The symplectic connections with a parallel Ricci curvature

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July, 2002

Abstract. A symplectic connection on a symplectic manifold, unlike the Levi-Civita connection on a Riemannian manifold, is not unique. However, some spaces admit a canonical one (symmetric symplectic spaces, Kähler manifolds ...); besides, some "preferred" symplectic connections can be defined in some situations (see [6]). These facts motivate a study of the symplectic connections *inducing a parallel Ricci tensor.* This paper gives the possible forms of the Ricci curvature on such manifolds and gives a decomposition theorem (linked with the holonomy decomposition) for them.

Keywords: Affine connection, Symplectic connection, Ricci curvature, Holonomy group, Kähler manifolds, parakähler manifolds, Einstein manifolds, pseudo-Riemannian manifolds.

AMS Classification: 53B05, 53B30, 53B35, 53C25, 53C55.

Introduction and motivation.

On a Riemannian or pseudo-Riemannian manifold is defined the Levi-Civita connection. The symplectic analog is the following. Let $(\mathcal{M}, \omega)$ be a symplectic manifold; a connection D on $\mathcal{M}$ is said to be symplectic when:

- D is torsion-free: for every vectorfields x and y, $D_x y - D_y x - [x, y] = 0$,
- the symplectic form is parallel for D : $D\omega = 0$.

To such a connection D is associated its (1,3)-curvature tensor R and its Ricci curvature tensor, here denoted by ric. Let us recall ric is the bilinear symmetric form defined on each tangent space by: $\operatorname{ric}(u, v) = \operatorname{tr} R(u, .)v$. Unlike in the (pseudo-)Riemannian situation, the set of symplectic connections is an affine space of infinite dimension (see 1**(b)** below). In some situations however, there is a privileged one.

In case $\mathcal{M}$ is a *symmetric* symplectic space for instance (in the natural sense introduced in [11]; see also [2] for the symplectic case), it has a canonical connection, which is symplectic. This connection is symmetric, so in particular its Ricci curvature is parallel.

In case $\mathcal{M}$ is a (pseudo-)Riemannian manifold carrying a parallel symplectic form — *e.g.* a Kähler manifold —, the Levi Civita connection is also symplectic with respect to it.

On a general symplectic manifold, F. Bourgeois and M. Cahen have introduced in [6] a variational principle distinguishing so-called "preferred" symplectic connections. The corresponding field equations are:

$$D_x \operatorname{ric}(y, z) + D_y \operatorname{ric}(z, x) + D_z \operatorname{ric}(x, y) = 0.$$

In particular, symplectic connections the Ricci curvature of which is parallel, *i.e.* such that $D \operatorname{ric} = 0$, are therefore preferred. More generally, they have a specific interest in this theory and were soon studied by M. Cahen and al. in [7]. Note also that the canonical connection of a symmetric symplectic manifold is thus preferred.

Besides, Riemannian manifolds the Ricci curvature of which is parallel, shortly called here Ricci-parallel, are, at least locally, products of Einstein manifolds. Pseudo-Riemannian Ricci-parallel manifolds admit an analogous, though slightly different decomposition; see [5]. We show here a similar result for Ricci parallel symplectic connections (Main Theorem in section 2). It shall be noticed that the algebraic part of the result is the same as in the pseudo-Riemannian case, the geometrical consequence being weaker in general.

The structure of the article is the following. After some lemmas and remarks given in section 1, the Main Theorem is stated and commented in section 2, then proven in section 3. Section 4 gives a refinement of the decomposition obtained in the Main Theorem and studies the subfactors. Finally section 5 provides some examples and last remarks.

Notations. On a symplectic manifold with a symplectic connection $(\mathcal{M}, \omega, D)$, we will denote by ric the Ricci tensor and by Ric the endomorphism induced by ric, *i.e.* the endomorphism such that $\mathrm{ric}(.,.) = \omega(., \mathrm{Ric}\,.)$.

We denote by H the holonomy group of $(\mathcal{M}, D)$, and classical Lie algebras by old German letters. For example, if $p \in \mathcal{M}$, the symplectic Lie algebra $\mathfrak{sp}(\omega_p)$ is the algebra of the ω_p-antiselfadjoint endomorphisms of $\mathrm{T}_p\mathcal{M}$; besides $\mathfrak{h}$ denotes the Lie algebra of the holonomy group.

1 Elementary facts about symplectic connections.

We need some basic facts in the following; pointing them out here together will also make symplectic connections more familiar.

(a) As hinted at above, the properties satisfied by a symplectic connection D are those that define the Levi-Civita connection of a Riemannian or pseudo-Riemannian metric g, if you replace ω by g. On any symplectic manifold, such a connection exists but it is *not* unique. The space of the symplectic connections associated with a given form ω is parametrized by $S^3\mathrm{T}^*\mathcal{M}$; let us recall the

Proposition 1 *let D be a symplectic connection on $(\mathcal{M}, \omega)$, then a connection Δ is symplectic* iff *: $\omega(D.\cdot, \cdot) - \omega(\Delta.\cdot, \cdot) \in S^3\mathrm{T}^*\mathcal{M}$.*

Proof. It is a straightforward remark, see [10] p.48.

(b) The curvature tensor R satisfies the usual algebraic properties :

- $R(x, y) = -R(y, x)$,
- $\omega(R(x, y)z, t) = \omega(R(x, y)t, z)$ *i.e.* all the $R(x, y)$ are ω-antiselfadjoint,
- $R(x, y).z + R(y, z).x + R(z, x).y = 0$ "Bianchi identity"

In the (pseudo-)Riemannian situation, an additional relation involving R and the metric g then follows:

$$g(R(x, y).z, t) = g(R(z, t).x, y) \tag{1}$$

It is *not* true with R and ω, ω being an *alternate* form. However, notice that, provided all the $R(x,y)$ for $x, y \in \mathrm{T}_p\mathcal{M}$ are antiselfadjoint with respect to a bilinear symmetric form g, we get (1) for R and g, whether g is degenerate or not. The proof does not need nondegeneracy, see [12] p.54 for example.

In particular, we get the following little

Lemma 1 *If g is a parallel symmetric bilinear form on $(\mathcal{M}, D)$, R and g satisfy (1).*

Note. Relation (1) between R and the metric g is one of the essential tools giving the pseudo-Riemannian result [5]. So is it here: the Main Theorem is based on the fact that R and ric satisfy (1), see the next remark.

(c) In the (pseudo-)Riemannian situation, ric is the only non-trivial invariant trace of R. In the symplectic case, there is *a priori* another one : $u, v \mapsto \mathrm{tr}_\omega[\omega(R(.,.)u,v)]$. However, it turns out that it is the same, up to a scalar; let us recall the (standard) little

Lemma 2 *If (M, ω, D) is a symplectic manifold with a symplectic connection:*

$$\mathrm{tr}_\omega[\omega(R(.,.)u,v)] = -2\,\mathrm{ric}(u,v).$$

Proof. It follows from Bianchi identity; as the result will be useful, let us recall its proof.

Let $2n$ be the dimension of $\mathcal{M}$, p be a point in $\mathcal{M}$ and $(e_i)_{i=1}^{2n}$ be a basis of $\mathrm{T}_p\mathcal{M}$ such that $\omega = \sum_{i\le n} e_i^* \wedge e_{n+i}^*$. For a and b in $\mathrm{T}_p\mathcal{M}$:

$$\begin{aligned}
& \mathrm{tr}_\omega[\omega(R(\cdot,\cdot)a,b)] \\
= \; & \textstyle\sum_{i\le n} \omega(R(e_i,e_{n+i})a,b) - \omega(R(e_{n+i},e_i)a,b) \\
= \; & 2\textstyle\sum_{i\le n} \omega(R(e_i,e_{n+i})a,b) \\
= \; & 2\left(\textstyle\sum_{i\le n} \omega(R(a,e_{n+i})e_i,b) + \omega(R(e_i,a)e_{n+i},b)\right) \quad \text{(Bianchi Identity)} \\
= \; & 2\left(\textstyle\sum_{i\le n} \omega(R(a,e_{n+i})b,e_i) - \omega(R(a,e_i)b,e_{n+i})\right) \\
= \; & -2\,\mathrm{tr}[R(a,\cdot)b] \\
= \; & -2\,\mathrm{ric}(a,b)
\end{aligned}$$

□

Important remark. The bilinear form ric is hence *symmetric* (what also holds for a Levi-Civita connection, but not in general for a torsion-free affine connection). In particular, Ric is ω-antiselfadjoint.

With lemma 1, if ric is parallel, it implies also that (1) holds for ric and more generally for all the bilinear symmetric forms $\omega(., \mathrm{Ric}\, P(\mathrm{Ric}^2).)$ where P is a polynomial.

(d) The endomorphism Ric in this framework.

A final preliminary is necessary before stating the theorem. It is some standard linear algebra but has to be precisely stated here.

Let p be a point of $\mathcal{M}$; Ric being parallel, its minimal polynomial (*i.e.* the monic generator of the ideal of the polynomials P of $\mathbb{R}[X]$ such that $P(\mathrm{Ric}) = 0$) is defined independently of the point. Now $\mathrm{Ric}_{|p} \in \mathfrak{sp}(\omega_{|p})$, so we can apply to the complexified endomorphism $\mathrm{Ric}^{\mathbb{C}}$ of $\mathrm{T}_p\mathcal{M} \otimes \mathbb{C}$ the following well-known

Lemma 3 *Let (E, ω) be a complex vectorspace endowed with a nondegenerate alternate form ω and U in $\mathfrak{sp}(\omega)$. The minimal polynomial μ of U then satisfies: $\mu(X) = \pm\mu(-X)$. There thus exists an $L \subset \mathbb{C}$ such that $L \cup (-L) = \{$nonzero eigenvalues of $U\}$ and $L \cap (-L) = \emptyset$; with such a L:*

$$E = \ker U^{\alpha_0} \overset{\perp}{\oplus} \left(\overset{\perp}{\bigoplus_{\lambda \in L}} (\ker(U - \lambda \operatorname{Id})^{\alpha_\lambda} \oplus \ker(U + \lambda \operatorname{Id})^{\alpha_\lambda}) \right),$$

where α_λ is the (common) power of $(X - \lambda)$ and $(X + \lambda)$ in μ. The decomposition is orthogonal with respect to ω and each space $\ker(U \pm \lambda \operatorname{Id})$ is ω-totally isotropic. Here α_0 may be zero.

Furthermore, Ric being real, its minimal polynomial μ is also invariant under complex conjugation; so taking for example $\Lambda = \{\text{eigenvalues of Ric}\} \cap (\mathbb{R}^+ \times i\mathbb{R}^+) \subset \mathbb{C}$, we get:

$$\mu = \prod_{\lambda \in \Lambda} P_\lambda^{\alpha_\lambda} \quad \text{with:} \begin{cases} P_0 = X & \text{appearing if } 0 \in \Lambda \\ P_\lambda = (X - \lambda)(X + \lambda) & \text{if } \lambda \in \mathbb{R}^* \cup i\mathbb{R}^* \\ P_\lambda = (X - \lambda)(X + \lambda)(X - \overline{\lambda})(X + \overline{\lambda}) & \text{otherwise.} \end{cases} \tag{2}$$

and the corresponding decomposition of $\mathrm{T}_p\mathcal{M}$:

$$\mathrm{T}_p\mathcal{M} = \overset{\perp}{\bigoplus_{\lambda \in \Lambda}} \ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric})). \tag{3}$$

Remark. This is the finest ω-orthogonal decomposition of $\mathrm{T}_p\mathcal{M}$ that is stable under the action of the centralisor of Ric. However, under this action and for example for $\lambda \in \mathbb{R}^*$:

- $\ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric})) = \ker(\mathrm{Ric} - \lambda \operatorname{Id})^{\alpha_\lambda} \oplus \ker(\mathrm{Ric} + \lambda \operatorname{Id})^{\alpha_\lambda}$; each factor being stable but ω-totally isotropic.
- $\ker(\mathrm{Ric} - \lambda \operatorname{Id})^{\alpha_\lambda}$ and $\ker(\mathrm{Ric} + \lambda \operatorname{Id})^{\alpha_\lambda}$ are irreducible *iff* $\alpha_\lambda = 1$.

2 The Main Theorem.

A Riemannian manifold with parallel Ricci curvature is, at least locally, a product of Einstein manifolds (its only a remark, see [5], pp.2 and 3). Let us recall a manifold is said Einstein if ric is proportional to the metric. In our situation, this notion has no sense, since ω is alternate and ric symmetric.

Nevertheless, being a local product of Einstein Riemannian manifolds can be stated in other terms: ric is parallel and the minimal polynomial of Ric has simple roots in $\mathbb{C}$ (then necessarily in $\mathbb{R}$, g being positive definite). That statement has a sense in our symplectic situation. Is it true? Yes, except possibly for the root zero. It is the same result as for a pseudo-Riemannian connection, the proof being quite different: see [5].

Remark. Before stating the theorem, let us recall a straightforward fact linking holonomy-stable subspaces with some foliations. If $(\mathcal{M}, D)$ is a manifold endowed with a torsion-free connection D, if p is a point of $\mathcal{M}$ and if the holonomy group stabilizes a subspace A of $\mathrm{T}_p\mathcal{M}$, then A can be extended by parallel transport to a (parallel) distribution on $\mathcal{M}$. The connection being torsion-free, this distribution is integrable; the leaves of the integral foliation are moreover totally geodesic.

Main Theorem. *Let (M,ω,D) be a symplectic manifold with a symplectic connection the Ricci curvature* ric *of which is parallel and $p\in\mathcal{M}$. Let μ be the minimal polynomial of* Ric *and $\mu=\prod_{\lambda\in\Lambda}P_\lambda^{\alpha_\lambda}$ the decomposition (2) of μ given in section 1. For simplicity of the statement, we set $0\in\Lambda$ and allow α_0 to be null. Let us also denote by M_λ the (parallel) distribution $\ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric}))$, by $\mathcal{M}_\lambda^q$ the integral leaf of M_λ through a point q and simply by $\mathcal{M}_\lambda$ the leaf $\mathcal{M}_\lambda^p$. Then:*

(i) *For each $\lambda\neq 0$, α_λ is equal to one and: $\alpha_0\leq 2$.*

(ii) *Setting, for each λ, $\omega_\lambda=\omega_{|\mathrm{T}\mathcal{M}_\lambda}$ and $D_\lambda=D_{|\mathrm{T}\mathcal{M}_\lambda}$, $(\mathcal{M}_\lambda,\omega_\lambda,D_\lambda)$ is a symplectic manifold with a symplectic connection. Now, the unique local diffeomorphism $\mathcal{M}\to\prod_\lambda\mathcal{M}_\lambda$ preserving the integral foliations of the M_λ and equal to identity on the $\mathcal{M}_\lambda$ identifies, on a suitable neighbourhood of p, $\mathcal{M}$ to $\prod_\lambda\mathcal{M}_\lambda$. With this identification:*

$$(\mathcal{M},\omega,D)\simeq\left(\textstyle\prod_\lambda\mathcal{M}_\lambda,\prod_\lambda\omega_\lambda,(\prod_\lambda D_\lambda)+S\right),$$

with S a (1,2)-tensor on $\mathcal{M}$. Moreover D and $\prod_\lambda D_\lambda$ have the same Ricci curvature and S satisfies the following properties:

- *$\omega(S(.,.),.)$ is completely symmetric,*
- *S is a section of $\pi_0^*(\mathrm{T}_2^1\mathcal{M}_0)$, where π_0 is defined, at each point q, as the canonical projection $\mathrm{T}_q\mathcal{M}=\oplus_\lambda\mathrm{T}_q\mathcal{M}_\lambda^q\to\mathrm{T}_q\mathcal{M}_0^q$,*
- *$\forall(x,y)\in\mathrm{T}_q\mathcal{M}$, $\mathrm{tr}[z\mapsto D_zS(x,y)]-\mathrm{tr}[z\mapsto S(x,S(y,z))]=0$,*
- *$\mathrm{Im}\,\mathrm{Ric}\subset\ker S$,*

the last property being a consequence of the third one.

We prove the main theorem in section 3. We make here some remarks.

(a) As the decomposition $\mathrm{T}_p\mathcal{M}=\oplus_\lambda M_\lambda$ is unique, either is the collection $((\mathcal{M}_\lambda,\omega_\lambda,D_\lambda)_{\lambda\in\Lambda},S)$.

(b) Once supposed that ric is parallel, the first point of the theorem is a purely "pointwise" consequence of the algebraic properties of the curvature tensor R,

The second one is a consequence of an adaptation of de Rham's decomposition theorem of Riemannian manifolds, see Proposition 2 below.

Point **(i)** will then give information on the factors $\mathcal{M}_\lambda$ given by point **(ii)**: see section 4, in particular Proposition 4 in section 4.2.

Proposition 2 *Let $(\mathcal{M},\omega,D)$ be a symplectic manifold with a symplectic connection and $p\in\mathcal{M}$. Suppose that the restricted holonomy group H^0 preserves an ω-orthogonal decomposition:*

$$\mathrm{T}_p\mathcal{M}=\overset{\perp}{\underset{0\leq i\leq k}{\oplus}}M_i$$

of $\mathrm{T}_p\mathcal{M}$. Then for each i, M_i induces by parallel transport a parallel, thus integrable, distribution on $\mathcal{M}$, also denoted by M_i.

Let $(\mathcal{M}_i)$ be the integral manifold through p of the distribution M_i. Then:

(i) *The* $(\mathcal{M}_i, \omega_i, D_i) = (\mathcal{M}_i, \omega_{|\mathrm{T}\mathcal{M}_i}, D_{|\mathrm{T}\mathcal{M}_i})$ *are symplectic manifolds with a symplectic connection.*

(ii) *The unique local diffeomorphism preserving the foliations induced by the* M_i *and equal to identity on the* $\mathcal{M}_i$ *identifies, on a suitable neighbourhood of* p*,* $\mathcal{M}$ *to* $\prod_i \mathcal{M}_i$*. On this neighbourhood:* $\omega = \prod_i \omega_i$*.*

(iii) *With this local identification* $\mathcal{M} \simeq \prod_i \mathcal{M}_i$*, there is* S *a (unique) (1,2)-tensor on* $\mathcal{M}$ *such that:* $D = (\prod_i D_i) + S$*.*

Moreover, S *satisfies the following conditions:*

- $\omega(S(.,.),.)$ *is symmetric,*
- $S = \sum_i S^i$ *where each* S^i *is a section of* $\pi_i^*(\mathrm{T}^1_2\mathcal{M}_i)$*, where* π_i *is defined, at each point* q*, as the canonical projection* $\mathrm{T}_q\mathcal{M} = \oplus_j \mathrm{T}_q\mathcal{M}_j^q \to \mathrm{T}_q\mathcal{M}_i^q$*,*
- *For each* i *and each* $q \in \mathcal{M}_i$*,* $S^i_q = 0$, i.e.: $S_{|(\mathrm{T}\mathcal{M}_i)^2}$ *is null on* $\mathcal{M}_i$*.*

This proposition is, adapted to a symplectic connection, the local (and easy) part of de Rham's theorem. Its proof will also be given in section 3. Two points of the Riemannian theorem fail here to be true:

- The result is weaker — and a little deceptive — because $(\mathcal{M}, D)$ is not a product for the affine structure: $\mathcal{M} \simeq \prod_i(\mathcal{M}_i, \omega_i)$ but $D \neq \prod_i D_i$. This is due to the non-uniqueness of a symplectic connection on a symplectic manifold.
- For a Riemannian manifold $\mathcal{M}$, $\mathrm{T}_p\mathcal{M}$ is the sum of a trivial subrepresentation of H and of a sum of irreducible subrepresentations; a consequence is the uniqueness of this decomposition. It is not the case here, since $\mathrm{T}_p\mathcal{M}$ may admit reducible-indecomposable factors. So there does not exist in general any *canonical* decomposition of $\mathrm{T}_p\mathcal{M}$ under the action of H (or of H^0).

Nevertheless, in case $(\mathcal{M}, \omega, D)$ is a *symmetric* symplectic space, a quite unexpected decomposition result holds, see [4], theorems 2.3 and 2.12. Besides, point **(iv)** of Proposition 2 will in fact not be used here; it is hewever mentioned as a natural part of the result.

(c) In general, the local symplectomorphism $(\mathcal{M}, \omega) \to \prod_\lambda(\mathcal{M}_\lambda, \omega_\lambda)$ of the Main Theorem is *not* an isomorphism of affine structure from $(\mathcal{M}, D)$ on $\prod_\lambda(\mathcal{M}_\lambda, D_\lambda)$. However, it *is* one in the case Ric is nondegenerate; the following decomposition holds then:

Corollary 1 *Let* (M, ω, D) *be a symplectic manifold with a symplectic connection, the Ricci curvature* ric *of which is parallel and nondegenerate. Let* μ *be the minimal polynomial of* Ric *and* $\mu = \prod_{\lambda\in\Lambda} P_\lambda^{\alpha_\lambda}$ *the decomposition* (2) *of* μ *given in section 1. Then:*

(i) *For each* λ*,* α_λ *is equal to one (and* $0 \notin \Lambda$ *since* ric *is nondegenerate).*

(ii) *There exists a unique family* $((\mathcal{M}_\lambda, \omega_\lambda, D_\lambda))_{\lambda\in\Lambda}$ *of symplectic manifolds with a symplectic connection such that*

- *for each* λ*, the minimal polynomial of* $\mathrm{Ric}_{\mathcal{M}_\lambda}$ *is* P_λ*,*
- $(\mathcal{M}, \omega, D)$ *is locally affinely symplectomorphic to* $\prod_\lambda(\mathcal{M}_\lambda, \omega_\lambda, D_\lambda)$*.*

(iii) *If $(\mathcal{M}, \omega, D)$ is moreover geodesically complete and simply connected, the isomorphism of point* **(ii)** *is global.*

Proof. Points **(i)** and **(ii)** are simply the case "$\alpha_0 = 0$, $\mathcal{M}_0$ reduced to a point" of the Main Theorem: then $S = 0$, what gives points **(i)** and **(ii)**. Point **(iii)** is an immediate consequence of the global part of Wu's theorem ([13], see here in section 4.1) applied to the pseudo-Riemannian manifold $(\mathcal{M}, \mathrm{ric})$. □

We can also easily understand autonomously the reason why it works. In that case indeed, $(\mathcal{M}, \mathrm{ric})$ turns out to be a pseudo-Riemannian manifold (which is moreover Einstein with constant 1 by definition). Ric being parallel, the decomposition

$$\mathrm{T}_p\mathcal{M} = \overset{\perp}{\underset{\lambda\in\Lambda}{\oplus}} \ker(P_\lambda^{\alpha_\lambda}(\mathrm{Ric}))$$

is holonomy stable. Applying Wu's theorem, the pseudo-Riemannian generalization of de Rham's theorem (see [13]), we get that $\mathcal{M}$ is isomorphic to the Riemannian product of the factors $\mathcal{M}_\lambda$. Besides, the symplectic connection D is torsion-free and satisfies $D\,\mathrm{ric} = 0$, so it is the Levi-Civita connection of the metric ric. Consequently, the Riemannian product is also a affine morphism $(\mathcal{M}, D) \simeq \prod_\lambda(\mathcal{M}_\lambda, D_\lambda)$. □

(d) Conversely, if $(\mathcal{M}_i, \omega_i, D_i)_{i=0}^k$ are symplectic manifolds with Ricci-parallel symplectic connections, with Ric_i nondegenerate except for $i = 0$, then a manifold of the type

$$(\textstyle\prod_i(\mathcal{M}_i, \omega_i), \prod_i D_i + S)$$

where S satisfying the properties listed in the Main Theorem, is Ricci-parallel. It is an immediate consequence of proposition 1 combined with lemma 6 below.

3 Proof of the Main Theorem.

Proof of Proposition 2 of section 2. We have to check that the Riemannian proof (see [Ko-No] pp.179 sq.) remains valid or can be adapted at each step. Let us do it for $k = 2$, the general case comes then by induction. We denote M_1, $\mathcal{M}_1$, M_2 and $\mathcal{M}_2$ by A, $\mathcal{A}$, B and $\mathcal{B}$ respectively. For another point q of $\mathcal{M}$, $\mathcal{A}_q$ (resp. $\mathcal{B}_q$) will stand for the integral leaf of A (resp. B) through q.

(i) At p, $\omega_{|A}$ and $\omega_{|B}$ are nondegenerate. Now $\mathcal{A}$ and $\mathcal{B}$ being integral leaves of *parallel* distributions and ω being parallel, $\omega_{|\mathrm{T}\mathcal{A}}$ and $\omega_{|\mathrm{T}\mathcal{B}}$ are nondegenerate; let us denote them by $\omega^\mathcal{A}$ and $\omega^\mathcal{B}$. $\mathcal{A}$ is totally geodesic, so the restriction $D^\mathcal{A}$ to $\mathrm{T}\mathcal{A}$ of the connection D is the connection induced by D on the submanifold $\mathcal{A}$. Hence similarly for $\mathcal{B}$. Eventually, as $D\omega = 0$, $D^\mathcal{A}\omega^\mathcal{A} = 0$ and $(\mathcal{A}, \omega^\mathcal{A}, D^\mathcal{A})$ is (locally) a symplectic submanifold of $\mathcal{M}$, with a symplectic connection (hence also for $(\mathcal{B}, \omega^\mathcal{B}, D^\mathcal{B})$).

(ii) The fact that $\mathcal{M}$ is locally canonically diffeomorphic to $\mathcal{A}\times\mathcal{B}$ is obvious and purely differential, see [Ko-No], lemma p.182, for a formal proof. We can then take local coordinates of $\mathcal{M}$ of the form $((a_i)_{i=1}^{d_\mathcal{A}}, (b_i)_{i=1}^{d_\mathcal{B}})$ such that, at every point q: $A_q = \mathrm{span}(\partial/\partial a_i)_{i=1}^{d_\mathcal{A}}$ and $B_q = \mathrm{span}(\partial/\partial b_i)_{i=1}^{d_\mathcal{B}}$.

Proving that ω is equal to the product form $\omega^\mathcal{A} \times \omega^\mathcal{B}$ is showing: for each (i, j, k),

$$L_{\partial/\partial b_i}[\omega(\partial/\partial a_j, \partial/\partial a_k)] = 0.$$

It follows ([Ko-No], prop. 5.2 p.182), from the fact that D is torsion-free. Indeed for each (i,j): $D_{\partial/\partial b_i}(\partial/\partial a_j) = D_{\partial/\partial a_j}(\partial/\partial b_i)$. Now, as the distributions A and B are parallel, $D_{\partial/\partial b_i}(\partial/\partial a_j) \in A$ and $D_{\partial/\partial a_j}(\partial/\partial b_i) \in B$. Thus:

$$D_{\partial/\partial b_i}(\partial/\partial a_j) = D_{\partial/\partial a_j}(\partial/\partial b_i) = 0. \tag{4}$$

Then:

$$L_{\partial/\partial b_i}[\omega(\partial/\partial a_j, \partial/\partial a_k)] =$$
$$\underbrace{(D_{\partial/\partial b_i}\omega)}_{=0}(\partial/\partial a_j, \partial/\partial a_k) + \omega(\underbrace{D_{\partial/\partial b_i}(\partial/\partial a_j)}_{=0}, \partial/\partial a_k) + \omega(\partial/\partial a_j, \underbrace{D_{\partial/\partial b_i}(\partial/\partial a_j)}_{=0}) = 0.$$

(iii) and the properties of S. The product connection $D^{\mathcal{A}} \times D^{\mathcal{B}}$ on $\mathcal{A} \times \mathcal{B}$ is a symplectic connection. Indeed, the local product structure of $(\mathcal{M}, \omega)$ induces a local diffeomorphism between $\mathcal{A}_p$ and $\mathcal{A}_q$ for each point q, preserving moreover ω and mapping $D^{\mathcal{A}}$ on $(D^{\mathcal{A}} \times D^{\mathcal{B}})_{|\mathrm{T}\mathcal{A}_q}$ by definition of $D^{\mathcal{A}} \times D^{\mathcal{B}}$. As $D^{\mathcal{A}}\omega = 0$ on $\mathcal{A}_p$, $(D^{\mathcal{A}} \times D^{\mathcal{B}})_{|\mathrm{T}\mathcal{A}_q}\omega = 0$. So by proposition 1, there exists a (2,1)-tensor S on $\mathcal{A}_q$ such that:

- $D_{|\mathrm{T}\mathcal{A}_q} = (D^{\mathcal{A}} \times D^{\mathcal{B}})_{|\mathrm{T}\mathcal{A}_q} + S^{\mathcal{A}_q}$
- $\omega(S^{\mathcal{A}_q}(.,.),.)$ is symmetric.

Now, as shown above, at any point of $\mathcal{M}$ and for any indexes (i,j), $D_{\partial/\partial b_i}(\partial/\partial a_j) = D_{\partial/\partial a_j}(\partial/\partial b_i) = 0$. The same equality is true for the procuct connection $D^{\mathcal{A}} \times D^{\mathcal{B}}$, by definition of it. So, for all (i,j), $S(\partial/\partial a_j, \partial/\partial b_i) = 0$. Therefore, as S is a tensor, $S_{|q}$ can be factored, pointwise at each point q, as $S_{|q} = (\pi_A)^*(S_{|q}^{\mathcal{A}_q}) + (\pi_B)^*(S_{|q}^{\mathcal{B}_q})$, where π_A is the projection $\mathrm{T}_q\mathcal{M} = A_q \oplus B_q \to A_q$ and similarly for π_B. The result follows. $\square$

In addition to proposition 2, we will also use three more lemmas. Point **(i)** of the Main Theorem is a consequence of an essential technical lemma we can state autonomously.

Lemma 4 *Let p be a point of $\mathcal{M}$, $U \in \mathfrak{sp}(\omega_{|p})$ (*i.e.* U is an ω-antiselfadjoint endomorphism of $\mathrm{T}_p\mathcal{M}$), commuting with all the $R(x,y)$ for $x, y \in \mathrm{T}_p\mathcal{M}$. Let us take $a, b \in \mathrm{T}_p\mathcal{M}$ with $b \in \mathrm{Im}\, U$. The bilinear form $\omega(R(.,.)a, b)$ is skew-symmetric; let us denote by $A_{a,b}$ the ω-selfadjoint endomorphism such that: $\omega(R(.,.)a,b) = \omega(., A_{a,b}.)$. Then:*

$$A_{a,b} = -U \circ R(a,c) = -R(a,c) \circ U, \text{ where } c \text{ is any antecedent of } b \text{ by } U.$$

Proof of the lemma. Let us simply write here $A = A_{a,b}$ and let us take c such that $U.c = b$. As $U \in \mathfrak{sp}(\omega_{|p})$, the bilinear form $u : (x,y) \mapsto \omega(x, Uy)$ is symmetric; as all the $R(x,y)$ are supposed to commute with U, notice they are all u-anti selfadjoint:

$$\begin{aligned} u(R(x,y)z, t) &= \omega(R(x,y)z, Ut) \\ &= \omega(R(x,y)Ut, z) \\ &= \omega(UR(x,y)t, z) \\ &= -\omega(z, UR(x,y)t) \\ &= -u(z, R(x,y)t). \end{aligned}$$

Consequently, by Lemma 1 of section 1, (1) holds for u:

$$\forall x, y, z, t,\ u(R(x,y).z, t) = u(R(z,t).x, y).$$

To prove the lemma it is sufficient to check: $\forall x, y \in \mathrm{T}_p\mathcal{M}$, $\omega(x, Ay) = \omega(x, U(R(a,c)y))$. Let x, y be any two vectors in $\mathrm{T}_p\mathcal{M}$. Then:

$$\begin{aligned}
\omega(x, Ay) &= \omega(R(x,y)a, b) \\
&= \omega(R(x,y), a, Uc) \\
&= u(R(x,y)a, c) \quad \text{by definition of } u \\
&= u(R(a,c)x, y) \quad \text{by (1)} \\
&= -u(x, R(a,c)y) \quad R(a,c) \text{ being } u\text{-anti selfadjoint} \\
&= -\omega(x, UR(a,c)y) \quad \text{by definition of } u. \qquad \Box
\end{aligned}$$

Let us also recall the following standard remark:

Lemma 5 *Let E be a real or complex vectorspace, $< .,. >$ a reflexive,* i.e. *symmetric or skew-symmetric form on E and U a $< .,. >$-antiselfadjoint endomorphism of E. Let $U = S + T$ be the decomposition of U into its semi-simple and nilpotent parts (unique such decomposition with $ST = TS$). Then S and T are $< .,. >$-antiselfadjoint.*

For point **(ii)** we will also need the following (standard) little

Lemma 6 *Let D and D' two symplectic connections on a symplectic manifold $(\mathcal{M}, \omega)$ and S the tensor such that $D' = D + S$. Let us denote by* ric *and* ric$'$ *the Ricci curvatures induced by D and D' respectively and by S_x the endomorphism $S(x,.)$. Then:*

$$\mathrm{ric}'(x,y) = \mathrm{ric}(x,y) - \mathrm{tr}[z \mapsto (D_z S)(x,y)] + \mathrm{tr}\, S_x S_y.$$

Proof. It is sufficient to do the proof with vectorfields which are coordinate vectorfields for some normal coordinate system at some point p in $\mathcal{M}$. For two distinct such vectors u and v: $D_u v = D_v u$ and, at p, $D_u v = 0$. With such vectors, a straightforward computation gives:

$$\begin{aligned}
R'(x,z)y &= (D+S)_z (D+S)_x y - (D+S)_x (D+S)_z y \\
&= R(x,z)y + (D_x S)_z y - (D_z S)_x y + S_x S_z y - S_z S_x y.
\end{aligned}$$

So: $\mathrm{ric}'(x,y) = \mathrm{ric}(x,y) + \mathrm{tr}[z \mapsto (D_x S)_z y - (D_z S)_x y - S_x S_z y + S_z S_x y]$. Now:

- $[z \mapsto S_z S_x y] = [z \mapsto S(z, S(x,y))] = S_{S(x,y)}$. But $\omega(S(.,.),.)$ is symmetric, so in particular: $\omega(S_u v, w) = \omega(S(u,v), w) = \omega(S(u,w), v) = -\omega(v, S(u,w)) = -\omega(v, S_u w))$ so the S_u are in $\mathfrak{sp}(\omega)$, thus trace-free. So $\mathrm{tr}[z \mapsto S_z S_x y] = 0$.
- For the same reason, $\mathrm{tr}(D_x S)_y = 0$. So by symmetry of S: $\mathrm{tr}[z \mapsto (D_x S)_z y] = \mathrm{tr}[(D_x S)_y] = 0$.
- By symmetry of S, $S_x S_z y = S_x S_y z$.

The result follows. $\Box$

We can now state the

Proof of the theorem.

(i) Let $\mathrm{Ric} = S + T$ be the decomposition of Ric into its semi-simple and nilpotent parts. As S and T are polynomials in Ric, they are themselves parallel. Let p be a point in $\mathcal{M}$, let us consider the endomorphism T acting on $\mathrm{T}_p\mathcal{M}$. Let us take $b \in \mathrm{Im}\, T$, say $b = T(c)$. By

Lemma 5, $T \in \mathfrak{sp}(\omega_p)$; T being parallel, it commutes with all the $R(x,y)$ for $x, y \in \mathrm{T}_p\mathcal{M}$, we can therefore apply Lemma 4. Combined with Lemma 2 it gives:

$$\begin{aligned} \forall a \in \mathrm{T}_p\mathcal{M},\ \mathrm{ric}(a,b) &= -\tfrac{1}{2}\,\mathrm{tr}_\omega[R(.,.)a,b] \quad \text{(Lemma 2)} \\ &= -\tfrac{1}{2}\,\mathrm{tr}[R(a,c)\circ T] \quad \text{(Lemma 4)}. \end{aligned}$$

But T is parallel so it commutes with $R(a,c)$; thus, T being nilpotent, so is $R(a,c)\circ T$. So $R(a,c)\circ T$ is trace-free, what means that: $\forall a \in \mathrm{T}_p\mathcal{M}$, $\mathrm{ric}(a,b) = 0$, that is to say: $b \in \ker \mathrm{Ric}$. So we get (at any point):

$$\mathrm{Im}\, T \subset \ker \mathrm{Ric}\,.$$

That is the wanted result. Indeed if μ is the minimal polynomial of Ric you can write:

$$\mu = X^{\alpha_0}.\prod_\lambda (X-\lambda)^{\alpha_\lambda}$$

where λ runs over the set of the nonzero eigenvalues of Ric and where α_0 is the — possibly null — power of X in μ. Then Ric is nondegenerate on $\ker\left[\prod_\lambda(\mathrm{Ric} - \lambda\,\mathrm{Id})^{\alpha_\lambda}\right]$ so on this space: $\mathrm{Im}\,T = \{0\}$ *i.e.* $T = 0$ *i.e.* all the α_λ for λ a nonzero eigenvalue of Ric are 1. On $\ker S = \ker \mathrm{Ric}^{\alpha_0}$, T is equal to Ric so $\mathrm{Im}\,T \subset \ker T$ *i.e.* $\alpha_0 \leq 2$.

(ii) The decomposition and the tensor S are given by Proposition 2. Let us denote by ric' the Ricci curvature of the product connection $D - S$ and let us prove that $\mathrm{ric}' = \mathrm{ric}$. Let us take $(x_i^\lambda)_{i=1}^{n_\lambda}$, where $n_\lambda = \dim \mathcal{M}_\lambda$, local coordinates on each $\mathcal{M}_\lambda$, in a neighbourhood of p; $((x_i^\lambda)_{i=1}^{n_\lambda})_{\lambda\in\Lambda}$ are coordinates of $\mathcal{M}$ in a neighbourhood of p. Let q be a point of $\mathcal{M}$ in such a neighbourhood and q_λ its projection on $\mathcal{M}_\lambda$, for any $\lambda \in \Lambda$.

• From the definition of the product connection follows: $\mathrm{ric}'_{|q}(\partial/\partial x_i^\lambda, \partial/\partial x_j^\lambda) = \mathrm{ric}'_{|q_\lambda}(\partial/\partial x_i^\lambda, \partial/\partial x_j^\lambda)$ for all i, j.

• By (4) applied to the distributions M_λ and the coordinate vectors $\partial/\partial x_i^\lambda$: $\lambda \neq \lambda' \Rightarrow \forall i, j$, $D_{\partial/\partial x_i^{\lambda'}}\partial/\partial x_j^\lambda = 0$. In particular, the vectorfields $\partial/\partial x_i^\lambda$ are D-parallel along any path tangent to $\oplus_{\lambda'\neq\lambda} M_{\lambda'}$. Now q and q_λ are joined by such a path so, as ric is D-parallel by assumption, the parallel transport along this path gives: $\mathrm{ric}_{|q}(\partial/\partial x_i^\lambda, \partial/\partial x_j^\lambda) = \mathrm{ric}_{|q_\lambda}(\partial/\partial x_i^\lambda, \partial/\partial x_j^\lambda)$ for all i, j.

Besides, by construction, D and $D - S$ coincide on each $\mathrm{T}\mathcal{M}_\lambda$, so either do their Ricci curvatures ric and ric' and therefore, for any λ and any (i,j), $\mathrm{ric}'_{|q}(\partial/\partial x_i^\lambda, \partial/\partial x_j^\lambda) = \mathrm{ric}_{|q}(\partial/\partial x_i^\lambda, \partial/\partial x_j^\lambda)$. Finally, as the M_λ are mutually ric-orthogonal (by definition of the M_λ) and ric'- orthogonal (by definition of the product connection), $\mathrm{ric} = \mathrm{ric}'$.

The first property of S comes from Proposition 1. Let us prove the factorization of S. On the similar integral manifold $\mathcal{M}_\lambda^q$ through any q, for $\lambda \neq 0$, the nondegenerate bilinear form $\mathrm{ric}_{\mathcal{M}_\lambda^q}$ is parallel for the product connection $(\prod_\lambda D_\lambda)_{|\mathrm{T}\mathcal{M}_\lambda^q}$ and for the original connection D of $\mathcal{M}$. So these connections are both equal to the Levi-Civita connection of $\mathrm{ric}_{\mathcal{M}_\lambda^q}$. So $S_{|\mathrm{T}\mathcal{M}_\lambda^q} = 0$. This gives, together with Proposition 2, the factorization of S.

After Lemma 6 above in this section, S satisfies the third property *iff* D and $D - S$ have the same Ricci curvature, what has been shown.

This implies finally that: $\mathrm{Im}\,\mathrm{Ric} \subset \ker S$. To see it, we show the following *Claim:* Let D' be a symplectic connection on some integral manifold $\mathcal{M}_0^q$ of $M_0 = \ker \mathrm{Ric}^2$

through some point q, inducing the same Ricci curvature as D and let S' be the tensor such that $D' = D + S'$. Then: $\operatorname{Im}\operatorname{Ric} \subset \ker S'$.

Let us indeed choose normal coordinates based at q. Then, for (x, y, z) any triple of coordinate vectors and ric being parallel:

$$2\operatorname{ric}(D'_x y, z) = L_x \operatorname{ric}(y, z) + L_y \operatorname{ric}(x, z) + L_z \operatorname{ric}(x, y),$$

by the same computations than those that give the expression of the Levi Civita connection of a metric g. So $\operatorname{ric}(D'_x y, z)$ is fixed *i.e.* is equal to $\operatorname{ric}(D_x y, z)$. Therefore, $\operatorname{ric}(S'(.,.),.) = 0$ or, equivalently: $\omega(S'(.,.), \operatorname{Im}\operatorname{Ric}) = 0$ by definition of Ric. By symmetry of S', it is again equivalent to: $S'(\operatorname{Im}\operatorname{Ric}, .) = 0$. So the claim holds, which completes the proof. □

4 Ricci decomposition and holonomy decomposition.

4.1 A refinement of the decomposition given by the Main Theorem.

The decomposition of $(\mathcal{M}, \omega, D)$ appearing in the Main Theorem may be refined. Let us introduce a definition.

Definition 1 *A pseudo-Riemannian manifold is said* weakly *irreducible if the holonomy group does not stabilize any* nondegenerate *proper subspace.*

Remark. Obviously, the holonomy representation is *weakly* irreducible *iff* it does not admit any decomposition into a direct *orthogonal* sum of stable subspaces.

De Rham's theorem on the decomposition of the Riemannian manifolds into a product of irreducible ones admits a pseudo-Riemannian generalization, in fact nearly the best that could be expected, *i.e.* the elementary factors are weakly irreducible. We recall the result of [13], appendix 1 p.389.

Theorem (de Rham, Wu) *Let $(\mathcal{M}, g)$ be a geodesically complete, simply connected Riemannian or pseudo-Riemannian manifold and $p \in \mathcal{M}$. We suppose that the maximal trivial subspace M_p^0 of H in $\mathrm{T}_p\mathcal{M}$ is nondegenerate. Then:*

(i) *$\mathrm{T}_p\mathcal{M}$ admits a decomposition, unique up to order: $\mathrm{T}_p\mathcal{M} = \overset{\perp}{\bigoplus\limits_{0\le i\le k}} M_p^i$, and H the decomposition: $H \simeq \prod_{1\le i\le k} H_i$, where each H_i acts weakly irreducibly on each M_p^i and trivially on the M_p^j for $j \neq i$.*

(ii) *$\mathcal{M}$ is isometric to the Riemannian product $\prod_{0\le i\le k} \mathcal{M}_i$, where each $\mathcal{M}_i$ is the maximal integral leaf through p of the parallel distribution M^i generated by M_p^i. $\mathcal{M}_0$ is flat.*

If $(\mathcal{M}, g)$ is not supposed to be geodesically complete and simply connected, the same result holds, for the full holonomy group H as well as for the restricted group H^0, except that the isometry of point **(ii)** *is only local.*

A consequence of this theorem in our situation is the following

Proposition 3 *Let $(\mathcal{M}, g)$ be a Riemannian or pseudo-Riemannian manifold and $p \in \mathcal{M}$. We suppose that the maximal trivial subspace M_p^0 of H in $\mathrm{T}_p\mathcal{M}$ is nondegenerate, and denote by $(\mathcal{M}, g) \simeq \prod_{0\le i\le k}(\mathcal{M}_i, g_i)$ Wu's decomposition of $\mathcal{M}$.*

Suppose that $(\mathcal{M}, g)$ admits a parallel and nondegenerate symplectic form ω. Then ω_i, the restriction of ω to $\mathrm{T}\mathcal{M}_i$, is nondegenerate and :

$$(\mathcal{M}, g, \omega) \simeq \prod_i (\mathcal{M}_i, g_i, \omega_i).$$

Proof. We use here the notations introduced in Wu's theorem above. It is sufficient to show that the M_p^i are in direct ω-orthogonal sum: the statement follows by parallel transport. Let us denote by Ω the element of $\mathfrak{so}(\mathrm{ric})$ such that: $\omega = g(.,\Omega.)$. By definition:

$$M_p^0 = \{x \in \mathrm{T}_p\mathcal{M} \ ; \ H.x = \{x\} \ \}.$$

So, with $x \in M_p^0$:

$$\begin{aligned} H.\Omega(x) &= \Omega(H.x) \quad \text{as } \Omega \text{ is parallel, so commutes with the action of } H, \\ &= \Omega(\{x\}) = \{\Omega(x)\}, \end{aligned}$$

therefore $\Omega(x) \in M_p^0$, hence: $\Omega(M_p^0) \subset M_p^0$, with equality as Ω is nondegenerate.

By point **(i)** of Wu's theorem, for $i \geq 1$:

$$M_p^i = (M_p^0)^\perp \cap \{x \in \mathrm{T}_p\mathcal{M} \ ; \ \forall j \neq i, H_j.x = \{x\} \ \}.$$

So similarly, for each $i \geq 1$: $\Omega(M_p^i) \subset M_p^0 \oplus M_p^i$. Now $\Omega \in \mathfrak{so}(\mathrm{ric})$ so:

$$g(\Omega(M_p^i), M_p^0) = -g(M_p^i, \Omega(M_p^0)) = -g(M_p^i, M_p^0) = \{0\},$$

so: $\Omega(M_p^i) \subset M_p^i$ (with equality). By definition of Ω, the wanted result follows. □

So Wu's holonomy decomposition provides a refinement of the Ricci decomposition given by the Main Theorem, at least a refinement of the decomposition of the factor on which ric is nondegenerate. Indeed, on this factor, ric, on the one hand, is parallel and nondegenerate, so is a (pseudo-)Riemannian metric, and, on the other hand, the trivial subspace of the action of the holonomy group is $\{0\}$, which is nondegenerate. So:

Corollary 2 *Let $(\mathcal{M}, \omega, D)$ a symplectic manifold with a symplectic connection D the Ricci curvature of which is parallel and nondegenerate. Then $(\mathcal{M}, \omega, D)$ admits a unique decomposition into a Riemannian product (with respect to* ric*, considered as a metric), such that each factor is weakly irreducible. Moreover, this decomposition holds also for ω:*

$$(\mathcal{M}, g, \omega) \simeq \prod_i (\mathcal{M}_i, g_i, \omega_i)$$

with g standing here for ric*, considered as the metric.*

Being unique and maximal, this decomposition is necessarily a refinement of that of the Main Theorem. Naturally, point **(i)** of the Main Theorem still applies and Ric is semi-simple on each factor (in fact, of minimal polynomial one of the P_λ).

4.2 A more precise description of the weakly irreducible factors.

Using the Main Theorem, we can now now give a more precise description of the weakly irreducible subfactors given by corollary 2. By the remark below, these factors are (pseudo-)Riemannian manifolds. We introduce also some vocabulary: paracomplex structures and related notions. Their names are chosen by analogy with the corresponding complex structures; other terminology is also used ("polarization" for a paracomplex structure for example).

Important remark. On these subfactors, as ric is parallel *and nondegenerate*, ric is a (pseudo-)Riemannian metric and D is its Levi-Civita connection. Moreover, such a manifold is obviously Einstein in that point of view, with Einstein constant 1. So in the following, symplectic manifolds with a symplectic connection such that ric is parallel and nondegenerate, will be viewed as Einstein non-Ricci flat manifolds admitting a parallel symplectic form.

Definition 2 *A paracomplex structure on a manifold $\mathcal{M}$ of dimension $2n$ is an endomorphism field L on $\mathcal{M}$, integrable, satisfying $L^2 = \mathrm{Id}$ with $\dim\ker(L-\mathrm{Id}) = \dim\ker(L+\mathrm{Id})$.*

If $(\mathcal{M}, g)$ is pseudo-Riemannian, a paracomplex structure on $\mathcal{M}$ satisfying: $g(Lx, y) = -g(x, Ly)$ is said to be parahermitian. If moreover $DL = 0$, it is said to be parakähler.

Remarks. A paracomplex structure gives therefore two complementary distributions of dimension n : $\ker(L \pm \mathrm{Id})$. Like L, these distributions are integrable. Equivalently, a paracomplex structure is given, up two sign, by the data of two such integrable distributions E and E' : $L = \pm(\mathrm{Id}_E \oplus -\mathrm{Id}_{E'})$.

For a parahermitian structure L, $\ker(L-\mathrm{Id})$ and $\ker(L+\mathrm{Id})$ are necessarily totally isotropic, so the signature of the metric is necessarily (n, n).

Vocabulary. Let us also recall that a pseudo-Kähler manifold is a pseudo-Riemannian manifold $(\mathcal{M}, g)$ admitting a g-orthogonal parallel complex structure J (in other words, a Kähler manifold with indefinite metric).

Now, a Riemannian or pseudo-Riemannian manifold admitting a parallel symplectic form is (pseudo-)Kähler or parakähler. The following proposition, using the Main Theorem (section 2), describes more precisely the situation when the manifold is Einstein, non Ricci-flat. The matrices of the different involved objects are also given, to make the situation clearer for the reader.

Notation. For each integer k, J_k will here denote the matrix: $\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$.

Proposition 4 *Let $(\mathcal{M}, g)$ be a weakly irreducible Einstein non Ricci-flat Riemannian or pseudo-Riemannian manifold and $p \in \mathcal{M}$. We suppose $(\mathcal{M}, g)$ admits a parallel symplectic form ω. Then, denoting $\dim\mathcal{M}$ by $2n$, $\mathcal{M}$ is in one of the three following situations:*

(i) *$(\mathcal{M}, g)$ has a parakähler structure L such that $\omega = \lambda g(., L.)$ with some λ in $\mathbb{R}^*$. In that case, g is of signature (n, n) and there is a basis of $\mathrm{T}_p\mathcal{M}$ in which:*

$$\mathrm{Mat}(g) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad \mathrm{Mat}(L) = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad \mathrm{Mat}(\omega) = \lambda \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

(ii) *$(\mathcal{M},g)$ has a (pseudo-)Kähler structure J such that $\omega=\lambda g(.,J.)$ with some λ in $\mathbb{R}^*$. In that case, g is of signature $(2p,2q)$ with $p+q=n$ and there is a basis of $\mathrm{T}_p\mathcal{M}$ in which:*

$$\mathrm{Mat}(g)=\begin{pmatrix} I_{2p} & 0 \\ 0 & -I_{2q}\end{pmatrix},\quad \mathrm{Mat}(J)=\begin{pmatrix} J_p & 0 \\ 0 & J_q\end{pmatrix},\quad \mathrm{Mat}(\omega)=\lambda\begin{pmatrix} J_p & 0 \\ 0 & -J_q\end{pmatrix}.$$

(iii) *$(\mathcal{M},g)$ has a pseudo-Kähler structure J and a parakähler structure L such that: $JL=LJ$ and that: $\omega=\alpha g(.,L.)+\beta g(.,J.)$ with $(\alpha,\beta)\in\mathbb{R}^{*2}$. In that case, n is even, g is of signature (n,n) and, setting $m=\frac{1}{2}n$, there is a basis of $\mathrm{T}_p\mathcal{M}$ in which:*

$$\mathrm{Mat}(g)=\begin{pmatrix} 0 & I_{2m} \\ I_{2m} & 0\end{pmatrix},\quad \mathrm{Mat}(L)=\begin{pmatrix} -I_{2m} & 0 \\ 0 & I_{2m}\end{pmatrix},\quad \mathrm{Mat}(J)=\begin{pmatrix} J_m & 0 \\ 0 & J_m\end{pmatrix},$$

$$\mathrm{Mat}(\omega)=\begin{pmatrix} 0 & \alpha I_{2m}+\beta J_m \\ -\alpha I_{2m}+\beta J_m & 0\end{pmatrix}.$$

Proof. After a possible rescaling, we may suppose that $g=\mathrm{ric}$. The decomposition (3), given in section 1 **(d)**, of $\mathrm{T}_p\mathcal{M}$ is stable under the action of H. So, by weak irreducibility of $\mathcal{M}$ and as ric is nondegenerate, the minimal polynomial of the endomorphism Ric is equal to a single factor $P_\nu^{\alpha_\nu}$ for some $\nu\in\mathbb{C}^*$ (with the definition given in (2), section 1). By point **(i)** of the Main Theorem in section 2, $\alpha_\nu=1$.

Let us discuss the situation for the different possible values of ν.

(i) If ν is real. Let us set $L=\frac{1}{\nu}\mathrm{Ric}$; L is a parallel endomorphism of $\mathfrak{so}(\mathrm{ric})$ with minimal polynomial $(X-1)(X+1)$, as $\alpha_\nu=1$. If $x,y\in\ker(L-\varepsilon\,\mathrm{Id})$ with $\varepsilon=\pm1$, $\mathrm{ric}(x,y)=\varepsilon\,\mathrm{ric}(x,Ly)=-\varepsilon\,\mathrm{ric}(Lx,y)=-\mathrm{ric}(x,y)$ so $\ker(L-\mathrm{Id})$ and $\ker(L+\mathrm{Id})$ are both ric-totally isotropic (that remark is also contained in lemma 3, section 1 **(d)**). As $\mathrm{T}_p\mathcal{M}=\ker(L-\mathrm{Id})\oplus\ker(L+\mathrm{Id})$ and ric is nondegenerate, these two spaces are of dimension n and ric is of signature (n,n); so L is a parakähler structure. Finally there is a basis of $\mathrm{T}_p\mathcal{M}$ as announced in the Proposition, with $\lambda=\frac{1}{\nu}$, and $\omega=\mathrm{ric}(.,\mathrm{Ric}^{-1}.)=\lambda\,\mathrm{ric}(.,L^{-1}.)=\lambda\,\mathrm{ric}(.,L.)$ as $L=L^{-1}$.

(ii) If ν is purely imaginary. Let us set $J=-\frac{1}{|\nu|}\mathrm{Ric}$; J is a parallel endomorphism of $\mathfrak{so}(\mathrm{ric})$ with minimal polynomial $(X-i)(X+i)=X^2-1$, as $\alpha_\nu=1$, so $J^2=-\mathrm{Id}$ and J is a Kähler or pseudo-Kähler structure (whether ric is definite or not). By the same computation as above or by lemma 3, and extending ric to a bilinear complex form on $\mathrm{T}_p\mathcal{M}\otimes\mathbb{C}$: $\ker(J-i\,\mathrm{Id})$ and $\ker(J+i\,\mathrm{Id})$ are both ric-totally isotropic; let n be their dimension ($\mathcal{M}$ is then of dimension $2n$). The complex conjugation $e\mapsto\overline{e}$ being a linear isomorphism of $\ker(J-i\,\mathrm{Id})$ to $\ker(J+i\,\mathrm{Id})$ and ric being nondegenerate, the sesquilinear form $h\ :(e,e')\mapsto\mathrm{ric}(e,\overline{e'})$ is nondegenerate on $\ker(J-i\,\mathrm{Id})$ and on $\ker(J+i\,\mathrm{Id})$. Its signature on each of these spaces is the same, let us denote it by (p,q). So if $(e_i)_{i=1}^n$ is a h-(pseudo-)orthonormal basis of $\ker(J-i\,\mathrm{Id})$, and setting $\beta=((e_i)_{i=1}^n(\overline{e_i})_{i=1}^n)$:

$$\mathrm{Mat}_\beta(\mathrm{ric})=\begin{pmatrix} 0 & I_{p,q} \\ I_{p,q} & 0\end{pmatrix}\quad\text{and: }\mathrm{Mat}_\beta(J)=\begin{pmatrix} iI & 0 \\ 0 & -iI\end{pmatrix}.$$

Now in the real basis $(f_i,f_i')_{i=1}^n$ of $\mathrm{T}_p\mathcal{M}$ defined by $f_i=\frac{1}{\sqrt{2}}(e_i+\overline{e_i})$ and $f_i'=\frac{1}{i\sqrt{2}}(e_i-\overline{e_i})$, the matrices of ric, J and ω have the announced form, with $\lambda=\frac{1}{|\nu|}$. Besides, $\omega=\mathrm{ric}(.,\mathrm{Ric}^{-1}.)=\lambda\,\mathrm{ric}(.,J.)$ as $J=-J^{-1}$.

Otherwise. Let us set $L = \frac{1}{2\Re\nu}(\mathrm{Ric} + |\nu|^2 \mathrm{Ric}^{-1})$ and $J = \frac{1}{2\Im\nu}(\mathrm{Ric} - |\nu|^2 \mathrm{Ric}^{-1})$. As $\alpha_\nu = 1$, a short computation gives:

$$L^2 - \mathrm{Id} = J^2 + \mathrm{Id} = P_\nu(\mathrm{Ric}) = 0.$$

Like in the previous point, using the nondegenerate hermitian form $h : e \mapsto \mathrm{ric}(e, \overline{e})$ of $\mathrm{T}_p\mathcal{M} \otimes \mathbb{C}$ and the fact that J and L commute, and denoting by n the dimension of $\ker(L - \mathrm{Id}) \cap \ker(J - i\,\mathrm{Id})$, we obtain a basis $(e_i)_{i=1}^{2m}$ of $\ker(J - i\,\mathrm{Id})$ such that, setting $\beta = ((e_i)_{i=1}^{m}, (\overline{e_i})_{i=1}^{m}, (\overline{e_i})_{i=m+1}^{2m}, (e_i)_{i=m+1}^{2m})$:

$$\mathrm{Mat}_\beta(\mathrm{ric}) = \begin{pmatrix} 0 & I_{2m} \\ I_{2m} & 0 \end{pmatrix}, \quad \mathrm{Mat}_\beta(L) = \begin{pmatrix} -I_{2m} & 0 \\ 0 & I_{2m} \end{pmatrix} \quad \text{and:}$$

$$\mathrm{Mat}_\beta(J) = \begin{pmatrix} iI_m & 0 & 0 & 0 \\ 0 & -iI_m & 0 & 0 \\ 0 & 0 & -iI_m & 0 \\ 0 & 0 & 0 & iI_m \end{pmatrix}.$$

$\mathcal{M}$ is of dimension $4m$ and ric of signature $(2m, 2m)$. As L and J are moreover in $\mathfrak{so}(\mathrm{ric})^{\mathfrak{h}}$, they are then, respectively, a parakähler and a pseudo-Kähler structure on $(\mathcal{M}, \mathrm{ric})$. Note also that they commute.

Now in the real basis $((f_i)_{i=1}^{n}, (f_i')_{i=1}^{n}, (f_i)_{i=n+1}^{2n}, (f_i')_{i=n+1}^{2n})$ of $\mathrm{T}_p\mathcal{M}$ defined by $f_i = \frac{1}{\sqrt{2}}(e_i + \overline{e_i})$ and $f_i' = \frac{1}{i\sqrt{2}}(e_i - \overline{e_i})$, the matrices of ric, L, J and ω have the announced form, with $\alpha + i\beta = \frac{1}{\nu}$. Besides, $\omega = \alpha\,\mathrm{ric}(., L.) + \beta\,\mathrm{ric}(., J.)$. $\square$

5 Some remarks and examples.

5.1 An example with $\mathbf{Ric}^2 = 0$ and $\mathbf{Ric} \neq 0$.

The Main Theorem requires that ric is nondegenerate to ensure that Ric has no nilpotent part. This assumption is necessary; it can be seen on a very simple example borrowed from [7] p.40. Take $(\mathcal{M}, \omega) = (\mathbb{R}^2, \mathrm{d}x \wedge \mathrm{d}y)$ and, denoting the coordinate vectors by X and Y, the connection defined by

$$D_X X = D_Y X = D_X Y = 0, \quad D_Y Y = xX.$$

In particular, X is stable by holonomy. By definition, D is torsion-free and we check:

$$\begin{aligned} (D_{aX+bY}\omega)(X,Y) &= a(D_X\omega)(X,Y) + b(D_Y\omega)(X,Y) \\ &= a[L_X(\omega(X,Y)) - \omega(X, D_X Y)] + b[L_Y(\omega(X,Y)) - \omega(X, D_Y Y)] \\ &= 0 \end{aligned}$$

so D is symplectic. Now $R(X,Y)X = 0$ and $R(X,Y)Y = -X$, so $\mathrm{ric}(Y,Y) = -1$, $\ker \mathrm{ric} = \mathrm{span}(X)$ and $D\,\mathrm{ric} = 0$. Actually, $DR = 0$ *i.e.* $(\mathcal{M}, \omega, D)$ is even symmetric. Now, $\mathrm{Ric}(X) = 0$ and $\mathrm{Ric}(Y) = X$ so $\mathrm{Ric} \neq 0$ and $\mathrm{Ric}^2 = 0$.

Remark. Examples where the minimal polynomial P_λ of Ric corresponds to a λ in $\mathbb{R}^*$, $i\mathbb{R}^*$ or $\mathbb{C}\backslash(\mathbb{R}^* \cup i\mathbb{R}^*)$ are numerous. They are the parakähler and (pseudo-)kähler manifolds, see prop. 4 in section 4.2. The next subsection gives symmetric examples of the three types.

5.2 The low-dimensional cases.

Let us recall the following fact:

Proposition 5 *Let* $(\mathcal{M}, g)$ *be*

- *either a Riemannian or pseudo-Riemannian manifold of dimension 3 or less,*
- *or a "complex Riemannian" manifold (*i.e. *a complex manifold with a complex bilinear — warning: not sesquilinear — symmetric form* g*) of complex dimension 3 or less,*

then the (real) curvature tensor R *of* $(\mathcal{M}, g)$ *is a linear function of* ric.

A proof can be found in [1] pp. 47–49. Consequently a Ricci-parallel manifold of low enough dimension, as required in the above proposition, is locally symmetric. So in the Main Theorem, the weakly indecomposable subfactors of the factor on which ric is nondegenerate are (locally) symmetric as soon as:

(i) they are of dimension two,

(ii) or they are of dimension four and admit a pseudo-Kähler structure J *and* a parakähler structure L. In this case indeed, $LJ = JL$ defines on $\mathcal{M}$ a complex structure with respect to which the complex form $h(.,.) = g(.,.) - ig(., LJ.)$ is $\mathbb{C}$-bilinear, symmetric. Besides, as after Proposition 4 of section 4.2, manifolds of this type are of (real) dimension multiple of 4, dimension 6 is here not concerned.

Then Berger's list — you can find its restriction to the symplectic case, with which we deal here, in [3] pp.267-268 — provides the list of the relevant simply connected symmetric spaces. We give each time the structure of the algebra $\mathfrak{so}(g)^{\mathfrak{h}}$ of the parallel endomorphisms, with the following convention: L denotes a parakähler structure ($L^2 = \mathrm{Id}$), J a (pseudo-)Kähler structure ($J^2 = -\mathrm{Id}$). The spaces are (see [2] p.315):

Space	Dimension	$\mathfrak{so}(g)^{\mathfrak{h}}$	sign(ric)
$SL(2,\mathbb{R})/\mathbb{R}^*$	2	$\mathbb{R}.L$	$(1,1)$
$SU(2)/SO(2)$	2	$\mathbb{R}.J$	$(2,0)$
$SL(2,\mathbb{R})/SO(2)$	2	$\mathbb{R}.J$	$(0,2)$
$SL(2,\mathbb{C})/\mathbb{C}^*$	4	$\mathbb{R}.L + \mathbb{R}.J$	$(2,2)$

Remark. To obtain the full list of the simply connected, simple, symplectic symmetric spaces of dimension 4 or less, one has to add the ones of dimension 4 and with $\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.L$ or $\mathfrak{so}(g)^{\mathfrak{h}} = \mathbb{R}.J$:

Space	Dimension	$\mathfrak{so}(g)^{\mathfrak{h}}$	sign(ric)
$SL(3,\mathbb{R})/(SL(2,\mathbb{R}) \times \mathbb{R}^*)$	4	$\mathbb{R}.L$	$(2,2)$
$SU(3)/(SU(2) \times SO(2))$	4	$\mathbb{R}.J$	$(4,0)$
$SU(1,2)/(SU(1,1) \times SO(2))$	4	$\mathbb{R}.J$	$(2,2)$
$SU(1,2)/(SU(2) \times SO(2))$	4	$\mathbb{R}.J$	$(0,4)$

Thanks. I thank M. Cahen for his quick answer to a technical question and the referee for his careful reading of the manuscript, which has helped me to make it clearer.

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