* J. Algebraic Combin. **1** (1992), 23–44.

[8] D. Blessenohl, C. Hohlweg and M. Schocker, *A symmetry of the descent algebra of a finite Coxeter group,* to appear in Adv. Math.

[9] D. Blessenohl and M. Schocker, *Noncommutative character theory of symmetric group,* preprint available on the Web page of the second author.

[10] C. Bonnafé and C. Hohlweg, *Generalized descent algebra and construction of irreducible characters of hyperoctahedral groups,* preprint arXiv:math.CO/0409199.

[11] C. W. Curtis and I. Reiner, *Methods of representation theory with applications to finite groups and orders, Vol. I,* Pure and Applied Mathematics. New York: John Wiley & Sons Inc., 1981.

[12] G. Duchamp, F. Hivert and J.-Y. Thibon, *Noncommutative symmetric functions. VI. Free quasi-symmetric functions and related algebras,* Internat. J. Algebra Comput. **12** (2002), 671–717.

[13] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions,* Adv. Math. **112** (1995), 218–348.

[14] I. M. Gessel, *Multipartite P-partitions and inner products of skew Schur functions,* in *Combinatorics and algebra (Boulder, 1984),* pp. 289–301, Contemp. Math., vol. 34, Providence: American Mathematical Society, 1984.

[15] F. Hivert, J.-C. Novelli and J.-Y. Thibon, *Representation theory of the 0-Ariki-Koike-Shoji algebras,* preprint arXiv:math.CO/0407218.

[16] A. Jöllenbeck, *Nichtkommutative Charaktertheorie der symmetrischen Gruppen,* Bayreuther Math. Schr. **56** (1999), 1–41.

[17] A. Jöllenbeck and C. Reutenauer, *Eine Symmetrieeigenschaft von Solomons Algebra und der höheren Lie-Charaktere,* Abh. Math. Sem. Univ. Hamburg **71** (2001), 105–111.

[18] R. Kilmoyer, *Some irreducible complex representations of a finte group with a BN-pair,* Ph. D. dissertation, M.I.T., Cambridge, 1969.

[19] D. E. Knuth, *Permutations, matrices, and generalized Young tableaux,* Pacific J. Math. **34** (1970), 709–727.

[20] J.-L. Loday and M. O. Ronco, *Hopf algebra of the planar binary trees,* Adv. Math. **139** (1998), 293–309.

[21] I. G. Macdonald, *Polynomial functors and wreath products,* J. Pure Appl. Algebra **18** (1980), 173–204.

[22] I. G. Macdonald, *Symmetric functions and Hall polynomials,* second ed., Oxford mathematical monographs, Oxford: Oxford University Press, 1995.

[23] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra,* J. Algebra **177** (1995), 967–982.

[24] R. Mantaci and C. Reutenauer, *A generalization of Solomon's algebra for hyperoctahedral groups and other wreath products,* Comm. Algebra **23** (1995), 27–56.

[25] S. Montgomery, *Hopf algebras and their actions on rings,* CBMS Regional Conference Series in Mathematics, vol. 82, Providence: American Mathematical Society, 1993.

[26] W. D. Nichols, *Bialgebras of type one,* Comm. Algebra **6** (1978), 1521–1552.

[27] J.-C. Novelli and J.-Y. Thibon, *Free quasi-symmetric functions of arbitrary level,* preprint arXiv:math.CO/0405597.

[28] S. Okada, *Wreath products by the symmetric groups and product posets of Young's lattices,* J. Combin. Theory Ser. A **55** (1990), 14–32.

[29] S. Poirier, *Cycle type and descent set in wreath products*, in *Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995),* Discrete Math. **180** (1998), 315–343.

[30] S. Poirier and C. Reutenauer, *Algèbres de Hopf de tableaux,* Ann. Sci. Math. Québec **19** (1995), 79–90.

[31] C. Reutenauer, *Free Lie algebras,* London Mathematical Society monographs new series, Oxford: Oxford University Press, 1993.

[32] M. Rosso, *Quantum groups and quantum shuffles,* Invent. Math. **133** (1998), 399–416.

[33] L. Solomon, *A Mackey formula in the group ring of a Coxeter group,* J. Algebra **41** (1976), 255–264.

[34] J. R. Stembridge, *Enriched P-partitions,* Trans. Amer. Math. Soc. **349** (1997), 763–788.

[35] J.-Y. Thibon, *Lectures on noncommutative symmetric functions,* in *Interaction of combinatorics and representation theory,* pp. 39–94, Math. Soc. Japan Memoirs, vol. 11, Tokyo: The Mathematical Society of Japan, 2001.

[36] A. V. Zelevinsky, *Representations of finite classical groups. A Hopf algebra approach.* Lecture Notes in Mathematics, vol. 869, Berlin and New York: Springer-Verlag, 1981.

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