Pseudomodular surfaces

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Contents

1 Introduction to the problem **2**

2 Construction of pseudomoduar groups **3**
2.1 The once-punctured torus group 3
2.2 Strategy to show that $\Delta(u^2, 2\tau)$ is pseudomodular 5
2.3 Index 2 supergroup of $\Delta(u^2, 2\tau)$ 6
2.4 Implementation and results 8

3 Infinite families of pseudomodular surfaces **8**

A Arithmeticity of fuchsian groups **10**
A.1 Quaternion algebras 10
A.2 Groups derived from a quaternion algebra 12
A.3 The case of $\Delta(u^2, 2\tau)$ 13

B Cusp density **17**

Introduction

This is a report about a question raised by Long and Reid in [1] in 2002 : they showed a result that they found surprising, which is that there exists fuchsian groups that have the same cusp set as the modular group $\mathrm{PSL}(2, \mathbb{Z})$, namely $\mathbb{Q} \cup \{\infty\}$. Such groups they call "pseudomodular groups", and the associated hyperbolic surfaces they call "pseudomodular surfaces". At the end of their article they ask a bunch of questions. Among others, they ask if there exists an infinity of noncommensurable pseudomodular examples in the family of groups they considered in their article, namely the once punctured torus groups. The first question was answered in 2018 by Lou, Tan and Vo in [2]. They applied the idea of Long and Reid to a wider family of fuchsian groups, whose fundamental domains are obtained by gluing together once-punctured torus domains. Such domains they call "hyperbolic jigsaws". They find infinitely many pseudomodular groups in this family, by increasing the size of the jigsaws. The second question remains open, and we tried, so far unsuccessfully, to answer a similar one : does there exist an infinite family of noncommensurable groups, i.e. whose associated hyperbolic surfaces are homeomorphic to each other ? This is still a research project.

In the first section of this report we expose the basic definitions required in the question initially asked by Long and Reid : does there exist finite covolume noncocompact fuchsian groups with the same cusp set that are not commensurable ? The proof by Long and Reid involving computational experiments that we have reproduced, we give some precisions about the implementation and the results of these computational experiments. In the second section we expose the the construction of pseudomodular groups by Long and Reid. The third section is devoted to the definition of hyperbolic jigsaws and the construction of an infinite family of hyperbolic jigsaws and the construction of an infinite family of hyperbolic jigsaw groups, by Lou, Tan and Vo.

All these sections rely on the same main ideas. In particular, in order to show that a group is not commensurable to the modular group $\mathrm{PSL}(2,\mathbb{Z})$, we show that it is not arithmetic.

1 Introduction to the problem

Definition 1. *Let Γ_1, Γ_2 be discrete subgroups of the group of orientation preserving isometries of the hyperbolic plane,* $\mathrm{PSL}(2,\mathbb{R})$*, Γ_1 and Γ_2 are said to be commensurable (or commensurable in a weak sense) if there exists G_1 subgroup of Γ_1, G_2 subgroup of Γ_2, both of finite index, such that*

$$\exists \gamma \in \mathrm{PSL}(2,\mathbb{R}), G_2 = \gamma^{-1} G_1 \gamma$$

The terminology "commensurable in a weak sense" is used because there is a stronger notion of commensurability, that we will use to introduce arithmetic groups (appendix A, see definition 15).

Definition 2. *Let Γ be a fuchsian group. The cusp set of Γ is the set of all fixed points of parabolic transformations, contained in $\mathbb{R} \cup \{+\infty\}$. It is denoted by* $\mathrm{Cusp}(\Gamma)$*.*

Remark 1. *The cusp set of a fuchsian group Γ is the orbit by Γ of the set of ideal vertices of any fundamental domain of Γ.*

Proposition 1.

$$\mathrm{Cusp}(\mathrm{PSL}(2,\mathbb{Z})) = \mathbb{Q} \cup \{+\infty\}$$

Proposition 2. *If Γ_1 and Γ_2 are two fuchsian groups that are commensurable, then there exists $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ such that* $\mathrm{Cusp}(\Gamma_1) = \gamma \cdot \mathrm{Cusp}(\Gamma_2)$*.*

Proof. Let G_1 be a finite index subgroup of Γ_1. It suffices to show that $\mathrm{Cusp}(G_1) = \mathrm{Cusp}(\Gamma_1)$. If $\gamma \in \Gamma_1$ is a parabolic element with fixed point $z \notin \mathrm{Cusp}(\Gamma_2)$, then for $n \in \mathbb{Z}$, each γ^n is in a different class of Γ_1/G_1, thus G_1 is not finite index. □

The question is the following : Is the other implication true ? In other words, does there exist Γ_1, Γ_2 fuchsian groups that are not commensurable and such that $\mathrm{Cusp}(\Gamma_1) = \mathrm{Cusp}(\Gamma_2)$?

The answer to the second question is obviously yes, it suffices to find two cocompact fuchsian groups that are not commensurable, or two noncommensurable fuchsian groups with empty cusp set. Thus we should refine the question to make it interesting : does there exist Γ_1, Γ_2 not cocompact and finite covolume fuchsian groups that are not commensurable and such that $\mathrm{Cusp}(\Gamma_1) = \mathrm{Cusp}(\Gamma_2)$?

2 Construction of pseudomoduar groups

Inthis section, we mostly follow the article [1] by D.D. Long and A.W. Reid.

Definition 3. *A pseudomodular group* $\Gamma \in \mathrm{PSL}(2,\mathbb{R})$ *is a fuchsian group for which* $\mathrm{Cusp}(\Gamma) = \mathbb{Q} \cup \{\infty\}$ *and that is not commensurable to* $\mathrm{PSL}(2,\mathbb{Z})$. *The complete hyperbolic surface* $\mathbb{H}^2/\Gamma$ *is then called a pseudomodular surface.*

Theorem 1. *(Long, Reid) There exists a pseudomodular group.*

2.1 The once-punctured torus group

One can exhibit a pseudomodular group by looking at the family of fuchsian groups giving rise to a once-punctured torus. Let $u^2 \in \mathbb{R}_+$ and consider the ideal quadrilateron whose vertices are $-1, 0, u^2, \infty$. A once-punctured torus is a hyperbolic surface obtained from this quadrilateron by gluing together the edges $(\infty, -1)$ and $(u^2, 0)$ and the edges $(-1, 0)$ and (∞, u^2).

Let us find Γ a fuchsian group such that $\mathbb{H}^2/\Gamma$ is the once-punctured torus. Γ has to contain $g_1, g_2 \in \mathrm{PSL}(2,\mathbb{R})$ such that

$$\begin{cases} g_1(0) = u^2 \\ g_1(-1) = \infty \\ g_2(\infty) = u^2 \\ g_2(-1) = 0 \end{cases}$$

A calculation gives :

$$g_2 \simeq \begin{pmatrix} au^2 & au^2 \\ a & \frac{1}{au^2} + a \end{pmatrix}$$

$$g1 \simeq \begin{pmatrix} bu^2 + \frac{1}{b} & bu^2 \\ b & b \end{pmatrix}$$

with $a, b \in \mathbb{R}$.

However the group $< g_1, g_2 >$ does not define a fuchsian group with fundamental domain the ideal quadrilateron $(\infty, -1, 0u^2)$ for all $a, b \in \mathbb{R}$. In order to find the good parameters, let us recall the Poincré theorem about hyperbolic surfaces obtained by gluing together edges of a polygon :

Theorem 2. *(Poincaré) Let P be an ideal polygon in $\mathbb{H}^2$ with a pair number of edges : $\{E_1, E_2, \dots E_{2k}\}$ Let $\phi_i \in \mathrm{PSL}(2,\mathbb{R})$ be such that $\phi_i(E_{2i-1}) = E_i$. Then the topological space X obtained from P by gluing together E_{2i-1} with E_i is a hyperbolic surface and it is complete if and only if there exists a family (C_p) of horodisks indexed by the vertices of P such that for all i if ϕ_i sends p to p' then $\phi_i(C_p) = C_{p'}$.*

Moreover, in that case, the group Γ generated by the ϕ_i's is discete and acts freely on $\mathbb{H}^2$. P is a fundamental polygon for Γ and the inclusion $P \hookrightarrow \mathbb{H}^2$ induces an isometry $X \simeq \mathbb{H}^2/\Gamma$

Thus, in order to have that $\mathbb{H}^2/ < g_1, g_2 >$ is the once-punctured torus we want, we need horocircles $C_\infty, C_{-1}, C_0, C_{u^2}$, respectively at ∞, -1, 0 and u^2, such that

$$\begin{aligned} g_2(C_\infty) &= C_{u^2} \\ g_1^{-1}(C_{u^2}) &= C_0 \\ g_2^{-1}(C_0) &= C_{-1} \\ g_1(C_{-1}) &= C_\infty \end{aligned}$$

Such horocircles exist if and only if for any horocircle C_∞ at ∞, we have $g_1 g_2^{-1} g_1^{-1} g_2(C_\infty) = C_\infty$, that is to say $g_1 g_2^{-1} g_1^{-1} g_2$ is a parabolic fixing ∞. We define τ such that

$$g_1 g_2^{-1} g_1^{-1} g_2 \simeq \begin{pmatrix} 1 & 2\tau \\ 0 & 1 \end{pmatrix}$$

Embedding in this equation the preceding calculus, we get $\tau \geq 1 + u^2$ and :

$$\begin{aligned} g_1 &= \frac{1}{\sqrt{\tau - 1 - u^2}} \begin{pmatrix} \tau - 1 & u^2 \\ 1 & 1 \end{pmatrix} \\ g_2 &= \frac{1}{\sqrt{\tau - 1 - u^2}} \begin{pmatrix} u & u \\ \frac{1}{u} & \frac{\tau - u^2}{u} \end{pmatrix} \end{aligned}$$

Then $< g_1, g_2 >$ is a group acting freely on $\mathbb{H}^2$ (thus containing only hyperbolic and parabolic isometries).

Let us now show that $< g_1, g_2 >$ is a free group with two generators. On figure 1 it is easy to show that the tiles with numbers $\geq n$ can only be reached from the orginal tile $(\infty, -1, 0, u^2)$ by words in $g_1^{\pm 1}, g_2^{\pm 1}$ of length $\geq n$.

Thus we see that composing at most n times g_1, g_2, g_1^{-1} and g_2^{-1} we get

$$5 + \sum_{k=1}^{n-1} 3^k$$

elements. Thus $< g_1, g_2 >$ is a free group. We also see that the group $< g_1, g_2 >$ is determined by only two parameters, 2τ and u^2. We thus denote the group $\Delta(g_1, g_2)$.

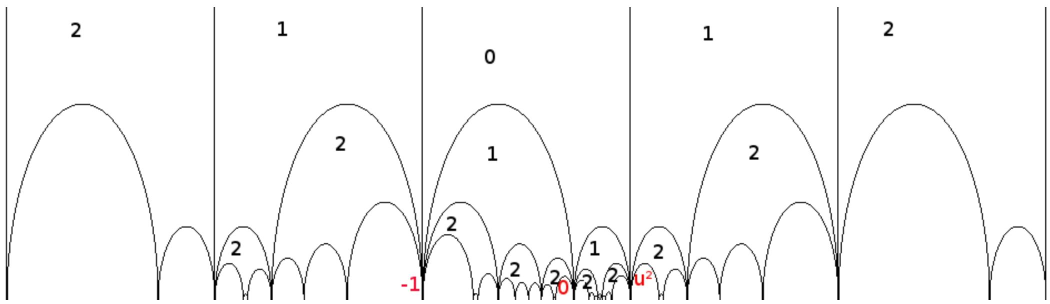

Figure 1: images of the fundamental domain of $\Delta(\frac{3}{8}, \frac{11}{2})$ by the words of length ≤ 2

2.2 Strategy to show that $\Delta(u^2, 2\tau)$ is pseudomodular

First of all, we notice that if u^2 and 2τ are in $\mathbb{Q}$, then it is easy to show that $\mathrm{Cusp}(\Delta(u^2, 2\tau)) \subset \mathbb{Q} \cup \{\infty\}$

Further, we see that since $\infty, -1, 0$ and u^2 are in the same orbit modulo $\Delta(u^2, 2\tau)$, and according to remark 1, we have

$$\mathrm{Cusp}(\Delta(u^2, 2\tau)) = \Delta(u^2, 2\tau) \cdot \infty$$

In order to show that $\Delta(u^2, 2\tau)$ is not pseudomodular, we thus need to check two things :

(i) First, $\Delta(u^2, 2\tau)$ is not commensurable to the modular group

(ii) Second, $\mathrm{Cusp}(\Delta(u^2, 2\tau)) = \mathbb{Q} \cup \{\infty\}$

To check (i), it is possible to prove that $< g_1, g_2 >$ is arithmetic if and only if $\mathrm{Tr}(g_1^2)$, $\mathrm{Tr}(g_2)^2$ and $\mathrm{Tr}(g_1^2 g_2^2)$ are in $\mathbb{Z}$ (see appendix A).

Let us now focus on the point (ii). Let $x \in \mathbb{Q} \cup \{\infty\}$ be a cusp of $\Delta(u^2, 2\tau)$. Then there exists $\gamma \in \Delta(u^2, 2\tau)$ such that $\gamma(\infty) = x = \frac{\alpha}{\beta}$. Take $\gamma \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ coprime. Then $\gamma^{-1} \simeq \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Take $\frac{p}{q} \in \mathbb{Q}$. We have

$$\gamma^{-1}\left(\frac{p}{q}\right) = \frac{dp - bq}{-cp + aq}$$

As a consequence, the denominator of $\gamma^{-1}\left(\frac{p}{q}\right)$ is a divisor of $-cp + aq$. Thus, in order to have the denominator of $\gamma^{-1}\left(\frac{p}{q}\right)$ smaller than q, it suffices to have

$$|-cp + aq| < q \iff \left|\frac{p}{q} - \frac{a}{c}\right| < \frac{1}{c}$$

The interval

$$\left(x - \frac{1}{c}, x + \frac{1}{c}\right)$$

is called the "killer interval" associated to γ.

Killer intervals are at the heart of the proof of pseudomodularity of some of the groups $\Delta(u^2, 2\tau)$, thanks to the following proposition :

Proposition 3. *The group $\Delta(u^2, 2\tau)$ is pseudomodular if and only if the union of the killer intervals associated to all the elements in $\Delta(u^2, 2\tau)$ contains $\mathbb{Q}$.*

Proof. If $\Delta(u^2, 2\tau)$ is pseudomodular, any rational is a cusp. Since any cusp $x \in \mathbb{Q}$ is the image of infinity by an isometry $\gamma \in \Delta(u^2, 2\tau)$, x is the centre of the killer interval associated to γ, $\mathbb{Q}$ is contained in the union of all killer intervals

Reciprocally, if any rational x is in a killer interval I, considering the isometry $\gamma \in \Delta(u^2, 2\tau)$ associated to I, we know that the denominator of $\gamma^{-1}(x)$ is strictly smaller than the denominator of γ. Iterating this process, we have a sequence of rationals in the same orbit whose denominators are decreasing, thus this sequence reaches ∞ at some point, showing that x is in the orbit of ∞, thus is a cusp of $\Delta(u^2, 2\tau)$. $\square$

We can notice that, since the translation $\begin{pmatrix} 1 & 2\tau \\ 0 & 1 \end{pmatrix}$ is in $\Delta(u^2, 2\tau)$, the translation $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ too, where α is the numerator of 2τ. Since the translation $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ does not change the denominator of rational numbers, if I is a closed interval of length α, any rational is in the orbit of some rational in I with same denominator. Thus, by the same argument as above, it is enough to cover any closed interval of length α with killer intervals in order to show that $\Delta(u^2, 2\tau)$ is a pseudomodular group. In fact, the following section will enable us to have a little bit weaker condition.

2.3 Index 2 supergroup of $\Delta(u^2, 2\tau)$

We will now introduce for any $u^2, 2\tau$ ($\tau \geq u^2 + 1$) a fuchsian group Γ of which $\Delta(u^2, 2\tau)$ is a index 2 subgroup and that contains the elliptic isometry $\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$. Since the cusp set of a finite index subgroup of a fuchsian group is the cusp set of this group, by the argument of the last paragraph in the above subsection, this gives us the proposition that we will use to search pseudomodular groups :

Proposition 4. *Let I be a closed interval of length δ where δ is the numerator of τ. Suppose there exists $\gamma_1, \ldots \gamma_n \in \Delta(u^2, 2\tau)$ such that the union of their killer intervals covers I. Then*

$$\mathrm{Cusp}(\Delta(u^2, 2\tau)) = \mathbb{Q} \cup \{\infty\}$$

Write $\Delta(u^2, 2\tau) =< g_1, g_2 >$ with g_1 and g_2 as defined in subsection 2.1. Let (x_1, y_1) be the axis of g_1 and (x_2, y_2) the axis of g_2. Let P_1 be the intersection point of (x_1, y_1) and (x_2, y_2) and let h_1 be the π-rotation around P_1.

Any π-rotation h with centre P in the axis (x_1, y_1) fixes this axis and interchanges x_1 and x_2. Thus hh_1 fixes x_1 and x_2 and it is not the identitity (if $P \neq P_1$), so it is a hyperbolic isometry and its translation distance is $2d(P_1, P)$. Let $P_2 \in (x_1, y_1)$ and h_2 the π-rotation around P_2 be such that $h_2 h_1 = g_1$ (so $2d(P_1, P_2)$ is the translation distance of g_1). Similarly, let $P_3 \in (x_2, y_2)$ and h_3 the π-rotation around P_3 be such that $h_1 h_3 = g_2$.

We have that the group $< h_1, h_2, h_3 >$ is an index 2 supergroup of $< g_1, g_2 >$. Indeed, let $\gamma \in < h_1, h_2, h_3 > \setminus < g_1, g_2 >$ and let us show that there exists $\alpha \in < g_1, g_2 >$ such that $\gamma = h_1 \alpha$. Write $\gamma = \gamma_1 \dots \gamma_n$ with $\gamma_i \in \{h_1, h_2, h_3\}$.

- If $n = 1$, then $\gamma = h_1$, or $\gamma = h_2 = h_1 h_1 h_2 = h_1 (h_2 h_1)^{-1} = h_1 g_1^{-1}$ or $\gamma = h_3 = h_1 h_1 h_3 = h_1 g_2$. In any case, $\gamma = h_1 \alpha$, $\alpha \in < g_1, g_2 >$.
- If $n \geq 2$, then several cases can occur. If $\gamma_n = \gamma_{n-1}$, then $\gamma = \gamma_1 \dots \gamma_{n-2} = h_1 \alpha, \alpha \in < g_1, g_2 >$ by induction. If $\gamma_n = h_1, \gamma_{n-2} = h_2$, then $\gamma = \gamma_1 \dots \gamma_{n-2} g_1^{-1} = h_1 \alpha' g_1^{-1}$ with $\alpha' \in < g_1, g_2$ by induction, thus $\gamma = h_1 \alpha$, with $\alpha = \alpha' g_1^{-1}$. If $\gamma_n = h_2$ and $\gamma_{n-1} = h_3$, then $\gamma = \gamma_1 \dots \gamma_{n-2} h_3 h_1 h_1 h_2 = h_1 \alpha' g_2^{-1} g_1^{-1}$ with $\alpha' \in < g_1, g_2 >$ by induction. The other cases are similar to one of the three cases that were just handled.

In order to have an index 2 fuchsian supergroup of $\Delta(u^2, 2\tau)$, it is now enough to prove the following lemma :

Lemma 1. *A finite index supergroup of a fuchsian group is a fuchsian group.*

Proof. Let Γ_1 and Γ_2 be subgroups of $\mathrm{PSL}(2, \mathbb{R})$ with Γ_1 a fuchsian group and a finite index subgroup of Γ_2. Let us show that Γ_2 acts properly discontinuously on $\mathbb{H}^2$. If $[\Gamma_2 : \Gamma_1] = n$, we have :

$$\Gamma_2 = \Gamma_1 \sqcup \Gamma_1 \cdot \gamma_1 \sqcup \dots \sqcup \Gamma_1 \cdot \gamma_n$$

Let $K \subset \mathbb{H}^2$ be compact. We have :

$$\begin{aligned} K \cap \Gamma_2 \cdot K &= (K \cap \Gamma_1 \cdot K) \cup (K \cap \Gamma_1 \cdot \gamma_1 K) \cup \dots \cup (K \cap \Gamma_1 \cdot \gamma_n K) \\ &= K \cap \Gamma_1 \cdot (K \cup \gamma_1 K \cup \dots \cup \gamma_n K) \\ &\subset (K \cup \gamma_1 K \cup \dots \cup \gamma_n K) \cap \Gamma_1 \cdot (K \cup \gamma_1 K \cup \dots \cup \gamma_n K) \end{aligned}$$

Writing $L = (K \cup \gamma_1 K \cup \dots \cup \gamma_n K)$, we have L is compact (finite union of compacts), and since Γ_1 is a fuchsian group, it acts properly discontinuously on $\mathbb{H}^2$, thus $L \cap \Gamma_1 \cdot L$ is finite, so $K \cap \Gamma_2 \cdot K$ is finite as well. This shows that Γ_2 acts properly discontinuously on $\mathbb{H}^2$, thus is a fuchsian group. $\square$

Now it only remains to show that $\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \in < h_1, h_2, h_3 >$. But we have :

$$\begin{aligned}\begin{pmatrix} 1 & 2\tau \\ 0 & 1 \end{pmatrix} &= g_1 g_2^{-1} g_1^{-1} g_2 \\ &= h_2 h_1 h_3 h_1 h_1 h_2 h_1 h_3 \\ &= (h_2 h_1 h_3)^2\end{aligned}$$

Necessarily, $h_2 h_1 h_3$ is the translation by τ.

2.4 Implementation and results

Using this technique, Long and Reid in [1] found four classes of pseudomodular groups $\Delta(\frac{5}{7}, 6)$, $\Delta(\frac{2}{5}, 4)$, $\Delta(\frac{3}{7}, 4)$, $\Delta(\frac{3}{11}, 4)$ and Ayaka in [3] found a fifth one which is $\Delta(\frac{5}{13}, 4)$. We focused on the case when $\tau = 2(u^2 + 2)$ and we found four more pseudomodular groups, for $u^2 \in \{\frac{2}{3}, \frac{3}{4}, \frac{3}{8}, \frac{5}{8}\}$. These four groups we found all have different cusp densities (see appendix B), so they are of four different commensurability classes. However we were not able to prove that those groups are not commensurable to the ones already found by Ayaka and Long and Reid.

3 Infinite families of pseudomodular surfaces

We now wonder if there exists infinitely many commensurability classes of pseudomodular groups. The answer is positive. In order to see it, we have to stop considering only ideal quadrilatera and take bigger polygons. In this section, we follow the presentation by Lou, Tan and Vo in [2].

Definition 4. *A marked triangle is an oriented triangle along with one point on each of its edges.*

Let T be such a triangle with oriented triple of ideal vertices $(v_1, v_2, v_3) \in (\mathbb{R} \cup \{\infty\})^3$, and marked points $x_1 \in (v_1, v_2)$, $x_2 \in (v_2, v_3)$ and $x_3 \in (v_3, v_1)$. Let p_1 (respectively p_2, p_3) be the orthogonal projection of v_3 on (x_1, x_2) (respectively of v_1 on (v_2, v_3), of v_2 on (v_2, v_1)) and for $i = 1, 2, 3$ let us call "parameter associated to the edge i" and write $k_i = e^{2d(p_i, x_i)} \in (0, +\infty)$, $d(p_i, x_i)$ being the oriented distance, calculated with respect from the orientation of the edge e_i containing x_i. The triple (k_1, k_2, k_3), up to cyclic permutation, characterizes T up to isometry of marked triangles. We thus write $\Delta(k_1, k_2, k_3)$ to denote T.

Let us write h_1, h_2, h_3 the π-rotations around x_1, x_2, x_3. The isometries $h_1 h_2$ and $h_2 h_3$ are side pairings of an ideal quadrilateron, just like g_1 and g_2 in section 2, and $< h_1, h_2, h_3 >$ is an index 2 supergroup of $< h_1 h_2, h_2 h_3 >$ as we saw in subsection 2.3. The Poincaré theorem says that $< h_1, h_2, h_3 >$ is a fuchsian group with the triangle $\Delta(k_1, k_2, k_3)$ as a fundamental domain if $h_1 h_2 h_3$ is parabolic, and a computation shows that $< h_1, h_2, h_3 >$ is a fuchsian group with the triangle $\Delta(k_1, k_2, k_3)$ as a fundamental domain if $h_1 h_2 h_3$ is parabolic, and a computation shows that this is equivalent to saying that $k_1k_2k_3 = 1$. We call "good" triangles marked triangles with $k_1k_2k_3 = 1$.

Definition 5. *Let T, T' be two marked triangles. We say that an edge e_i of T is compatible with an edge e'_j of T' if e_i and e'_j have the same parameter ($i, j = 1, 2, 3$).*

Let T, T' be two marked triangles, e_i, e'_j sides respectively of T and T' that are compatible. Let us glue T and T' along e_i and e'_j, such that the marked point x of e_i and e'_j are identified. Then the π rotation around x sends the (unmarked) triangle T on T' and the (unmarked) triangle T' on T.

Definition 6. *A jigsaw is defined by induction :*

- *A marked triangle $\Delta(k_1, k_2, k_3)$ is a jigsaw.*
- *A triangle glued with a jigsaw along an exterior side of the jigsaw and a compatible side of the triangle is a jigsaw.*

Remark 2. *A jigsaw with n triangles is a $(n+2)$-agon.*

Definition 7. *The group Γ_J associated to a jigsaw J is the group generated by the π-rotations around the marked points of the exterior sides of J.*

Proposition 5. *If a jigsaw J is made only with "good" triangles, then*

(1) Γ_J is a fuchsian group with fundamental domain the polygon defined by the jigsaw, and $\mathrm{Cusp}(\Gamma_J) = \Gamma_J \cdot \infty$

(2) If there are n triangles in J, then Γ_J is an order 2-supergroup of a free group defined by side pairings of a $(2n+2)$-agon.

Proof. (2) is a consequence of the Poincaré theorem applied to $J \cup h(J)$, where h is a π-rotation around an exterior side of J. (1) is a consequence of (2). $\square$

From now on, we will consider only jigsaws made up of "good" hyperbolic triangles.

Such a jigsaw J gives rise, when considering $\Gamma_J \cdot J$, to a triangulation of the hyperbolic plane with marked triangles, each edge of the triangulation having a parameter associated to it.

Definition 8.

- *An integral triangle is a marked triangle that is isometric to $\Delta(1, \frac{1}{n}, n)$ for some $n \in \mathbb{N}$.*
- *An integral jigsaw is a jigsaw made with integral triangles, among which there is a $\Delta(1, 1, 1)$ triangle.*
- *An integral jigsaw is in standard position if it contains a $\Delta(1, 1, 1)$ traingle with vertices in position $(\infty, -1, 0)$.*

Proposition 6. *Let J be an integral jigsaw in standard position. Let T_J be the triangulation of $\mathbb{H}^2$ associated to J. Let Δ be a $\Delta(1, \frac{1}{n}, n)$-triangle that has an ideal vertex at ∞.*

- *If $n = 1$, then the two other ideal vertices are m and $m+1$ with $m \in \mathbb{Z}$.*
- *If $n \neq 1$ and the edge of Δ opposite to ∞ has parameter 1, then the two other vertices are m and $m+n$ with $m \in \mathbb{Z}$.*
- *If $n \neq 1$ and the edge of Δ opposite to ∞ has parameter n or $\frac{1}{n}$, then the two other vertices of Δ are m and $m+1$ with $m \in \mathbb{Z}$.*

Proof. It is an induction starting at the $\Delta(1,1,1)$ triangle of J with vertices ∞, -1 and 0. $\square$

Proposition 7. *Let J be an intergral jigsaw in standard position. Let T_J be the triangulation of $\mathbb{H}^2$ associated to J and let Δ be an integral triangle of T_J with one vertex at ∞. Let $m \in \mathbb{Z}$ be another ideal vertex of Δ. Then there is $\gamma \in \Gamma_J$ such that $\gamma(\infty) = m$ and such that the killer interval associated to γ is $(m-1, m+1)$.*

Proof. We have $m \in \mathrm{Cusp}(\Gamma_J)$, so there exists $\gamma \in \Gamma_J$ such that $\gamma(\infty) = m$. Let Δ' be the triangle of T_J that is the preimage of Δ by γ : $\Delta = \gamma(\Delta')$. Since γ preserves orientation, the relations $\Delta = \gamma(\Delta')$, $m = \gamma(\infty)$ is enough to compute γ knowing the vertices of Δ and Δ'. According to proposition 6, there are only few cases for the vertices of Δ and Δ'. In each case, the computation gives $(m-1, m+1)$ as a killer interval for γ. $\square$

Proposition 8. *If J is an integral jigsaw in standard position that contains only $\Delta(1,1,1)$ and $\Delta(1,\frac{1}{2},2)$ triangles (and at least one of each type), then Γ_J is nonarithmetic.*

Proposition 9. *Let J_n be the integral jigsaw in standard position with a $\Delta(1,1,1)$ triangle and n $\Delta(1,\frac{1}{2},2)$-triangles glued, each having a vertex at ∞ and positive ideal vertices. Then the Γ_{J_n} are noncommensurable with each other.*

A Arithmeticity of fuchsian groups

There is a canonical surjection $\pi : \mathrm{SL}(2,\mathbb{R}) \twoheadrightarrow \mathrm{PSL}(2,\mathbb{R})$. Reciprocally, to any group $\Gamma < \mathrm{PSL}(2,\mathbb{R})$, we can associate the group $\pi^{-1}(\Gamma) < \mathrm{SL}(2,\mathbb{R})$. In this section, thanks to these two correspondances, we will consider without distinction subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and the corresponding subgroups in $\mathrm{SL}(2,\mathbb{R})$.

A.1 Quaternion algebras

Definition 9. *(Quaternion algebra) A quaternion algebra over a field k is an algebra over k whose dimension as a k-vector space is 4 and which is a simple central algebra, i.e.*

- *The centre of A is $k \cdot 1$*
- *For any ideal $R \subset A$ such that there exists $e \in \mathbb{N}$ with $R^e = \{0\}$, $R = \{0\}$*

Proposition 10. *Let A be a quaternion algebra. There exists $i, j \in A$ such that the family $(1, i, j, ij)$ is a basis of the k-vector space A and $i^2 \in k$, $j^2 \in k$, $ij = -ji$.*

Let $i, j \in A$ be as in the proposition 10. Then the operations on A are totally determined by the values of $i^2 \in k$ and $j^2 \in k$. Furthermore, if B is another quaternion algebra over the same field k, and $i', j' \in B$ as in the proposition 10, if $i^2 = i'^2$ and $j^2 = j'^2$, then A and B are isomorphic as k-algebras. Thus we can denote quaternion algebras by the notation

$$\left(\frac{a\ b}{k}\right)$$

k being the base field and $a, b \in k$ being the respective values of i^2, j^2.

For example, the hamiltonian quaternions

$$\mathbb{H} = \left(\frac{-1\ -1}{\mathbb{R}}\right)$$

and the matrices of size 2 with coefficients in a field k

$$M(2,k) = \left(\frac{1\ 1}{k}\right)$$

are quaternion algebras, with $i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the case of $M(2, k)$.

Let $A = \left(\frac{a\,b}{k}\right)$ be a quaternion algebra over a field k, and $i, j \in A$ as in proposition 10. We can define on A a reduced trace Trd and a reduced norm Nrd, by setting for $x = x_1 \cdot 1 + x_2 \cdot i + x_3 \cdot j + x_4 \cdot ij$:

$$\begin{aligned} \mathrm{Trd}(x) &= 2x_1 \\ \mathrm{Nrd}(x) &= x_1^2 - ax_2^2 - bx_3^2 + x_4^2 \end{aligned}$$

We can check that these two functions don't depend on the choice of i and j. In the case the the matrices of size 2, the reduced trace corresponds to the trace and the reduced norm to the determinant. One can check that for any quaternion algebra, the reduced trace is additive and the reduced norm is multiplicative.

Definition 10. *A quaternion algebra A is called a division algebra if any element in A is invertible.*

In particular, the hamiltonian quaternions $\mathbb{H}$ form a division algebra, but $M(2, k)$ is not a division algebra. In fact we ave the following proposition :

Proposition 11. *Let A be a quaternion algebra over a field k. If A is not a division algebra, then A is isomorphic to $M(2, k)$.*

A.2 Groups derived from a quaternion algebra

Let us now define arithmetic fuchsian groups and state the main theorems that enable us to characterize them. First we need to define a very special case of quaternion algebras, that are the ones involved in the definition of arithmetic fuchsian groups.

Definition 11. *A totally real number field is a number field k such that for any embedding $\sigma : k \hookrightarrow \mathbb{C}$, one has $\sigma(k) \subset \mathbb{R}$.*

Let $A = \left(\frac{a\, b}{k}\right)$ be a quaternion algebra over a number field k, and let $k \hookrightarrow \mathbb{C}$ be an embedding. Then $\left(\frac{\sigma(a)\ \sigma(b)}{\sigma(k)}\right)$ defines a quaternion algebra over the number field $\sigma(k)$, that we write A^σ.

Definition 12. *Let A be a quaternion algebra over a totally real number field k, and let $id = \sigma_1, \ldots, \sigma_n$ be the embeddings $k \hookrightarrow \mathbb{C}$. A is said to be a "good" quaternion algebra if*

$$\begin{cases} A \otimes \mathbb{R} \simeq M(2, \mathbb{R}) \\ A^{\sigma_i} \otimes \mathbb{R} \simeq \mathbb{H}\ (2 \leq i \leq n) \end{cases}$$

$A \simeq B$ meaning "A and B isomorphic".

Definition 13. *Let A be a quaternion algebra over k totally real number field, and write k^0 for the ring of integers of k (i.e. elements in k that are roots of unitary polynomials with coefficients in $\mathbb{Z}$). An order of A is a subring $\mathcal{O}$ of A that generates A as a k-vector space and such that for all $x \in \mathcal{O}$, $\mathrm{Trd}(x) \in k^0$ and $\mathrm{Nrd}(x) \in k^0$.*

Proposition 12. *If A is a "good" quaternion algebra and $\mathcal{O}$ is an order of A, then $\iota(\mathcal{O})$ is a fuchsian group, where ι is the isomorphism $\iota : A \otimes \mathbb{R} \simeq M(2, \mathbb{R})$ and*

$$\mathcal{O}^1 = \{x \in \mathcal{O}, \mathrm{Nrd}(x) = 1\}$$

Definition 14. *A group $\Gamma < \mathrm{SL}(2, \mathbb{R})$ is derived from a quaternion algebra if there exists a "good" quaternion algebra A and an order $\mathcal{O}$ of A such that Γ is a finite index subgroup of $\iota(\mathcal{O}^1)$ where ι is the isomorphism $\iota : A \otimes \mathbb{R} \simeq M(2, \mathbb{R})$ and $\mathcal{O}^1 = \{x \in \mathcal{O}, \mathrm{Nrd}(x) = 1\}$*

Definition 15. *Two subgroups Γ_1 and Γ_2 of $\mathrm{SL}(2, \mathbb{R})$ are said to be* commensurable in a strong sense *if $\Gamma_1 \cap \Gamma_2$ is finite index in both Γ_1 and Γ_2.*

Definition 16. *A group $\Gamma < \mathrm{SL}(2, \mathbb{R})$ is said to be arithmetic if it is commensurable in a strong sense to a group derived from a quaternion algebra.*

Here are now the two main theorems of this subsection :

Theorem 3. *A group $\Gamma < \mathrm{SL}(2, \mathbb{R})$ is derived from a quaternion algebra if and only if*

(i) $k = \mathbb{Q}[\mathrm{Tr}(\Gamma)]$ is a totally real number field, where $\mathrm{Tr}(\Gamma) = \{\mathrm{Tr}(x), x \in \Gamma\}$

(ii) $\mathrm{Tr}(\Gamma) \subset k^0$

(iii) *If $\phi : k \hookrightarrow \mathbb{R}$ is an embedding that is not identity, then $\phi(\mathrm{Tr}(\mathbb{R}))$ is bounded in $\mathbb{R}$.*

Proof. (Sketch) Let $A = k[\Gamma]$ (i.e. linear combinations of elements in Γ with elements in k) and $\mathcal{O} = k^0[\Gamma]$. The following statements need to be proven :

(1) A is a quaternion algebra

(2) A is "good"

(3) $\mathcal{O}$ is an order of A

(4) Γ is a finite order subgroup of $\mathcal{O}^1$.

- Point (4) is easy : it is obvious that Γ is a subgroup of $\mathcal{O}^1$. Moreover, according to proposition 12, $\mathcal{O}^1$ is a fuchsian group. Since Γ is of finite coarea, $\mathcal{O}^1$ is of finite coarea and $\mathrm{Area}(\mathbb{H}^2/\Gamma) = [\Gamma : \mathcal{O}^1] \cdot \mathrm{Area}(\mathbb{H}^2/\mathcal{O}^1)$. So Γ is a finite order subgroup of $\mathcal{O}^1$
- The proofs of (1) and (3) are quite elementary, though a little bit technical. It is in the proof of (3) that hypothesis (ii) is involved.
- The proof of (2) is more difficult. It is the one that involves (iii).

□

Theorem 4. *If $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ is finitely generated and finite covolume, then Γ is arithmetic if and only if Γ^2 is derived from a quaternion algebra.*

A.3 The case of $\Delta(u^2, 2\tau)$

In the case of $\Delta(u^2, 2\tau) =< g_1, g_2 >$ with $u^2, \tau \in \mathbb{Q}$, we know that $\Delta(u^2, 2\tau)^{(2)} < \mathrm{PSL}(2,\mathbb{Q})$, thus $\mathrm{Tr}(\Delta(u^2, 2\tau)) \subset \mathbb{Q}$. As a consequence, the theorems 3 and 4 can be stated as follows :

Theorem 5. *For $u^2, 2\tau \in \mathbb{Q}$, $\Delta(u^2, \tau)$ is an arithmetic fuchsian group if and only if for $\gamma \in \Delta(u^2, 2\tau)^{(2)}$, $\mathrm{Tr}(\gamma) \in \mathbb{Z}$.*

In our research of pseudomodular groups, we want to find groups that are *not* pseudomodular to $\mathrm{PSL}(2,\mathbb{Z})$. In fact it is enough to find groups that are not arithmetic thanks to the following proposition :

Proposition 13. *If Γ is a finite covolume fuchsian group that is commensurable (in a weak sense) to an arithmetic group, then Γ is arithmetic.*

Proof. If Γ is commensurable to Γ' with Γ' arithmetic, then Γ is commensurable in a strong sense to $\gamma\Gamma'\gamma^{-1}$ with $\gamma \in \mathrm{PSL}(2,\mathbb{R})$. Then it is enough to show that $\gamma\Gamma'\gamma^{-1}$ is arithmetic. Since Γ is finite covolume, Γ' too and $\gamma\Gamma'\gamma^{-1}$ too. Moreover $\mathrm{Tr}((\gamma\Gamma'\gamma^{-1})^2) = \mathrm{Tr}(\gamma\Gamma'^{(2)}\gamma^{-1}) = \mathrm{Tr}(\Gamma^{(2)})$. So since $\mathrm{Tr}(\Gamma'^{(2)})$ verifies the conditions of the theorem 3, $\mathrm{Tr}((\gamma\Gamma'\gamma^{-1})^{(2)})$ too, thus $(\gamma\Gamma'\gamma^{-1})^{(2)}$ is derived from a quaternion algebra. Finally, $\gamma\Gamma'\gamma^{-1}$ is arithmetic and so is Γ. □

$\mathrm{PSL}(2,\mathbb{Z})$ is obviously arithmetic since $M(2,\mathbb{Z})$ is an order of the "good" quaternion algebra $M(2,\mathbb{Q})$. Thus it is enough to show that a group $\Delta(u^2,2\tau)$ is nonarithmetic in order to show that it is not commensurable to $\mathrm{PSL}(2,\mathbb{Z})$. Knowing this is enough for our purpose because there are very few arithmetic groups among the family $\Delta(u^2,2\tau)$. But we have a stronger fact :

Proposition 14. *The group $\Delta(u^2,2\tau)$ is commensurable (in a weak sense) to $\mathrm{PSL}(2,\mathbb{Z})$ if and only if it is arithmetic.*

In order to prove this proposition we need two hard theorems :

Theorem 6. *Any order of $M(2,\mathbb{Q})$ is embedded into a maximal order.*

Theorem 7. *Any maximal order of $M(2,\mathbb{Q})$ is conjugated to $M(2,\mathbb{Z})$.*

Proof. of proposition 14 Suppose $\Delta(u^2,2\tau)$ is arithmetic. Then $\Delta(u^2,2\tau)^{(2)}$ is derived from a quaternion algebra. As a consequence of the proof of theorem 3, we have $A=\mathbb{Q}[\Delta(u^2,2\tau)^{(2)}]\subset M(2,\mathbb{Q})$ is a quaternion algebra and $\mathcal{O}=\mathbb{Z}[\Delta(u^2,2\tau)^{(2)}]$ is an order of A. Since A is a $\mathbb{Q}$-vector space of dimension 4, subspace of $M(2,\mathbb{Q})$, $A=M(2,\mathbb{Q})$ and $\mathcal{O}$ is an order of $M(2,\mathbb{Q})$. Thus $\mathcal{O}$ is included into a maximal order $\widetilde{\mathcal{O}}$ of $M(2,\mathbb{Q})$ (theorem 6), which is conjugated to $M(2,\mathbb{Z})$ by theorem 7 :

$$\mathcal{O}\subset\widetilde{\mathcal{O}}=\gamma M(2,\mathbb{Z})\gamma^{-1}$$

with $\gamma\in GL(2,\mathbb{Q})$. Thus we have :

$$\Delta(u^2,2\tau)^{(2)}<\mathcal{O}^1<\widetilde{\mathcal{O}}^1=\gamma\,\mathrm{SL}(2,\mathbb{Z})\gamma^{-1}$$

Since $\Delta(u^2,2\tau)^{(2)}$ and $\mathrm{PSL}(2,\mathbb{Z})$ are finite covolume fuchsian groups, $\Delta(u^2,2\tau)^{(2)}$ is a finite index subgroup of $\gamma\,\mathrm{PSL}(2,\mathbb{Z})\gamma^{-1}$, so it is commensurable to $\mathrm{PSL}(2,\mathbb{Z})$. But $\Delta(u^2,2\tau)^{(2)}$ is a finite index subgroup of $\Delta(u^2,2\tau)$, so $\Delta(u^2,2\tau)$ is commensurable to $\mathrm{PSL}(2,\mathbb{Z})$. □

Let us now come back to practical purposes. Instead of computing the whole set of traces of $\Delta(u^2,2\tau)^{(2)}$ in order to see if $\Delta(u^2,2\tau)$ is arithmetic, we only need to compute 3 of them :

Proposition 15. *If $\gamma\in\Delta(u^2,2\tau)$, then $\mathrm{Tr}(\gamma)$ is a polynomial in $\mathrm{Tr}(g_1^2)$, $\mathrm{Tr}(g_2^2)$ and $\mathrm{Tr}(g_1^2g_2^2)$ with integer coefficients.*

Once again, for our purpose only one implication is really useful and it is the easy one. Indeed, since we want to prove that groups are *not* arithmetic, it suffices to have *one* trace that is not in $\mathbb{Z}$. Let us see, however, the techniques that allow to prove this proposition.

Proposition 16. *Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Q})$. Let $\alpha_1,\dots\alpha_n$ be generators of Γ. Let*

$$E=\{\alpha_{i_1}\cdots\alpha_{i_k},i_1<\cdots<i_k,1\leq k\leq n\}$$

Then the trace of any element $\gamma\in\Gamma$ can be expressed as a polynomial in elements of E with integer coefficients.

Proof. Let us show this proposition when $n = 2$, α_1 and α_2 are generators of Γ. The proof for bigger n's uses the same idea. Let us show that for any word with letters in $\{\alpha_1, \alpha_2, \alpha_1^{-1}, \alpha_2^{-1}\}$, $\mathrm{Tr}(w)$ is a polynomial in $\mathrm{Tr}(\alpha_1)$, $\mathrm{Tr}(\alpha_2)$ and $\mathrm{Tr}(\alpha_3)$ with integer coefficients. Let us proceed by induction. The proof only uses the Cayley-Hamilton identity :

$$\gamma^2 - \mathrm{Tr}(\gamma)\gamma + I_2 = 0$$

- For words of length 1, i.e. α_1, α_2, α_1^{-1} and α_2^{-1}, we have have for instance

  $$\alpha_1 - \mathrm{Tr}(\alpha_1)I_2 + \alpha_1^{-1} = 0$$

  thus taking the trace :

  $$\mathrm{Tr}(\alpha_1) = \mathrm{Tr}(\alpha_1^{-1})$$

  and similarly $\mathrm{Tr}(\alpha_2) = \mathrm{Tr}(\alpha_2^{-1})$.

- Let w be a word of length $k \geq 2$. If $w = w'\alpha_1^2$, we have by multiplicating the Cayley-Hamilton identity by w' :

  $$w'\alpha_1 - \mathrm{Tr}(\alpha_1)w'\alpha_1 - w' = 0$$

  thus, taking the trace

  $$\mathrm{Tr}(w) = \mathrm{Tr}(\alpha_1)\,\mathrm{Tr}(w'\alpha_1) - \mathrm{Tr}(w')$$

  by induction, $\mathrm{Tr}(w'\alpha_1)$ and $\mathrm{Tr}(w')$ are polynomials in $\mathrm{Tr}(\alpha_1)$, $\mathrm{Tr}(\alpha_2)$ and $\mathrm{Tr}(\alpha_1\alpha_2)$ with integer coefficients. The same arguments works if $w = w'\alpha_2^2$, $w' = \alpha_1^{-2}$ or $w' = \alpha_2^{-2}$. Since $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$, we can reduce the problem to one of these cases as soon as w is not of the form $\gamma_1 \cdots \gamma_k$ with $\gamma_1 = \gamma_k$ or $\gamma_i = \gamma_{i+1}$ for some i. In this last case, w is in fact an integer polynomial in words that are strictly shorter than w and a word of the form $(\alpha_1\alpha_2)^l$. Indeed, if $w = w'\gamma$, then still by multiplicating by w' and taking the trace in Cayley-Hamilton, we get

  $$\mathrm{Tr}(w) = \mathrm{Tr}(w')\,\mathrm{Tr}(\gamma) - \mathrm{Tr}(w'\gamma^{-1})$$

  With this relation we can "change" any occurence of α_1^{-1} or α_2^{-1} in w in respectively α_1 or α_2. Finally, it suffices to notice that by multiplicating by γ^{l-2} in Cayley-Hamilton, we have

  $$\mathrm{Tr}(\gamma^l) - \mathrm{Tr}(\gamma)\,\mathrm{Tr}(\gamma^{l-1}) + \mathrm{Tr}(\gamma^{l-2}) = 0$$

  Thus $\mathrm{Tr}((\alpha_1\alpha_2)^l)$ is an integer polynomial in $\mathrm{Tr}(\alpha_1\alpha_2)$.

□

Proposition 17. *The group $\Delta(u^2, 2\tau)^{(2)} =< g_1, g_2 >^{(2)}$ is generated by words of length 2 and 4 in which the sum of all the occurences of g_1 (respectively g_2) (counting negatively for occurences of g_1^{-1}) is even.*

Now that we have a finite family of generators for $< g_1, g_2 >^{(2)}$, we can compute all the traces associated to the set E of the proposition 16. Such a computation (for instance with a formal calculus software) shows that they are all polynomials in $\mathrm{Tr}(g_1^2)$, $\mathrm{Tr}(g_2^2)$ and $\mathrm{Tr}(g_1^2 g_2^2)$, proving proposition 15. In order to prove proposition 17, we first need a lemma that gives us explicitely the group $< g_1, g_2 >^{(2)}$.

Lemma 2. *As a set, the group* $< g_1, g_2 >^{(2)}$ *is*

$$\mathcal{E} = \left\{ (g_2^{\eta_1}) g_1^{\epsilon_1} g_2^{\eta_2} \cdots g_1^{\epsilon_k} (g_2^{\eta_{k+1}}), \epsilon_i, \eta_i \in \{\pm 1\}, \sum \eta_i \equiv \sum \epsilon_i \pmod 2 \right\}$$

As a consequence, proposition 17 says that $< g_1, g_2 >^{(2)}$ is generated by words of $< g_1, g_2 >^{(2)}$ of length ≤ 4.

Proof. (of the lemma) The only hard inclusion is $\mathcal{E} \subset < g_1, g_2 >^{(2)}$. Let us pick $w \in \mathcal{E}$ and let us show that w is a product of squares of elements of $< g_1, g_2 >$ by induction on the length of w.

- If w is of length 0 or 2, it is obvious.
- If w is of length ≥ 3, then we can suppose g_1 is the first letter of w. If g_1^2 are the first two letters of w, then $w = g_1 w'$ and we conclude by induction. Else, since there is an even number of elements in $\{g_1, g_1^{-1}\}$ in w, there is another occurence of $g_1^{\pm 1}$ further in the word : $w = g_1 w' g_1^{\pm 1} w''$. A multiplication by $(g_1^2 w')^{-2}$ gives $(g_1 w')^{-2} w = w'^{-1} g_1^{-1} g_1^{\pm 1} w''$. We can iterate this process untill we get a smaller word or a word of the type $\gamma_1^2 \cdots \gamma_k^2$ with $\gamma_i \in \{g_1, g_1^{-1}, g_2, g_2^{-1}\}$. This proves the lemma by induction.

□

Proof. (of proposition 17) Let $\mathcal{G}$ be the set defined in the proposition 17, and let w be a word in $< g_1, g_2 >^{(2)}$. Once again we proceed by induction.

- By the lemma, if the length of w is ≤ 4, $w \in \mathcal{E}$.
- If the length of w is bigger than 4, let us focus on the last 4 letters of w : $w = w_1 w_2$ with w_2 of length 4. If $w_2 \in \mathcal{G}$, we can already conclude by induction. We can also conclude if the 2 last letters of w_2 are the same. The cases that are left can all be handled with the same trick. Let us consider the case where $w_2 = g_1^2 g_2 g_1$. Then $w_2 \cdot (g_1^{-1} g_2^{-1} g_1^{-1} g_2) = g_1 g_2$, which is of length < 4. Since $g_1^{-1} g_2^{-1} g_1^{-1} g_2 \in \mathcal{G}$, we can also conclude by induction in this case.

□

B Cusp density

Definition 17. *Let Γ be a fuchsian group. The commensurator of Γ is the set*

$$\left\{\gamma \in \mathrm{PSL}(2, \mathbb{R} : \gamma^{-1}\Gamma\gamma \textit{ is commensurable to } \Gamma)\right\}$$

Proposition 18. *Let Γ, Γ' be fuchsian groups*

(1) $\Gamma \subset \mathrm{Comm}(\Gamma)$

(2) $\mathrm{Comm}(\Gamma)$ *is a group*

(3) If Γ *and* Γ' *are commensurable, then* $\mathrm{Comm}(\Gamma)$ *and* $\mathrm{Comm}(\Gamma')$ *are conjugated*

Theorem 8. *(Margulis)*

- *If* Γ *is an arithmetic fuchsian group, then* $\mathrm{Comm}(\Gamma)$ *is dense in* $\mathrm{PSL}(2, \mathbb{Z})$.
- *If* Γ *is a nonarithmetic fuchsian group, then* $\mathrm{Comm}(\Gamma)$ *is a fuchsian group and* $[\Gamma : \mathrm{Comm}(\Gamma)] < +\infty$

Take $u_1, \tau_1, u_2, \tau_2 \in \mathbb{R}$ so that $\Gamma_1 = \Delta(u_1^2, 2\tau_1)$ and $\Gamma_2 = \Delta(u_2^2, 2\tau_2)$ are nonarithmetic and commensurable to each other. We have the following commutative diagram, with G their common commensurator.

$$\begin{tikzcd}
\mathbb{H}^2 \arrow[r, dashed, "\exists\tilde{\gamma}"] \arrow[d, "\pi_1"] & \mathbb{H}^2 \arrow[d, "\pi_2"] \\
\mathbb{H}^2/\Gamma_1 \arrow[d, "\sigma_1"] & \mathbb{H}^2/\Gamma_2 \arrow[d, "\sigma_2"] \\
\mathbb{H}^2/\mathrm{Comm}(\Gamma_1) \arrow[r, "\gamma"] & \mathbb{H}^2/\mathrm{Comm}(\Gamma_2)
\end{tikzcd}$$

The existence of $\tilde{\gamma}$ is ensured by the proposition 18 : since $\mathrm{Comm}(\Gamma_1)$ and $\mathrm{Comm}(\Gamma_2)$ are conjugated in $\mathrm{PSL}(2, \mathbb{R})$, the isometry γ lifts to an isometry $\tilde{\gamma} \in \mathrm{PSL}(2, \mathbb{R})$.

Now consider a horoball D_1 in $\mathbb{H}^2$ at a cusp p of $\Gamma_1 =< g_1, g_2 >$, and then its orbit P_1 under the action of Γ_1. The condition on $g_1 g_2^{-1} g_1^{-1} g_2$ being a parabolic ensures that there is in P_1 at most one horoball per cusp of Γ_1 ; and the fact that $\mathbb{H}^2/\Gamma_1$ has only one cusp ensures that there is at least one horocircle in P_1 at each cusp of Γ_1. Finally there is in P_1 exactly one horocircle per cusp of Γ_1. Moreover, if D_1 is small enough for the disks of P_1 corresponding to the 4 cusps of a fundamental domain of Γ_1, then all the horoballs in P_1 are disjoint. By increasing D_1 continuously, we get a smallest D_1, called D_1^0, such that the horoballs in P_1^0 are not disjoint anymore. This P_1 is called the "maximal circle packing" associated to Γ_1. We denote it by P_1^0.

We can consider similarly a horoball D_2 at the cusp $\tilde{\gamma}$ of Γ_2 and the circle packing P_2 associated. Since $\tilde{\gamma}$ is an isometry, the horoball D_2^0 corresponding to the maximal circle P_2^0 verifies $D_2^0 = \tilde{\gamma}(D_1^0)$ and $P_2^0 = \tilde{\gamma}(P_1^0)$.

We call "cusp density" of Γ_1 and we write $\mathrm{Dens}(\Gamma_1)$ the quantity

$$\mathrm{Dens}(\Gamma_1) = \frac{\mathrm{Area}(\pi_1(P_1^0))}{\mathrm{Area}(\mathbb{H}^2/\Gamma_1)}$$

Let us now show that the cusp density is a commensurability invariant, i.e., in our context, that $\mathrm{Dens}(\Gamma_1) = \mathrm{Dens}(\Gamma_2)$. We have, by definition of $\tilde{\gamma}$, $\sigma_2 \circ \pi_2(P_2^0) = \gamma(\sigma_1 \circ \pi_1(P_1^0))$. Thus $\mathrm{Area}(\sigma_1 \circ \pi_1(P_1^0)) = \mathrm{Area}(\sigma_2 \circ \pi_2(P_2^0))$ But σ_1 and σ_2 are finite coverings, say d_1 and d_2 coverings. So we have :

$$\begin{aligned}
\mathrm{Area}(\mathbb{H}^2/\Gamma_1) &= d_1 \cdot \mathrm{Area}(\mathbb{H}^2/\operatorname{Comm}(\Gamma_1)) \\
\mathrm{Area}(\pi_1(P_1^0)) &= d_1 \cdot \mathrm{Area}(\sigma_1 \circ \pi_1(P_1^0)) \\
\mathrm{Area}(\mathbb{H}^2/\Gamma_2) &= d_2 \cdot \mathrm{Area}(\mathbb{H}^2/\operatorname{Comm}(\Gamma_2)) \\
\mathrm{Area}(\pi_2(P_2^0)) &= d_2 \cdot \mathrm{Area}(\sigma_2 \circ \pi_2(P_2^0))
\end{aligned}$$

Since $\mathrm{Area}(\mathbb{H}^2/\operatorname{Comm}(\Gamma_1)) = \mathrm{Area}(\mathbb{H}^2/\operatorname{Comm}(\Gamma_2))$, this shows $\mathrm{Dens}(\Gamma_1) = \mathrm{Dens}(\Gamma_2)$

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